Higher dimensional Dedekind sums in finite fields

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Abstract. We introduce Dedekind sums of new type defined over finite fields. These are similar to higher dimensional Dedekind sums of Zagier. The main result is the reciprocity law for them.

Key words: Dedekind sums, finite fields.

1 Introduction

Let $c > 0, a$ be relatively prime rational integers. The classical Dedekind sums is defined as

$$s(a,c) = \frac{1}{4c}\sum_{k=1}^{c-1} \cot\left(\frac{\pi k}{c}\right)\cot\left(\frac{\pi k a}{c}\right).$$

It satisfies a famous relation called the *reciprocity law*, i.e., for relatively prime positive integers a, c,

$$s(a,c) + s(c,a) = \frac{a^2 + c^2 + 1 - 3ac}{12ac}.$$

We refer to Rademacher-Grosswald [6] for its proofs. One knows a higher generalization for Dedekind sums due to Zagier [7]. Let p be a positive integer, and $a_1, \ldots, a_n$ integers prime to p. We assume that n is odd. Zagier defines a higher dimensional Dedekind sum as follows:

$$d(a_1, \ldots, a_{n-1}; p) := (-1)^{(n-1)/2}\frac{1}{p}\sum_{k=1}^{p-1} \cot\left(\frac{\pi k a_1}{p}\right)\cdots\cot\left(\frac{\pi k a_{n-1}}{p}\right).$$

For pairwise coprime positive integers $a_1, \ldots, a_{n-1}$ (n odd), this sum satisfies the reciprocity law

$$\sum_{j=1}^{n} d(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n; a_j) = 1 - \frac{l_n(a_1, \ldots, a_n)}{a_1 \cdots a_n},$$

where $l_n(a_1, \ldots, a_n)$ is the polynomial in $a_1, \ldots, a_n$ defined as the coefficient of t^n in the power series expansion of

$$\prod_{j=1}^{n} \frac{a_j t}{\tanh(a_j t)} = \prod_{j=1}^{n} \left(1 + \frac{1}{3}a_j^2 t^2 - \frac{1}{45}a_j^4 t^4 + \frac{2}{945}a_j^6 t^6 - \cdots\right).$$

It should be noted that Beck [1] generalized Zagier's higher dimensional Dedekind sum.

It is known that $\pi \cot \pi z$ has the following expression:

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n}\right). \tag{1}$$

In finite fields, we have periodic functions that have analogous expressions to (1). From this point of view, in [3], [4] and [5], we introduced Dedekind sums in finite fields, and established reciprocity laws for them. These sums are like Apostol-Dedekind sums defined by

$$s_n(a, c) = \sum_{k=1}^{c-1} \frac{k}{c} \overline{B}_n \left(\frac{ka}{c}\right),$$

where $\overline{B}_n(x)$ denotes the nth Bernoulli function. In [4], we posed a question: can we define higher dimensional Dedekind sums defined over finite fields as Zagier did in [7]?

The goal of our paper is to introduce new kinds of Dedekind sums defined over finite fields. Our Dedekind sums are similar to higher dimensional Dedekind sums. As the main theorem, we establish the reciprocity law for them.

Notation.
$\sum'$ = the sum over non-vanishing elements
$\prod'$ = the product over non-vanishing elements

2 Lattices

We recall some facts about lattices and periodic polynomials. We refer to Gekeler [2] for details.

For $K = \mathbb{F}_q$, the finite field with q elements, $\overline{K}$ denotes a fixed algebraic closure of K. Let Λ be a subset in $\overline{K}$. We call Λ a *lattice* if it is a linear K-subspace in $\overline{K}$ of finite dimension. For such a lattice Λ, define the product

$$e_\Lambda(z) = z {\prod_{\lambda \in \Lambda}}' \left(1 - \frac{z}{\lambda}\right).$$

The map $e_\Lambda : \overline{K} \to \overline{K}$ satisfies the following properties:
• e_Λ is K-linear and Λ-periodic.
• If $\dim_K \Lambda = r$, then $e_\Lambda(z)$ has the form

$$e_\Lambda(z) = \sum_{i=0}^{r} \alpha_i(\Lambda) z^{q^i},$$

where $\alpha_0(\Lambda) = 1, \alpha_r(\Lambda) \neq 0$.
• $e_\Lambda(z)$ has simple zeros at the points of Λ and no other zeros.
• $de_\Lambda(z)/dz = e'_\Lambda(z) = 1$. Hence we have

$$\frac{1}{e_\Lambda(z)} = \frac{e'_\Lambda(z)}{e_\Lambda(z)} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}.$$

For a positive integer k,

$$E_k(\Lambda) = \sum_{\lambda \in \Lambda}{}' \lambda^{-k}$$

is called the *Eisenstein series of weight* k for Λ. We use the convention $E_0(\Lambda) = -1$. The function $z/e_\Lambda(z)$ has the following expression as a formal series:

$$\frac{z}{e_\Lambda(z)} = -\sum_{k=0}^{\infty} E_k(\Lambda) z^k.$$

3 Higher dimensional Dedekind sums

Let Λ be a lattice. We introduce Dedekind sums for Λ. Let $K(\Lambda)$ denote the field generated by Λ over K and assume $n \geq 2$. We pick up $a_1, \ldots, a_n \in \overline{K} \setminus \{0\}$ satisfying

$$\frac{a_i}{a_n} \notin K(\Lambda) \quad \text{if} \quad i \neq n.$$

Definition 3.1 *We define*

$$s_\Lambda(a_1, \ldots, a_{n-1}; a_n) = (-1)^{n-1} \frac{1}{a_n} \sum_{\lambda \in \Lambda}{}' e_\Lambda \left(\frac{a_1 \lambda}{a_n} \right)^{-1} \cdots e_\Lambda \left(\frac{a_{n-1} \lambda}{a_n} \right)^{-1}.$$

Remark 3.2 In case $n = 2$, $q = 2$, our Dedekind sum coincides with one of Dedekind sums defined in [4], [5].

The Dedekind sum $s_\Lambda(a_1, \ldots, a_{n-1}; a_n)$ has similar properties to those of Zagier's Dedekind sum. More precisely we have:

Proposition 3.3 (i) $s_\Lambda(a_1, \ldots, a_{n-1}; a_n)$ *only depends on* $a_i + a_n K$*,*
(ii) $s_\Lambda(a_1, \ldots, a_{n-1}; a_n)$ *is symmetric in the* $n - 1$ *arguments* $a_1, \ldots, a_{n-1}$*,*
(iii) $s_\Lambda(\zeta a_1, \ldots, a_{n-1}; a_n) = \zeta^{-1} s_\Lambda(a_1, \ldots, a_{n-1}; a_n)$ *for any* $\zeta \in K \setminus \{0\}$*,*
(iv) $s_\Lambda(\zeta a_1, \ldots, \zeta a_{n-1}; a_n) = s_\Lambda(a_1, \ldots, a_{n-1}; a_n)$ *for any* $\zeta \in K \setminus \{0\}$*,*
(v) *The Dedekind sum* $s_\Lambda(a_1, \ldots, a_{n-1}; a_n)$ *is rational i.e.,* $s_\Lambda(a_1, \ldots, a_{n-1}; a_n) \in K(\Lambda)(a_1, ..., a_n)$.

Proof. The proof of the properties (i)—(iv) is trivial, so that we omit it.

Now we prove the rationality of the sum $s_\Lambda(a_1, \ldots, a_{n-1}; a_n)$: we recall from Section 2 that if $\dim_K \Lambda = r$, then $e_\Lambda(z)$ has the form

$$e_\Lambda(z) = \sum_{i=0}^{r} \alpha_i(\Lambda) z^{q^i}.$$

It is easy to see that the coefficients $\alpha_i(\Lambda)$, $0 \leq i \leq r$, are elements of $K(\Lambda)$. Therefore, for any $0 \leq i \leq r$ we have $e_\Lambda \left(\frac{a_i \lambda}{a_n} \right) \in K(\Lambda)(a_i, a_n)$. This yields the property

$$s_\Lambda(a_1, \ldots, a_{n-1}; a_n) \in K(\Lambda)(a_1, ...a_n).$$

□

Remark 3.4 We present an easy case here. This example will be much more developed in section 4. Let $\Lambda = K = \mathbb{F}_q$, $n = 2$, $q = 2$. Suppose $\frac{a_1}{a_2} \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Then $\frac{a_1}{a_2}$ is a primitive element of $\mathbb{F}_{q^2}$ and $\frac{a_1}{a_2} - \frac{a_1^q}{a_2^q} = \frac{a_1}{a_2} + \frac{a_1^q}{a_2^q} \in \mathbb{F}_q \setminus \{0\}$. Therefore

$$s_K(a_1; a_2) = \frac{1}{a_2\left(\frac{a_1}{a_2} - \frac{a_1^q}{a_2^q}\right)} \in \mathbb{F}_q(a_2) \cap \mathbb{F}_q(a_1, a_2).$$

But if we take a primitive element a_2 of $\mathbb{F}_{q^4}$ and $a_1 = ta_2$ with t a primitive element of $\mathbb{F}_{q^2}$, it is obvious to see that $\mathbb{F}_q(a_2) \cap \mathbb{F}_q(a_1, a_2) = \mathbb{F}_q(a_1, a_2) = \mathbb{F}_{q^4}$. This example shows us that, in general, for the definition field of the Dedekind sum $s_\Lambda(a_1, \ldots, a_{n-1}; a_n)$ we must take $K(\Lambda)(a_1, \ldots a_n)$.

Remark 3.5 As is well known, Zagier's Dedekind sum $d(a_1, \ldots, a_{n-1}; p)$ satisfies

$$d(xa_1, \ldots, xa_{n-1}; p) = d(a_1, \ldots, a_{n-1}; p)$$

for any integer x prime to p. In our situation, if we suppose $\Lambda\Lambda \subset \Lambda$, that is, Λ is a finite field, then

$$s_\Lambda(xa_1, \ldots, xa_{n-1}; a_n) = s_\Lambda(a_1, \ldots, a_{n-1}; a_n)$$

for any $x \in \Lambda \setminus \{0\}$. Moreover, $s_\Lambda(a_1, \ldots, a_{n-1}; a_n)$ only depends on $a_i + a_n\Lambda$.

We now state the reciprocity law for our Dedekind sums.

Theorem 3.6 *For $a_1, \ldots, a_n \in \overline{K} \setminus \{0\}$ such that*

$$\frac{a_i}{a_j} \notin K(\Lambda) \quad \textit{if} \quad i \neq j$$

holds, we have

$$\sum_{i=1}^{n} s_\Lambda(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n; a_i) = \sum_{\substack{i_1+\cdots+i_n=n-1 \\ i_1,\ldots,i_n \geq 0}} \frac{a_1^{i_1} \cdots a_n^{i_n}}{a_1 \cdots a_n} E_{i_1}(\Lambda) \cdots E_{i_n}(\Lambda).$$

4 Example

Let Λ be a fixed lattice. Firstly, we give the value of the sum of Dedekind sums for small n.

$$\begin{aligned}
&s_\Lambda(a_1; a_2) + s_\Lambda(a_2; a_1) = -\left(\frac{1}{a_1} + \frac{1}{a_2}\right) E_1(\Lambda), \\
&s_\Lambda(a_1, a_2; a_3) + s_\Lambda(a_1, a_3; a_2) + s_\Lambda(a_2, a_3; a_1) \\
&\quad = \frac{a_1^2 + a_2^2 + a_3^2}{a_1a_2a_3} E_2(\Lambda) - \frac{a_1a_2 + a_2a_3 + a_3a_1}{a_1a_2a_3} E_1(\Lambda)^2, \\
&s_\Lambda(a_1, a_2, a_3; a_4) + s_\Lambda(a_1, a_2, a_4; a_3) + s_\Lambda(a_1, a_3, a_4; a_2) + s_\Lambda(a_2, a_3, a_4; a_1) \\
&\quad = -\frac{a_1^3 + a_2^3 + a_3^3 + a_4^3}{a_1a_2a_3a_4} E_3(\Lambda) \\
&\qquad + \frac{a_1^2a_2 + a_1^2a_3 + a_1^2a_4 + a_1a_2^2 + a_2^2a_3 + a_2^2a_4}{a_1a_2a_3a_4} E_1(\Lambda)E_2(\Lambda) \\
&\qquad + \frac{a_1a_3^2 + a_2a_3^2 + a_3^2a_4 + a_1a_4^2 + a_2a_4^2 + a_3a_4^2}{a_1a_2a_3a_4} E_1(\Lambda)E_2(\Lambda) \\
&\qquad - \frac{a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4}{a_1a_2a_3a_4} E_1(\Lambda)^3.
\end{aligned}$$

We next consider the special case $\Lambda = K = \mathbb{F}_q$. Then $e_K(z) = z - z^q$. Since

$$\frac{z}{e_K(z)} = \sum_{k=0}^{\infty} z^{k(q-1)},$$

$E_n(K)$ is -1 (resp. 0) if $q-1$ divides $n-1$ (resp. otherwise). One can see

$$s_K(a_1,\dots,a_{n-1};a_n) = \begin{cases} (-1)^n \frac{1}{a_n} \prod_{i=1}^{n-1} \left(\frac{a_i}{a_n} - \frac{a_i^q}{a_n^q} \right)^{-1} & (q-1|n-1) \\ 0 & (q-1 \not| n-1) \end{cases}.$$

Case 1. $q=2$.

$$\begin{aligned}
&s_K(a_1;a_2) + s_K(a_2;a_1) = \frac{1}{a_1} + \frac{1}{a_2}, \\
&s_K(a_1,a_2;a_3) + s_K(a_1,a_3;a_2) + s_K(a_2,a_3;a_1) = \frac{a_1^2+a_2^2+a_3^2+a_1a_2+a_2a_3+a_3a_1}{a_1a_2a_3}, \\
&s_K(a_1,a_2,a_3;a_4) + s_K(a_1,a_2,a_4;a_3) + s_K(a_1,a_3,a_4;a_2) + s_K(a_2,a_3,a_4;a_1) \\
&= \frac{1}{a_1a_2a_3a_4} \left(a_1^3+a_2^3+a_3^3+a_4^3+a_1^2a_2+a_1^2a_3+a_1^2a_4+a_1a_2^2+a_2^2a_3+a_2^2a_4+a_1a_3^2 \right. \\
&\qquad \left. +a_2a_3^2+a_3^2a_4+a_1a_4^2+a_2a_4^2+a_3a_4^2+a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4 \right).
\end{aligned}$$

Case 2. $q=3$.

$$\begin{aligned}
&s_K(a_1;a_2) + s_K(a_2;a_1) = 0, \\
&s_K(a_1,a_2;a_3) + s_K(a_1,a_3;a_2) + s_K(a_2,a_3;a_1) = -\frac{a_1^2+a_2^2+a_3^2}{a_1a_2a_3}, \\
&s_K(a_1,a_2,a_3;a_4) + s_K(a_1,a_2,a_4;a_3) + s_K(a_1,a_3,a_4;a_2) + s_K(a_2,a_3,a_4;a_1) = 0.
\end{aligned}$$

Case 3. $q \geq 3$, $q > n \geq 2$.

$$\sum_{i=1}^{n} s_\Lambda(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n;a_i) = 0.$$

$n=q$.

$$\sum_{i=1}^{n} s_\Lambda(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n;a_i) = (-1)^q \frac{a_1^{q-1}+\cdots+a_n^{q-1}}{a_1\cdots a_n}.$$

5 Proof of Theorem 3.6

Let us consider the rational function $F(z) = e_\Lambda(a_1z)^{-1}\cdots e_\Lambda(a_nz)^{-1}$. By assumption on $a_1,\dots,a_n$, we have $a_i^{-1}\Lambda \cap a_j^{-1}\Lambda = \{0\}$ if $i \neq j$. This implies that $F(z)$ has a simple pole at any nonzero element of $\bigcup_{i=1}^{n} a_i^{-1}\Lambda$. For any nonzero element $\lambda \in \Lambda$, we have

$$\mathrm{Res}_{\lambda/a_i}(F(z)dz) = \mathrm{Res}_{\lambda/a_i}(e_\Lambda(a_iz)^{-1}dz) \prod_{j\neq i} e_\Lambda\left(\frac{a_j\lambda}{a_i}\right)^{-1}.$$

Since $e_\Lambda(a_i z)^{-1} = a_i^{-1} \sum_{\lambda\in\Lambda} (z - \lambda/a_i)^{-1}$,

$$\mathrm{Res}_{\lambda/a_i}(e_\Lambda(a_i z)^{-1} dz) = 1/a_i.$$

Hence

$$\mathrm{Res}_{\lambda/a_i}(F(z)dz) = \frac{1}{a_i} \prod_{j\neq i} e_\Lambda\left(\frac{a_j\lambda}{a_i}\right)^{-1}.$$

To compute the left hand side of Theorem 3.6, we make use of Residue Theorem. It should be noted that though $e_\Lambda(z)$ is Λ-periodic, each $e_\Lambda(a_i z)$ is not. From this $F(z)$ is *not* Λ-periodic. Our rational function $F(z)$ has the form

$$F(z) = \frac{1}{G(z)}$$

where

$$G(z) = e_\Lambda(a_1 z)\cdots e_\Lambda(a_n z),$$

which is a polynomial in z with degree equal to $n\#\Lambda$ (> 1). To obtain our desired theorem 3.6 we need the following elementary lemma.

Lemma 5.1 *Let $G(z)$ be a polynomial, over a field L, of degree > 1, and R the set of all roots of $G(z)$. Then we have*

$$\sum_{a\in R} \mathrm{Res}_a\left(\frac{1}{G(z)}dz\right) = 0.$$

Proof of lemma. The partial fraction decomposition of $1/G(z)$ can be expressed as

$$\frac{1}{G(z)} = \sum_{a\in R}\sum_{n=1}^{\mathrm{ord}(a)} \frac{C_a, n}{(z-a)^n}.$$

Then for any $a \in R$, we have $\mathrm{Res}_a(1/G(z)) = C_{a,1}$. In other hand, it is easy to see that $1/G(z)$ can be rewritten as follows

$$\frac{1}{G(z)} = \frac{\left(\sum_{a\in R} C_{a,1}\right) z^{m-1}}{G(z)} + \frac{\text{a polynomial in } z \text{ with degree less than } m-1}{G(z)}$$

where m is the degree of the polynomial $G(z)$. Hence,

$$1 = \left(\sum_{a\in R} C_{a,1}\right) z^{m-1} + \text{a polynomial in } z \text{ with degree less than } m-1.$$

But the degree m of the polynomial $G(z)$ is > 1, thus we can easily by identification obtain that $\sum_{a\in R} C_{a,1} = 0$. □

The set of all poles of $F(z)$ is $\displaystyle\bigcup_{i=1}^{n} a_i^{-1}\Lambda$. By the above Lemma, we have

$$(-1)^{n-1}\sum_{i=1}^{n} s_\Lambda(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n;a_i)+\mathrm{Res}_0(F(z)dz)$$
$$=\sum_{i=1}^{n}{\sum_{\lambda\in\Lambda}}'\mathrm{Res}_{\lambda/a_i}(F(z)dz)+\mathrm{Res}_0(F(z)dz)=0.$$

Since

$$\frac{a_iz}{e_\Lambda(a_iz)}=-\sum_{k=0}^{\infty}E_k(\Lambda)a_i^kz^k,$$

we have the expression

$$F(z)=\frac{(-1)^n}{a_1\cdots a_nz^n}\prod_{i=1}^{n}\left(\sum_{k=0}^{\infty}E_k(\Lambda)a_i^kz^k\right).$$

Hence

$$\mathrm{Res}_0(F(z)dz)=\frac{(-1)^n}{a_1\cdots a_n}\sum_{i_1+\cdots+i_n=n-1}a_1^{i_1}\cdots a_n^{i_n}E_{i_1}(\Lambda)\cdots E_{i_n}(\Lambda).$$

This completes the proof.

6 Concluding remark

Finally, we would like to make the following remark.

The classical Dedekind sum $d(a_1,\dots,a_{n-1};p)$ can be defined for any integer $n\geq 2$. However, for n even the sum is zero, so that n is assumed to be odd in general. The same thing holds in our setting, that is to say, $s_\Lambda(a_1,\dots,a_{n-1};a_n)$ can be defined for any integer $n\geq 2$. If $\mathrm{Char}\,\mathbb{F}_q\neq 2$ and $2|n$, then our sum is equal to zero because

$$(-1)^{n-1}s_\Lambda(a_1,\dots,a_{n-1};a_n)=s_\Lambda(a_1,\dots,a_{n-1};a_n)$$

by Proposition 3.3 (ii), (iii), (iv). Moreover, it may be possible to impose the condition about n according to Λ. For instance, as seen in Section 4, $s_K(a_1,\dots,a_{n-1};a_n)$ is zero if $q-1$ is not a factor of $n-1$. Hence we can impose the assumption that $q-1$ divides $n-1$.

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References

[1] M. Beck, Dedekind cotangent sums, Acta Arith. **109** (2003), 109-130.

[2] E. -U. Gekeler, Finite modular forms, Finite Fields and Their Applications **7** (2001), 553-572.

[3] Y. Hamahata, Dedekind sums for finite fields, In: Diophantine Analysis and Related Fields: DARF 2007/2008. AIP Conference Proceedings, vol **976**, pp. 96-102 (2008).

[4] Y. Hamahata, Finite Dedekind sums, In: Arithmetic of Finite Fields, Lecture Notes in Comput. Sci. **5130**, pp. 11-18, Springer, Berlin, 2008.

[5] Y. Hamahata, Dedekind sums in finite characteristic, Proc. Japan Academy **84**, Ser. A (2008), 89-92.

[6] H. Rademacher and E. Grosswald, Dedekind Sums, The Mathematical Association of America, Washington (1972).

[7] D. Zagier, Higher dimensional Dedekind sums, Math. Ann. **202** (1973), 149-172.