# IDENTITIES ON $q$-HYPERGEOMETRIC BERNOULLI NUMBERS AND POLYNOMIALS

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Abstract. In this paper, we introduce new $q$-hypergeometric Bernoulli polynomials and numbers. Some basic and important identities about them are presented. In particular, we give a relation among two kinds of $q$-hypergeometric Bernoulli polynomials, from which a Bernoulli polynomial version of Kaneko-Momiyama relations among Bernoulli numbers is obtained.

Contents



## 1. Introduction and main results

Throughout this paper we fix $q \in \mathbb{C}$, which is different from the roots of unity. We recall some usual notions and notations used in $q$-theory (see [1], [2], [6]).

1.1. **Notation and preliminaries.** Let $a \in \mathbb{C}$. The *$q$-shifted factorials* are defined by

$$(a,q)_0 = 1, \quad (a,q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad (n = 1, 2, \ldots).$$

If $|q| < 1$, then we define

$$(a,q)_\infty = \lim_{n\to\infty}(a,q)_n = \prod_{k=0}^{\infty}(1-aq^k).$$

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We also denote

$$\begin{aligned}
[x]_q &= \frac{1-q^x}{1-q}, \quad x \in \mathbb{C}, \\
[n]_q! &= \frac{(q,q)_n}{(1-q)^n}, \quad n \in \mathbb{N}, \\
\begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad k,n \in \mathbb{N}, \\
\begin{bmatrix} n \\ i_1,\cdots,i_m \end{bmatrix}_q &= \frac{[n]_q!}{[i_1]_q!\cdots[i_m]_q!}, \quad n,i_1,\ldots,i_m \in \mathbb{N}.
\end{aligned}$$

The $q$-*exponential functions* are given by

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}$$

and

$$e_{q^{-1}}(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q^{-1}}!}.$$

It is easy to see that $[n]_{q^{-1}}! = q^{-\binom{n}{2}}[n]_q!$. Hence

$$e_{q^{-1}}(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}.$$

See [3], [5] for related topics. By $q$-binomial theorem [2], we have

$$e_q(z) = \frac{1}{((1-q)z,q)_\infty}, \qquad e_{q^{-1}}(z) = (-(1-q)z,q)_\infty .$$

This yields $e_q(z)e_{q^{-1}}(-z) = 1$.

1.2. **The $q$-derivative and $q$-integral.** The $q$-*derivative* of a function $f$ is given by

$$D_q f(x) := \frac{f(x)-f(qx)}{(1-q)x} \qquad (x \neq 0, q \neq 1),$$

where $x$ and $qx$ should be in the domain of $f$. If $f$ is differentiable on an open set $I$, then for all $x \in I$,

$$\lim_{q\to 1} D_q f(x) = f'(x).$$

Besides, for all $n \in \mathbb{N}$,

$$\begin{aligned}
D_q(x^n) &= [n]_q x^{n-1}, \\
D_q(x,q)_n &= -[n]_q(xq,q)_{n-1}, \\
D_{q^{-1}}(x,q)_n &= -[n]_q(x,q)_{n-1}, \\
D_q\left(\frac{x^n}{[n]_q!}\right) &= \frac{x^{n-1}}{[n-1]_q!}.
\end{aligned}$$

From the last identity, for instance, we have $D_q e_q(z) = e_q(z)$. For the product of two functions $f$ and $g$, the following formula holds:

$$\begin{aligned}
D_q(f\cdot g)(x) &= g(x)D_q(x) + f(qx)D_q g(x) \\
&= f(x)D_q g(x) + g(qx)D_q f(x).
\end{aligned}$$

We next treat the composition of $f(x)$ and $g(x)$. When $g(x) = -x$, the following chain rule for the $q$-derivative is valid:

$$D_q(f \circ g)(x) = D_q f(g(x)) D_q g(x),$$

which will be used in the proofs of Theorems 1.6 and 1.7. However, in general, the rule above does not hold. If we modify the definition of the composition of two functions, then a new chain rule for the $q$-derivative is gained. We refer to Gessel [3] for this topic.

The $q$-*Jackson integrals* from 0 to $a$ is defined by

$$\int_0^a f(x) d_q x \quad := \quad (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n$$

provided the infinite sums converge absolutely. The $q$-Jackson integral in the generic interval $[a, b]$ is given by

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

For any function $f$ we have

$$D_q \int_0^x f(t) d_q t = f(x).$$

1.3. **$q$-Hypergeometric Bernoulli polynomials and numbers.** Let $q \in \mathbb{C}, |q| < 1$. We define two different $q$-hypergeometric Bernoulli polynomials $B_n(x, q)$, $C_n(x, q^{-1})$ in the variable $x$ by

$$\frac{te_q(xt)}{e_q(t) - 1} \quad = \quad \sum_{n=0}^{\infty} B_n(x, q) \frac{t^n}{[n]_q!}, \tag{1.1}$$

$$\frac{te_q(xt)}{e_{q^{-1}}(t) - 1} \quad = \quad \sum_{n=0}^{\infty} C_n(x, q^{-1}) \frac{t^n}{[n]_q!}. \tag{1.2}$$

We call $B_n(0, q)$ (resp. $C_n(0, q^{-1})$) the *first* (resp. *second*) *$q$-hypergeometric Bernoulli numbers, respectively.*

1.4. **The $q$-binomial formula.** Let $q \in \mathbb{C}$, and take two $q$-commuting variables $x$ and $y$ which satisfy the relation

$$xy = q^{-1} yx.$$

Let $\mathbb{C}_q[x, y]$ be the complex associative algebra with 1 generated by $x$ and $y$. Then the following identity is valid in the algebra $\mathbb{C}_q[x, y]$:

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}, \qquad n \in \mathbb{N},$$

or alternatively,

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} y^k x^{n-k}, \qquad n \in \mathbb{N}.$$

For details, we refer to [1], [2].

1.5. **The $q$-exponential identity.** Let $\mathbb{C}_q[[x,y]]$ be the complex associative algebra with 1 of formal power series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n}x^n y^n$$

with arbitrary complex coefficients $a_{m,n}$. One knows in [1], [2] that in $\mathbb{C}_q[[x,y]]$, we have the following identity

$$e_q(x+y) = e_q(x)e_q(y).$$

We can easily verify the following identities:

$$\begin{aligned}
\lim_{q\to 1} e_q(z) &= \lim_{q\to 1} e_{q^{-1}}(z) = e^z,\\
\lim_{q\to 1} [n]_q &= n,\\
\lim_{q\to 1} [n]_q! &= n!,\\
\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \binom{n}{k},\\
\lim_{q\to 1} \begin{bmatrix} n \\ i_1,\cdots,i_m \end{bmatrix}_q &= \begin{pmatrix} n \\ i_1,\cdots,i_m \end{pmatrix} := \frac{n!}{i_1!\cdots i_m!},\\
\lim_{q\to 1} B_n(x,q) &= \lim_{q\to 1} C_n(x,q^{-1}) = B_n(x),
\end{aligned}$$

where $B_n(x)$ is the $n$-th classical Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

1.6. **Purpose and main results.** In this paper we deal with very interesting identities satisfied by our $q$-hypergeometric Bernoulli numbers and polynomials. We state our results.

**Theorem 1.1** (Sums of products)**.** *Let $m$ be a positive integer. For any $n \geq m$, we have*

$$\begin{aligned}
&(-1)^n \sum_{i_1+\cdots+i_m=n} \begin{bmatrix} n \\ i_1,\cdots,i_m \end{bmatrix}_q C_{i_1}(-x,q^{-1})\cdots C_{i_m}(-x,q^{-1})\\
&\quad = \sum_{j=0}^{m}\binom{m}{j} \sum_{\substack{k_1+\cdots+k_m\\ =n-m+j}} \begin{bmatrix} n \\ k_1,\ldots,k_m \end{bmatrix}_q B_{k_1}(x,q)\cdots B_{k_m}(x,q)x^{m-(k_1+\cdots+k_j)}.
\end{aligned}$$

*In particular, if $m=1$, then*

$$(-1)^n C_n(-x,q^{-1}) = B_n(x,q) + [n]_q x^{n-1}.$$

*If $m=2$, then*

$$\begin{aligned}
&(-1)^n \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q C_i(-x,q^{-1})C_{n-i}(-x,q^{-1})\\
&\quad = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q B_i(x,q)B_{n-i}(x,q) + 2\sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix}_q B_i(x,q)x^{n-1-i} + \left(\sum_{i=0}^{n-2} \begin{bmatrix} n \\ i \end{bmatrix}_q\right) x^n.
\end{aligned}$$

As $q\to 1$ in the above theorem, we have

**Theorem 1.2.** *If $n \geq m$, then*

$$(-1)^n \sum_{i_1+\cdots+i_m=n} \left( \begin{array}{c} n \\ i_1, \ldots, i_m \end{array} \right) B_{i_1}(-x) \cdots B_{i_m}(-x)$$
$$= \sum_{j=0}^{m} \binom{m}{j} \sum_{\substack{k_1+\cdots+k_m \\ =n-m+j}} \left( \begin{array}{c} n \\ k_1, \ldots, k_m \end{array} \right) B_{k_1}(x) \cdots B_{k_m}(x) x^{m-(k_1+\cdots+k_m)}.$$

Putting $x = 0$ and $m = 2$, we get Euler's identity:

**Theorem 1.3.** *For any odd integer $n \geq 2$,*

$$\sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} = -n B_{n-1}.$$

**Theorem 1.4** ($q$-Derivative formula)**.** *We have*

$$\begin{aligned} D_q B_0(x,q) &= D_q C_0(x,q^{-1}) = 0, \\ D_q B_{n+1}(x,q) &= [n+1]_q B_n(x,q), \quad (n \geq 0) \\ D_q C_{n+1}(x,q^{-1}) &= [n+1]_q C_n(x,q^{-1}). \end{aligned}$$

**Theorem 1.5** ($q$-Addition formula)**.** *Let $x, y$ be two $q$-commuting variables which satisfy $xy = q^{-1}yx$. Then for any nonnegative integer $n$, we have*

$$B_n(x+y,q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q B_k(x,q) y^{n-k}.$$

*In particular,*

$$B_n(y,q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q B_k(0,q) y^{n-k}.$$

**Theorem 1.6** ($q$-Integral formula)**.** *For all nonnegative integers $n$,*

$$\int_a^x B_n(t,q) d_q t = \frac{B_{n+1}(x,q) - B_{n+1}(a,q)}{[n+1]_q}.$$

**Theorem 1.7** ($q$-Symmetry)**.** *For any $n, m \in \mathbb{N}$, we have*

$$(1.3) \quad (-1)^n \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q B_{m+k}(x,q) q^{-km} =$$
$$(-1)^m \sum_{k=0}^{m} \left[ \begin{array}{c} n \\ k \end{array} \right]_q C_{n+k}(-x,q^{-1}) q^{\binom{m-k}{2}-mn}.$$

By $q$-differentiation, we obtain

**Theorem 1.8** ($q$-Recurrence formula 1)**.** *For $n, m \in \mathbb{N}$,*

$$(-1)^n \sum_{k=0}^{n+1} \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_q [m+k+1]_q B_{m+k}(x,q) q^{-k(m+1)}$$
$$+ (-1)^m \sum_{k=0}^{m+1} \left[ \begin{array}{c} m+1 \\ k \end{array} \right]_q [n+k+1]_q C_{n+k}(-x,q^{-1}) q^{\binom{m+1-k}{2}-(m+1)(n+1)} = 0.$$

As an application, as $q \to 1$, we get from Theorem 1.5 and Theorem 1.6 the results

**Theorem 1.9.**

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{n+k}(x) = (-1)^m \sum_{k=0}^{m} \binom{m}{k} B_{n+k}(-x),$$

$$(-1)^n \sum_{k=0}^{n+1} \binom{n+1}{k} (m+k+1) B_{m+k}(x) + (-1)^m \sum_{k=0}^{m+1} \binom{m+1}{k} (n+k+1) B_{n+k}(-x) = 0.$$

The second identity is a polynomial version of Kaneko-Momiyama's formulae [4], [7], [8] on Bernoulli numbers. Indeed, to obtain Kaneko's formula, we set $m = n$ and $x = 0$:

**Theorem 1.10.**

$$\sum_{k=0}^{n+1} \binom{n+1}{k} (n+k+1) B_{n+k} = 0.$$

For Momiyama's formula, put $x = 0$:

**Theorem 1.11.**

$$(-1)^n \sum_{k=0}^{n+1} \binom{n+1}{k} (m+k+1) B_{m+k} + (-1)^m \sum_{k=0}^{m+1} \binom{m+1}{k} (n+k+1) B_{n+k} = 0.$$

Now, by $q$-integration, we derive

**Theorem 1.12** ($q$-Recurrence formula 2)**.** *For any $n, m \in \mathbb{N}$, $a, b \in \mathbb{R}$,*

$$(-1)^n \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \frac{B_{m+k+1}(a,q) - B_{m+k+1}(b,q)}{[m+k+1]_q} q^{-km}$$
$$+ (-1)^m \sum_{k=0}^{m} \left[ \begin{array}{c} m \\ k \end{array} \right]_q \frac{C_{n+k+1}(-a,q^{-1}) - C_{n+k+1}(-b,q^{-1})}{[n+k+1]_q} q^{\binom{m-k}{2} - mn} = 0.$$

## 2. Proofs of main results

*Proof of Theorem 1.1:*
Using $e_q(t) e_{q^{-1}}(-t) = 1$, we have

$$\frac{-t}{e_{q^{-1}}(-t) - 1} = t + \frac{t}{e_q(t) - 1}.$$

From this identity,

$$\left( t e_q(xt) + \frac{t e_q(xt)}{e_q(t) - 1} \right)^m = \left( \frac{-t}{e_{q^{-1}}(-t) - 1} \cdot e_q\left((-x)(-t)\right) \right)^m.$$

The right hand side is equal to

$$\sum_{n=0}^{\infty} \sum_{i_1 + \cdots + i_m = n} \left[ \begin{array}{c} n \\ i_1, \ldots, i_m \end{array} \right] C_{i_1}(-x, q^{-1}) \cdots C_{i_m}(-x, q^{-1}) \frac{t^n}{[n]_q!},$$

while the left hand side is

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}\left(\frac{te_q(xt)}{e_q(t)-1}\right)^j t^{m-j}e_q(xt)^{m-j}\\&=\sum_{j=0}^{m}\binom{m}{j}\sum_{n=m-j}^{\infty}\sum_{\substack{i_1+\cdots+i_m\\=n-m+j}}\left[\begin{array}{c}n\\i_1,\ldots,i_m\end{array}\right]B_{i_1}(x,q)\cdots B_{i_m}(x,q)x^{m-(i_1+\cdots+i_m)}\frac{t^n}{[n]_q!}.\end{aligned}$$

This completes the proof of the theorem. □

*Proof of Theorem 1.4:*
Operating $D_q$, the $q$-derivative with respect to variable $x$, on

$$\frac{te_q(xt)}{e_q(t)-1}=\sum_{n=0}^{\infty}B_n(x,q)\frac{t^n}{[n]_q!},$$

we obtain

$$\frac{t^2e_q(xt)}{e_q(t)-1}=\sum_{n=0}^{\infty}D_qB_n(x,q)\frac{t^n}{[n]_q!}.$$

From this, we have

$$\begin{aligned}D_qB_0(x,q)&=0,\\D_qB_{n+1}(x,q)&=[n+1]_qB_n(x,q),\qquad n\in\mathbb{N}.\end{aligned}$$

Similarly we obtain

$$\begin{aligned}D_qC_0(x,q^{-1})&=0,\\D_qC_{n+1}(x,q^{-1})&=[n+1]_qC_n(x,q^{-1}),\qquad n\in\mathbb{N}.\end{aligned}$$

Thus this proves the theorem. □

*Proof of Theorem 1.5:*
Let $x, y$ be two $q$-commuting variables satisfying $xy=q^{-1}yx$. We have

$$\frac{t}{e_q(t)-1}e_q((x+y)t)=\frac{t}{e_q(t)-1}e_q(xt)e_q(yt),$$

where $t$ is a variable commuting with $x$ and $y$. If $wt=tw$, then

$$e_q(wt)=\sum_{n=0}^{\infty}\frac{w^nt^n}{[n]_q}.$$

Hence

$$\sum_{n=0}^{\infty}B_n(x+y,q)\frac{t^n}{[n]_q!}=\left(\sum_{m=0}^{\infty}B_m(x,q)\frac{t^m}{[m]_q!}\right)\left(\sum_{k=0}^{\infty}\frac{y^kt^k}{[k]_q!}\right),$$

which yields the first identity.

For $z=xt_1$ ($t_1\in\mathbb{C}$), we have $zy=q^{-1}yz$. Then

$$B_n(z+y,q)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q B_k(z,q)y^{n-k}.$$

As $t_1\to 0$, we get the second identity. □

*Proof of Theorem 1.6:*
By Theorem 1.2,

$$\int_a^x B_n(t,q)d_qt = \int_a^x \frac{D_qB_{n+1}(t,q)}{[n+1]_q}d_qt.$$

This yields the result. □

*Proof of Theorem 1.7:*
For two $q$-commuting variables $x, y$ satisfying $xy = q^{-1}yx$, we now consider the generating functions for $q$-hypergeometric Bernoulli polynomials:

$$\begin{aligned} L(w,x,y) &:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q B_{m+k}(w,q)q^{m(n-k)}\frac{x^m}{[m]_q!}\frac{y^n}{[n]_q!}, \\ R(w,x,y) &:= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{m} \left[ \begin{array}{c} m \\ k \end{array} \right]_{q^{-1}} C_{n+k}(-w,q^{-1})q^{kn-\binom{n+k}{2}}\frac{x^m}{[m]_{q^{-1}}!}\frac{y^n}{[n]_{q^{-1}}!}, \end{aligned}$$

where $w$ is a commuting variable with $x$ and $y$. We calculate them:

$$\begin{aligned} L(w,x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q B_{m+k}(w,q)\frac{y^{n-k}(-x)^m y^k}{[m]_q![n]_q!} \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} B_{m+k}(w,q)\sum_{k=0}^{n}\frac{y^{n-k}}{[n-k]_q!}\frac{(-x)^m}{[m]_q!}\frac{y^k}{[k]_q!} \\ &= e_q(y)\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} B_{m+k}(w,q)\frac{(-x)^m}{[m]_q!}\frac{y^k}{[k]_q!} \\ &= e_q(y)\sum_{j=0}^{\infty} B_j(w,q)\sum_{k=0}^{j}\frac{(-x)^{j-k}}{[j-k]_q!}\frac{y^k}{[k]_q!} \\ &= e_q(y)\sum_{j=0}^{\infty} B_j(w,q)\frac{(y-x)^j}{[j]_q!} \\ &= e_q(y)\cdot\frac{y-x}{e_q(y-x)-1}\cdot e_q(w(y-x)). \end{aligned}$$

$$\begin{aligned}
R(w,x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n+k}(-w,q^{-1}) q^{kn-\binom{n+k}{2}} \sum_{k=0}^{m} \frac{x^{m-k}}{[m-k]_{q^{-1}}!}\frac{x^k}{[k]_{q^{-1}}!}\frac{(-y)^n}{[n]_{q^{-1}}!} \\
&= e_{q^{-1}}(x)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n+k}(-w,q^{-1}) q^{-\binom{n+k}{2}} q^{kn} \frac{x^k}{[k]_{q^{-1}}!}\frac{(-y)^n}{[n]_{q^{-1}}!} \\
&= e_{q^{-1}}(x)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_{n+k}(-w,q^{-1}) q^{-\binom{n+k}{2}} \frac{(-y)^n}{[n]_{q^{-1}}!}\frac{x^k}{[k]_{q^{-1}}!} \\
&= e_{q^{-1}}(x)\sum_{j=0}^{\infty} C_{j}(-w,q^{-1}) q^{-\binom{j}{2}} \sum_{k=0}^{j} \frac{(-y)^{j-k}}{[j-k]_{q^{-1}}!}\frac{x^k}{[k]_{q^{-1}}!} \\
&= e_{q^{-1}}(x)\sum_{j=0}^{\infty} C_{j}(-w,q^{-1}) q^{-\binom{j}{2}} \frac{(x-y)^j}{[j]_{q^{-1}}!} \\
&= e_{q^{-1}}(x)\sum_{j=0}^{\infty} C_{j}(-w,q^{-1}) \frac{(x-y)^j}{[j]_{q}!} \\
&= e_{q^{-1}}(x)\cdot \frac{x-y}{e_{q^{-1}}(x-y)-1}\cdot e_q(-w(x-y)).
\end{aligned}$$

Using $e_q(z)e_{q^{-1}}(-z) = e_q(-z)e_{q^{-1}}(z) = 1$, we obtain $L(w,x,y) = R(w,x,y)$. From this,

$$\begin{aligned}
(-1)^m \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{m+k}(w,q) q^{m(n-k)} \frac{1}{[m]_q![n]_q!} \\
= (-1)^n \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_{q^{-1}} C_{n+k}(-w,q^{-1}) q^{kn-\binom{n+k}{2}} \frac{1}{[m]_{q^{-1}}![n]_{q^{-1}}!}.
\end{aligned}$$

Using

$$\begin{aligned}
[m]_{q^{-1}}! &= q^{-\binom{m}{2}}[m]_q!, \\
\begin{bmatrix} m \\ k \end{bmatrix}_{q^{-1}} &= q^{-\binom{m}{2}+\binom{k}{2}+\binom{m-k}{2}} \begin{bmatrix} m \\ k \end{bmatrix}_q,
\end{aligned}$$

we get our identity. □

*Proof of Theorem 1.8:*
Applying the $q$-differentiation by $x$ to the identity (1.3). We obtain from Theorem 1.4 our theorem. □

*Proof of Theorem 1.12:*
Applying the $q$-integration by $x$ to the identity (1.3). We obtain from Theorem 1.6 our theorem.

□

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