Some results concerning q-Bernstein and q-Laguerre polynomials

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Abstract In this paper, we consider q-Laguerre polynomials related to q-Bernstein polynomials. From these q-Laguerre polynomials, we derive some interesting relations between q-Bernstein polynomials and q-Laguerre polynomials.

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1. Introduction

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. In [2], Bernstein introduced the following well known linear positive operator:

$$\begin{aligned}\mathbb{B}_n(f \mid x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \\ &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right)B_{k,n}, \text{ for } f \in C[0,1].\end{aligned} \tag{1}$$

Here $\mathbb{B}_n(f \mid x)$ is called the Bernstein operator of order n for f. The Bernstein polynomials of degree n is defined by

$$B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}, \text{ for } n,k \in \mathbb{Z}_+, x \in [0,1]. \tag{2}$$

For example, $B_{0,1}(x) = 1-x, B_{1,1}(x) = x, B_{0,2}(x) = (1-x)^2, B_{1,2}(x) = 2x(1-x), B_{2,2}(x) = x^2, B_{0,3}(x) = (1-x)^3, B_{1,3}(x) = 3x(1-x)^2, B_{2,3}(x) = 3x^2(1-x), B_{3,3}(x) = x^3, \cdots$,(see [1-3, 8-12]).

Many researchers have studied the Bernstein polynomials in the area of approximation theory(see [1-15]). As well known equation, the Laguerre differential equation is given by

$$xy'' + (1-x)y' + ny = 0, (\text{see } [7, 16]). \tag{3}$$

From (3), the Laguerre polynomials are defined by the solutions of (3). That is, Laguerre polynomials are given by

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n). \tag{4}$$

By (4) and Leibniz formula, we get

$$L_n(x) = \sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{x^l}{l!}, (\text{see } [1, 7, 16]). \tag{5}$$

From (4) and (5), we note that

$L_0(x) = 1, L_1(x) = 1-x$, and $L_{n+1}(x) = \dfrac{1}{n+1}((2n+1-x)L_n(x) - nL_{n-1}(x))$.

In the viewpoint of (4), the generalized Laguerre polynomials are considered as follows:

$$\frac{e^x}{n!x^k}\frac{d^n}{dx^n}(e^{-x}x^{n+k}) = L_n^{(k)}(x). \tag{6}$$

Note that $L_n^{(0)}(x) = L_n(x)$. By (6), we can derive the following formula for the generalized Laguerre polynomials:

$$L_n^{(k)}(x) = \sum_{l=0}^{n}\binom{n+k}{n-l}(-1)^l\frac{x^l}{l!}, (\text{see } [1, 7, 16]). \tag{7}$$

For example, $L_0^{(k)}(x)=1, L_1^{(k)}(x)=1+k-x, L_2^{(k)}(x)=(2+k)(1+k)-(2+k)x+\frac{x^2}{2},\cdots$.

Throughout this paper, we assume that $q\in\mathbb{C}$ with $|q|<1$. The q-number is defined by

$$[x]_q=\frac{1-q^x}{1-q},\ \text{(see [1-15])}.$$

Note that $\lim_{q\to 1}[x]_q=x$.

Recently, q-Bernstein polynomials of degree n are defined by

$$B_{k,n}(x\mid q)=\binom{n}{k}[x]_q^k[1-x]_{1/q}^{n-k},\ \text{where } n,k\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}, \text{(see [8])}. \tag{8}$$

From (8), we note that the reflection symmetric property is given by

$$B_{k,n}(x\mid q)=B_{n-k,n}(1-x\mid \frac{1}{q}), \text{(see [9])}. \tag{9}$$

In this paper we consider q-Laguerre polynomials related to q-Bernstein polynomials. From these q-Laguerre polynomials, we give some identities between q-Bernstein polynomials and q-Laguerre polynomials.

2. q-Laguerre polynomials and q-Bernstein polynomials

Let us consider the q-Laguerre polynomials in the viewpoint of the q-extension of (5). For $q\in\mathbb{C}$ with $|q|<1$, we define the q-Laguerre polynomials as follows:

$$\frac{e^{-\frac{[x]_q t}{1-t}}}{1-t}=\sum_{n=0}^{\infty}L_{n,q}(x)t^n. \tag{10}$$

From (10), we can derive the following equation.

$$\begin{aligned}
\frac{e^{-\frac{[x]_q t}{1-t}}}{1-t}&=\frac{1}{1-t}\sum_{l=0}^{\infty}(-1)^l\frac{[x]_q^l}{(1-t)^l}\frac{t^l}{l!}\\
&=\sum_{l=0}^{\infty}(-1)^l[x]_q^l\frac{t^l}{(1-t)^{l+1}}\frac{1}{l!}\\
&=\left(\sum_{l=0}^{\infty}(-1)^l[x]_q^l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\binom{l+m}{m}t^m\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}(-1)^l\frac{[x]_q^l}{l!}\binom{n}{l}\right)t^n.
\end{aligned} \tag{11}$$

By comparing the coefficients on the both sides of (10) and (11), we obtain the following proposition.

Proposition 1. For $n \in \mathbb{Z}_+$, we have

$$L_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{[x]_q^l}{l!}.$$

As an analogue of (6), we consider the generalized q-Laguerre polynomials which are given by

$$\frac{e^{-\frac{[x]_q t}{1-t}}}{(1-t)^{k+1}} = \sum_{n=0}^{\infty} L_{n,q}^{(k)}(x) t^n. \tag{12}$$

For the left side of (12), we have

$$\begin{aligned}
\frac{e^{-\frac{[x]_q t}{1-t}}}{(1-t)^{k+1}} &= \frac{1}{(1-t)^{k+1}} \sum_{l=0}^{\infty} (-1)^l \frac{[x]_q^l}{l!} \frac{t^l}{(1-t)^l} \\
&= \left(\sum_{l=0}^{\infty} (-1)^l \frac{[x]_q^l}{l!} t^l \right) \left(\sum_{m=0}^{\infty} \binom{m+k+l}{m} t^m \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} (-1)^l \frac{[x]_q^l}{l!} \binom{n+k}{n-l} \right) t^n.
\end{aligned} \tag{13}$$

By comparing the coefficients on the both sides of (12) and (13), we obtain the following proposition.

Proposition 2. For $n, k \in \mathbb{Z}_+$, we have

$$L_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \binom{n+k}{n-l} (-1)^l \frac{[x]_q^l}{l!}.$$

The q-Laguerre polynomials also aries in quantum mechanics in the radial part of the solution of the Schrödinger equation for one-electron atom:

$$\begin{aligned}
\frac{e^{-\frac{[x]_q t}{1-t}}}{(1-t)^{k+1}} &= \frac{1}{(1-t)^k} \frac{e^{-\frac{[x]_q t}{1-t}}}{1-t} \\
&= \left(\sum_{m=0}^{\infty} \binom{m+k-1}{m} t^m \right) \left(\sum_{l=0}^{\infty} L_{l,q}(x) \frac{t^l}{l!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n+k-l-1}{n-l} \frac{L_{l,q}(x)}{l!} \right) t^n.
\end{aligned} \tag{14}$$

By comparing the coefficients on the both sides of (12) and (14), we obtain the following corollary.

Corollary 3. For $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, we have

$$L_{n,q}^{(k)}(x) = \sum_{l=0}^{n} \binom{n+k-l-1}{n-l} \frac{1}{l!} L_{l,q}(x).$$

For example, $L_{0,q}^{(k)}(x) = 1, L_{1,q}^{(k)}(x) = 1+k-[x]_q, L_{2,q}^{(k)}(x) = (2+k)(1+k)-(2+k)[x]_q + \frac{[x]_q^2}{2}, \cdots$.

By (11), we get

$$L_{n,\frac{1}{q}}(1-x) = \sum_{l=0}^{n} (-1)^l \frac{[1-x]_{\frac{1}{q}}^l}{l!} \binom{n}{l} = \sum_{l=0}^{n} (-1)^l [x]_q^{n-l} [1-x]_{\frac{1}{q}}^l \binom{n}{l} \frac{[x]_q^{l-n}}{l!}. \tag{15}$$

From (8), (9) and (15), we can derive the following equation.

$$L_{n,\frac{1}{q}}(1-x) = \sum_{l=0}^{n} (-1)^l B_{n-l,n}(x \mid q) \frac{[x]_q^{l-n}}{l!} = \sum_{l=0}^{n} (-1)^l B_{l,n}(1-x \mid \frac{1}{q}) \frac{[x]_q^{l-n}}{l!}. \tag{16}$$

Therefore, we obtain the following proposition.

Proposition 4. For $n \in \mathbb{Z}_+$, we have

$$[1-x]_{\frac{1}{q}}^n L_{n,q}(x) = \sum_{l=0}^{n} B_{l,n}(x \mid q) \frac{([x]_q - 1)^l}{l!}, \text{ where } x \in [0,1].$$

From Proposition 2, we have

$$\begin{aligned} [x]_q^k L_{n,q}^{(k)}(x) &= \sum_{l=0}^{n} (-1)^l \binom{n+k}{l+k} \frac{[x]_q^{l+k}}{l!} \\ &= \sum_{l=0}^{n} (-1)^l \binom{n+k}{l+k} \frac{[x]_q^{l+k}}{l!} [1-x]_q^{n-l} \left(\frac{1}{[1-x]_q^{n-l}} \right) \\ &= \sum_{l=0}^{n} (-1)^l B_{l+k,n+k}(x \mid q) \left(\frac{1}{[1-x]_{\frac{1}{q}}^{n-l}} \right) \frac{1}{l!}. \end{aligned} \tag{17}$$

By (9) and (17), we obtain the following theorem.

Theorem 5. For $n,k\in\mathbb{Z}_+$, we have

$$[x]_q^k[1-x]_{\frac{1}{q}}^n L_{n,q}^{(k)}(x)=\sum_{l=0}^n B_{l+k,n+k}(x\mid q)\frac{([x]_q-1)^l}{l!}$$
$$=\sum_{l=0}^n B_{n-l,n+k}(1-x\mid \frac{1}{q})\frac{([x]_q-1)^l}{l!}.$$

For example, $[x]_q^k L_{0,q}^{(k)}(x)=B_{k,k}(x\mid q)=[x]_q^k$. So, $L_{0,q}^{(k)}(x)=1$.
For $n=1$, we have

$$[x]_q^k[1-x]_{\frac{1}{q}} L_{1,q}^{(k)}(x)=\sum_{i=0}^1 B_{i+k,1+k}(x\mid q)\frac{([x]_q-1)^i}{i!}$$
$$=B_{k,1+k}(x\mid q)+B_{1+k,1+k}(x\mid q)([x]_q-1)$$
$$=(1+k)[x]_q^k[1-x]_{\frac{1}{q}}-[x]_q^{1+k}[1-x]_{\frac{1}{q}}.$$

Thus, $L_{1,q}^{(k)}(x)=1+k-[x]_q$, by the same method $L_{2,q}^{(k)}(x)=(2+k)(1+k)-(2+k)[x]_q+\frac{[x]_q^2}{2},\cdots$.

By (8), we get

$$[1-x]_{\frac{1}{q}}B_{k,n-1}(x\mid q)+[x]_qB_{k-1,n-1}(x\mid q)=\binom{n}{k}[x]_q^k[1-x]_{\frac{1}{q}}^{n-k}=B_{k,n}(x\mid q). \tag{18}$$

From (18), we note that

$$B_{l+k,n+k}(x\mid q)=[1-x]_{\frac{1}{q}}B_{l+k,n-1+k}(x\mid q)+[x]_qB_{l+k-1,n+k-1}(x\mid q). \tag{19}$$

Thus, by (17) and (19), we have

$$\begin{aligned}&[1-x]_{\frac{1}{q}}\sum_{l=0}^n B_{l+k,n-1+k}(x\mid q)\frac{([x]_q-1)^l}{l!}+[x]_q\sum_{l=0}^n B_{l+k-1,n+k-1}(x\mid q)\frac{([x]_q-1)^l}{l!}\\&=\sum_{l=0}^n B_{l+k,n+k}(x\mid q)\frac{([x]_q-1)^l}{l!}.\end{aligned} \tag{20}$$

Therefore, by Theorem 5 and (20), we obtain the following theorem.

Theorem 6. For $n,k\in\mathbb{Z}_+$ and $x\in[0,1]$, we have

$$[1-x]_{\frac{1}{q}}L_{n-1,q}^{(k)}(x)+[x]_qL_{n-1,q}^{(k-1)}(x)=[x]_q^k[1-x]_{\frac{1}{q}}^nL_{n,q}^{(k)}(x).$$

From (8) and (9), we note that

$$B_{k,n}(x \mid q) = \sum_{i=k}^{n} \binom{i}{k}\binom{n}{i}(-1)^{i-k}[x]_q^i. \tag{21}$$

By Theorem 5 and (21), we get

$$[x]_q^k[1-x]_{\frac{1}{q}}^n L_{n,q}^{(k)}(x) = \sum_{l=0}^{n}\left(\sum_{i=l+k}^{n+k}\binom{i}{l+k}\binom{n+k}{i}(-1)^{i-l-k}[x]_q^i\right)\frac{([x]_q-1)^l}{l!}.$$

Therefore, we obtain the following theorem.

Theorem 7. For $n, k \in \mathbb{Z}_+$, we have

$$[x]_q^k[1-x]_{\frac{1}{q}}^n L_{n,q}^{(k)}(x) = \sum_{l=0}^{n}\left(\sum_{i=l+k}^{n+k}\binom{i}{l+k}\binom{n+k}{i}(-1)^{i-l-k}[x]_q^i\right)\frac{(-1)^l[1-x]_{\frac{1}{q}}^l}{l!}.$$

By (8), we get

$$\begin{aligned}
&\left(\frac{n-k+1}{k}\right)\left(\frac{[x]_q}{[1-x]_{\frac{1}{q}}}\right)B_{k-1,n}(x \mid q)\\
&=\left(\frac{n-k+1}{k}\right)\left(\frac{[x]_q}{[1-x]_{\frac{1}{q}}}\right)\binom{n}{k-1}[x]_q^{k-1}[1-x]_{\frac{1}{q}}^{n-k+1}\\
&=\frac{n!}{k!(n-k)!}[x]_q^k[1-x]_{\frac{1}{q}}^{n-k} = B_{k,n}(x \mid q).
\end{aligned} \tag{22}$$

From (22) and Theorem 5, we can derive the following equation.

$$\begin{aligned}
&[1-x]_{\frac{1}{q}}^{n+1}[x]_q^{k-1}L_{n,q}^{(k)}(x) = \sum_{l=0}^{n}\left(\frac{n-l+1}{l+k}\right)B_{l+k-1,n+k}(x \mid q)(-1)^l\frac{[1-x]_{\frac{1}{q}}^l}{l!}\\
&=(n+k+1)\sum_{l=0}^{n}\left(\frac{1}{l+k}\right)B_{l+k-1,n+k}(x \mid q)\frac{([x]_q-1)^l}{l!} - [x]_q^{k-1}[1-x]_q^n L_{n,q}^{(k-1)}(x)\\
&=(n+k+1)\sum_{l=0}^{n}\left(\frac{1}{l+k}\right)B_{l+k-1,n+k}(x \mid q)\frac{([x]_q-1)^l}{l!} - [x]_q^{k-1}[1-x]_{\frac{1}{q}}^n L_{n,q}^{(k-1)}(x).
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 8. For $n \in \mathbb{Z}_+, k \in \mathbb{N}$ and $x \in [0,1]$, we have

$$\begin{aligned}
&[1-x]_{\frac{1}{q}}^{n+1}[x]_q^{k-1}L_{n,q}^{(k)}(x) + [x]_q^{k-1}[1-x]_{\frac{1}{q}}^n L_{n,q}^{(k-1)}(x)\\
&=(n+k+1)\sum_{l=0}^{n}\left(\frac{B_{l+k-1,n+k}(x \mid q)}{l+k}\right)\frac{([x]_q-1)^l}{l!}.
\end{aligned}$$

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