VALUES OF TWISTED BARNES ZETA FUNCTIONS AT NEGATIVE INTEGERS

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Abstract. In this paper we study the analytical and arithmetical properties of the twisted zeta function given by the following expression

$$\frac{1}{\Gamma(s)}\int_0^{\infty} e^{-xt}t^{s-1}\prod_{j=1}^{N}\frac{a_jt-log(w^{a_j})}{1-w^{a_j}e^{-a_jt}}dt \tag{0.1}$$

where $Re(s)>N$,$Re(x)>0$, $w\in\mathbb{C}\backslash\{0\}$, $N\in\mathbb{N}$ and $a_1,\cdots,a_N$ have positive real part. These functions have many interesting properties. We prove a collection of fundamental identities satisfied by this zeta function. For instance, the special values of these zeta functions are connected to twisted Barnes numbers and polynomials. This give us an elementary new approach to show various new and known results concerning the Barnes zeta functions. We derive from our study all known results on Hurwitz zeta functions.

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1. Introduction and motivation

In this paper our main motivations come from various identities satisfied by the Riemann, Hurwitz and Barnes zeta functions. These identities are at the origin of numerous applications in various areas in mathematics and physics. Especially, the identities on Riemann-Hurwitz's zeta function occurs in a variety of disciplines. Most commonly, it occurs in number theory, where its theory is the deepest and most developed. In number theory the Riemann and Hurwitz zeta funtions are closely connected to Dedekind zeta and Artin L-funtion which play a central role in this discipline. However, it also occurs in the study of fractals and dynamical systems. In applied statistics, it occurs in Zipf's law and the Zipf-Mandelbrot law. In particle physics, it occurs in a formula by Julian Schwinger [?] giving an exact result for the pair production rate of a Dirac electron in a uniform electric field. On the other hand, we know less well all the properties satisfied by Barnes zeta function as well as their applications which can ensue from it. Let us recall briefly only their fundamental analytical properties.

1.1. **Hurwitz zeta function:** The Hurwitz zeta function is one of the many zeta functions. It is formally defined for complex arguments s with $Re(s)>1$ and x with $Re(x)>0$ by

$$\zeta(s,x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^s}. \tag{1.1}$$

The most important properties of $\zeta(s,x)$ are:

(1) The function $s\to\zeta(s,x)$ has meromorphic continuation to whole complex plane $\mathbb{C}$, whose only poles at $s=1$. At $s=1$ it has a simple pole with residue 1. The constant term is given by

$$\lim_{s\to 1}\left[\zeta(s,x)-\frac{1}{s-1}\right]=-\psi_0(x), \tag{1.2}$$

where $\psi_0(x)=\frac{d}{dx}ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function.

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(2) It has an integral representation in terms of the Mellin-Laplace transform as

$$\zeta(s,x)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-xt}}{1-e^{-t}}dt \tag{1.3}$$

for $Re(s)>1$ and $Re(x)>0$.

(3) The values of $\zeta(s,x)$ at $s=-n$, $n\in\mathbb{N}$ are given

$$\zeta(-n,x)=-\frac{B_{n+1}(x)}{n+1}, \tag{1.4}$$

and the coefficients of the polynomial $\zeta(-n,x)$ are rational. The Bernoulli polynomials are given by the generating function

$$\frac{t}{e^t-1}e^{tx}=\sum_{n=0}^\infty B_n(x)\frac{t^n}{n!}\ ,\ |t|<2\pi, \tag{1.5}$$

and $B_n:=B_n(0)$ are the Bernoulli numbers. These numbers are rational numbers.

(4) There is the multiplication theorem

$$\sum_{p=0}^{q-1}\zeta(s,x+p/q)=q^s\,\zeta(s,qx),\ q\in\mathbb{N}^*, \tag{1.6}$$

(5) There is the difference theorem

$$\zeta(s,x+1)-\zeta(s,x+1)=-x^{-s}. \tag{1.7}$$

(6) At $s=-n$, $n\in\mathbb{N}$, there is a symmetry relation:

$$\zeta(-n,1-x)=(-1)^{n+1}\zeta(-n,x). \tag{1.8}$$

,

(7) The derivative of the Hurwitz zeta in the second argument is a shift:

$$\frac{\partial}{\partial x}\zeta(s,x)=-s\zeta(s+1,x). \tag{1.9}$$

(8) It has an addition identity:

$$\zeta(s,x+y)=\sum_{k=0}^\infty\binom{s+k-1}{s-1}(-y)^k\zeta(s+k,x). \tag{1.10}$$

See [**?**, **?**, **?**] for basic properties of Riemann and Hurwitz zeta functions.

1.2. **Barnes multiple zeta function.** The Barnes multiple zeta function is a multidimensional generalization of the Hurwitz zeta function. For this section we refer to the remarkable works of Friedman and Ruijsenaars [**?**, **?**]. The Barnes multiple zeta is given by

$$\zeta_N(s,x|a_1,...,a_N)=\sum_{n_1,...,n_N\geq 0}\frac{1}{(x+n_1a_1+\cdots+n_Na_N)^s} \tag{1.11}$$

where x and a_j have positive real part and s has real part greater than the positive integer N. For $N=a_1=1$ it is the Hurwitz zeta function.

$\zeta_N(s,x|a_1,...,a_N)$ has the following properties:

(1) It has a meromorphic continuation to all complex s, whose only singularities are simple poles at $s=1,2,...,N$.

(2) It has an integral representation in terms of the Mellin-Laplace transform as

$$\zeta_N(s,x|a_1,...,a_N)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{e^{-xt}t^{s-1}}{\prod_{j=1}^N(1-e^{-a_jt})}dt\ ,\ Re(s)>N,\ Re(x)>0. \tag{1.12}$$

(3) The values of $\zeta_N(s,x|a_1,...,a_N)$ at $s=-n$, $n\in\mathbb{N}$ are given by

$$\zeta_N(-n,x|a_1,...,a_N)=\frac{(-1)^N n!}{(n+N)!}B_{n+N}(x|a_1,...,a_N), \tag{1.13}$$

where $B_n(x|a_1,...,a_N)$ are Barnes multiple polynomials defined by

$$\frac{t^N}{\prod_{j=1}^{N}(e^{a_jt}-1)}e^{xt}=\sum_{n=0}^{\infty}B_n(x|a_1,...,a_N)\frac{t^n}{n!},\ |t|<\min\left(\frac{2\pi}{a_1},\cdots,\frac{2\pi}{a_N}\right). \tag{1.14}$$

(4) The derivative of the Barnes multiple zeta in the second argument is a shift:

$$\frac{\partial}{\partial x}\zeta_N(s,x|a_1,...,a_N)=-s\zeta_N(s+1,x|a_1,...,a_N). \tag{1.15}$$

(5) It has a difference-recurrence formula

$$\zeta_{N+1}(s,x+a_{N+1}|a_1,...,a_{N+1})-\zeta_{N+1}(s,x|a_1,...,a_{N+1})=-\zeta_N(s,x|a_1,...,a_N) \tag{1.16}$$

with $\zeta_0(s,x)=-x^{-s}$.

(6) It has an addition identity:

$$\zeta_N(s,x+y|a_1,...,a_N)=\sum_{k=0}^{\infty}\binom{s+k-1}{s-1}(-y)^k\zeta_N(s+k,x|a_1,...,a_N). \tag{1.17}$$

Regrettably, we did not find in the literature the multipcation and the symmetry formulae for Barnes multiple zeta function. We remedy it in our work.

In this paper we study new zeta function (**??**). It's obvious to note that our zeta recovers Hurwitz and Barnes multiple zeta functions. According to the expressions (**??**) and (**??**) if we take $N=1, a_1=1$ and $w=1$ we obtain Hurwitz zeta function. From (**??**) and (**??**) if we take $w=1$ we obtain Barnes multiple zeta function. Incidentally it recovers Riemann zeta: if we take $N=1, a_1=1$ and $x=w=1$.

In this paper we prove extension of all properties to those quoted above for the zeta (**??**). For instance, the properties (**??**),(**??**),(**??**),(**??**),(**??**). Our study allows us to investigate new properties satisfied by this new zeta.

2. Identities on twisted Barnes zeta functions

In this section we construct and investigate properties of twisted Barnes zeta functions and twisted Barnes polynomials and numbers. Let $v\in\mathbb{N}$, $\vec{a}=(a_1,...,a_v)$, where $a_1,\cdots,a_v$ are strictly positive real numbers.

Let $w\in\mathbb{C}\backslash\{0\}$. We consider $F_w^{(v)}(t\mid\vec{a})$ the function

$$F_w^{(v)}(t\mid\vec{a})=\prod_{j=1}^{v}\frac{a_jt+log(w^{a_j})}{(we^t)^{a_j}-1},\ |t+\log(|w|)|<\min\left(\frac{\pi}{a_1},\cdots,\frac{\pi}{a_v}\right)\ , \tag{2.1}$$

it is a continuous function on $[0,+\infty)$ with worst polynomial growth at $t\to\infty$.

We start by studying the Mellin-Laplace transform of this function

$$Z_w^{(v)}(s,x\mid\vec{a})=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}F_w^{(v)}(-t\mid\vec{a})e^{-xt}dt\ .$$

It is easily verified that $Z_w^{(v)}(s,x\mid\vec{a})$ is a well-defined analytic function for $\Re(x)>0$ and either $|w|\le 1, w\neq 1$, $\Re(s)>0$ or $w=1$, $\Re(s)>v$. We call it twisted Barnes zeta function. This function satisfies the functional equation:

$$\frac{\partial Z_w^{(v)}}{\partial x}(s+1,x\mid\vec{a})=-sZ_w^{(v)}(s,x\mid\vec{a}) \tag{2.2}$$

Taking $w=1$, we can get originally Barnes multiple zeta function $\zeta_v(s,x|a_1,...,a_v)$. In fact,

$$Z_w^{(v)}(s,x \mid \overrightarrow{a}) = \frac{a_1\cdots a_v}{\Gamma(s)}\int_0^\infty t^{s+v-1}\prod_{j=1}^{v}\left(1-e^{-a_jt}\right)e^{-xt}dt = a_1\cdots a_v\frac{\Gamma(s+v)}{\Gamma(s)}\zeta_v(s+v,x|a_1,...,a_v),$$

Finally,

$$\zeta_v(s,x|a_1,...,a_N) = \frac{1}{a_1\cdots a_v}\frac{\Gamma(s-v)}{\Gamma(s)}Z_{w=1}^{(v)}(s-v,x \mid \overrightarrow{a}). \tag{2.3}$$

In this case, according to the **??** (**??**), $Z_w^{(v)}(s,x \mid \overrightarrow{a})$ has a holomorphic continuation to all complex s (we delete all the singularities at s).

Now we define the twisted Barnes polynomials $B_{n,w}^{(v)}(x \mid \overrightarrow{a})$ as follows:

$$\frac{(t+log(w))^v}{\prod_{j=1}^{v}((we^t)^{a_j}-1)}e^{tx} = \sum_{n=0}^{\infty}B_{n,w}^{(v)}(x \mid \overrightarrow{a})\frac{t^n}{n!},\ |t+\log(|w|)| < \min\left(\frac{\pi}{a_1},\cdots,\frac{\pi}{a_v}\right). \tag{2.4}$$

$B_{n,w}^{(v)}(\overrightarrow{a}) := B_{n,w}^{(v)}(0 \mid \overrightarrow{a})$ are the twisted Barnes numbers.
Taking $w=1$ we obtain the originally Barnes multiple polynomials $B_n^{(v)}(x \mid a_1,...,a_v)$ given by

$$\frac{t^v}{\prod_{j=1}^{v}(e^{a_jt}-1)}e^{xt} = \sum_{n=0}^{\infty}B_n^{(v)}(x \mid a_1,...,a_v)\frac{t^n}{n!}. \tag{2.5}$$

We have the fundamental result:

Lemma 1 (Fundamental lemma)**.** *The function $s \to Z_w^{(v)}(s,x \mid \overrightarrow{a})$ has analytic continuation to whole complex s-plane.*

Proof. We have

$$\int_0^\infty t^{s+k-1}e^{-xt}\ dt = x^{-s-k}\int_0^\infty u^{s+k-1}e^{-u}\ du = x^{-s-k}\Gamma(s+k). \tag{2.6}$$

Fixing $N\in\mathbb{N}$, we therefore obtain

$$\Gamma(s)Z_w^{(v)}(s,x \mid \overrightarrow{a}) = \sum_{k=0}^{N}\frac{B_k^{(v)}(x \mid a_1,...,a_v)}{k!}x^{-s-k}\Gamma(s+k) +$$
$$\int_0^\infty t^{s-1}\left(F_w^{(v)}(-t \mid \overrightarrow{a}) - \sum_{k=0}^{N}B_k^{(v)}(x \mid a_1,...,a_v)\frac{t^k}{k!}\right)e^{-xt}dt. \tag{2.7}$$

The term in brackets is $O(t^{N+1})$ for $t\to 0$, so the integral yields a function that analytic for $\Re(s) > -N-1$. The remaining terms have simple poles for $s=-n-k, n\in\mathbb{N}$. Therefore the function $s \to Z_w^{(v)}(s,x \mid \overrightarrow{a})$ has analytic continuation to whole complex s-plane. ∎

Hence , we can derive the following theorem

Theorem 1 (Values at negative integers)**.** *Let $v\in\mathbb{N}$, $\overrightarrow{a}=(a_1,...,a_v)$, where $a_1,\cdots,a_v$ are strictly positive real numbers. Let $n\in\mathbb{N}$, $x>0$. The function $s\to Z_w^{(v)}(s,x \mid \overrightarrow{a})$ has analytic continuation to an entire function on the whole complex s-plane and*

$$Z_w^{(v)}(-n,x \mid \overrightarrow{a}) = a_1\cdots a_vB_{n,w}^{(v)}(x \mid \overrightarrow{a}).$$

Proof. From the Lemma **??** we get the analytic continuation. Therefore it remains to verify the residue assertion. To this end we splits up $Z_w^{(v)}(s,x \mid \overrightarrow{a})$ as the sum of two integrals :

$$Z_w^{(v)}(s,x \mid \overrightarrow{a}) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} F_w^{(v)}(-t \mid \overrightarrow{a}) e^{-xt} dt + \frac{1}{\Gamma(s)} \int_1^{\infty} t^{s-1} F_w^{(v)}(-t \mid \overrightarrow{a}) e^{-xt} dt$$

It's easy to see that the second integral converges absolutely for all $s \in \mathbb{C}$ and $\Re(x) > 0$ and cancels at negative integers. For $\Re(s) > 0$, the first integral can be written :

$$\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} F_w^{(v)}(-t \mid \overrightarrow{a}) e^{-xt} dt = a_1 \cdots a_v \frac{1}{\Gamma(s)} \int_0^1 \sum_{m=0}^{\infty} B_{m,w}^{(v)}(x \mid \overrightarrow{a}) \frac{(-1)^m t^{m+s-1}}{m!} dt \ .$$

the integral converges absolutely and then we can make inversion of $\int$ and $\sum$. Hence

$$\frac{1}{\Gamma(s)} \int_0^1 t^{s-1} F_w^{(v)}(-t \mid \overrightarrow{a}) e^{-xt} dt \quad = \quad \frac{a_1 \cdots a_v}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m B_{m,w}^{(v)}(x \mid \overrightarrow{a})}{m!} \frac{1}{m+s} \ .$$

from which follows that :

$$\begin{aligned} \lim_{s \to -n} Z_w^{(v)}(s,x \mid \overrightarrow{a}) &= \lim_{s \to -n} \frac{a_1 \cdots a_v (-1)^n}{n! \Gamma(s)(n+s)} B_{n,w}^{(v)}(x \mid \overrightarrow{a}) \\ &= a_1 \cdots a_v B_{n,w}^{(v)}(x \mid \overrightarrow{a}) \ . \end{aligned}$$

Thus completes the proof of this theorem. ∎

From our Theorem **??**, relations (**??**) and(**??**) , we get he following results

Corollary 1.

(1) *If we take $w = 1$, then we have*

$$\zeta_v(-n, x|a_1, ..., a_v) = \frac{(-1)^v n!}{(n+v)!} B_{n+v}(x|a_1, ..., a_v).$$

(2) *For any $n \geq 1$ we have:*

$$\frac{dB_{n,w}^{(v)}}{dx}(x \mid \overrightarrow{a}) = n B_{n-1,w}^{(v)}(x \mid \overrightarrow{a}). \tag{2.8}$$

Our zeta function $Z_w^{(v)}(s,x \mid \overrightarrow{a})$ can be rewritten as a power series

Proposition 1 (Explicit formula)**.** *Let $v \in \mathbb{N}$, $\overrightarrow{a} = (a_1, ..., a_v)$, where $a_1, \cdots, a_v$ are strictly positive real numbers. Let $n \in \mathbb{N}, x > 0$. Then we have*

$$Z_w^{(v)}(s,x \mid \overrightarrow{a}) = a_1 \cdots a_v \sum_{k=0}^{v} \left(\begin{array}{c} v \\ k \end{array} \right) \left(log(w^{-1}) \right)^{v-k} \frac{\Gamma(s+k)}{\Gamma(s)} \zeta_v(s+k, w, x \mid \overrightarrow{a}) \ . \tag{2.9}$$

where

$$\zeta_v(s, w, x \mid \overrightarrow{a}) = \sum_{m_1, \cdots, m_v \geq 0} \frac{w^{a_1 m_1 + \cdots + a_v m_v}}{(x + a_1 m_1 + \cdots + a_v m_v)^s} \ . \tag{2.10}$$

Proof. From equation [**??**), we write

$$F_w^{(v)}(-t \mid \overrightarrow{a}) = \prod_{j=1}^{v} \frac{-a_j t + log(w^{a_j})}{(we^{-t})^{a_j} - 1} = a_1 \cdots a_v \frac{\left(t + log(w^{-1}) \right)^v}{\prod_{j=1}^{v} (1 - w^{a_j} e^{-a_j t})} . \tag{2.11}$$

By using the binomial formula in equation (**??**, we have

$$\frac{1}{\Gamma(s)}t^{s-1}F_w^{(v)}(-t \mid \overrightarrow{a})e^{-xt} = a_1\cdots a_v \sum_{k=0}^{v} \binom{v}{k} \left(log(w^{-1})\right)^{v-k} \frac{1}{\Gamma(s)} \frac{t^{s+k-1}}{\prod_{j=1}^{v}(1-w^{a_j}e^{-a_j t})} e^{-xt} \ .$$

The zeta function can be rewritten

$$\begin{aligned} Z_w^{(v)}(s,x \mid \overrightarrow{a}) &= \frac{1}{\Gamma(s)}a_1\cdots a_v \int_0^\infty t^{s-1}F_w^{(v)}(-t \mid \overrightarrow{a})e^{-xt}dt \\ &= a_1\cdots a_v \sum_{k=0}^{v} \binom{v}{k} \left(log(w^{-1})\right)^{v-k} \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s+k-1}}{\prod_{j=1}^{v}(1-w^{a_j}e^{-a_j t})} e^{-xt}dt \ . \end{aligned}$$

It can be rewritten as a power series by using

$$\frac{e^{-xt}}{\prod_{j=1}^{v}(1-w^{a_j}e^{-a_j t})} = \sum_{m_1,\cdots,m_v\geq 0} w^{a_1m_1+\cdots+a_vm_v}e^{-(x+a_1m_1+\cdots+a_vm_v)} \ . \tag{2.12}$$

$$\begin{aligned} &\frac{Z_w^{(v)}(s,x \mid \overrightarrow{a})}{a_1\cdots a_v} \\ &= \sum_{m_1,\cdots,m_v\geq 0}\sum_{k=0}^{v} \binom{v}{k} \left(log(w^{-1})\right)^{v-k} \frac{1}{\Gamma(s)} \int_0^\infty t^{s+k-1}w^{a_1m_1+\cdots+a_vm_v}e^{-(x+a_1m_1+\cdots+a_vm_v)}dt \\ &= \sum_{m_1,\cdots,m_v\geq 0}\sum_{k=0}^{v} \binom{v}{k} \left(log(w^{-1})\right)^{v-k} w^{a_1m_1+\cdots+a_vm_v} \frac{1}{\Gamma(s)} \int_0^\infty t^{s+k-1}e^{-(x+a_1m_1+\cdots+a_vm_v)}dt \ . \end{aligned}$$

Let $z=(x+a_1m_1+\cdots+a_vm_v)t$. Then we obtain

$$\begin{aligned} &\frac{Z_w^{(v)}(s,x \mid \overrightarrow{a})}{a_1\cdots a_v} \\ &= \sum_{m_1,\cdots,m_v\geq 0}\sum_{k=0}^{v} \binom{v}{k} \left(log(w^{-1})\right)^{v-k} w^{a_1m_1+\cdots+a_vm_v} \frac{1}{\Gamma(s)} \int_0^\infty \frac{z^{s+k-1}e^{-z}}{(x+a_1m_1+\cdots+a_vm_v)^{s+k}}dz \\ &= \sum_{k=0}^{v} \binom{v}{k} \left(log(w^{-1})\right)^{v-k} \left(\sum_{m_1,\cdots,m_v\geq 0} \frac{w^{a_1m_1+\cdots+a_vm_v}}{(x+a_1m_1+\cdots+a_vm_v)^{s+k}} \right) \frac{1}{\Gamma(s)} \int_0^\infty z^{s+k-1}e^{-z}dz \\ &= \sum_{k=0}^{v} \binom{v}{k} \left(log(w^{-1})\right)^{v-k} \zeta_v(s+k,w,x \mid \overrightarrow{a}) \frac{\Gamma(s+k)}{\Gamma(s)} \ . \end{aligned}$$

This yields the explicit formula. ∎

Using our Proposition **??**, we obtain

Corollary 2 (Special cases)**.**

(1) *If* $v=a_1=1$ *we obtain*

$$Z_w^{(1)}(s,x \mid 1) = -log(w)\Phi(w,s,x) + s\Phi(w,s+1,x) \ .$$

(2) *If $v = w = a_1 = 1$ we obtain*

$$Z_w^{(1)}(s, x \mid 1) = s\Phi(1, s+1, x) \ ,$$

where

$$\Phi(w, s, x) = \sum_{m \geq 0} \frac{w^m}{(x+m)^s} \ ,$$

is the Lerch transcendent function [**?**].

Theorem 2 (Addition identity)**.** *For any $x, y \in \mathbb{C}$, $Z_w^{(v)}(s, x+y \mid \overrightarrow{a})$ has the addition identity:*

$$Z_w^{(v)}(s, x+y \mid \overrightarrow{a}) = \sum_{k=0}^{\infty} \binom{s+k-1}{s-1} (-y)^k Z_w^{(v)}(s+k, x \mid \overrightarrow{a}). \tag{2.13}$$

Proof.

$$\begin{aligned}
Z_w^{(v)}(s, x+y \mid \overrightarrow{a}) &= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_w^{(v)}(-t \mid \overrightarrow{a}) e^{-(x+y)t} dt \\
&= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-xt} F_w^{(v)}(-t \mid \overrightarrow{a}) e^{-yt} dt \\
&= \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{(-y)^k}{k!} \int_0^{\infty} t^{s+k-1} e^{-xt} F_w^{(v)}(-t \mid \overrightarrow{a}) e^{-yt} dt \\
&= \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{(-y)^k}{k!} \Gamma(s+k) Z_w^{(v)}(s+k, x \mid \overrightarrow{a}) \\
&= \sum_{k=0}^{\infty} \frac{(-y)^k}{k!} \frac{\Gamma(s+k)}{\Gamma(s)} Z_w^{(v)}(s+k, x \mid \overrightarrow{a}) \\
&= \sum_{k=0}^{\infty} \binom{s+k-1}{s-1} (-y)^k Z_w^{(v)}(s+k, x \mid \overrightarrow{a}).
\end{aligned}$$

Thus finished the proof. ∎

By using the fundamental Lemma **??**, Theorems **??** and **??** at $s = -n$ we derive

Corollary 3. *For any $x, y \in \mathbb{C}$, we have*

$$B_{n,w}^{(v)}(x+y \mid \overrightarrow{a}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k,w}^{(v)}(x \mid \overrightarrow{a}) y^k. \tag{2.14}$$

Theorem 3 (Rationality theorem)**.** *Let $a_1, ..., a_v$ positive rationals numbers. We have*

$$B_{n,w}^{(v)}(\overrightarrow{a}) \in \mathbb{Q}(w)[log(w)]. \tag{2.15}$$

Indeed, if $w = 1$ the numbers $B_{n,w=1}^{(v)}(\overrightarrow{a}) \in \mathbb{Q}$ are rationals numbers.

Proof. Set $z = t + log(w)$. Let us write

$$\prod_{j=1}^{v} \frac{a_j t + log(w^{a_j})}{(we^t)^{a_j} - 1} = \prod_{j=1}^{v} \frac{a_j z}{e^{a_j z} - 1} = \prod_{j=1}^{v} \left(\sum_{m_j \geq 0} a_j^{m_j} B_{m_j} \frac{z^{m_j}}{m_j!} \right) \tag{2.16}$$

where B_{m_k} are the classical rational Bernoulli numbers. Furthermore,

$$\prod_{j=1}^{v} \left(\sum_{m_j \geq 0} a_j^{m_j} B_{m_j} \frac{z^{m_j}}{m_j!} \right) = \sum_{m \geq 0} \left(\sum_{\substack{m_1, \ldots, m_v \geq 0 \\ m_1 + \ldots + m_v = m}} \frac{a_1^{m_1} ... a_v^{m_v} B_{m_1} \cdots B_{m_v}}{m_1! ... m_v!} \right) z^m \tag{2.17}$$

Now by using the identity

$$z^m = \sum_{k=0}^{m} \binom{m}{k} (log(w))^k t^{m-k}$$

we obtain

$$\begin{aligned} \prod_{j=1}^{v} \tfrac{a_j t + log(w^{a_j})}{(we^t)^{a_j}-1} &= \sum_{m\geq 0} \Big(\sum_{\substack{m_1,\ldots,m_v\geq 0\\ m_1+\ldots+m_v=m}} \tfrac{a_1^{m_1}\ldots a_v^{m_v} B_{m_1}\cdots B_{m_v}}{m_1!\ldots m_v!} \Big) z^m \\ &= \sum_{m\geq 0} \Big(\sum_{\substack{m_1,\ldots,m_v\geq 0\\ m_1+\ldots+m_v=m}} \tfrac{a_1^{m_1}\ldots a_v^{m_v} B_{m_1}\cdots B_{m_v}}{m_1!\ldots m_v!} \Big) \sum_{k=0}^{m} \tbinom{m}{k} (log(w))^k t^{m-k} \end{aligned} \tag{2.18}$$

$$\sum_{m\geq 0} \left(\sum_{\substack{m_1,\ldots,m_v\geq 0\\ m_1+\ldots+m_v=m}} \frac{a_1^{m_1}\ldots a_v^{m_v} B_{m_1}\cdots B_{m_v}}{m_1!\ldots m_v!} \right) \sum_{k=0}^{m} \binom{m}{k} (log(w))^k t^{m-k} =$$

$$\sum_{n\geq 0} \left(\sum_{k\geq 0} \sum_{\substack{m_1,\ldots,m_v\geq 0\\ m_1+\ldots+m_v=n+k}} \frac{a_1^{m_1}\ldots a_v^{m_v} B_{m_1}\ldots B_{m_v}}{m_1!\ldots m_v!} \binom{n+k}{k} (log(w))^k \right) t^n.$$

Then

$$\frac{B_{n,w}^{(v)}(\vec{a})}{n!} = \sum_{k\geq 0} \left(\sum_{\substack{m_1,\ldots,m_v\geq 0\\ m_1+\ldots+m_v=n+k}} \frac{a_1^{m_1}\ldots a_v^{m_v} B_{m_1}\cdots B_{m_v}}{m_1!\ldots m_v!} \right) \binom{n+k}{k} (log(w))^k . \tag{2.19}$$

This give our theorem. ∎

Let us prove the homogeneity property of $Z_w^{(v)}(s, x \mid \vec{a})$.

Theorem 4 (Homogeneity identity). *Let $v \in \mathbb{N}$, $\vec{a} = (a_1, ..., a_v)$, where $a_1, \cdots, a_v$ are strictly positive real numbers. Let $\lambda > 0$ and $x > 0$. Then we have*

$$Z_w^{(v)}(s, \lambda x \mid \lambda\vec{a}) = \lambda^{-s} Z_{w^\lambda}^{(v)}(s, x \mid \vec{a})$$

Proof. It's easy to see that

$$\begin{aligned} F_w^{(v)}(t \mid \lambda\vec{a}) &= \prod_{j=1}^{v} \frac{\lambda t + log(w^\lambda)}{(we^t)^{\lambda a_j} - 1} \\ &= \prod_{j=1}^{v} \frac{(\lambda t) + log(w^\lambda)}{(w^\lambda e^{\lambda t})^{a_j} - 1} \\ &= F_{w^\lambda}^{(v)}(\lambda t \mid \vec{a}) . \end{aligned}$$

Then we obtain

$$\begin{aligned} Z_w^{(v)}(s, \lambda x \mid \lambda\vec{a}) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} F_w^{(v)}(-t \mid \lambda\vec{a}) e^{-\lambda x t} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} F_{w^\lambda}^{(v)}(-\lambda t \mid \vec{a}) e^{-\lambda x t} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \lambda^{-s+1} t^{s-1} F_{w^\lambda}^{(v)}(-t \mid \vec{a}) e^{-x t} \frac{dt}{\lambda} \\ &= \lambda^{-s} Z_{w^\lambda}^{(v)}(s, x \mid \vec{a}) . \end{aligned}$$

This yields the theorem. ∎

We obtain immediately from Theorems **??** and **??** the identity

Corollary 4. *Let $v \in \mathbb{N}$, $\overrightarrow{a} = (a_1, ..., a_v)$, where $a_1, \cdots, a_v$ are strictly positive real numbers. For all $\lambda > 0$, we have*

$$B_{n,w}^{(v)}(\lambda x \mid \lambda \overrightarrow{a}) = \lambda^n B_{n,w^\lambda}^{(v)}(x \mid \overrightarrow{a}) \ . \tag{2.20}$$

Using homogeneity Theorem **??**, without loss of generality, we can assume $\sum_{i=1}^{v} a_i = 1$.

By using the Theorem **??** at $s = -n$ we get the following symmetry formula for the twisted Barnes polynomials:

Theorem 5 (Symmetry identity)**.** *Let $v \in \mathbb{N}$, $\overrightarrow{a} = (a_1, ..., a_v)$, where $a_1, \cdots, a_v$ are strictly positive real numbers. Assume $\sum_{i=1}^{v} a_i = 1$. Then for any $n \in \mathbb{N}$, we have the symmetry identity*

$$B_{n,w}^{(v)}(x \mid \overrightarrow{a}) = (-1)^n w^{-1} B_{n,w^{-1}}^{(v)}\left(1 - x \mid \overrightarrow{a}\right) \ . \tag{2.21}$$

Proof. We study the generating function of the right hand side of the equality (**??**). We set

$$\begin{aligned}
R.H.S(x,t) &= \sum_{n=0}^{\infty}\left((-1)^n w^{-\sum_{i=1}^{v} a_i} B_{n,w^{-1}}^{(v)}\left(\sum_{i=1}^{v} a_i - x \mid \overrightarrow{a}\right)\right)\frac{t^n}{n!} \\
&= w^{-\sum_{i=1}^{v} a_i}\sum_{n=0}^{\infty}\left(B_{n,w^{-1}}^{(v)}\left(\sum_{i=1}^{v} a_i - x \mid \overrightarrow{a}\right)\right)\frac{(-t)^n}{n!} \\
&= w^{-\sum_{i=1}^{v} a_i}\left(\prod_{j=1}^{v}\frac{a_j(-t) + log(w^{-a_j})}{(w^{-1}e^{-t})^{a_j} - 1}\right)e^{xt + (\sum_{i=1}^{v} a_i)(-t)} \\
&= w^{-\sum_{i=1}^{v} a_i} e^{(\sum_{i=1}^{v} a_i)(-t)}\left(\prod_{j=1}^{v}\frac{a_j(-t) + log(w^{-a_j})}{(w^{-1}e^{-t})^{a_j} - 1}\right)e^{xt} \\
&= \left(\prod_{j=1}^{v}\frac{a_j(-t) + log(w^{-a_j})}{w^{a_j}e^{a_j t}\left((w^{-1}e^{-t})^{a_j} - 1\right)}\right)e^{xt} \\
&= \left(\prod_{j=1}^{v}\frac{a_j t + log(w^{a_j})}{(we^t)^{a_j} - 1}\right)e^{tx}.
\end{aligned}$$

Finally, we have

$$R.H.S(x,t) = \sum_{n=0}^{\infty} B_{n,w}^{(v)}(x \mid \overrightarrow{a})\frac{t^n}{n!}.$$

Thus we obtain our symmetry identity. ∎

Next we state the distribution identity identity

Theorem 6 (Distribution identity)**.** *Let $v \in \mathbb{N}$, $\overrightarrow{a} = (a_1, ..., a_v)$, where $a_1, \cdots, a_v$ are strictly positive real numbers. Let $m \in \mathbb{N}$. Then we have*

$$Z_w^{(v)}(s, mx \mid \overrightarrow{a}) = m^{-s-v}\sum_{t_1,\cdots,t_v=0}^{m-1} w^{\sum_{i=1}^{v} a_i t_i} Z_{w^m}^{(v)}\left(s, x + \sum_{i=1}^{v} a_i \frac{t_i}{m} \mid \overrightarrow{a}\right).$$

Proof. Writing

$$
\begin{aligned}
& m^{-s-v}\sum_{t_1,\cdots,t_v=0}^{m-1} w^{\sum_{i=1}^v a_i t_i} Z_{w^m}^{(v)}\left(s, x+\sum_{i=1}^v a_i \frac{t_i}{m} \mid \overrightarrow{a}\right) \\
= \; & m^{-s-v}\sum_{t_1,\cdots,t_v=0}^{m-1} w^{\sum_{i=1}^v a_i t_i} \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1} e^{-(x+\sum_{i=1}^v a_i \frac{t_i}{m})u} F_{w^m}^{(v)}(-u \mid \overrightarrow{a})du \\
= \; & m^{-s-v}\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}e^{-xu}\prod_{i=1}^v \sum_{t_i=0}^{m-1}\left(w^{a_i}e^{-\frac{a_i u}{m}}\right)^{t_i}\prod_{j=1}^v \frac{-a_j u+log(w^{ma_j})}{(w^m e^{-u})^{a_j}-1}\,du \\
= \; & m^{-s-v}\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}e^{-xu}\prod_{j=1}^v \frac{w^{ma_j}e^{-a_j u}-1}{w^{a_j}e^{-\frac{a_j u}{m}}-1}\prod_{j=1}^v \frac{-a_j u+log(w^{ma_j})}{(w^m e^{-u})^{a_j}-1}\,du \\
= \; & m^{-s-v}\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}e^{-xu}m^v\prod_{j=1}^v \frac{-a_j\frac{u}{m}+log(w^{a_j})}{w^{a_j}e^{-\frac{a_j u}{m}}-1}\,du,
\end{aligned}
$$

we change u by mt, we obtain

$$
\begin{aligned}
& m^{-s-v}\frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}e^{-xu}m^v\prod_{j=1}^v \frac{-a_j\frac{u}{m}+log(w^{a_j})}{w^{a_j}e^{-\frac{a_j u}{m}}-1}\,du \\
= \; & \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-xmt}\prod_{j=1}^v \frac{-a_j t+log(w^{a_j})}{w^{a_j}e^{-a_j t}-1}\,dt \\
= \; & Z_w^{(v)}(s, mx \mid \overrightarrow{a}).
\end{aligned}
$$

This implies the theorem. ∎

From Theorems **??** and **??**, at $s=-n$, we obtain Raabe type formula for twisted Barnes polynomials

Corollary 5 (Raabe type identity)**.** *Let $v\in\mathbb{N}$, $\overrightarrow{a}=(a_1,...,a_v)$, where $a_1,\cdots,a_v$ are strictly positive real numbers. Let $m\in\mathbb{N}$. Then we have*

$$
B_{n,w}^{(v)}(mx \mid \overrightarrow{a}) = m^{n-v}\sum_{t_1,\cdots,t_v=0}^{m-1} w^{\sum_{i=1}^v a_i t_i} B_{n,w^m}^{(v)}\left(x+\sum_{i=1}^v a_i\frac{t_i}{m} \mid \overrightarrow{a}\right).
$$

Theorem 7 (Difference equation)**.** *Put $\overrightarrow{a}_{v+1}=(a_1,...,a_{v+1})$ and $\overrightarrow{a}_v=(a_1,...,a_v)$. Then we have the difference identity*

$$
\begin{aligned}
& Z_w^{(v+1)}(s, x+a_{v+1} \mid \overrightarrow{a}_{v+1}) - w^{-a_{v+1}}Z_w^{(v+1)}(s, x \mid \overrightarrow{a}_{v+1}) \\
& = -a_{v+1}w^{-a_{v+1}}sZ_w^{(v)}(s+1, x \mid \overrightarrow{a}_v) + w^{-a_{v+1}}log(w^{a_{v+1}})Z_w^{(v)}(s, x \mid \overrightarrow{a}_v).
\end{aligned}
\tag{2.22}
$$

Proof. We write

$$
\begin{aligned}
& t^{s-1}e^{-(x+a_{v+1})t}F_w^{(v+1)}(-t \mid \overrightarrow{a}_{v+1}) - w^{-a_{v+1}}e^{-xt}F_w^{(v)}(-t \mid \overrightarrow{a}_{v+1}) \\
= \; & t^{s-1}e^{-xt}F_w^{(v+1)}(-t \mid \overrightarrow{a}_{v+1})(e^{-a_{v+1}t}-w^{-a_{v+1}}) \\
= \; & w^{-a_{v+1}}t^{s-1}e^{-xt}F_w^{(v+1)}(-t \mid \overrightarrow{a}_{v+1})(w^{a_{v+1}}e^{-a_{v+1}t}-1) \\
= \; & w^{-a_{v+1}}t^{s-1}e^{-xt}F_w^{(v)}(-t \mid \overrightarrow{a}_v)(-a_{v+1}t+log(w^{a_{v+1}})) \\
= \; & -a_{v+1}w^{-a_{v+1}}t^s e^{-xt}F_w^{(v)}(-t \mid \overrightarrow{a}_{v+1}) + w^{-a_{v+1}}log(w^{a_{v+1}})t^{s-1}e^{-xt}F_w^{(v)}(-t \mid \overrightarrow{a}_{v+1}).
\end{aligned}
$$

From this relation it's easy to obtain

$$
\begin{aligned}
& Z_w^{(v+1)}(s, x+a_{v+1} \mid \overrightarrow{a}_{v+1}) - w^{-a_{v+1}}Z_w^{(v+1)}(s, x \mid \overrightarrow{a}_{v+1}) \\
& = -a_{v+1}w^{-a_{v+1}}sZ_w^{(v)}(s+1, x \mid \overrightarrow{a}_v) + w^{-a_{v+1}}\log(w^{a_{v+1}})Z_w^{(v)}(s, x \mid \overrightarrow{a}_v).
\end{aligned}
$$

This completes the proof. ∎

We obtain from Theorems **??** and **??** at $s = -n$ the corollary

Corollary 6. *For any $a_1, \cdots, a_{v+1}$ reals positive numbers and $n \in \mathbb{N}$, we have*

$$B_{n,w}^{(v+1)}(x + a_{v+1} \mid \overrightarrow{a}_{v+1}) - w^{-a_{v+1}} B_{n,w}^{(v+1)}(x \mid \overrightarrow{a}_{v+1})$$
$$= n a_{v+1} w^{-a_{v+1}} B_{n-1,w}^{(v)}(x \mid \overrightarrow{a}_v) + w^{-a_{v+1}} log(w^{a_{v+1}}) B_{n,w}^{(v)}(x \mid \overrightarrow{a}_v).$$

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