Identities on q-Euler numbers and polynomials

Abdelmejid Bayad[a], Yoshinori Hamahata[b,*]

[a] *Département de mathématiques, Université d'Évry Val d'Essonne, Bd. F. Mitterrand, 91025 Évry Cedex, France*
[b] *Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo, 153-8914, Japan*

Abstract

We introduce new q-Euler polynomials and numbers. Some identities about them are presented. In particular, we give a relation among two kinds of q-Euler polynomials, from which an Euler polynomial version of Kaneko-Momiyama relations among Bernoulli numbers is given.

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Contents

1. Introduction

The Euler numbers E_n ($n = 0, 1, 2, \ldots$) are defined by the generating function

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

The first few values are $E_0 = 1$, $E_1 = -1/2$, $E_2 = 0$, $E_3 = 1/4$, and it holds that $E_{2k} = 0$ ($k = 1, 2, 3, \ldots$). The Euler polynomials $E_n(x)$ ($n = 0, 1, 2, \ldots$) are defined by the generating series

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

The first few values are

$$\begin{aligned} E_0(x) &= 1, \\ E_1(x) &= x - \frac{1}{2}, \\ E_2(x) &= x^2 - x, \\ E_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{4}. \end{aligned}$$

*Corresponding author
Email addresses: `abayad@maths.univ-evry.fr` (Abdelmejid Bayad), `hamahata@ms.u-tokyo.ac.jp` (Yoshinori Hamahata)

In this present paper, we introduce new q-Euler polynomials and numbers. We will give q-analogues for the following formulae involving the ordinary Euler polynomials:

$$
\begin{aligned}
&(-1)^n E_n(-x) + E_n(x) = 2x^n,\\
&(-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{n+k}(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} E_{m+k}(-x),\\
&(-1)^m \sum_{k=0}^{m+1} \binom{m+1}{k} (n+k+1) E_{n+k}(x) + (-1)^n \sum_{k=0}^{n+1} \binom{n+1}{k} (m+k+1) E_{m+k}(x) = 0.
\end{aligned}
$$

The last identity is an Euler polynomial version of Kaneko-Momiyama relations among Bernoulli numbers ([6], [8], [10]). We also establish a relation between sums of products of our q-Euler polynomials. It should be noted that Simsek [9] found formulae for sums of products of ordinary q-Euler polynomials.

2. Preliminaries

2.1. Notation

Let $a \in \mathbb{C}$. The *q-shifted factorials* are defined by

$$(a,q)_0 = 1, \quad (a,q)_n = \prod_{k=0}^{n-1} (1 - aq^k) \quad (n = 1, 2, \ldots).$$

If $|q| < 1$, then we define

$$(a,q)_\infty = \lim_{n\to\infty} (a,q)_n = \prod_{k=0}^{\infty} (1 - aq^k).$$

We also denote

$$
\begin{aligned}
[x]_q &= \frac{1-q^x}{1-q}, \quad x \in \mathbb{C},\\
[n]_q! &= \frac{(q,q)_n}{(1-q)^n}, \quad n \in \mathbb{N},\\
\begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad k, n \in \mathbb{N},\\
\begin{bmatrix} n \\ i_1, \cdots, i_m \end{bmatrix}_q &= \frac{[n]_q!}{[i_1]_q! \cdots [i_m]_q!}, \quad n, i_1, \ldots, i_m \in \mathbb{N}.
\end{aligned}
$$

2.2. q-Exponential functions

The *q-exponential functions* are given by

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}$$

and

$$e_{q^{-1}}(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q^{-1}}!}.$$

It is easy to see that $[n]_{q^{-1}}! = q^{-\binom{n}{2}}[n]_q!$. Hence

$$e_{q^{-1}}(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}.$$

See [3], [5] for related topics. As is well known,

$$e_q(z) = \frac{1}{((1-q)z, q)_\infty}, \qquad e_{q^{-1}}(z) = (-(1-q)z, q)_\infty .$$

This yields $e_q(z)e_{q^{-1}}(-z) = 1$.

2.3. q-Euler polynomials and numbers

Definition 2.1. We define two kinds of q-Euler polynomials $E_n(x,q)$ and $F_n(x,q^{-1})$ ($n = 0, 1, 2, \ldots$) by

$$\begin{aligned}\frac{2e_q(xt)}{e_q(t)+1} &= \sum_{n=0}^{\infty} E_n(x,q)\frac{t^n}{[n]_q!},\\ \frac{2e_q(xt)}{e_{q^{-1}}(t)+1} &= \sum_{n=0}^{\infty} F_n(x,q^{-1})\frac{t^n}{[n]_q!}.\end{aligned}$$

We call $E_n(x,q)$ (resp. $F_n(x,q^{-1})$) the *first* (resp. *second*) *q-Euler polynomials*. In particular, we call $E_n(0,q)$ (resp. $F_n(0,q^{-1})$) the *first* (resp. *second*) *q-Euler numbers*.

Example 2.2.

$$\begin{aligned}E_0(x,q) &= 1,\\ E_1(x,q) &= x-\frac{1}{2},\\ E_2(x,q) &= x^2-\frac{[2]_q}{2}x-\frac{1}{2}+\frac{[2]_q}{4},\\ E_3(x,q) &= x^3-\frac{[3]_q}{2}x^2+\left(\frac{[3]_q[2]_q}{4}-\frac{[3]_q}{2}\right)x-\frac{1}{2}-\frac{[3]_q[2]_q}{8}+\frac{[3]_q}{2}.\end{aligned}$$

$$\begin{aligned}F_0(x,q^{-1}) &= 1,\\ F_1(x,q^{-1}) &= x-\frac{1}{2},\\ F_2(x,q^{-1}) &= x^2-\frac{[2]_q}{2}x-\frac{q}{2}+\frac{[2]_q}{4},\\ F_3(x,q^{-1}) &= x^3-\frac{[3]_q}{2}x^2+\left(\frac{[3]_q[2]_q}{4}-\frac{[3]_q}{2}q\right)x-\frac{q^3}{2}-\frac{[3]_q[2]_q}{8}+\frac{[3]_q}{2}q.\end{aligned}$$

Proposition 2.3 (q-Recurrence formula). *For any $n \geq 1$, we have*

$$\begin{aligned}E_n(x,q) &= x^n-\frac{1}{2}\sum_{i=0}^{n-1}\begin{bmatrix} n\\ i\end{bmatrix}_q E_i(x,q),\\ F_n(x,q^{-1}) &= x^n-\frac{1}{2}\sum_{i=0}^{n-1}\begin{bmatrix} n\\ i\end{bmatrix}_q F_i(x,q^{-1})q^{\binom{n-i}{2}}.\end{aligned}$$

Proof. As for the first identity, we make use of

$$\left(\sum_{n=0}^{\infty} E_n(x,q)\frac{t^n}{[n]_q!}\right)(e_q(t)+1) = 2e_q(xt).$$

We deduce from this identity

$$\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\begin{bmatrix} n\\ i\end{bmatrix}_q E_i(x,q)+E_n(x,q)\right)\frac{t^n}{[n]_q!} = \sum_{n=0} 2x^n\frac{t^n}{[n]_q!},$$

which yields the result. We get the second result in the similar way. □

As $q \to 1$, one has a recurrence formula for the ordinary Euler polynomials:

$$E_n(x) = x^n-\frac{1}{2}\sum_{i=0}^{n-1}\binom{n}{i}E_i(x) \qquad (n \geq 1).$$

2.4. q-Derivative and q-integral

The *q-derivative* of a function f is given by

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x} \qquad (x \neq 0, q \neq 1),$$

where x and qx should be in the domain of f. If f is differentiable on an open set I, then for all $x \in I$,

$$\lim_{q \to 1} D_q f(x) = f'(x).$$

Besides, for all $n \in \mathbb{N}$,

$$\begin{aligned} D_q(x^n) &= [n]_q x^{n-1}, \\ D_q(x,q)_n &= -[n]_q (xq,q)_{n-1}, \\ D_{q^{-1}}(x,q)_n &= -[n]_q (x,q)_{n-1}, \\ D_q\left(\frac{x^n}{[n]_q!}\right) &= \frac{x^{n-1}}{[n-1]_q!}. \end{aligned}$$

From the last identity, for instance, we have $D_q e_q(x) = e_q(x)$.

Our q-Euler polynomials form "q-Appell sequences":

Proposition 2.4 (q-Derivative formula). *For any $n \geq 0$, we have*

$$\begin{aligned} D_q E_{n+1}(x,q) &= [n+1]_q E_n(x,q), \\ D_q F_{n+1}(x,q^{-1}) &= [n+1]_q F_n(x,q^{-1}). \end{aligned}$$

Proof. Since

$$\sum_{n=0}^{\infty} D_q E_n(x,q) \frac{t^n}{[n]_q!} = \frac{2te_q(xt)}{e_q(t)+1} = \sum_{n=1}^{\infty} [n]_q E_{n-1}(x,q) \frac{t^n}{[n]_q!},$$

we have the first identity. The second identity can be obtained similarly. □

As $q \to 1$, we have the identities of Appell sequences of the ordinary Euler polynomials:

$$\frac{d}{dx} E_{n+1}(x) = (n+1) E_n(x).$$

For the product of two functions f and g, the following formula holds:

$$\begin{aligned} D_q(f \cdot g)(x) &= g(x) D_q(x) + f(qx) D_q g(x) \\ &= f(x) D_q g(x) + g(qx) D_q f(x). \end{aligned}$$

We next treat the composition of $f(x)$ and $g(x)$. When $g(x) = -x$, the following chain rule for the q-derivative is valid:

$$D_q(f \circ g)(x) = D_q f(g(x)) D_q g(x),$$

which will be used in the proofs of Theorems 3.5 and 3.9. However, in general, the rule above does not hold. If we modify the definition of the composition of two functions, then a new chain rule for the q-derivative is gained. We refer to Gessel [3] for this topic.

The *q-Jackson integral* from 0 to a is defined by

$$\int_0^a f(x) d_q x \quad := \quad (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n$$

provided the infinite sums converge absolutely. The q-Jackson integral in the generic interval $[a,b]$ is given by

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

For any function f we have

$$D_q \int_0^x f(t) d_q t = f(x).$$

Proposition 2.5 (q-Integral formula). *For any $n \geq 0$,*

$$\begin{aligned}\int_a^x E_n(t,q)d_qt &= \frac{E_{n+1}(x,q)-E_{n+1}(a,q)}{[n+1]_q},\\ \int_a^x F_n(t,q^{-1})d_qt &= \frac{F_{n+1}(x,q^{-1})-F_{n+1}(a,q^{-1})}{[n+1]_q}.\end{aligned}$$

This result follows from q-derivative formula. As $q \to 1$, we have integral formula for the classical Euler polynomials:

$$\int_a^x E_n(t)dt = \frac{E_{n+1}(x)-E_{n+1}(a)}{n+1}.$$

2.5. q-Binomial formula

Let $q \in \mathbb{C}$, and take two q-commuting variables x and y which satisfy the relation

$$xy = q^{-1}yx.$$

Let $\mathbb{C}_q[x,y]$ be the complex associative algebra with 1 generated by x and y. Then the following identity is valid in the algebra $\mathbb{C}_q[x,y]$:

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}, \qquad n \in \mathbb{N},$$

or alternatively,

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} y^k x^{n-k}, \qquad n \in \mathbb{N}.$$

For details, we refer to [1], [2].

2.6. q-Exponential identity

Let x, y be the q-commuting variables satisfying the relation $xy = q^{-1}yx$. Let $\mathbb{C}_q[[x,y]]$ be the complex associative algebra with 1 of formal power series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n}x^n y^n$$

with arbitrary complex coefficients $a_{m,n}$. One knows in [1], [2] that in $\mathbb{C}_q[[x,y]]$, we have the following identity

$$e_q(x+y) = e_q(x)e_q(y).$$

Proposition 2.6 (q-Addition formula). *For any $n \geq 0$, we have*

$$\begin{aligned}E_n(x+y,q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_k(x,q)y^{n-k},\\ F_n(x+y,q^{-1}) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_k(x,q^{-1})y^{n-k}.\end{aligned}$$

Particularly, it follows that

$$\begin{aligned}E_n(y,q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_k(0,q)y^{n-k},\\ F_n(y,q^{-1}) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_k(0,q^{-1})y^{n-k}.\end{aligned}$$

Proof. The first identity follows from

$$\frac{2e_q((x+y)t)}{e_q(t)+1} = \frac{2e_q(xt)}{e_q(t)+1} \cdot e_q(yt).$$

One can easily prove the remaining identities. $\square$

As $q \to 1$, we have the classical formula:

$$E_{n+1}(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}.$$

Particularly, it holds that

$$E_n(y) = \sum_{k=0}^{n} \binom{n}{k} E_k y^{n-k}.$$

At the end of this section, we give a list of limit of q-analogues.

$$\begin{aligned}
\lim_{q\to 1} e_q(z) &= \lim_{q\to 1} e_{q^{-1}}(z) = e^z, \\
\lim_{q\to 1} [n]_q &= n, \\
\lim_{q\to 1} [n]_q! &= n!, \\
\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \binom{n}{k}, \\
\lim_{q\to 1} \begin{bmatrix} n \\ i_1, \cdots, i_m \end{bmatrix}_q &= \binom{n}{i_1, \cdots, i_m} := \frac{n!}{i_1! \cdots i_m!}, \\
\lim_{q\to 1} E_n(x,q) &= \lim_{q\to 1} F_n(x, q^{-1}) = E_n(x).
\end{aligned}$$

3. Main results

3.1. Sums of products

Theorem 3.1 (Sums of products). *Let m be a given positive integer. Then for any $n \geq 0$,*

$$\begin{aligned}
&(-1)^n \sum_{i_1+\cdots+i_m=n} \begin{bmatrix} n \\ i_1, \ldots, i_m \end{bmatrix}_q F_{i_1}(-x, q^{-1}) \cdots F_{i_m}(-x, q^{-1}) \\
&\qquad = \sum_{j=0}^{m} (-1)^j 2^{m-j} \binom{m}{j} \sum_{k_1+\cdots k_m=n} \begin{bmatrix} n \\ k_1, \ldots, k_m \end{bmatrix}_q E_{k_1}(x,q) \cdots E_{k_j}(x,q) x^{n-(k_1+\cdots+k_j)}.
\end{aligned}$$

In particular, if $m = 1$, then

$$(-1)^n F_n(-x, q^{-1}) + E_n(x,q) = 2x^n.$$

If $m = 2$, then

$$\begin{aligned}
&(-1)^n \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q F_i(-x, q^{-1}) F_{n-i}(-x, q^{-1}) \\
&\qquad = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q E_i(x,q) E_{n-i}(x,q) - 4 \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q E_i(x,q) x^{n-i} + 4x^n \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q.
\end{aligned}$$

Proof. In view of $e_q(t) e_{q^{-1}}(-t) = 1$, we have

$$\frac{1}{e_{q^{-1}}(-t) + 1} = 1 - \frac{1}{e_q(t) + 1}.$$

Hence for $m \geq 1$,

$$\left(\frac{2e_q((-x)(-t))}{e_{q^{-1}}(-t) + 1} \right)^m = \left(2e_q(xt) - \frac{2e_q(xt)}{e_q(t) + 1} \right)^m.$$

The left hand side of the identity is

$$\sum_{n=0}^{\infty} (-1)^n \sum_{i_1+\cdots+i_m=n} \begin{bmatrix} n \\ i_1, \ldots, i_m \end{bmatrix}_q F_{i_1}(-x, q^{-1}) \cdots F_{i_m}(-x, q^{-1}) \frac{t^n}{[n]_q!}.$$

The right hand side becomes

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(\frac{2e_q(xt)}{e_q(t)+1}\right)^j 2^{m-j}e_q(xt)^{m-j}$$
$$=\sum_{j=0}^{m}(-1)^j 2^{m-j}\binom{m}{j}\sum_{n=0}^{\infty}\sum_{k_1+\cdots+k_m=n}\begin{bmatrix} n \\ k_1,\ldots,k_m\end{bmatrix}_q E_{k_1}(x,q)\cdots E_{k_j}(x,q)x^{k_{j+1}}\cdots x^{k_m}\frac{t^n}{[n]_q!}.$$

□

As $q\to 1$ in the formula of Theorem 3.1, we have

Theorem 3.2. *Let m be a given positive integer. Then for any $n\geq 0$,*

$$(-1)^n\sum_{i_1+\cdots+i_m=n}\binom{n}{i_1,\ldots,i_m}E_{i_1}(-x)\cdots E_{i_m}(-x)$$
$$=\sum_{j=0}^{m}(-1)^j 2^{m-j}\binom{m}{j}\sum_{k_1+\cdots k_m=n}\binom{n}{k_1,\ldots,k_m}E_{k_1}(x)\cdots E_{k_j}(x)x^{n-(k_1+\cdots+k_j)}.$$

Especially in the cases $m=1,2$, the following results hold:

(1) *For any $n\geq 0$, we have $(-1)^nE_n(-x)+E_n(x)=2x^n$.*

(2) *For any $k\geq 1$, $E_{2k}=0$.*

(3) *For any $n\geq 0$,*

$$(-1)^n\sum_{i=0}^{n}\binom{n}{i}E_i(-x)E_{n-i}(-x)=\sum_{i=0}^{n}\binom{n}{i}E_i(x)E_{n-i}(x)-4\sum_{i=0}^{n}\binom{n}{i}E_i(x)x^{n-i}+2^{n+2}x^n.$$

(4) *If n is an odd positive integer, then*

$$\sum_{i=0}^{n}\binom{n}{i}E_iE_{n-i}=2E_n.$$

3.2. q-Symmetry

Theorem 3.3 (q-Symmetry 1). *For any $m,n\in\mathbb{N}$, we have*

$$(-1)^m\sum_{k=0}^{m}\begin{bmatrix} m \\ k\end{bmatrix}_q E_{n+k}(x,q)q^{-kn+mn}=(-1)^n\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_{q^{-1}}F_{m+k}(-x,q^{-1})q^{\binom{n}{2}-\binom{k}{2}}. \tag{1}$$

Proof. Let x,y be two q-commuting variables with $xy=q^{-1}yx$. We compute the generating functions

$$\begin{aligned}
L(w,x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m\sum_{k=0}^{m}\begin{bmatrix} m \\ k\end{bmatrix}_q E_{n+k}(w,q)q^{-kn+mn}\frac{x^m}{[m]_q!}\frac{y^n}{[n]_q!},\\
R(w,x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q F_{m+k}(-w,q^{-1})q^{\binom{n}{2}-\binom{k}{2}}\frac{x^m}{[m]_q!}\frac{y^n}{[n]_q!},
\end{aligned}$$

where w is a commuting variable with x and y.

$$\begin{aligned}
L(w,x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m\sum_{k=0}^{m}\begin{bmatrix} m \\ k\end{bmatrix}_q E_{n+k}(w,q)q^{-kn}\frac{y^n}{[n]_q!}\frac{x^m}{[m]_q!}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m\sum_{k=0}^{m}E_{n+k}(w,q)q^{-kn}\frac{y^n}{[n]_q!}\frac{x^k}{[k]_q!}\frac{x^{m-k}}{[m-k]_q!}\\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}E_{n+k}(w,q)\frac{(-x)^k}{[k]_q!}\frac{y^n}{[n]_q!}\frac{(-x)^j}{[j]_q!}\\
&= \left(\sum_{i=0}^{\infty}E_i(x,q)\sum_{k=0}^{i}\frac{(-x)^k}{[k]_q!}\frac{y^{i-k}}{[i-k]_q!}\right)e_q(-x)\\
&= \frac{2e_q(w(y-x))}{e_q(y-x)+1}e_q(-x).
\end{aligned}$$

$$
\begin{aligned}
R(w,x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\sum_{k=0}^{n}F_{m+k}(-w,q^{-1})\frac{x^m}{[m]_q!}\frac{y^k}{[k]_q!}\frac{q^{\binom{n-k}{2}}y^{n-k}}{[n-k]_q!}\\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{j+k}F_{m+k}(-w,q^{-1})\frac{x^m}{[m]_q!}\frac{y^k}{[k]_q!}\frac{y^j}{[j]_{q^{-1}}!}\\
&= \left(\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}F_{m+k}(-w,q^{-1})\frac{x^m}{[m]_q!}\frac{(-y)^k}{[k]_q!}\right)e_q(-y)\\
&= \frac{2e_q(-w(y-x))}{e_{q^{-1}}(x-y)+1}e_{q^{-1}}(-y).
\end{aligned}
$$

Hence it follows that

$$
\begin{aligned}
R(w,x,y)e_q(y) &= \frac{2e_q(w(y-x))}{e_{q^{-1}}(x-y)+1}\\
&= \frac{2e_q(w(y-x))}{e_q(y-x)+1}e_q(y-x)\\
&= L(w,x,y)e_q(y),
\end{aligned}
$$

which provides $R(w,x,y)=L(w,x,y)$. Therefore we can complete the proof. □

As $q\to 1$ in (1) of Theorem 3.3, we have a symmetric relation for the ordinary Euler polynomials:

Theorem 3.4. *For any $m,n\in\mathbb{N}$, we have*

$$(-1)^m\sum_{k=0}^{m}\binom{m}{k}E_{n+k}(x)=(-1)^n\sum_{k=0}^{n}\binom{n}{k}E_{m+k}(-x).$$

It is known that this result follows from Theorem 7.4 in Gessel [4].

Theorem 3.5 (q-Symmetry 2). *For any $m,n\in\mathbb{N}$, we have*

$$
\begin{aligned}
&(-1)^m\sum_{k=0}^{m+1}\begin{bmatrix}m+1\\k\end{bmatrix}_q[n+k+1]_qE_{n+k}(x,q)q^{-k(n+1)-\binom{n}{2}+1}\\
&\qquad+(-1)^n\sum_{k=0}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}_{q^{-1}}[m+k+1]_{q^{-1}}F_{m+k}(-x,q^{-1})q^{k(m+1)+\binom{m}{2}}=0. \qquad (2)
\end{aligned}
$$

Proof. Applying q-derivative formula to the identity (1) in Theorem 3.3 replaced m, n by $m+1$, $n+1$, respectively, we have the result. □

As $q\to 1$ in (2) of Theorem 3.5, we have another symmetric formula for the ordinary Euler polynomials:

Theorem 3.6. *For any $m,n\in\mathbb{N}$, we have*

$$(-1)^m\sum_{k=0}^{m+1}\binom{m+1}{k}(n+k+1)E_{n+k}(x)+(-1)^n\sum_{k=0}^{n+1}\binom{n+1}{k}(m+k+1)E_{m+k}(-x)=0. \qquad (3)$$

This can be regarded as an Euler polynomial version of Kaneko-Momiyama formulae for Bernoulli numbers. To be precise, put $m=n$ and $x=0$ in (3). Then we have an analogue of Kaneko's formula:

Theorem 3.7. *For any $n\in\mathbb{N}$,*

$$\sum_{k=0}^{n+1}\binom{n+1}{k}(n+k+1)E_{n+k}=0.$$

Put $x=0$ in (3). Then we have an analogue of Momiyama's formula:

Theorem 3.8. *For any $m,n\in\mathbb{N}$, we have*

$$(-1)^m\sum_{k=0}^{m+1}\binom{m+1}{k}(n+k+1)E_{n+k}+(-1)^n\sum_{k=0}^{n+1}\binom{n+1}{k}(m+k+1)E_{m+k}=0.$$

Using q-integral formula to (1) in Theorem 3.3, we have

Theorem 3.9 (q-Symmetry 3). *For any $m, n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,*

$$(-1)^m \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \frac{E_{n+k+1}(a,q) - E_{n+k+1}(b,q)}{[n+k+1]_q} q^{-kn-\binom{n}{2}+mn} + (-1)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} \frac{F_{m+k+1}(-a,q^{-1}) - F_{m+k+1}(-b,q^{-1})}{[m+k+1]_q} q^{-km+\binom{m}{2}} = 0.$$

As $q \to 1$, we get

Theorem 3.10. *For any $m, n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,*

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} \frac{E_{n+k+1}(a) - E_{n+k+1}(b)}{n+k+1} + (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{E_{m+k+1}(-a) - E_{m+k+1}(-b)}{m+k+1} = 0.$$

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