RESULTS ON VALUES OF BARNES POLYNOMIALS

A. BAYAD AND T. KIM

Abstract In this paper, we prove rationality and Von Staudt type result for the denominators of Barnes numbers. In addition, we investigate Fourier expansion of Barnes polynomials and from this study we connect generalized Barnes numbers to values of Dirichlet L-function at non-negative positive integers.

1. Introduction and statement of results

Throughout this paper we use the following notation: $\mathbb{N} = \{0, 1, ...\}$ set of naturals numbers. Bernoulli numbers and polynomials are connected to many areas in mathematics and physics. Their most important properties are: Bernoulli polynomials have Fourier series, Bernoulli numbers are rationals numbers, their denominators are controlled by Von Staudt-Kummer congruences, are values of Riemann zeta function at positive integers, the values of L-Dirichlet functions at negative integers give us the generalized Bernoulli numbers. In this paper, we prove the extension of these properties to Barnes numbers and polynomials. Let us recall the definitions of Bernoulli and Barnes polynomials and numbers. The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad , \ |t| < 2\pi. \tag{1}$$

$B_n := B_n(0)$ is the n-th Bernoulli number. Let $\{x\}$ be the fractional part of the real number x. Then the Bernoulli functions $\bar{B}_n(x)$ are defined by

$$\bar{B}_n(x) = \begin{cases} 0, & \text{if } n = 1, x \in \mathbb{Z}, \\ B_n(\{x\}) & \text{otherwise.} \end{cases} \tag{2}$$

For more informations about properties of Bernoulli numbers and their generalization, see [1, 4, 8, 9, 11, 14] Let N positive integer and $a_1, ..., a_N$ complex with positive real part. The Barnes polynomials and numbers are given by

$$\frac{t^N}{\prod_{j=1}^{N}\left(e^{a_j t} - 1\right)} e^{xt} = \sum_{n=0}^{\infty} B_n(x|a_1, ..., a_N)\frac{t^n}{n!}, \ |t| < \min\left(\frac{2\pi}{a_1}, \cdots, \frac{2\pi}{a_N}\right).(\text{ See } [2, 3, 5, 7, 12, 13, 15]\ ) \tag{3}$$

Writing

$$e(x) = e^{2\pi i x}, \ x \in \mathbb{C}.$$

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The essential series is

$$2\pi i \frac{e(uv)}{e(u)-1} = \sum_{n\in\mathbb{Z}} \frac{e(nv)}{u-n} \tag{4}$$

As Kronecker emphasized [10, 17], this identity is the foundation of the theory of classical Bernoulli functions and their relation to special values of the Riemann zeta function and Dirichlet L-functions. More precisely, let χ be a Dirichlet character with conductor $f = f_\chi$. The *generalized Bernoulli numbers* $B_{m,\chi} \in \mathbb{Q}\big(\chi(1), \chi(2), ...\big)$ associated to χ $(m = 0, 1, \ldots)$ are defined by the generating function

$$\sum_{a=1}^{f} \chi(a) \frac{t}{e^{ft}-1} e^{at} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}, \ \mid t \mid < \frac{2\pi}{f}. \tag{5}$$

The main interest of these numbers is that they give the values at negative integers of Dirichlet L-series: if $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ $(\mathrm{Re}(s) > 1)$ is the L-series attached to χ, then we have the formula

$$L(-n,\chi) = -\frac{B_{n+1,\chi}}{n+1} \qquad (n \geq 0).(\text{ see [16] Theorem 4.2 }) \tag{6}$$

Indeed, in the equation (4) expanding the left and the right side into a Laurent series in u, yields at once the most important property Fourier expansion of the classical Bernoulli

$$\bar{B}_n(x) = \frac{-n!}{(2\pi i)^n} \sum_{k\in\mathbb{Z}}^{*} \frac{e(kx)}{k^n}, \ n \geq 1. \tag{7}$$

where the sum $\sum_{k\in\mathbb{Z}}^{*}$ means that $k = 0$ is to be omitted (and in the non-absolutely convergent case $n = 1$ the sum to be interpreted as a Cauchy principal value). From this, it is not difficult to relate Bernoulli numbers and polynomials to special values of zeta and partial zeta functions defined over the field of rational numbers. This paper can now be summarized as a generalization of these facts to the Barnes polynomials and numbers.

For $\lambda \in \mathbb{C}\backslash\{0\}$, note that

$$\frac{t^N}{\prod_{j=1}^{N} (e^{\lambda a_j t} - 1)} e^{\lambda x t} = \lambda^{-N} \frac{(\lambda t)^N}{\prod_{j=1}^{N} \Big(e^{a_j(\lambda t)} - 1\Big)} e^{x(\lambda t)}. \tag{8}$$

Then, we obtain

$$\sum_{n=0}^{\infty} B_n(\lambda x | \lambda a_1, ..., \lambda a_N) \frac{t^n}{n!} = \lambda^{-N} \sum_{n=0}^{\infty} B_n(x|a_1, ..., a_N) \frac{\lambda^n t^n}{n!}. \tag{9}$$

Therefore by comparing the coefficients of both sides of the equation (9), it's easy to see that

Proposition 1 (Homogeneity). *For any $a_1, ...a_N$ non nuls positive real numbers and $\lambda \in \mathbb{C}\backslash\{0\}$, we have*

$$B_n(\lambda x \mid \lambda a_1, .., \lambda a_N) = \lambda^{n-N} B_n(x \mid a_1, .., a_N), \ (n \geq 1). \tag{10}$$

Now we state our main results.

Theorem 2 (Explicit formula and rationality). *Let $a_1, ...a_N$ be non nuls positive real numbers. Then*

$$\frac{B_n(x \mid a_1, .., a_N)}{n!} = \sum_{m_1+..+m_N=n} a_1^{m_1-1}...a_N^{m_N-1} \frac{B_{m_1}(X)}{m_1!}...\frac{B_{m_N}(X)}{m_N!}, \ (n \in \mathbb{N}), \quad (11)$$

where $X = \frac{x}{A_N}$ and $A_N = a_1 + ... + a_N$. In addition if $a_1, ...a_N$ are rationals numbers then $\frac{B_n(x|a_1,..,a_N)}{n!}$ is a polynomial with rational coefficients.

Theorem 3 (Fourier expansion). *Let $a_1, ...a_N$ non nuls positive real numbers and set $A_N = a_1 + ... + a_N$ and $X = \frac{x}{A_N}$. Then for any $n \geq 1$ and $|X| < 1$ we have*

$$B_n(x \mid a_1, .., a_N) = \frac{(-1)^N n!}{(2\pi i)^n} \sum^{*}_{m_1+..+m_N=n} a_1^{m_1-1}...a_N^{m_N-1} \sum^{\prime}_{k_1,...,k_N \in \mathbb{Z}\setminus\{0\}} \frac{e\left((k_1 + ... + k_N)X\right)}{k_1^{m_1}...k_N^{m_N}}.$$

Here $\sum'_{k_1,...,k_N\in\mathbb{Z}\setminus\{0\}}$ means that $k_1, ..., k_N \in \mathbb{Z}\setminus\{0\}$ and, in the non-absolutely convergent case $m_i = 1$, for any $1 \leq i \leq N$, the sum $\sum'_{k_i\in\mathbb{Z}\setminus\{0\}}$ to be interpreted as a Cauchy principal value for each i, and $\sum^*_{m_1+..+m_N=n}$ means that $m_1, ..., m_N \in \mathbb{N}$ with the usual convention the sum $\sum'_{k_i\in\mathbb{Z}\setminus\{0\}} e\left(k_i X\right) = -1$ if $m_i = 0$.

Theorem 4 (Von Staut type identity). *Let $a_1, ...a_N$ non nuls positive integers. Put*

$$\mu(n) = \prod_{p \ prime \ \leq n} p^{\left[\frac{n}{p-1}\right]}, \ (n \geq 1).$$

Then the denominator of $\dfrac{B_n(0 \mid a_1, .., a_N)}{n!}$ divides $\mu(n)$.

Let q an integer ≥ 2 and χ a Dirichlet character modulo q. As usual,

$$L(s, \chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s}, \ G_\chi = \sum_{t=1}^{q} \chi(t) e(\frac{t}{q})$$

where s is a complex variable.
By homogeneity Proposition 1, without loss of generality we can assume for the following theorems that: $a_1 + \cdots + a_N = 1$.

Theorem 5 (Values of L-function at non-negative integers). *Let $q \geq 2$ be a natural number, $a_1, ...a_N$ non nuls positive real numbers with $a_1 + ... + a_N = 1$, χ a non trivial Dirichlet character modulo $q \geq 2$. Then we have*

$$\sum_{t=1}^{q} \bar{\chi}(t) B_n\left(\frac{t}{q} \mid a_1, .., a_N\right) =$$

$$\begin{cases} \frac{2^N(-1)^N(n!)}{(2\pi i)^n} G_{\bar{\chi}} \displaystyle\sum_{\substack{m_1+..+m_N=n \\ \chi(-1)=(-1)^{m_i}, m_1,...,m_N \geq 0}} a_1^{m_1-1}...a_N^{m_N-1} L(m_1, \chi)...L(m_N, \chi), & \text{if } \chi(-1) = (-1)^n, \\ 0 & \text{otherwise.} \end{cases}$$

with the usual convention $L(0, \chi) = \frac{-1}{2}$.

From the above Theorem 5 we get immediately the following corollaries.

Corollary 6 (Values of L-function for even character). *Let $q \geq 2, n$ be a natural numbers, $a_1, ...a_N$ non nuls positive real numbers with $a_1 + ... + a_N = 1$, χ an even non trivial Dirichlet character modulo $q \geq 2$. Then we have*

$$\sum_{t=1}^{q} \bar{\chi}(t) B_{2n}\left(\frac{t}{q} \mid a_1, .., a_N\right) =$$
$$\frac{2^N(-1)^N(2n)!}{(2\pi i)^{2n}}\ G_{\bar{\chi}} \sum_{\substack{j_1+..+j_N=n \\ j_1,...,j_N \geq 0}} a_1^{2j_1-1}...a_N^{2j_N-1} L(2j_1,\chi)...L(2j_N,\chi),$$

with the usual convention $L(0,\chi) = \frac{-1}{2}$.

Corollary 7 (Values of L-function for odd character). *Let $q \geq 2, n$ be a natural numbers, $a_1, ...a_N$ non nuls positive real numbers with $a_1 + ... + a_N = 1$ and χ an odd non trivial Dirichlet character modulo $q \geq 2$. Then we have*

$$\sum_{t=1}^{q} \bar{\chi}(t) B_{2n+1}\left(\frac{t}{q} \mid a_1, .., a_N\right) =$$
$$\frac{2^N(-1)^N(2n+1)!}{(2\pi i)^{2n+1}}\ G_{\bar{\chi}} \sum_{\substack{m_1+..+m_N=2n+1 \\ m_1,...,m_N \ odd}} a_1^{m_1-1}...a_N^{m_N-1} L(m_1,\chi)...L(m_N,\chi).$$

Corollary 8. *Let $a_1, ...a_N$ non nuls positive real numbers with $a_1 + ... + a_N = 1$ and χ a non trivial Dirichlet character modulo $q \geq 2$. If $\chi(-1) \neq (-1)^n$, we have*

$$\sum_{t=1}^{q} \bar{\chi}(t) B_n(\frac{t}{q} \mid a_1, ..., a_N) = 0.$$

Remark 1. *Here we show how to use our Theorem 5 to obtain*

$$L(-n,\chi) = -\frac{B_{n+1,\chi}}{n+1} \qquad (n \geq 0).$$

Set

$$\delta = \begin{cases} 0 & \textit{if } \chi(-1) = 1, \\ 1 & \textit{if } \chi(-1) = -1. \end{cases}$$

Using the following functional equations

$$G_{\bar{\chi}} = \chi(-1)q/G_{\chi},\ \Gamma(s)\cos(\pi\frac{s-\delta}{2}) = \frac{G_{\chi}}{2i^{\delta}}\left(\frac{2\pi}{q}\right)^n L(1-s,\chi),$$

(see [16] *chap.4 p.29-37), at $s = n$ we obtain*

$$-\frac{2(n!)G_{\bar{\chi}}L(n,\chi)}{(2\pi i)^n} = -nq^{1-n}L(1-n,\chi).$$

Take $N = 1$ and $a_1 = 1$, our Theorem 5 gives us

$$\sum_{t=1}^{q} \bar{\chi}(t) B_n\left(\frac{t}{q}\right) = -\frac{2(n!)G_{\bar{\chi}}L(n,\chi)}{(2\pi i)^n} = -nq^{1-n}L(1-n,\chi).$$

Hence,

$$L(1-n,\chi) = -\frac{1}{n} q^{n-1} \sum_{t=1}^{q} \bar{\chi}(t) B_n\left(\frac{t}{q}\right)$$

and from the definition (5), it 's easy to see that

$$q^{n-1}\sum_{t=1}^{q}\bar{\chi}(t)B_n\left(\frac{t}{q}\right)=B_{n,\chi}.$$

Therefore, we get the equality (6).

2. Proofs of main results

Proof of Theorem 2: Writing $X=\frac{x}{A_N}$. We have

$$\begin{aligned}\frac{t^N}{\prod_{j=1}^{N}\left(e^{a_jt}-1\right)}e^{xt} &= \frac{1}{a_1...a_N}\prod_{i=1}^{N}\frac{a_ite^{X(a_it)}}{e^{a_it}-1}\\ &= \sum_{n=0}^{\infty}\left(\sum_{m_1+\cdots+m_N=n}a_1^{m_1-1}\cdots a_N^{m_N-1}\frac{B_{m_1}(X)}{m_1!}\cdots\frac{B_{m_N}(X)}{m_N!}\right)t^n.\end{aligned}$$

In other hand

$$\frac{t^N}{\prod_{j=1}^{N}\left(e^{a_jt}-1\right)}e^{xt}=\sum_{n=0}^{\infty}\frac{B_n}{n!}(x|a_1,...,a_N)t^n.$$

By comparing the right members of both equations above we obtain

$$\frac{B_n(x\mid a_1,..,a_N)}{n!}=\sum_{m_1+..+m_N=n}a_1^{m_1-1}...a_N^{m_N-1}\frac{B_{m_1}(X)}{m_1!}...\frac{B_{m_N}(X)}{m_N!},\ (n\in\mathbb{N}).$$

Let $a_1,...a_N$ be rationals numbers. By using the fact that the classical Bernoulli polynomials are in $\mathbb{Q}[x]$, we obtain that $\frac{B_n(x|a_1,..,a_N)}{n!}$ is a polynomial with rational coefficients. This completes the proof of our theorem. □

Proof of Theorem 3: Using the equation 7 and Theorem 2, we can write

$$\begin{aligned}\frac{B_n(x\mid a_1,..,a_N)}{n!} &= \sum_{\substack{m_1+\cdots+m_N=n\\ m_1,\cdots,m_N\geq 0}}^{*}\frac{a_1^{m_1-1}\cdots a_N^{m_N-1}}{m_1!\cdots m_N!}\frac{(-1)^N m_1!\cdots m_N!}{(2\pi i)^{m_1+\cdots+m_N}}\\ &\qquad\sum_{k_1,\cdots,k_N\in\mathbb{Z}\setminus\{0\}}^{\prime}\frac{e\left((k_1+\cdots+k_N)X\right)}{k_1^{m_1}\cdots k_N^{m_N}},\\ &= \frac{(-1)^N}{(2\pi i)^n}\sum_{\substack{m_1+\cdots+m_N=n\\ m_1,\cdots,m_N\geq 0}}^{*}a_1^{m_1-1}\cdots a_N^{m_N-1}\sum_{k_1,\cdots,k_N\in\mathbb{Z}\setminus\{0\}}^{\prime}\frac{e\left((k_1+\cdots+k_N)X\right)}{k_1^{m_1}\cdots k_N^{m_N}}.\end{aligned}$$

This, yields the theorem. □

Proof of Theorem 4 .□

Let n be non-negative integers, p be a prime number. We denote by $v_p(n)$ the p-adic valuation of n. In order to prove our theorem we need the following two lemmas:

Lemma 9. *For any $n \in \mathbb{N}$, $n \geq 1$ and p prime number. We have*

$$v_p(n!) \leq \begin{cases} \left[\frac{n}{p-1}\right] & \text{if } p-1 \not| n, \\ \left[\frac{n}{p-1}\right] - 1 & \text{if } p-1 \mid n, \end{cases}$$

For this, we set $K = [logn/logp]$. It's easy to see that $v_p(n!) = \sum_{i=1}^{K}[n/p^i]$.

(1) If $p-1 \not| n$, we have $v_p(n!) = \sum_{i=1}^{K}[n/p^i] \leq n\sum_{i=1}^{\infty} n/p^i = \frac{n}{p-1}$.
(2) If $p-1|n$, we write $n = m(p-1)$ with $m \geq 1$. Then we have $v_p(n!) \leq n\sum_{i=1}^{K} n/p^i = m(1-p^{-K}) < m$.

This proves the lemma 9.

Lemma 10. *For any $n \in \mathbb{N}$, $n \geq 1$ and p prime number. Let d_n denotes the denominator of $\frac{B_n}{n!}$. We have*

$$v_p(D_n) \leq \left[\frac{n}{p-1}\right].$$

From Von Staudt Theorem [6] p.233, we know that

$$v_p(denominator(B_n)) = \begin{cases} 0 & \text{if } p-1 \not| n, \\ 1 & \text{if } p-1 \mid n, \end{cases}$$

From this and Lemma 9, we obtain

$$v_p(d_n) = v_p(n!) + v_p(denominator(B_n)) \leq \left[\frac{n}{p-1}\right].$$

Then we get our Lemma 10.

Now we are ready to prove our theorem. Let D_n the denominator of $\dfrac{B_n(0 \mid a_1, .., a_N)}{n!}$.
By Theorem 2, we get

$$\frac{B_n(0 \mid a_1, .., a_N)}{n!} = \sum_{m_1+..+m_N=n} a_1^{m_1-1} ... a_N^{m_N-1} \frac{B_{m_1}}{m_1!} ... \frac{B_{m_N}}{m_N!}, \ (n \in \mathbb{N}).$$

Therefore, we can write

$$v_p(D_n) \leq \max_{m_1+..+m_N=n} v_p(d_{m_1} \cdots d_{m_N})$$

From this relation and Lemma 10, we obtain

$$v_p(D_n) \leq \max_{m_1+..+m_N=n}\left(\left[\frac{m_1}{p-1}\right] + \cdots + \left[\frac{m_N}{p-1}\right]\right) \leq \max_{m_1+..+m_N=n}\left[\frac{m_1+\cdots+m_N}{p-1}\right] = \left[\frac{n}{p-1}\right].$$

Hence the denominator D_n of $\dfrac{B_n(0 \mid a_1, .., a_N)}{n!}$ divides $\mu(n) = \prod_{p \text{ prime } \leq n} p^{\left[\frac{n}{p-1}\right]}$.

This gives our theorem. □
Proof of Theorem 5

Using Theorem 3, we have

$$\sum_{t=1}^{q}\bar{\chi}(t)B_n\left(\frac{t}{q}\mid a_1,..,a_N\right)=$$

$$\tfrac{(-1)^N n!}{(2\pi i)^n}\sum_{m_1+..+m_N=n}^{*}a_1^{m_1-1}...a_N^{m_N-1}\sum_{k_1,\ldots,k_N\in\mathbb{Z}\setminus\{0\}}^{\prime}\frac{1}{k_1^{m_1}...k_N^{m_N}}\sum_{t=1}^{q}\bar{\chi}(t)e\left((k_1+...+k_N)\frac{t}{q}\right).$$

Since

$$\sum_{t=1}^{q}\bar{\chi}(t)e\left(k\frac{t}{q}\right)=\chi(k)G_{\bar{\chi}}$$

we have

$$=\tfrac{(-1)^N n!}{(2\pi i)^n}G_{\bar{\chi}}\sum_{m_1+..+m_N=n}^{*}a_1^{m_1-1}...a_N^{m_N-1}\sum_{k_1,\ldots,k_N\in\mathbb{Z}\setminus\{0\}}^{\prime}\frac{1}{k_1^{m_1}...k_N^{m_N}}\chi(k_1+...+k_N).$$

$$=\tfrac{(-1)^N n!}{(2\pi i)^n}G_{\bar{\chi}}\sum_{m_1+..+m_N=n}^{*}a_1^{m_1-1}...a_N^{m_N-1}\sum_{k_1,\ldots,k_N\in\mathbb{Z}\setminus\{0\}}^{\prime}\frac{\chi(k_1)}{k_1^{m_1}}\cdots\frac{\chi(k_N)}{k_1^{m_N}}$$

While

$$\sum_{k\in\mathbb{Z}\setminus\{0\}}^{\prime}\frac{\chi(k_i)}{k^{m_i}}=\left(1+\chi(-1)(-1)^{m_i}\right)L(m_i,\chi)$$

Therefore

$$\sum_{t=1}^{q}\bar{\chi}(t)B_n\left(\frac{t}{q}\mid a_1,..,a_N\right)=\frac{(-1)^N(n!)}{(2\pi i)^n}\,G_{\bar{\chi}}\sum_{\substack{m_1+..+m_N=n\\ m_1,\ldots,m_N\geq 0}}a_1^{m_1-1}...a_N^{m_N-1}\prod_{i=1}^{N}\left(1+\chi(-1)(-1)^{m_i}\right)L(m_i,\chi)$$

$$=\frac{2^N(-1)^N(n!)}{(2\pi i)^n}\,G_{\bar{\chi}}\sum_{\substack{m_1+..+m_N=n\\ \chi(-1)=(-1)^{m_i},m_1,\ldots,m_N\geq 0}}a_1^{m_1-1}...a_N^{m_N-1}L(m_1,\chi)...L(m_N,\chi).$$

Since the summation is over $m_1,...,m_N\geq 0, m_1+..+m_N=n, \chi(-1)=(-1)^{m_i}$, this implies that $\chi(-1)\neq(-1)^n$. otherwise the sum is zero.
Hence, we obtain our desired theorem. □

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Abdelmejid Bayad. Département de mathématiques, Université d'Evry Val d'Essonne, Bd. F. Mitterrand, 91025 Evry Cedex, France,
E-mail address: abayad@maths.univ-evry.fr

Taekyun Kim. Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea,
E-mail address: tkkim@kw.ac.kr