ARITHMETIC PROPERTIES OF q-BARNES POLYNOMIALS

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Abstract In this paper, we introduce and investigate the q-analogues of Barnes numbers and polynomials. The main purpose of this paper is to establish Fourier expansion of these q-Barnes polynomials and from this study we connect q-Barnes numbers to values of Dirichlet-Hurwitz L-function evaluating at non-negative positive integers.

1. Introduction and preliminaries

Throughout this paper we use the following notation: $\mathbb{N} = \{0, 1, ...\}$ set of naturals numbers. Let $q \in \mathbb{R}$ and x a variable, the q-Bernoulli polynomials $B_n(x;q)$ are defined by the generating function

$$\frac{t}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x;q) \frac{t^n}{n!}, \ (q = 1, |t| < 2\pi), (q \neq 1, |t| < |log(-q)|). \tag{1}$$

The q-Bernoulli numbers $B_n(q)$ are given by $B_n(q) := B_n(0;q)$. These polynomials were introduced by Apostol, see [1, 13] . These polynomials are a natural extension of the classical Bernoulli polynomials : $B_n(x) = B_n(x;1)$, see [12] . They have many applications in mathematics. Recently, first author proves their most important property Fourier expansion which is given by

$$q^x B_n(x;q) = \frac{-n!}{(2\pi i)^n} \sum_{k\in\mathbb{Z}}^{*} \frac{e^{2\pi i k x}}{\left(k - \frac{log(q)}{2\pi i}\right)^n}, \tag{2}$$

for $q \in \mathbb{C}\backslash\{0\}$ and , for $0 < x < 1$ if $n = 1$, $0 \leq x \leq 1$ if $n \geq 2$. Here $\sum_{k\in\mathbb{Z}}^{*} = \sum_{k\in\mathbb{Z}\backslash\{0\}}$ if $q = 1$ and $\sum_{k\in\mathbb{Z}}^{*} = \sum_{k\in\mathbb{Z}}$ if $q \neq 1$. See [2, 3, 13]. This identity is the foundation of the theory of q-Bernoulli polynomials and their relations to special values of the Riemann zeta function and Dirichlet L-functions.

Let us define the Barnes and the q-Barnes polynomials and numbers. Let N positive integer and $\overrightarrow{a}_N = (a_1, ..., a_N)$, where $a_1, ..., a_N$ are complex with strictly

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2010 Mathematics Subject Classification : 11B68, 11B83, 11B99, 11M32 .

Key words and phrases : Bernoulli numbers and polynomials, q-Bernoulli numbers and polynomials, Barnes numbers and polynomials, q-Barnes numbers and polynomials, Dirichlet-Hurwitz L-function.

positive real part. The Barnes polynomials and numbers are given by

$$\frac{t^N}{\prod_{j=1}^{N}\left(e^{a_j t}-1\right)}e^{xt}=\sum_{n=0}^{\infty}B_n(x|\overrightarrow{a}_N)\frac{t^n}{n!},\ |t|<\min\left(\frac{2\pi}{a_1},\cdots,\frac{2\pi}{a_N}\right),\ \text{see [5, 6, 8, 9, 14, 15, 17]}\ .$$

The main interest of these numbers is that they give the values at non negative integers of Dirichlet L-series: if $L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}$ $(\mathrm{Re}(s)>1)$ is the L-series attached to χ of conductor f, then we have the formula in [4]

$$\begin{aligned}&\sum_{m_1,\ldots,m_N}^{*} a_1^{m_1-1}...a_N^{m_N-1}L(m_1,\chi)...L(m_N,\chi)=\\&\frac{(2\pi i)^n(-1)^N}{n!}\frac{\chi(-1)G_\chi}{f}\sum_{t=1}^{f}\bar{\chi}(t)B_n\left(\frac{t}{f}\mid\overrightarrow{a}_N\right),\end{aligned}\tag{3}$$

where $\sum\limits_{m_1,\ldots,m_N}^{*}=\sum\limits_{\substack{m_1+..+m_N=n,m_1,\ldots,m_N\geq 0\\(-1)^{m_1}=\ldots=(-1)^{m_N}=\chi(-1)}}$.

Note that in case $N=1,a_1=1$, the numbers in the right side of the equality (13) correspond to the *generalized Bernoulli numbers* $B_{m,\chi}$ which are defined by the generating function

$$\sum_{a=1}^{f}\chi(a)\frac{t}{e^{ft}-1}e^{at}=\sum_{n=0}^{\infty}B_{n,\chi}\frac{t^n}{n!},\ \mid t\mid<\frac{2\pi}{f}.\tag{4}$$

From the equation (3) we have , for $n\geq 0$, the well-known formula

$$L(-n,\chi)=-\frac{B_{n+1,\chi}}{n+1},\ \text{see [4, 18]}\ .\tag{5}$$

Let N positive integer , $\overrightarrow{a}_N=(a_1,...,a_N)$, where $a_1,...,a_N$ are complex with strictly positive real part and let $q\in\mathbb{C}$, $\mid q\mid<1$. We introduce and investigate the following q-Barnes polynomials $B_{n,q}(x|\overrightarrow{a}_N)$ defined by

$$\frac{t^N}{\prod_{j=1}^{N}\left(qe^{a_j t}-1\right)}e^{xt}=\sum_{n=0}^{\infty}B_{n,q}(x|\overrightarrow{a}_N)\frac{t^n}{n!},\ |t+log(q)|<\min\left(\frac{2\pi}{a_1},\cdots,\frac{2\pi}{a_N}\right).\tag{6}$$

and $B_{n,q}(\overrightarrow{a}_N)=B_{n,q}(0|\overrightarrow{a}_N)$ are the so called q-Barnes numbers.
This paper can now be summarized as a generalization of these facts to the q-Barnes polynomials and numbers. More precisely, the main purpose of this paper is to prove the extension of the properties (2) and (3) to q-Barnes numbers and polynomials.

2. Statement and proof of main results

For $\lambda \in \mathbb{C}\backslash\{0\}$ and $q \in \mathbb{C}$, we can write

$$\frac{t^N}{\prod_{j=1}^{N}\left(qe^{\lambda a_j t}-1\right)}e^{\lambda xt} = \lambda^{-N}\frac{(\lambda t)^N}{\prod_{j=1}^{N}\left(qe^{a_j(\lambda t)}-1\right)}e^{x(\lambda t)}. \tag{7}$$

Their Taylor expansions are given as follows

$$\sum_{n=0}^{\infty} B_{n,q}(\lambda x|\lambda \overrightarrow{a}_N)\frac{t^n}{n!} = \lambda^{-N}\sum_{n=0}^{\infty} B_{n,q}(x|\overrightarrow{a}_N)\frac{\lambda^n t^n}{n!}. \tag{8}$$

Then, by comparing the coefficients of both sides of the equation (8), we obtain the homogeneity equation

Proposition 1 (Homogeneity). *For any $a_1, ..., a_N$ are complex with strictly positive real part and $\lambda \in \mathbb{C}\backslash\{0\}$, we have*

$$B_{n,q}(\lambda x \mid \lambda \overrightarrow{a}_N) = \lambda^{n-N}B_{n,q}(x \mid \overrightarrow{a}_N),\ (n \geq 1). \tag{9}$$

Now we state our main results.

Theorem 2. *Let $a_1, ...a_N$ are complex with strictly positive real part. Then*

$$\frac{B_{n,q}(x \mid \overrightarrow{a}_N)}{n!} = \sum_{\substack{m_1+..+m_N=n \\ m_1,\cdots,m_N\geq 0}} a_1^{m_1-1}...a_N^{m_N-1}\frac{B_{m_1}(X;q)}{m_1!}...\frac{B_{m_N}(X;q)}{m_N!}. \tag{10}$$

where $X = \frac{x}{A_N}$ and $A_N = a_1 + ... + a_N$.

Proof of Theorem 2:

Writing $X = \frac{x}{A_N}$ where $A_N = a_1 + ... + a_N$. We have

$$\frac{t^N}{\prod_{j=1}^{N}\left(qe^{a_j t}-1\right)}e^{xt} = \frac{1}{a_1...a_N}\prod_{i=1}^{N}\frac{a_i te^{X(a_i t)}}{qe^{a_i t}-1}$$

Then we get

$$\frac{t^N}{\prod_{j=1}^{N}\left(qe^{a_j t}-1\right)}e^{xt} =$$

$$\sum_{n=0}^{\infty}\left(\sum_{m_1+\cdots+m_N=n} a_1^{m_1-1}\cdots a_N^{m_N-1}\frac{B_{m_1}(X;q)}{m_1!}\cdots\frac{B_{m_N}(X;q)}{m_N!}\right)t^n. \tag{11}$$

In other way

$$\frac{t^N}{\prod_{j=1}^{N}\left(qe^{a_j t}-1\right)}e^{xt} = \sum_{n=0}^{\infty}\frac{B_{n,q}}{n!}(x|\overrightarrow{a}_N)t^n. \tag{12}$$

By comparing the right sides of the equations (11) and (11) we obtain

$$\frac{B_{n,q}(x \mid \overrightarrow{a}_N)}{n!} = \sum_{m_1+..+m_N=n} a_1^{m_1-1}...a_N^{m_N-1}\frac{B_{m_1}(X;q)}{m_1!}...\frac{B_{m_N}(X;q)}{m_N!},\ (n \in \mathbb{N}).$$

This yields our theorem. □

Theorem 3 (Fourier expansion)**.** *Let $a_1, ..., a_N$ are complex with strictly positive real part and set $A_N = a_1 + ... + a_N$ and $X = \frac{x}{A_N}$. Then for any $n \geq 1$ and $|X| < 1$ we have*

$$q^x B_{n,q}(x \mid \overrightarrow{a}_N) =$$

$$\frac{(-1)^N n!}{(2\pi i)^n} \sideset{}{^*}\sum_{m_1+..+m_N=n} a_1^{m_1-1}...a_N^{m_N-1} \sideset{}{^{**}}\sum_{k_1,...,k_N\in\mathbb{Z}} \frac{e\left((k_1+...+k_N)X\right)}{\left(k_1-\frac{log(q)}{2\pi i}\right)^{m_1}...\left(k_N-\frac{log(q)}{2\pi i}\right)^{m_N}}.$$

Here $\sideset{}{^{**}}\sum_{k_1,...,k_N\in\mathbb{Z}}$ *means that $k_1, ..., k_N \in \mathbb{Z}\backslash\{\frac{log(q)}{2\pi i}\}$ and, in the non-absolutely convergent case $m_i = 1$, for any $1 \leq i \leq N$, the sum* $\sideset{}{^{**}}\sum_{k_i\in\mathbb{Z}}$ *to be interpreted as a Cauchy principal value for each i, and* $\sideset{}{^*}\sum_{m_1+..+m_N=n}$ *means that $m_1, ..., m_N \in \mathbb{N}$ with the usual convention the sum* $\sideset{}{'}\sum_{k_i\in\mathbb{Z}\backslash\{0\}} e\left(k_iX\right) = -1$ *if $m_i = 0$.*

Proof of Theorem 3:

Using the equation (2) and Theorem 2, we can write

$$q^x \tfrac{B_{n,q}(x|\overrightarrow{a}_N)}{n!}$$

$$= \sideset{}{^*}\sum_{m_1+\cdots+m_N=n} \frac{a_1^{m_1-1}\cdots a_N^{m_N-1}}{m_1!\cdots m_N!}\frac{(-1)^N m_1!\cdots m_N!}{(2\pi i)^{m_1+\cdots+m_N}} \sideset{}{^{**}}\sum_{k_1,\cdots,k_N\in\mathbb{Z}} \frac{e\left((k_1+\cdots+k_N)X\right)}{\left(k_1-\frac{log(q)}{2\pi i}\right)^{m_1}\cdots\left(k_N-\frac{log(q)}{2\pi i}\right)^{m_N}},$$

$$= \frac{(-1)^N}{(2\pi i)^n} \sideset{}{^*}\sum_{m_1+\cdots+m_N=n} a_1^{m_1-1}\cdots a_N^{m_N-1} \sideset{}{^{**}}\sum_{k_1,\cdots,k_N\in\mathbb{Z}} \frac{e\left((k_1+\cdots+k_N)X\right)}{\left(k_1-\frac{log(q)}{2\pi i}\right)^{m_1}\cdots\left(k_N-\frac{log(q)}{2\pi i}\right)^{m_N}}.$$

This gives the theorem. □

Let f an integer ≥ 2 and χ a Dirichlet character modulo f. As usual we define the L-series by

$$L(s,x,\chi) = \sum_{k\in\mathbb{Z},k\neq -x} \frac{\chi(k)}{(x+k)^s},\ \Re(s) > 1.$$

In this L-series we relax the summation over all $k \in \mathbb{Z}, k \neq -x$. But it's easy to see that is related to the classical Dirichlet L-series where the summation is over

$k \in \mathbb{N}, k \neq -x$.
We recall the definition of the Gauss sum associated to the character χ is

$$G_\chi = \sum_{t=1}^{f} \chi(t) e(\frac{t}{f}).$$

By homogeneity Proposition 1, without loss of generality we can assume for the following theorems that: $a_1 + \cdots + a_N = 1$.

Theorem 4 (Values of L-function at non-negative integers)**.** *Let $f \geq 2$ be a natural number, $a_1, ...a_N$ with real part strictly positive real and $a_1 + ... + a_N = 1$, χ a non trivial Dirichlet character modulo $f \geq 2$. Then we have*

$$\sum_{\substack{m_1+...+m_N=n \\ m_1,...,m_N \geq 0}} a_1^{m_1-1} ... a_N^{m_N-1} L(m_1, \alpha, \chi) ... L(m_N, \alpha, \chi) =$$

$$\frac{(2\pi i)^n (-1)^N}{n!} \frac{\chi(-1) G_\chi}{f} \sum_{t=1}^{f} \bar{\chi}(t) q^{\frac{t}{f}} B_{n,q} \left(\frac{t}{f} \mid \vec{a}_N \right), \tag{13}$$

where $\alpha = \frac{log(q)}{2\pi i}$, with the usual convention $L(0, \chi) = \frac{-1}{2}$.

Proof of Theorem 4:

Using Theorem 3, we have

$$\sum_{t=1}^{f} \bar{\chi}(t) q^{\frac{t}{f}} B_n \left(\frac{t}{f} | \vec{a}_N \right)$$

$$= \frac{(-1)^N n!}{(2\pi i)^n} \sum^{*}_{m_1+..+m_N=n} a_1^{m_1-1} ... a_N^{m_N-1} \sum^{**}_{k_1,...,k_N \in \mathbb{Z}} \frac{1}{\left(k_1 - \frac{log(q)}{2\pi i}\right)^{m_1} ... \left(k_N - \frac{log(q)}{2\pi i}\right)^{m_N}}$$
$$\sum_{t=1}^{f} \bar{\chi}(t) e\left((k_1 + ... + k_N) \frac{t}{f} \right) .$$

Since

$$\sum_{t=1}^{f} \bar{\chi}(t) e\left(k \frac{t}{f} \right) = \chi(k) G_{\bar{\chi}}$$

we have

$$\sum_{t=1}^{f} \bar{\chi}(t) q^{\frac{t}{f}} B_n \left(\frac{t}{f} | \vec{a}_N \right)$$

$$= \frac{(-1)^N n!}{(2\pi i)^n} G_{\bar{\chi}} \sum^{*}_{m_1+..+m_N=n} a_1^{m_1-1} ... a_N^{m_N-1} \sum^{**}_{k_1,...,k_N \in \mathbb{Z}} \frac{\chi(k_1 + ... + k_N)}{\left(k_1 - \frac{log(q)}{2\pi i}\right)^{m_1} ... \left(k_N - \frac{log(q)}{2\pi i}\right)^{m_N}}.$$

$$= \frac{(-1)^N n!}{(2\pi i)^n} G_{\bar{\chi}} \sum^{*}_{m_1+..+m_N=n} a_1^{m_1-1} ... a_N^{m_N-1} \sum^{'}_{k_1,...,k_N \in \mathbb{Z}\backslash\{0\}} \frac{\chi(k_1)}{\left(k_1 - \frac{log(q)}{2\pi i}\right)^{m_1}} \cdots \frac{\chi(k_N)}{\left(k_N - \frac{log(q)}{2\pi i}\right)^{m_N}}$$

While

$$\sum_{k\in\mathbb{Z}\setminus\{0\}}' \frac{\chi(k_i)}{\left(k_1-\frac{log(q)}{2\pi i}\right)^{m_i}} = L(m_i,\alpha,\chi), \qquad \text{where } \alpha = \frac{log(q)}{2\pi i}.$$

Therefore, we arrive at

$$\sum_{t=1}^{f} \bar{\chi}(t) q^{\frac{t}{f}} B_n\left(\frac{t}{f}|\overrightarrow{a}_N\right) = \frac{(-1)^N(n!)}{(2\pi i)^n}\, G_{\bar{\chi}} \sum_{\substack{m_1+..+m_N=n\\ m_1,\dots,m_N\geq 0}} a_1^{m_1-1}\dots a_N^{m_N-1}\prod_{i=1}^{N} L(m_i,\alpha,\chi)$$

$$= \tfrac{2^N(-1)^N(n!)}{(2\pi i)^n}\, G_{\bar{\chi}} \sum_{\substack{m_1+..+m_N=n\\ \chi(-1)=(-1)^{m_i},m_1,\dots,m_N\geq 0}} a_1^{m_1-1}\dots a_N^{m_N-1} L(m_1,\chi)\dots L(m_N,\chi).$$

Using the relation

$$G_{\bar{\chi}} = \chi(-1)q/G_{\chi},$$

see [18] chap.4 p.29-37), we obtain the following formula

$$\sum_{\substack{m_1+\dots+m_N=n\\ m_1,\dots,m_N\geq 0}} a_1^{m_1-1}\dots a_N^{m_N-1} L(m_1,\alpha,\chi)\dots L(m_N,\alpha,\chi) =$$

$$\tfrac{(2\pi i)^n(-1)^N}{n!}\, \tfrac{\chi(-1)G_\chi}{f} \sum_{t=1}^{f} \bar{\chi}(t) q^{\frac{t}{f}} B_{n,q}\left(\frac{t}{f}\mid \overrightarrow{a}_N\right).$$

Hence, we obtain our desired theorem. □

Remark 1.

(1) *Taking $q=1$ we recover the main results of Bayad and Kim in* [4].
(2) *If we take $N=1$ and $q=1$ we obtain results of Bayad* [2, 3].
(3) *In case $N=1$, $q=a_1=1$ the Theorem 4 and functional equation of L-series gives us the main property on the values of Dirchlet L-series (5).*

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