HIGHER RECURRENCES FOR q-BERNOULLI AND q-EULER NUMBERS

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Abstract In this work we present a very explicit formulas for sums of products of q-Bernoulli and q-Euler numbers of the forms:

$$\sum_{\substack{m_1+\cdots+m_N=n\\ m_1,\cdots,m_N\geq 0}} \binom{n}{m_1,\cdots,m_N} B_{m_1}(q)\cdots B_{m_N}(q),$$

and

$$\sum_{\substack{m_1+\cdots+m_N=n\\ m_1,\cdots,m_N\geq 0}} \binom{n}{m_1,\cdots,m_N} E_{m_1}(q)\cdots E_{m_N}(q)$$

($N, n \geq 1$) respectively, where $B_m(q)$ is the q-Bernoulli numbers and $E_m(q)$ is the q-Euler numbers and $\binom{n}{m_1,\cdots,m_N} = \frac{n!}{m_1!\cdots m_N!}$. Our formulas involves Stirling numbers of first Kind. From these we derived results for q-Bernoulli and q-Euler polynomials. As application when $q = 1$ we recover and complete the results of Dilcher [6]. Our method is different to Dilcher's one.

Contents

1. Introduction and preliminaries

Throught this paper we use the following notation: $\mathbb{N} = \{0, 1, ...\}$ set of naturals numbers.

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1.1. **Euler's identities for Bernoulli numbers: An overview.** The Bernoulli numbers B_n are defined by the generating function

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\ |t|<2\pi. \tag{1}$$

One of the most remarkable identities for Bernoulli numbers is Euler's well-known nonlinear relation

$$\sum_{j=0}^{n}\binom{n}{j}B_jB_{n-j}=-nB_{n-1}-(n-1)B_n\ (n\geq 1). \tag{2}$$

This identity has been generalized and extended in different directions, see [3-17]. First, Eie [7] and Sitaramachandra and Davis [18] considered the sum of the products of three for Bernoulli numbers and proved that

$$\sum_{\substack{j_1+j_2+j_3=n\\ j_1,j_2,j_3\geq 1}}\binom{2n}{2j_1,2j_2,2j_3}B_{2j_1}B_{2j_2}B_{2j_3}= \\ (n+1)(2n+1)B_{2n}+n\left(n-\frac{1}{2}\right)B_{2n-2}, \tag{3}$$

and

$$\sum_{\substack{j_1+j_2+j_3+j_4=n\\ j_1,j_2,j_3,j_4\geq 1}}\binom{2n}{2j_1,2j_2,2j_3,2j_4}B_{2j_1}B_{2j_2}B_{2j_3}B_{2j_4}= \\ -\binom{2n+3}{3}B_{2n}-\frac{4}{3}n^2(2n-1)B_{2n-2}. \tag{4}$$

This was further extented to $N=5$ by Sankaranarayan [16], to $N\geq 7$ by Zhang [19]. Recently, Dilcher [6] and Petojevic and Srivastava [15] considered the sums of the products of N Bernoulli numbers in the form:

$$\sum_{\substack{m_1+\cdots+m_N=n\\ m_1,\cdots,m_N\geq 0}}\binom{2n}{2m_1,\cdots,2m_N}B_{2m_1}\cdots B_{2m_N}, \tag{5}$$

and

$$\sum_{\substack{m_1+\cdots+m_N=n\\ m_1,\cdots,m_N\geq 1}}\binom{2n}{2m_1,\cdots,2m_N}B_{2m_1}\cdots B_{2m_N} \tag{6}$$

and established interesting identities. The sums (5) and (6) are different. The Dilcher's sums (5) include the Bernoulli number B_0, Dilcher [6] p.27 remarked that these some are equivalent if we take into account the slightly different ranges of summation. As an application of our study we will give an easy and complete formula for the general sums in form:

$$\sum_{\substack{m_1+\cdots+m_N=n\\ m_1,\cdots,m_N\geq 0}}\binom{n}{m_1,\cdots,m_N}B_{m_1}\cdots B_{m_N}, \tag{7}$$

here the sums include the Bernoulli numbers B_0 and B_1.

1.2. **q-Bernoulli and q-Euler numbers and polynomials.** Let $q \in \mathbb{R}$, the q-Bernoulli polynomials $B_n(x|q)$ can be defined by the generating function

$$\frac{t}{qe^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x|q)\frac{t^n}{n!}, \ (q = 1, |t| < 2\pi), (q \neq 1, |t| < |log(-q)|). \tag{8}$$

The q-Bernoulli numbers $B_n(q)$ is given by $B_n(q) := B_n(0|q)$.
The q-Euler polynomials $E_n(x|q)$ can be defined by the generating function

$$\frac{2}{qe^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x|q)\frac{t^n}{n!}, \ q \neq -1, |t| < |log(-q)|. \tag{9}$$

The q-Euler numbers $E_n(q)$ are given by $E_n(q) := E_n(0|q)$.
In the litterature, these numbers are usually called "Apostol-Bernoulli" and "Apostol-Euler" numbers (see [2, 3, 4, 11, 13]) respectively. These numbers generalized Bernoulli and Euler numbers and have many interesting properties and numerous important applications in number theory and other areas.

1.3. **Stirling numbers of first kind.** For this subsection we refer to Chapter V of the book [5] written by L. Comtet. The Stirling numbers of the first kind, frequently denoted as $s(n, k)$ or $\left[{n \atop k}\right]$, $\quad k, n \in \mathbb{N}, \quad 1 \leq k \leq n$, are the coefficients in the expansion

$$(x)_n = \sum_{k=0}^{n} s(n, k)x^k,$$

where $(x)_n$ is the falling factorial

$$(x)_n = x(x - 1)(x - 2)\cdots(x - n + 1).$$

We give the following table of first values and recurrence relation for $s(n, k)$:

$$s(1, 1) = 1, s(2, 1) = -1, s(2, 2) = 1, s(3, 1) = 2, s(3, 2) = -3, s(3, 3) = 1,$$
$$s(4, 1) = -6, s(4, 2) = 11, s(4, 3) = -6, s(4, 4) = 1,$$
$$s(5, 1) = 24, s(5, 2) = -50, s(5, 3) = 35, s(5, 4) = -10, s(5, 5) = 1.$$

These numbers satisfy the recurrence formula:

$$s(n + 1, k) = s(n, k - 1) - ns(n, k), \qquad 1 \leq k < n, \tag{10}$$

with the following initial conditions:

$$s(n, 0) = 0, \qquad s(1, 1) = 1. \tag{11}$$

Moreover they have the following generating function:

$$(1 + t)^u = \sum_{n=0}^{\infty}\sum_{k=1}^{n} s(n, k)\frac{t^n}{n!}u^k. \tag{12}$$

The purpose of this paper is to obtain Euler's type explicit formulas corresponding for q-Bernoulli and q-Euler numbers. Our nonlinear relations involve Stirling numbers. We will do this in Section 2. In Section 3 we prove our main results. In section 4 we give corresponding explicits formulas for the sums of N products of q-Bernoulli $B_n(x|q)$ and q-Euler polynomials $E_n(x|q)$.

2. Statement of main results

Let $q \in \mathbb{R}$. We consider the function

$$F_q(t) = \frac{1}{qe^t + 1}.$$

Then we have the following important identity

Theorem 1 (**Fundamental identity).** *For any $N \geq 1$ and $q \in \mathbb{R}$ we have the key identity*

$$(N-1)!F_q^N = \sum_{k=1}^{N} a_k(N) F_q^{(k-1)}, \tag{13}$$

where $a_k(N)$ can be given by two differents ways:

i)

$$a_k(N) = \begin{cases} \frac{N!}{k!} \sum\limits_{\substack{l_1,\cdots,l_k \geq 1 \\ l_1+\cdots+l_k=N}} \frac{1}{l_1 \cdots l_k}, & \text{if } N \geq k \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

ii) *$a_k(N) = (-1)^{N+k} s(N,k)$, $s(N,k)$ is the Stirling number of the first kind. Note that $a_k(N)$ are the unsigned Stirling numbers.*

Remark 1. *Remark that the coefficents $a_k(N)$ are independent to the choice of q.*

Now, we can derive the following interesting results.

Theorem 2. *For $N \geq 1$ and $q \in \mathbb{R}\backslash\{-1\}$, we have*

$$\sum_{\substack{m_1,\cdots,m_N \geq 0 \\ m_1+\cdots+m_N=n}} \binom{n}{m_1,\cdots,m_N} E_{m_1}(q)\cdots E_{m_N}(q) = \tag{15}$$

$$\frac{(-1)^{N-1}2^{N-1}}{(N-1)!} \sum_{k=0}^{N-1} (-1)^k s(N,k+1) E_{k+n}(q). \tag{16}$$

Theorem 3. *For $N \geq 1$ and $q \in \mathbb{R}$, we have*

$$\sum_{\substack{m_1,\cdots,m_N \geq 0 \\ m_1+\cdots+m_N=n}} \binom{n}{m_1,\cdots,m_N} B_{m_1}(q)\cdots B_{m_N}(q) = \tag{17}$$

$$\begin{cases} \frac{s(N,N-n)}{\binom{N-1}{n}} \delta_{1,q} & \text{if} \quad n \leq N-1, \\ n\binom{n-1}{N-1} \sum\limits_{k=1}^{N} (-1)^{k-1} s(N,k) \frac{B_{n-N+k}(q)}{n-N+k} & \text{if} \quad n \geq N, \end{cases}$$

where

$$\delta_{1,q} = \begin{cases} 1 & \text{if } q = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the identities (68) and (15) can be rewritten in terms of polylogarithms functions. Moreover, if we replace $q = 1$ in Theorem 3 we obtain a complete explicit formulation of Dilcher's result [6].

Corollary 4 ([6]). *For $N, n \geq 1$, we have*

$$\sum_{\substack{m_1,\cdots,m_N \geq 0 \\ m_1+\cdots+m_N=n}} \binom{n}{m_1,\cdots,m_N} B_{m_1}\cdots B_{m_N} =$$

$$\begin{cases} \frac{s(N,N-n)}{\binom{N-1}{n}} & \text{if} \quad n \leq N-1, \\ n\binom{n-1}{N-1}\displaystyle\sum_{k=1}^{N}(-1)^{k-1}s(N,k)\frac{B_{n-N+k}}{n-N+k} & \text{if} \quad n \geq N. \end{cases} \tag{18}$$

Remark that Dilcher's results are obtained under the assumption $n \geq N$, see [6] p.32. Our method is fundamentally different from that used by Dilcher.
More general, in section 4 we derive from our Theorem 1 Euler's type identities for q-Bernoulli and q-Euler polynomials. Taking $q = 1$ we obtain an easy and complete polynomial version of Dilcher's results [6] about Euler's type sums for the classical Bernoulli and Euler polynomials.

3. Proofs of main results

Proof of Theorem 1:
It's easy to see that the function $F_q(t) = \frac{1}{qe^t+1}$ satisfies the following differential equation

$$F_q^2 = F_q + F_q'. \tag{19}$$

By the derivative of (19), we have

$$2F_qF_q' = F_q'' + F_q', \ F_q' = F_q^2 - F_q$$

Implies that $2F_q(F_q^2 - F_q) = F_q'' + F_q'$ and $2F_q^3 = F_q^{"} + F_q' + 2F_q^2$. From (19) we get

$$2!F_q^3 = 2F_q + 3F_q' + F_q^{"}. \tag{20}$$

Continuing this process, we can set

$$(N-1)!F_q^N = \sum_{k=1}^{N} a_k(N)F_q^{(k-1)}, \ N \geq 1. \tag{21}$$

Thus, using derivation we get

$$N!F_q^{N-1}F_q' = \sum_{k=1}^{N} a_k(N)F_q^{(k)}. \tag{22}$$

and by equation 19 we have

$$N!F_q^{N-1}F_q' = N!F_q^{N-1}(F_q^2 - F_q) = N!F_q^{N+1} - N!F_q^N. \tag{23}$$

By equations (21) and (24), we get

$$N!F_q^{N+1} = N(N-1)!F_q^N + \sum_{k=1}^{N} a_k(N)F_q^{(k)} \tag{24}$$

$$= N\sum_{k=1}^{N} a_k(N)F_q^{(k-1)} + \sum_{k=1}^{N} a_k(N)F_q^{(k)}. \tag{25}$$

Since (21), we see that

$$N!F_q^{N+1} = \sum_{k=1}^{N+1} a_k(N+1)F_q^{(k)}. \tag{26}$$

By (21) and (24), we get

$$\begin{aligned}\sum_{k=1}^{N+1} a_k(N+1)F_q^{(k)} &= N\sum_{k=1}^{N} a_k(N)F_q^{(k-1)} + \sum_{k=1}^{N} a_k(N)F_q^{(k)} \\ &= Na_1(N)F_q + a_N(N)F_q^N + \sum_{k=2}^{N} \left(Na_k(N) + a_k(N)\right) F_q^{(k)}.\end{aligned}$$

By comparing coefficients on the both sides of the above equation, we see that

$$a_1(N+1) = Na_1(N), \quad a_{N+1}(N+1) = a_N(N+1) = a_2(2) = a_1(1) = 1,$$

then we obtain

(1)

$$a_1(N) = (N-1)!, \quad a_N(N) = 1, \ N \geq 1. \tag{27}$$

(2) $2 \leq k \leq N$, we have

$$a_{k+1}(N+1) = Na_{k+1}(N) + a_k(N). \tag{28}$$

(3) $a_k(N) = 0$ if $k > N, k < 1$.

Let us consider the function for variables t, u as follows:

$$f(t,u) = \sum_{N,k\geq 1} a_k(N)\frac{t^N}{N!}u^{k-1} = \sum_{\substack{N\geq 1\\ 1\leq k\leq N}} a_k(N)\frac{t^N}{N!}u^{k-1}, \ |t| < 1. \tag{29}$$

Then from the recurrence equation (28) we can write

$$\sum_{N,k\geq 1} a_{k+1}(N+1)\frac{t^N}{N!}u^{k-1} = \sum_{N,k\geq 1} Na_{k+1}(N)\frac{t^N}{N!}u^{k-1} + \sum_{N,k\geq 1} a_k(N)\frac{t^N}{N!}u^{k-1} \tag{30}$$

$$= \sum_{N,k\geq 1} Na_{k+1}(N)\frac{t^N}{N!}u^{k-1} + f(t,u). \tag{31}$$

For $\sum_{N,k\geq 1} Na_{k+1}(N)\frac{t^N}{N!}u^{k-1}$, we have

$$\begin{aligned}
\sum_{N,k\geq 1} Na_{k+1}(N)\frac{t^N}{N!}u^{k-1} &= \frac{1}{u}\sum_{N,k\geq 1} Na_{k+1}(N)\frac{t^N}{N!}u^{k} \\
&= \frac{1}{u}\sum_{\substack{N\geq 1\\ 2\leq k\leq N+1}} Na_{k}(N)\frac{t^N}{N!}u^{k-1} \\
&= \frac{1}{u}\sum_{\substack{N\geq 1\\ 2\leq k\leq N+1}} a_{k}(N)\frac{t^N}{(N-1)!}u^{k-1} \\
&= \frac{1}{u}\sum_{N\geq 1}\left(\sum_{1\leq k\leq N+1} a_{k}(N)\frac{t^N}{(N-1)!}u^{k-1} - a_1(N)\frac{t^N}{(N-1)!}\right) \\
&= \frac{1}{u}\left(\sum_{N\geq 1}\sum_{1\leq k\leq N+1} a_{k}(N)\frac{t^N}{(N-1)!}u^{k-1} - \sum_{N\geq 1} t^N\right).
\end{aligned}$$

We obtain

$$\begin{aligned}
\sum_{N,k\geq 1} Na_{k+1}(N)\frac{t^N}{N!}u^{k-1} &= \frac{1}{u}\left(\sum_{N\geq 1}\sum_{1\leq k\leq N+1} a_{k}(N)\frac{t^N}{(N-1)!}u^{k-1} - \frac{t}{1-t}\right) \\
&= \frac{t}{u}\left(f'(t,u) - \frac{1}{1-t}\right). \qquad (32)
\end{aligned}$$

By (30) and (32), we get

$$\sum_{N,k\geq 1} a_{k+1}(N+1)\frac{t^N}{N!}u^{k-1} = f(t,u) + \frac{t}{u}\left(f'(t,u) - \frac{1}{1-t}\right). \qquad (33)$$

For the left sides of (33), we have

$$\begin{aligned}
\sum_{N,k\geq 1} a_{k+1}(N+1)\frac{t^N}{N!}u^{k-1} &= \sum_{\substack{N\geq 1\\ 1\leq k\leq N}} a_{k+1}(N+1)\frac{t^N}{N!}u^{k-1} \qquad (34) \\
&= \sum_{\substack{N\geq 2\\ 1\leq k\leq N-1}} a_{k+1}(N)\frac{t^{N-1}}{(N-1)!}u^{k-1} \\
&= \sum_{\substack{N\geq 2\\ 2\leq k\leq N}} a_{k}(N)\frac{t^{N-1}}{(N-1)!}u^{k-2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{u}\sum_{\substack{N\geq 2\\ 2\leq k\leq N}} a_k(N)\frac{t^{N-1}}{(N-1)!}u^{k-1}\\
&= \frac{1}{u}\sum_{N\geq 2}\left(\sum_{2\leq k\leq N} a_k(N)\frac{t^{N-1}}{(N-1)!}u^{k-1} - a_1(N)\frac{t^{N-1}}{(N-1)!}\right)\\
&= \frac{1}{u}\left(\sum_{N\geq 2}\sum_{2\leq k\leq N} a_k(N)\frac{t^{N-1}}{(N-1)!}u^{k-1} - \frac{t}{1-t}\right)\\
&= \frac{1}{u}\left(\sum_{N\geq 1}\sum_{2\leq k\leq N} a_k(N)\frac{t^{N-1}}{(N-1)!}u^{k-1} - a_1(1) - \frac{t}{1-t}\right)\\
&= \frac{1}{u}\left(\sum_{N\geq 1}\sum_{2\leq k\leq N} a_k(N)\frac{t^{N-1}}{(N-1)!}u^{k-1} - \frac{1}{1-t}\right)\\
&= \frac{1}{u}\left(f'(t,u) - \frac{1}{1-t}\right). \qquad (35)
\end{aligned}
$$

By (33) and (34) we get

$$f(t,u) + \frac{t}{u}\left(f'(t,u) - \frac{1}{1-t}\right) = \frac{1}{u}\left(f'(t,u) - \frac{1}{1-t}\right) \qquad (36)$$

It's implies that

$$f(t,u) + \frac{t-1}{u}f'(t,u) = -\frac{1}{u}. \qquad (37)$$

To solve (37), we first consider the solution of homogeous differential equation:

(1) **Step 1:** The equation

$$f(t,u) + \frac{t-1}{u}f'(t,u) = 0 \qquad (38)$$

is equivalent to

$$f(t,u) = e^{-u\log(1-t)}\lambda(u). \qquad (39)$$

(2) **Step 2:** Variation of constant $\lambda(u) = \lambda(t,u)$.

$$f'(t,u) = \lambda'(t,u)e^{-u\log(1-t)} + \lambda(t,u)e^{-u\log(1-t)}\frac{u}{1-t}. \qquad (40)$$

We multiply by $\frac{t-1}{u}$ on both sides of (39), we see that

$$\frac{t-1}{u}f'(t,u)f(t,u) = \lambda'(t,u)\frac{t-1}{u}e^{-u\log(1-t)} - f(t,u). \qquad (41)$$

From (41) we have

$$\frac{t-1}{u}f'(t,u)f(t,u) + f(t,u) = \lambda'(t,u)\frac{t-1}{u}e^{-u\log(1-t)}. \qquad (42)$$

Therefore, by (37) and (41), we get

$$\lambda'(t,u) = \frac{e^{-u\log(1-t)}}{1-t} = (1-t)^{u-1}, \qquad (43)$$

From the above equality we obtain

$$\lambda(t,u) = -\frac{(1-t)^u}{u} + C(u). \tag{44}$$

By (37) and (44), we see that

$$f(t,u) = e^{-u\log(1-t)}\left[-\frac{(1-t)^u}{u} + C(u)\right]. \tag{45}$$

In (45), let $t=0$. Then

$$0 = f(0,u) = \left[-\frac{1}{u} + C(u)\right], \tag{46}$$

hence $C(u) = \frac{1}{u}$. So

$$f(t,u) = \frac{(1-t)^{-u}-1}{u} = \frac{e^{-u\log(1-t)}-1}{u}. \tag{47}$$

By (47), we get

$$\begin{aligned} f(t,u) &= \frac{e^{-u\log(1-t)}-1}{u} \\ &= \frac{1}{u}\sum_{n\geq 1}\frac{u^n}{n!}\left(-\log(1-t)\right)^n \\ &= \sum_{n\geq 1}\frac{u^{n-1}}{n!}\left(\sum_{l\geq 1}\frac{t^l}{l}\right)^n. \end{aligned} \tag{48}$$

We observe that

$$\left(\sum_{l\geq 1}\frac{t^l}{l}\right)^n = \sum_{N\geq n}\left(\sum_{l_1+\cdots+l_n=N}\frac{1}{l_l}\cdots\frac{1}{l_n}\right)t^N. \tag{49}$$

By (48) and (49), we get

$$f(t,u) = \sum_{n\geq 1}\frac{u^{n-1}}{n!}\left(\sum_{N\geq n}\left(\sum_{l_1+\cdots+l_n=N}\frac{1}{l_l}\cdots\frac{1}{l_n}\right)t^N\right). \tag{50}$$

Hence,w e derive

$$f(t,u) = \sum_{k\geq 1}\frac{u^{k-1}}{k!}\sum_{n\geq k}\left(\sum_{l_1+\cdots+l_k=N}\frac{1}{l_l}\cdots\frac{1}{l_k}\right)t^N. \tag{51}$$

By comparing the coefficients of on the both sides of (29) and (51), we get our desired equality (14).
On other hand, we set $b(k,N) = (-1)^{k+N}a_k(N)$. From the recurrence relation (28), we deduce that

$$\begin{aligned} b(k+1,N+1) &= b(k,N) - Nb(k+1,N),\ 0\leq k\leq N, \\ b(n,n) &= 1,\ n\geq 0,\ b(k,N)=0,\ k>N. \end{aligned} \tag{52}$$

Recall that The Stirling numbers of the first kind satisfy the same recurrence relation (10) and (11). Then

$$s(N+1,k+1) = s(N,k-1) - Ns(N,k), \qquad 0\leq k\leq N, \tag{53}$$

with the following initial conditions:

$$s(N,0)=0, N\geq 1, \qquad s(1,1)=1. \tag{54}$$

Therefore, for any N,k intgers, we have

$$b(k,N)=s(N,k).$$

This yields the formula $a_k(N)=(-1)^{k+N}s(N,k)$. Thus completes the proof of our Theorem 1. □

Proof of Theorem 2:
From the Theorem 1, we obtain

$$(N-1)!(2F_q)^N = 2^{N-1}\sum_{k=1}^{N} a_k(N)(2F_q)^{(k-1)},\ N\geq 1,$$

and by the definition (9) we get

$$(N-1)!\sum_{n\geq 0}\left(\sum_{\substack{m_1,\cdots,m_N\geq 0\\ m_1+\cdots+m_N=n}}\binom{n}{m_1,\cdots,m_N}E_{m_1}(q)\cdots E_{m_N}(q)\right)t^n =$$
$$2^{N-1}\sum_{n\geq 0}\left(\frac{(-1)^{N-1}}{(N-1)!}\sum_{k=0}^{N-1}(-1)^k s(N,k+1)E_{k+n}(q)\right)t^n. \tag{55}$$

This yields our theorem. □
Proof of Theorem 3:
Writting

$$F_B(t)=\frac{t}{qe^t-1}=-tF_{-q}(t),\ \text{where } F_q(t)=\frac{1}{qe^t-1}. \tag{56}$$

Then by using the equation (13), we get

$$(N-1)!F_B^N(t)=(-1)^{N-1}\sum_{k=1}^{N}a_k(N)\left(-F_{-q}\right)^{(k-1)}(t)t^N. \tag{57}$$

By (8), we have

$$-F_{-q}(t)=\frac{B_0(q)}{t}+\sum_{n=0}^{\infty}\frac{B_{n+1}(q)}{(n+1)!}t^n, \tag{58}$$

where $B_0(q)=\delta_{1,q}$.

$$\left(-F_{-q}\right)^{(k-1)}(t)=\frac{B_0(q)(-1)^{k-1}(k-1)!}{t^k}+\sum_{n=0}^{\infty}\frac{B_{n+k}(q)}{n!(n+k)}t^n. \tag{59}$$

Hence

$$\left(-F_{-q}\right)^{(k-1)}(t)t^N=B_0(q)(-1)^{k-1}(k-1)!t^{N-k}+\sum_{n=0}^{\infty}\frac{B_{n+k}(q)}{n!(n+k)}t^{N+n}, \tag{60}$$

Now, using relations (57), (58), (59) and (61) we obtain the equation

$$(N-1)F_B^N(t) = B_0(q)\sum_{n=0}^{N-1} a_{N-n}(N)(-1)^n(N-n-1)!t^n + \sum_{n\geq N}\left((-1)^{N-1}\sum_{k=1}^{N}\frac{a_k(N)B_{n-N+k}}{(n-N)!(n-N+k)}\right)t^n. \tag{61}$$

This give us our theorem. □

4. q-Bernoulli and q-Euler polynomials

Let $x_1,\cdots,x_N$ variables and $y = x_1+\cdots+x_N$.

Theorem 5. *For $N \geq 1$ and $q \in \mathbb{R}\backslash\{-1\}$. Then we have*

$$\sum_{\substack{m_1,\cdots,m_N\geq 0\\ m_1+\cdots+m_N=n}}\binom{n}{m_1,\cdots,m_N}E_{m_1}(x_1|q)\cdots E_{m_N}((x_N|q) = \tag{62}$$
$$\frac{(-1)^{N-1}2^{N-1}}{(N-1)!}\sum_{k=0}^{N-1}(-1)^k s(N,k+1)\left(\sum_{j=0}^{k}\binom{k}{j}(-1)^j y^j E_{k-j+n}(y|q)\right).$$

Proof of Theorem 5:
Put

$$F_E(t,x_i) = \frac{2}{qe^t+1}e^{x_i t} = 2F_q(t)e^{x_i t}.$$

Multiplying by $2^N e^{x_1 t}\cdots e^{x_N t}$ both sides of the fondamental identity (13) we obtain

$$\begin{aligned} &(N-1)!F_E(t,x_1)\cdots F_E(t,x_N) \qquad (63)\\ =\;& 2^{N-1}\sum_{k=1}^{N}a_k(N)\left(2F_q(t)\right)^{(k-1)}e^{yt}\\ =\;& 2^{N-1}\sum_{k=1}^{N}a_k(N)\left(F_E(t,y)e^{-yt}\right)^{(k-1)}e^{yt}\\ =\;& 2^{N-1}\sum_{k=1}^{N}a_k(N)\sum_{j=0}^{k-1}\binom{k-1}{j}F_E^{(k-1-j)}(t,y)(-1)^j y^j e^{-yt}e^{yt}\\ =\;& 2^{N-1}\sum_{k=1}^{N}a_k(N)\sum_{j=0}^{k-1}\binom{k-1}{j}F_E^{(k-1-j)}(t,y)(-1)^j y^j\\ =\;& 2^{N-1}\sum_{k=1}^{N}a_k(N)\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j y^j F_E^{(k-1-j)}(t,y). \end{aligned}$$

On other hand, we have

$$F_E^{(k-1-j)}(t,y) = \sum_{n\geq 0}\frac{E_{k-j-1+n}(y|q)}{n!}t^n. \tag{64}$$

By substituting the formula (64) in the equation (63), we obtain

$$\begin{aligned}
& F_E(t,x_1)\cdots F_E(t,x_N) \qquad (65)\\
= & \frac{2^{N-1}}{(N-1)!}\sum_{n\geq 0}\left(\sum_{k=1}^{N} a_k(N)\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j y^j \frac{E_{k-j-1+n}(y|q)}{n!}\right)t^n\\
= & \frac{2^{N-1}}{(N-1)!}\sum_{n\geq 0}\left(\sum_{k=0}^{N-1} a_{k+1}(N)\sum_{j=0}^{k}\binom{k}{j}(-1)^j y^j \frac{E_{k-j+n}(y|q)}{n!}\right)t^n\\
= & \frac{2^{N-1}}{(N-1)!}\sum_{n\geq 0}\left(\sum_{k=0}^{N-1} (-1)^{k+1+N}s(N,k+1)\sum_{j=0}^{k}\binom{k}{j}(-1)^j y^j \frac{E_{k-j+n}(y|q)}{n!}\right)t^n.
\end{aligned}$$

Indeed, from the formula

$$F_E(t,y)=\sum_{n\geq 0}\frac{E_n(y|q)}{n!}t^n, \qquad (66)$$

we get

$$\begin{aligned}
& F_E(t,x_1)\cdots F_E(t,x_N)\\
= & \sum_{n\geq 0}\left(\sum_{\substack{m_1,\cdots,m_N\geq 0\\ m_1+\cdots+m_N=n}}\frac{E_{m_1}(x_1|q)}{m_1!}\cdots\frac{E_{m_N}(x_N|q)}{m_N!}\right)t^n. \qquad (67)
\end{aligned}$$

Then by comparing the coefficients of t^n, in the right of sides in the equations (65) and (67) we get our theorem. □

Theorem 6. *For $N\geq 1$ and $q\in\mathbb{R}$. Let $y=x_1+\cdots+x_N$. Then we have*

$$\sum_{\substack{m_1,\cdots,m_N\geq 0\\ m_1+\cdots+m_N=n}}\binom{n}{m_1,\cdots,m_N}B_{m_1}(x_1|q)\cdots B_{m_N}(x_N|q)= \qquad (68)$$

$$\begin{cases}
\delta_{1,q}\frac{n!}{(N-1)!}\sum_{k=N-n}^{N}s(N,k)\frac{(k-1)!}{(n-N+k)!}y^{n-N+k} & \text{if } n\leq N-1,\\[2ex]
\delta_{1,q}\frac{n!}{(N-1)!}\sum_{k=1}^{N}s(N,k)\frac{(k-1)!}{(n-N+k)!}y^{n-N+k} \quad + & \\
\frac{n!}{(N-1)!}\sum_{k=1}^{N}(-1)^{k-1}s(N,k)\sum_{m=0}^{n-N}\frac{B_{m+k}(q)y^{n-m-N}}{m!(n-m-N)!(m+k)} & \text{if } n\geq N,
\end{cases}$$

Proof of Theorem 6:
Set

$$F_B(t,x_i)=\frac{t}{qe^t-1}e^{x_i t}=-tF_{-q}(t)e^{x_i t}.$$

Multiplying by $(-t)^N e^{x_1 t}\cdots e^{x_N t}$ both sides of the fondamental identity (13) we obtain

$$\begin{aligned}
& (N-1)!F_B(t,x_1)\cdots F_B(t,x_N) \\
= \quad & (-1)^{N-1}\sum_{k=1}^{N} a_k(N)\left(-F_{-q}(t)\right)^{(k-1)} t^N e^{yt} \\
= \quad & (-1)^{N-1}\sum_{k=1}^{N} a_k(N)\left(\sum_{n\geq 0}\frac{B_n(q)}{n!}t^{n-1}\right)^{(k-1)} t^N e^{yt} \\
= \quad & (-1)^{N-1}\sum_{k=1}^{N} a_k(N)\left(\frac{B_0(q)}{t}+\sum_{n\geq 0}\frac{B_{n+1}(q)}{(n+1)!}t^{n}\right)^{(k-1)} t^N e^{yt} \\
= \quad & (-1)^{N-1}\sum_{k=1}^{N} a_k(N)\left(\frac{B_0(q)(-1)^{k-1}(k-1)!}{t^k}+\sum_{n\geq k-1}\frac{B_{n+1}(q)}{(n+1)!}\frac{n!}{(n-k+1)!}t^{n-k+1}\right) t^N e^{yt} \\
= \quad & (-1)^{N-1}\sum_{k=1}^{N} a_k(N)\left(B_0(q)(-1)^{k-1}(k-1)!t^{N-k}+\sum_{n\geq k-1}\frac{B_{n+1}(q)}{(n+1)!}\frac{n!}{(n-k+1)!}t^{N+n-k+1}\right) e^{yt}
\end{aligned}$$

Then we can write

$$\begin{aligned}
& (N-1)!F_B(t,x_1)\cdots F_B(t,x_N) \\
= \quad & (-1)^{N-1}\sum_{k=1}^{N} a_k(N)\left(B_0(q)(-1)^{k-1}(k-1)!t^{N-k}+\sum_{n\geq 0}\frac{B_{n+k}(q)}{(n+k)!}\frac{(n+k-1)!}{n!}t^{N+n}\right) e^{yt} \\
= \quad & (-1)^{N-1}\sum_{k=1}^{N} a_k(N)\left(B_0(q)(-1)^{k-1}(k-1)!t^{N-k}+\sum_{n\geq 0}\frac{B_{n+k}(q)}{n!(n+k)}t^{N+n}\right) e^{yt} \\
= \quad & (-1)^{N-1}B_0(q)\sum_{k=1}^{N} a_k(N)(-1)^{k-1}(k-1)!t^{N-k}e^{yt}+(-1)^{N-1}\sum_{k=1}^{N} a_k(N)\left(\sum_{n\geq 0}\frac{B_{n+k}(q)}{n!(n+k)}t^{N+n}e^{yt}\right).
\end{aligned}$$

On other hand, we have

$$\begin{aligned}
t^{N-k}e^{yt} &= \sum_{n\geq 0}\frac{y^n}{n!}t^{n+N-k} \\
&= \sum_{n\geq N-k}\frac{y^{n-N+k}}{(n-N+k)!}t^n \qquad (69)
\end{aligned}$$

$$\begin{aligned}
\sum_{m\geq 0}\frac{B_{m+k}(q)}{m!(m+k)}t^{N+m}e^{yt} &= \sum_{m\geq 0}\frac{B_{m+k}(q)}{m!(m+k)}\sum_{l\geq 0}\frac{y^l}{l!}t^{l+m+N} \\
&= \sum_{m\geq 0}\frac{B_{m+k}(q)}{m!(m+k)}\sum_{n\geq m+N}\frac{y^{n-m-N}}{(n-m-N)!}t^n \\
&= \sum_{n\geq N}\left(\sum_{m=0}^{n-N}\frac{B_{m+k}(q)}{m!(m+k)}\frac{y^{n-m-N}}{(n-m-N)!}\right)t^n. \qquad (70)
\end{aligned}$$

Finally from (69) and (70), we obtain the following formula

$$\begin{aligned}
& (N-1)!F_B(t,x_1)\cdots F_B(t,x_N) \qquad (71)\\
= & \sum_{n=0}^{N-1}\left((-1)^{N-1}B_0(q)\sum_{k=N-n}^{N}a_k(N)\frac{(-1)^{k-1}(k-1)!y^{n-N+k}}{(n-N+k)!}\right)t^n\\
& +\sum_{n\geq N}\left((-1)^{N-1}B_0(q)\sum_{k=1}^{N}a_k(N)\frac{(-1)^{k-1}(k-1)!y^{n-N+k}}{(n-N+k)!}\right)t^n\\
& +\sum_{n\geq N}\left((-1)^{N-1}\sum_{k=1}^{N}a_k(N)\sum_{m=0}^{N-n}\frac{B_{m+k}(q)y^{n-m-N}}{m!(n-m-N)!(m+k)}\right)t^n
\end{aligned}$$

The equalities (69),(69), (70) and (71) give us our theorem.□
One can get an easy complete version of Dilcher's results [6] by replacing $q = 1$ in the above theorems.

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