Homogeneous and inhomogeneous elliptic Dedekind-Rademacher Sums

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Abstract

We introduce multiple homogeneous elliptic Dedekind-Rademacher sums, in terms of special values of Jacobi forms in two variables, that generalize the elliptic Dedekind-Sczech sums [19]. These sums give us an elliptic analogue to various classical Dedekind sums introduced by Beck in [7], Berndt in [8], Dieter in [9] and Hall-Lewis-Zagier in [11]. We prove their reciprocity laws. The second aim is to show that our result contains an elliptic analogous to the main results of Beck in [7], Berndt in [8], Dieter in [9] and Hall-Lewis-Zagier in [11]. Our elliptic Dedekind -Rademacher sums are connected to many interesting invariants.

1 Introduction

We use throughout the notation : $e(z) = e^{2\pi i z}$ $(z \in \mathbb{C})$.

1.1 Sczech's formula

The elliptic Dedekind-Rademacher sums that we introduce in this work are a generalization for the elliptic Dedekind-Rademacher sums studied by Sczech in his paper [19].
Let L be a lattice in the complex plane $\mathbb{C}$ with $\{\omega_1, \omega_2\}$ an $\mathbb{Z}$-oriented basis of L and with the multiplicator ring

$$O_L = \{m \in \mathbb{C} \mid mL \subset L\}$$

We define the Eisenstein series

$$E_k(z; L) := \sum_{\substack{\omega \in L \\ \omega + z \neq 0}}^{(e)} (\omega + z)^{-k} |\omega + z|^{-s} \Big|_{s=0}, \quad k = 0, 1, 2, ...$$

where $\sum_{\omega \in L}^{(e)}$ is the Eisenstein summation defined by

$$\sum_{\omega \in L}^{(e)} = \lim_{M, N \to \infty} \sum_{m=-M}^{m=M} \sum_{n=-N}^{n=N}, \text{ Where } \omega = m\omega_1 + n\omega_2.$$

If a, c be coprime elements in O_L, R. Sczech in [19] defined the following elliptic Dedekind sums

$$D(a,c) = \frac{1}{c} \sum_{k \in L/cL} E_1\left(\frac{k}{c}\right) E_1\left(\frac{ak}{c}\right) \tag{1.1.1}$$

and stated his reciprocity formula

$$D(a,c) + D(c,a) = 2iE_2(0) \operatorname{Im}\left(\frac{a}{c} + \frac{1}{ac} + \frac{c}{a}\right), c \neq 0 \tag{1.1.2}$$

From this result, he construct a non trivial element in $H^1\Big(\Big(\mathrm{SL}_2(O_K), \times\Big), (\mathbb{C}, +)\Big)$.

In this paper we study the elliptic Dedekind-Rademacher sums in the general frame of Jacobi forms (expressed in terms of Jacobi forms). We will prove their reciprocity law. The first aim of this paper is to show how to deduce the formula 1.1.2 (stated in Sczech's paper [19]) from our reciprocity theorem 2.2.1.
The second aim is to explain that our second main theorem 2.2.1 can be regarded as an ellitpic version of the main results of Beck in [7], Berndt in [8], Dieter in [9] and Hall-Lewis-Zagier in [11].
The third aim is to give several interesting invariants linking different imaginary quadratic fields. These invariants provided from our elliptic Dedekind sums.
Various elliptic Dedekind sums are the main objects of this paper.

1.2 Notations and definitions

Throughout this paper, we fix L be a lattice in the complex plane $\mathbb{C}$ with the multiplicator ring O_L and we fix $\{\omega_1, \omega_2\}$ an $\mathbb{Z}$-oriented basis of L. i.e

$$Im\left(\frac{\omega_1}{\omega_2}\right) > 0, L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

We define the $\mathbb{R}$-alternating bilinear form

$$E_L(z, \varphi) = \frac{\bar{z}\varphi - z\bar{\varphi}}{\omega_1\bar{\omega_2} - \bar{\omega_1}\omega_2} = \frac{\bar{z}\varphi - z\bar{\varphi}}{2i|\omega_2|^2 Im\left(\frac{\omega_1}{\omega_2}\right)}$$

which is a symplectic form on $\mathbb{C}$ associated to the oriented complex lattice L.
We note that E_L :

i) satistfies $E_{\lambda L}(\lambda z, \lambda\varphi) = E_L(z, \varphi)$; for all $\lambda \in \mathbb{C}^*$;

ii) $E_L(L \times L) \subset \mathbb{Z}$;

iii) $E_L(w_1, w_2) = -1$ for all $\mathbb{Z}$-oriented basis (w_1, w_2) of L;

iv) For $z, \varphi \in \mathbb{C}$, $z = a\omega_2 + b\omega_1, \varphi = c\omega_2 + d\omega_1$ with $a, b, c, d \in \mathbb{R}$, we have

$$E_L(z, \varphi) = \frac{\operatorname{Im}(\bar{z}\varphi)}{a(L)} = ad - bc.$$

v) for two complex lattices $L \subset \Lambda$ we have

$$E_\Lambda = [\Lambda : L] E_L$$

where $[\Lambda : L]$ indicates the number of elements of Λ/L.

We introduce the following Eisenstein-Kronecker series

$$(1.2.3) \qquad d_k(\varphi; L) = \begin{cases} -\sum\limits_{\substack{\omega\in L \\ \omega\neq 0}}^{(e)} \dfrac{e(E_L(\omega,\varphi))}{\omega^k} & \text{If} \quad k = 1,2,3,\ldots \\ 1 & \text{If} \quad k = 0 \end{cases}$$

2 Main results

Now, we are going to state our main results.

2.1 Elliptic Dedekind-Rademacher-Sczech sums

We now introduce the elliptic Dedekind-Rademacher-Sczech sums, for a, b, c in O_L,

$$(2.1.4) \qquad D\left(a,b,c|x,y,z\right) = \frac{1}{c}\sum_{\bar{k}\in L/cL} E_1\left(a\frac{k+z}{c} - x; L\right) E_1\left(b\frac{k+z}{c} - y; L\right)$$

In particular, the sums

$$D(a,b,c) := D\left(a,b,c|x=0,y=0,z=0\right) = \frac{1}{c}\sum_{\bar{k}\in L/cL} E_1\left(\frac{ak}{c}; L\right) E_1\left(\frac{bk}{c}; L\right)$$

are the Rademacher homogeneous analoguous to the elliptic Dedekind-Sczech's sums (1.1.1), the fraction $\frac{k}{c}$ in the left-hand Eisenstein serie E_1 in the sums (1.1.1) becomes $\frac{bk}{c}$.
Now, we state our first main result

Theorem 2.1.1 *(**First main result**) Let a_1, a_2, a_3 be three elements in O_L being pairwise coprime, z_1, z_2 and z_3 be complex numbers. Then, we have*

$$\begin{aligned} &D\left(a_1,a_2,a_3|z_1,z_2,z_3\right) + D\left(a_2,a_3,a_1|z_2,z_3,z_1\right) + D\left(a_3,a_1,a_2|z_3,z_1,z_2\right) = \\ &\sum_{k(mod\ 3)} \left(\frac{\bar{a}_k}{\bar{a}_{k+1}\bar{a}_{k+2}} - \frac{a_k}{a_{k+1}a_{k+2}}\delta_{1,2,3}\right) d_2(a_{k+1}z_{k+2} - a_{k+2}z_{k+1}; L) + \\ &\qquad \sum_{k(mod\ 3)} \left(\delta_{k+1,k+2} - \delta_{1,2,3}\right) \frac{a_k}{a_{k+1}a_{k+2}} E_2\left(z_k - a_k Z_{k+1,k+2}; L\right). \end{aligned}$$

where $Z_{j,k}$ is a common pole of $D_L\left(\dfrac{\varphi_j}{\bar{a}_j}; a_j z - z_j\right)$ and $D_L\left(\dfrac{\varphi_k}{\bar{a}_k}; a_k z - z_k\right)$

$$\begin{aligned} \delta_{j,k} &= \begin{cases} 1 & \textit{if } a_j z_k - a_k z_j \in L \\ 0 & \textit{Otherwise} \end{cases}, \\ \delta_{1,2,3} &= \begin{cases} 1 & \textit{if } \ a_j z_k - a_k z_j \in L, \ \forall\ 1 \leqslant j \neq k \leqslant 3 \\ 0 & \textit{Otherwise} \end{cases} \\ a_k &= a_{k(mod\ 3)}, \bar{a}_k = \bar{a}_{k(mod\ 3)}, z_k = z_{k(mod\ 3)}, \\ Z_{k,k+1} &= Z_{k(mod\ 3),k+1(mod\ 3)}, \delta_{k,k+1} = \delta_{k(mod\ 3),k+1(mod\ 3)}, \quad \forall\ k \in \mathbb{N}. \end{aligned}$$

Corollary 2.1.2 *(***R. Sczech [19]***)*
For any a_1, a_2, a_3 three elements in O_L being pairwise coprime and z_1, z_2, z_3 complex numbers such that:

$$a_j z_k - a_k z_j \in L, \ \forall\ 1 \leqslant j \neq k \leqslant 3$$

Then, we have

$$D\left(a_1, a_2, a_3 | z_1, z_2, z_3\right) + D\left(a_2, a_3, a_1 | z_2, z_3, z_1\right) + D\left(a_3, a_1, a_2 | z_3, z_1, z_2\right) = -d_2(0; L)\ \mathbf{I}\left(\frac{a_1}{a_2 a_3} + \frac{a_2}{a_1 a_3} + \frac{a_3}{a_1 a_2}\right)$$

where $\mathbf{I}(z) = z - \bar{z}$

The result of this corollary 2.1.2 is an homogenization of the main Theorem of Sczech in [19]. Precisely, for any a_1, a_2, a_3 three elements in O_L being pairwise coprime and $z_1 = z_2 = z_3 = 0$ we obtain

$$D(a_1, a_2, a_3) + D(a_2, a_3, a_1) + D(a_3, a_1, a_2) = 2iE_2(0; L)\ \mathbf{Im}\left(\frac{a_1}{a_2 a_3} + \frac{a_2}{a_1 a_3} + \frac{a_3}{a_1 a_2}\right)$$

This reciprocity formula is not implied by Sczech's (1.1.2).

Corollary 2.1.3 *Let a_1, a_2, a_3 be three elements in O_L, pairwise coprime and z_1, z_2, z_3 complex numbers such that:*

$$a_j z_k - a_k z_j \notin L, \ \forall\ 1 \leqslant j \neq k \leqslant 3$$

Then, we have

$$D\left(a_1, a_2, a_3 | z_1, z_2, z_3\right) + D\left(a_2, a_3, a_1 | z_2, z_3, z_1\right) + D\left(a_3, a_1, a_2 | z_3, z_1, z_2\right) = \sum_{k=1}^{3} \frac{\bar{a}_k}{\bar{a}_{k+1}\bar{a}_{k+2}} d_2(a_{k+1}z_{k+2} - a_{k+2}z_{k+1}; L)$$

Corollary 2.1.4 *Assume that*

$$a_1 z_2 - a_2 z_1 \in L \quad \textit{and } a_1 z_3 - a_3 z_1, a_2 z_3 - a_3 z_2 \notin L.$$

Then, we have

$$D\left(a_1, a_2, a_3 | z_1, z_2, z_3\right) + D\left(a_2, a_3, a_1 | z_2, z_3, z_1\right) + D\left(a_3, a_1, a_2 | z_3, z_1, z_2\right) = \sum_{k (mod\ 3)} \frac{\bar{a}_k}{\bar{a}_{k+1}\bar{a}_{k+2}} d_2(a_{k+1}z_{k+2} - a_{k+2}z_{k+1}; L) + \frac{a_3}{a_1 a_2} E_2\left(z_3 - a_3 Z_{2,3}; L\right).$$

Corollary 2.1.5 *Assume that*

$$a_1 z_2 - a_2 z_1, a_1 z_3 - a_3 z_1 \in L \quad \textit{and } a_2 z_3 - a_3 z_2 \notin L.$$

Then, we have

$$D\left(a_1, a_2, a_3 | z_1, z_2, z_3\right) + D\left(a_2, a_3, a_1 | z_2, z_3, z_1\right) + D\left(a_3, a_1, a_2 | z_3, z_1, z_2\right) = \sum_{k (mod\ 3)} \frac{\bar{a}_k}{\bar{a}_{k+1}\bar{a}_{k+2}} d_2(a_{k+1}z_{k+2} - a_{k+2}z_{k+1}; L) + \frac{a_3}{a_1 a_2} E_2\left(z_3 - a_3 Z_{2,3}; L\right) + \frac{a_2}{a_1 a_3} E_2\left(z_2 - a_2 Z_{1,3}; L\right)$$

Remark 2.1.6
The results of corollary 2.1.2, corollary 2.1.3,corollary 2.1.4 and corollary 2.1.5 are completely independent and together represents a complete reciprocity law for our shifted **elliptic Dedekind-Rademacher-Sczech sums***.*

2.2 Multiple elliptic Dedekind-Rademacher sums

We state a higher version of the theorem 2.1.1 in terms of Jacobi forms.
More precisely, our theorem 2.1.1 can be derived from the following generalized Dedekind reciprocity law in terms of Jacobi forms $D_L(z;\varphi)$ which is defined in the next section 3.
For $k = 1,\ldots,n$ we consider the sets

$$\begin{aligned} \mathrm{S}_k &= \left\{ \bar{t} \in Ł/a_k L : \ \frac{z_k+t}{a_k} \ \text{ is a simple pole of } z \to \prod_{j=1}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j z - z_j\right) \right\} \\ \mathrm{M}_k &= \left\{ \bar{t} \in Ł/a_k L : \ \frac{z_k+t}{a_k} \ \text{ is a multiple pole of } z \to \prod_{j=1}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j z - z_j\right) \right\} \end{aligned}$$

We introduce the **multiple elliptic Dedekind-Rademacher** sums

$$\begin{gathered} d(a_k; a_1, ..., \check{a_k}, ..., a_n | z_k; z_1, ..., \check{z_k}, ..., z_n | \varphi_k; \varphi_1, ..., \check{\varphi_k}, ..., \varphi_n) := \\ \frac{1}{a_k} \sum_{\bar{t}\in S_k} \prod_{1\leqslant j\neq k\leqslant n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k+t}{a_k} - z_j\right) \end{gathered}$$

We state now, our reciprocity formula concerning our multiple elliptic Dedekind-Rademacher sums

Theorem 2.2.1 *(***Second main result***) Let $n \in \mathbb{N}$, $n \geqslant 3$, $a_1,\ldots,a_n$ be elements in O_L, $z_1,\ldots,z_n$ complex numbers and $\varphi_1,...,\varphi_n$ complex variables such that $\sum_{j=1}^{n} \varphi_j = 0$. We obtain the following reciprocity laws*

$$\begin{gathered} \sum_{k=1}^{n} d(a_k; a_1, ..., \check{a_k}, ..., a_n | z_k; z_1, ..., \check{z_k}, ..., z_n | \varphi_k; \varphi_1, ..., \check{\varphi_k}, ..., \varphi_n) = \\ -\sum_{k=1}^{n} \sum_{\bar{t}\in M_k} Res\left(\prod_{j=1}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j z - z_j\right); z = \frac{z_k+t}{a_k} \right) \end{gathered}$$

We state the following special cases of our theorem 2.2.1

Corollary 2.2.2 *(***Explicit formula 1***)*

Let $n \in \mathbb{N}$, $n \geqslant 3$, $\varphi_1,...,\varphi_n$ complex variables with sum zero and $a_1,\ldots,a_n$ be elements in O_L, $z_1,\ldots,z_n$ complex numbers such that

$$a_j z_k - a_k z_j \notin L, \quad \forall\ 1 \leqslant j \neq k \leqslant n$$

Then we have $S_k = L/a_k L$ and

$$\sum_{k=1}^{n} d(a_k; a_1, ..., \check{a_k}, ..., a_n | z_k; z_1, ..., \check{z_k}, ..., z_n | \varphi_k; \varphi_1, ..., \check{\varphi_k}, ..., \varphi_n) = 0$$

Corollary 2.2.3 *(***Explicit formula 2***)*

Let $n \in \mathbb{N},\ n \geqslant 3$, $a_1, \ldots, a_n$ be elements in O_L and $\varphi_1, ..., \varphi_n$ complex variables with sum zero. Then **for** *$z_1 = \cdots = z_n = 0$ we have $S_k = L/a_k L\backslash\{\overline{0}\}$ and*

$$\sum_{k=1}^{n} d(a_k; a_1, ..., \check{a_k}, ..., a_n | z_k; z_1, ..., \check{z_k}, ..., z_n | \varphi_k; \varphi_1, ..., \check{\varphi_k}, ..., \varphi_n) =$$

$$- \sum_{\substack{m_1,\ldots,m_n \geqslant 0 \\ m_1+\ldots+m_n = n-1}} a_1^{m_1-1} \ldots a_n^{m_n-1} d_{m_1}\left(\frac{\varphi_1}{\bar{a}_1}; L\right) \ldots d_{m_n}\left(\frac{\varphi_n}{\bar{a}_n}; L\right)$$

In the section 4 of this paper we give the complete proofs of our results mainly theorem 2.1.1 and theorem 2.2.1. Besides, we show how to deduce the Sczech's formula 1.1.2 from theorem 2.2.1

2.3 Some invariants connected to our multiple elliptic Dedekind-Rademacher sums

In this section the complex lattice L will be O_K which is the ring of integers of the imaginary quadratic number field K.
Now, our main results in the previous subsection produced some applications in the following areas: Eisenstein Cohomology of the groups $\mathrm{SL}_2(O_K)$, are connected to special values of Hecke L-functions at $s = 1$ associated to some quadratic forms, also related to quadratic residue Legendre symbols to elliptic Dedekind sums. Finally we precise the result on the problem of the density of elliptic Dedekind sums. These applications which we will recall here are already well known. Indeed, there are obtained by H. Ito in [16, 18] and R. Sczech in [19]. In particular, they study only the simplest form of our elliptic Dedekind sums and they obtain the following theorems.

Let $K = \mathbb{Q}, \mathbb{Q}(\sqrt{D})$, D negative integer.
Put

$$D(a_1, a_2) := \frac{1}{a_2} \sum_{\mathrm{tmod}(a_2 L)} E_1\left(a_1 \frac{t}{a_2}; L\right) E_1\left(\frac{t}{a_2}; L\right)$$

Let $A = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}$ in $\mathrm{SL}_2(O_K)$.
We define an elliptic analogous to Rademacher function

$$\Phi : A \mapsto \begin{cases} E_2(0)\mathbf{I}\left(\frac{a_1+a_4}{a_2}\right) - D(a_1, a_2) & \textbf{If } a_2 \neq 0 \\ E_2(0)\mathbf{I}\left(\frac{a_3}{a_4}\right) & \textbf{Otherwise} \end{cases}$$

In 1984 R. Sczech in his paper [19] proved the following result

Theorem 2.3.1 (R. Sczech [19]) *Φ is a group morphism: $(SL_2(O_K), \times) \to (\mathbb{C}, +)$ Non trivial iff $K \neq \mathbb{Q}, \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$.*

Then, we obtain a cocycle in $H^1(\mathrm{SL}_2(O_K), \mathbb{C})$ which is represented a non trivial Eisenstein cohomology class of $\mathrm{SL}_2(O_K)$.

For $A = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}$ in $\mathrm{SL}_2(O_K)$ with $a_1 + a_4 \neq 0, \pm 1, \pm 2$.
Then the equation $a_2 X^2 + (a_4 - a_1)X - a_2 = 0$ has two distincts solutions α, α' in $\mathbb{C}$. We define the quadratic form

$$Q(m,n) := (m\alpha + n)(m\alpha' + n)$$

it satisfies

$$Q\Big((m,n)A\Big) = Q(m,n)$$

Then A operates on the complex lattice O_K.
Define the Hecke L-function

$$L(s,A) = \sum_{(m,n)}{}' \frac{\overline{Q(m,n)}}{Q(m,n)}, \quad \mathrm{Re}(s) > \frac{3}{2}.$$

In 1987 H. Ito proved the result [16]

Theorem 2.3.2 (H. Ito [16]) *Let $\epsilon = a_3\alpha + a_4$. Then*

$$\Phi(A) = sgn\,(log|\epsilon|)\,.(\alpha - \alpha').L(1,A)$$

This result can be interpreted as a periods of some differential forms on the upper half space which consists of all quaternion numbers. The relation between these periods and special values of L-functions has already been observed by Harder [12, 13, 14].

Now, we set $\Psi(A) = \frac{\Phi(A)}{\sqrt{D}E_2(0)}$, $K = \mathbb{Q}(\sqrt{D})$, D negative integer.
In 1990 H. Ito proved in [17] the following result

Theorem 2.3.3 (H. Ito [17]) *There exists explicit group morphism*

$$\chi : (SL_2(O_K), \times) \to (Z/8\mathbb{Z}, +)$$

such that

$$\Psi(A) = \chi(A) mod(8O_K)$$

This homomorphism describes the eighth roots of unity which occur in the transformation formula of certain theta series. $\chi(A)$ is a class of $\mathbb{Z}/8\mathbb{Z}$. For more details we refer to R. Sczech [20]. Then we get cohomology class in $H^1\left(\Gamma(8), \mathbb{Z}/2\mathbb{Z}\right)$.

In 2004 H. Ito again proved the following result [18]

Theorem 2.3.4 (H. Ito [18]) *If K is euclidian $\neq \mathbb{Q}, \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$. Then the set*

$$\left\{ \left(\frac{a_1}{a_2}, \frac{D(a_1,a_2)}{\sqrt{D}E_2(0)} \right) : \frac{a_1}{a_2} \in K \right\}$$

is dense in the space $\mathbb{C} \times \mathbb{R}$.

This result is an elliptic analogue to the main result of Hickerson in [15]. However, this result does not mean that points $\left(\frac{a_1}{a_2}, \frac{D(a_1,a_2)}{\sqrt{D}E_2(0)}\right)$ distribute uniformly.
We note also, that the treatement of Ito does not work in non-Euclidian case. It will be interesting to solve these questions.

3 Eisenstein-Kronecker numbers $d_m(\varphi, L)$

3.1 An overview on the Jacobi form $D_L(z, \varphi)$

Our references for this subsection are [2, 3, 4, 5, 6].
For $\tau \in \mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ the upper half plane, we consider the following Jacobi's Theta function

$$\theta_\tau(z) = \sum_{n\in\mathbb{Z}} e\left(\frac{1}{2}(n+\frac{1}{2})^2\tau + (n+\frac{1}{2})(z+\frac{1}{2})\right)$$

Or by Jacobi Triple product formula

(3.1.5) $\theta_\tau(z) = iq_\tau^{1/8}\left(e(z/2) - e(-z/2)\right)\prod_{n=1}^{\infty}\left(1-q_\tau^n\right)\left(1-q_\tau^n e(z)\right)\left(1-q_\tau^n e(-z)\right)$

We shall use the following notation

$$\varphi = \varphi_1\tau + \varphi_2, (\varphi_1, \varphi_2) \in \mathbb{R}^2, \forall\varphi \in \mathbb{C},$$

because $\{\tau, 1\}$ is an $\mathbb{R}$-basis of $\mathbb{C}$.
Now, for each complex lattice L , we fix $\{\omega_1, \omega_2\}$ an $\mathbb{Z}$-oriented basis of L .
We associate to L a Jacobi form of two variables

(3.1.6) $$D_L(z;\varphi) = \frac{1}{\omega_2} e\left(\frac{z}{\omega_2}\varphi_1\right) \frac{\theta_\tau'(0)\theta_\tau\left(\frac{z+\varphi}{\omega_2}\right)}{\theta_\tau\left(\frac{z}{\omega_2}\right)\theta_\tau\left(\frac{\varphi}{\omega_2}\right)}$$

where $\tau = \frac{\omega_1}{\omega_2}$.
We quote from [2, 3, 4, 5, 6] the following fondamental properties of $D_L(z;\varphi)$:

Theorem 3.1.1 *(***Properties of** $D_L(z;\varphi)$*)*

i) D_L *is meromorphic in the first variable* z*, and only real analytic on the second variable* φ*.*

ii) (**Periodicity** *of* $D_L(z;\varphi)$*):* $\forall\rho \in L$

$$\begin{cases} D_L(z;\varphi+\rho) = D_L(z;\varphi) \\ D_L(z+\rho;\varphi) = e(E_L(\rho,\varphi))D_L(z;\varphi) \end{cases}$$

*iii) (***Functional Equation***):* $D_L(z;\varphi)$ *satisfies* $D_L(z;\varphi)e(-E_L(z,\varphi)) = D_L(\varphi;z)$.

iv) The **Laurent expansion** *of the Jacobi form* $D_L(z,\varphi)$ *is given by*

$$D_L(z;\varphi) = \sum_{m\geqslant 0} d_m(\varphi;L)z^{m-1}, \qquad \forall\ z,\varphi \in \mathbb{C}\backslash L$$

$$d_0(\varphi;L) = 1, d_1(\varphi;L) = E_1(\varphi;L), \quad d_2(\varphi;L) = \frac{1}{2}E_1(\varphi,L)^2 - \frac{1}{2}\wp_L(\varphi)$$

where $\wp_L(\varphi)$ *is the Weierstrass* $\wp_L$*-function*

$$\wp_L(\varphi) = \frac{1}{z^2} + \sum_{w\in L\backslash\{0\}}\left[\frac{1}{(z-w)^2} - \frac{z}{w^2}\right]$$

v) **(Distribution Formula for $D_L(z;\varphi)$) :**
For L, Λ complex lattices such that : $L \subset \Lambda$, $[\Lambda : L] = l$. We have for all $\forall\ z, \varphi \in \mathbb{C}\backslash\Lambda$

$$\sum_{\bar{t}\in\Lambda/L} D_L(lz;\varphi+t) = D_\Lambda(z;\varphi),$$

and

$$\sum_{\bar{t}\in\Lambda/L\backslash\{\bar{0}\}} D_L(lz;t) = \zeta(z;\Lambda) - \zeta(lz, L),$$

where

$$\zeta(z, L) = \frac{1}{z} + \sum_{w\in L\backslash\{0\}} \left[\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right]$$

In the next section we study the properties of $d_m(\varphi; L)$ coefficents of Laurent expansion of $D_L(z, \varphi)$ in the first variable z.

3.2 Properties of the Eisenstein-Kronecker numbers $d_m(\varphi, L)$

In the following we precise the most important properties of Eisenstein-Kronecker numbers $d_m(\varphi, L)$

Theorem 3.2.1 *(***properties of** $d_m(\varphi, L)$*)*

i) *(***Homogeneity and symmetry***) For each $m \in \mathbb{N}^*$, $d_m(\varphi, L_\tau)$ is homogenous of degre $-m$ i.e*

$$d_m(\lambda\varphi, \lambda L) = \lambda^{-m} d_m(\varphi, L), \forall\lambda \in \mathbb{C}\backslash\{0\}.$$

In particular,

$$d_m(-\varphi; L) = (-1)^{m-1} d_m(\varphi; L)$$

ii) *(***Periodicity***):*

$$d_m(\varphi+\rho; L) = d_m(\varphi; L), \forall\rho \in L$$

iii) *(***Modularity***): We let $d_m(\varphi, \tau) := d_m(\varphi, L_\tau)$ where $L_\tau = \mathbb{Z}\tau + \mathbb{Z}, \tau \in \mathcal{H}$. $d_m(\varphi, \tau)$ is a modular form for $SL_2(\mathbb{Z})$, with index 0 and weight m i.e*

$$d_m\left(\frac{\varphi}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^m d_m(\varphi;\tau), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

iv) **Elliptic Raabe Formula**
For L, Λ complex lattices such that : $L \subset \Lambda$. Then, we have for all $m \geqslant 1$

$$\sum_{\bar{t}\in\Lambda/L} d_m(\varphi+t; L) = [\Lambda : L]^{1-m} d_m(\varphi;\Lambda), \quad \forall\varphi \in \mathbb{C}\backslash\Lambda$$

and

$$\sum_{\bar{t}\in\Lambda/L\backslash\{\bar{0}\}} d_m(\varphi+t; L) = E_m(0; L) - [\Lambda : L]^{1-m} E_m(0;\Lambda), \quad \forall\varphi \in \Lambda$$

Proof:

This theorem 3.2.1 is a direct consequence of the theorem 3.1.1. The idea consists in extracting the coefficients of Laurent series of the Jacobi forms in the statement of the previous theorem 3.1.1.

4 Proofs of main results

4.1 Proof of Theorem 2.2.1

To prove the Theorem 2.2.1, we start with some preliminaries results.
We consider the function

$$F(z, \overrightarrow{\Phi}, \overrightarrow{A}) = \prod_{j=1}^{n} D_L\left(a_j z - z_j; \frac{\varphi_j}{\bar{a}_j}\right)$$

Where

$$\overrightarrow{A} = (a_1 \ldots, a_n)\,, \overrightarrow{\Phi} = (\varphi_1, \ldots, \varphi_n)$$

$z_1, ..., z_n$ are complex numbers and $\varphi_1, ..., \varphi_n$ complex variables with sum zero. Now, we have the following interesting properties of F

Proposition 4.1.1 *The function*

$$F : z \to F(z, \overrightarrow{\Phi}, \overrightarrow{A})$$

i) F is a meromorphic, with poles only at

$$z = \frac{z_k + t}{a_k}, k = 1, \ldots, n,\ t \in L$$

ii) F is periodic with periods the lattice L i.e

$$F(z + \rho, \overrightarrow{\Phi}, \overrightarrow{A}) = F(z, \overrightarrow{\Phi}, \overrightarrow{A}), \forall \rho \in L$$

iii)

$$\prod_{j=1}^{n} D_L\left(a_j z - z_j; \frac{\varphi_j}{\bar{a}_j}\right) = e\left(-\sum_{j=1}^{n}\left(z_j, \frac{\varphi_j}{\bar{a}_j}\right)\right)\prod_{j=1}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j z - z_j\right)$$

iv) For all $\bar{t} \in S_k$ we have

$$\frac{1}{a_k}\prod_{1 \leqslant j \neq k \leqslant n}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j\frac{z_k + t}{a_k} - z_j\right) =$$

$$e\left(\sum_{j=1}^{n}\left(z_j, \frac{\varphi_j}{\bar{a}_j}\right)\right) Res\left(F(z, \overrightarrow{\Phi}, \overrightarrow{A})dz; z = \tfrac{z_k+t}{a_k}\right)$$

Proof :

From the theorem 3.1.1, we know that the Jacobi form D_L is meromorphic in the first variable z, and only real analytic on the second variable φ. Then the function F is meromorphic. Now, from the Triple Jacobi formula 3.1.5 and the definition of the Jacobi form $D_L(z;\varphi)$ 3.1.6, we obtain that the poles of $z \to F(z, \overrightarrow{\Phi}, \overrightarrow{A})$ are exactly the elements

$$z = \frac{z_k + t}{a_k}, k = 1, \ldots, n,\ \forall\ t \in L$$

We deduce the property i).
The property ii) can be obtained by using the periodicity equation of the Jacobi form $D_L(z;\varphi)$. Indeed,

$$D_L(z+\rho,\varphi) = e\left(E_L(\rho,\varphi)\right) D_L(z,\varphi)$$

and thanks to the equality $\displaystyle\sum_{i=1}^{n} \varphi_i = 0$, then we obtain for all $\rho \in L$

$$\begin{aligned} F(z+\rho, \overrightarrow{\Phi}, \overrightarrow{A}) &= e\left(\sum_{i=1}^{n} E_L\left(a_i\rho, \frac{\varphi_i}{\bar{a}_i}\right)\right) F(z, \overrightarrow{\Phi}, \overrightarrow{A}) \\ &= e\left(E_L\left(\rho, \sum_{i=1}^{n} \varphi_i\right)\right) F(z, \overrightarrow{\Phi}, \overrightarrow{A}) \\ &= F(z, \overrightarrow{\Phi}, \overrightarrow{A}), \quad \text{thanks to: } \sum_{i=1}^{n} \varphi_i = 0. \end{aligned}$$

The property iii) can be obtained by using the functional equation of the Jacobi form $D_L(z;\varphi)$. Indeed,

$$D_L(z;\varphi) = e(E_L(z,\varphi)) D_L(\varphi; z)$$

Then

$$\begin{aligned} \prod_{j=1}^{n} D_L\left(a_j z - z_j; \frac{\varphi_j}{\bar{a}_j}\right) &= \prod_{j=1}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j z - z_j\right) e\left(E_L\left(a_j z - z_j; \frac{\varphi_j}{\bar{a}_j}\right)\right) \\ &= e\left(-\sum_{j=1}^{n}\left(z_j, \frac{\varphi_j}{\bar{a}_j}\right)\right) \prod_{j=1}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j z - z_j\right) \end{aligned}$$

Hence, we have the property iii).
We prove property iv). We compute the residu of the function F at the pole $z = \frac{z_k+t}{a_k}$ which is of order 1 (i.e simple pole). Then

$$\begin{aligned} \text{Res}\left(F(z, \overrightarrow{\Phi}, \overrightarrow{A})dz; z = \frac{z_k+t}{a_k}\right) &= e\left(-\sum_{j=1}^{n}\left(z_j, \frac{\varphi_j}{\bar{a}_j}\right)\right) \text{Res}\left(\prod_{j=1}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j z - z_j\right) dz; z = \frac{z_k+t}{a_k}\right) \\ &= e\left(-\sum_{j=1}^{n}\left(z_j, \frac{\varphi_j}{\bar{a}_j}\right)\right) \frac{1}{a_k} \prod_{1\leqslant j\neq k\leqslant n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k+t}{a_k} - z_j\right) \end{aligned}$$

Thus we obtain our desired property iv). $\square$

Now, we are going to prove the theorem 2.2.1.
By applying the Liouville's residue theorem to the elliptic function F we obtain

$$\sum_{k=1}^{n} \sum_{t\in L/a_k L} \text{Res}\left(F(z, \overrightarrow{\Phi}, \overrightarrow{A})dz; z = \frac{z_k+t}{a_k}\right) = 0$$

That implies

$$\sum_{k=1}^{n}\sum_{\bar{t}\in \mathrm{S}_k} \operatorname{Res}\left(F(z,\overrightarrow{\Phi},\overrightarrow{A})dz; z=\frac{z_k+t}{a_k}\right) = -\sum_{k=1}^{n}\sum_{\bar{t}\in \mathrm{M}_k} \operatorname{Res}\left(\prod_{j=1}^{n} D_L\left(a_j z - z_j, \frac{\varphi_j}{\bar{a}_j}\right); z=\frac{z_k+t}{a_k}\right)$$

By using property iv) of proposition 4.1.1, we obtain

$$\sum_{k=1}^{n}\frac{1}{a_k}\sum_{\bar{t}\in \mathrm{S}_k}\prod_{1\leqslant j\neq k\leqslant n}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j\frac{z_k+t}{a_k}-z_j\right) = -\sum_{k=1}^{n}\sum_{\bar{t}\in \mathrm{M}_k} \operatorname{Res}\left(\prod_{j=1}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j z - z_j\right); z=\frac{z_k+t}{a_k}\right)$$

Hence, we deduce

$$\sum_{k=1}^{n} d(a_k; a_1,\ldots,\check{a}_k,\ldots,a_n|z_k; z_1,\ldots,\check{z}_k,\ldots,z_n|\varphi_k;\varphi_1,\ldots,\check{\varphi}_k,\ldots,\varphi_n) =$$
$$-\sum_{k=1}^{n}\sum_{\bar{t}\in \mathrm{M}_k} \operatorname{Res}\left(\prod_{j=1}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j z - z_j\right); z=\frac{z_k+t}{a_k}\right)$$

Thus, we obtain our desired theorem 2.2.1.□

Proof : We prove corollaries 2.2.2, 2.2.3.

For the **corollary 2.2.2**, we have to assume that

$$\frac{z_j+t_j}{a_j}-\frac{z_k+t_k}{a_k}\notin L\ ,\ \forall\ t_j,t_k\in L,\ 1\leqslant j\neq k\leqslant n$$

which is equivalent to

$$a_j z_k - a_k z_j \notin L,\quad \forall\ 1\leqslant j\neq k\leqslant n$$

This implies that $\mathrm{S}_k = L/a_kL,\ \mathrm{M}_k=\emptyset,\ \forall\ k=1,\ldots,n.$ Then, from theorem 2.2.1, we obtain

$$\sum_{k=1}^{n}\frac{1}{a_k}\sum_{t\in L/a_kL}\prod_{1\leqslant j\neq k\leqslant n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j\frac{z_k+t}{a_k}-z_j\right)=0$$

It is the corollary 2.2.2.

Now, to prove the **corollary 2.2.3**, we have to assume thathat $z_1 = \ldots = z_n = 0$ **.** Then, $\mathrm{S}_k = L/a_kL\backslash\{\bar{0}\}, \mathrm{M}_k=\{\bar{0}\},\ \forall\ k=1,\ldots,n.$ and $z=0$ is a unique pole of F with order n. Hence

$$\sum_{k=1}^{n} d(a_k; a_1,\ldots,\check{a}_k,\ldots,a_n|z_k; z_1,\ldots,\check{z}_k,\ldots,z_n|\varphi_k;\varphi_1,\ldots,\check{\varphi}_k,\ldots,\varphi_n) =$$
$$\sum_{k=1}^{n}\frac{1}{a_k}\sum_{\bar{t}\in L/a_kL\backslash\{\bar{0}\}}\prod_{1\leqslant j\neq k\leqslant n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j\frac{z_k+t}{a_k}-z_j\right) = -\operatorname{Res}\left(\prod_{j=1}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j z\right); z=0\right)$$

Now to compute this residue, we use the Laurent expansion of Jacobi forms

$$D_L\left(a_j z; \frac{\varphi_j}{\bar{a}_j}\right) = \sum_{m_j\geqslant 0} d_{m_j}\left(\frac{\varphi_j}{\bar{a}_j}; L\right) a_j^{m_j-1} z_j^{m_j-1}$$

Hence

$$\operatorname{Res}\left(\prod_{j=1}^{n} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j z\right); z=0\right) = \sum_{\substack{m_1,\dots,m_n \geqslant 0 \\ m_1+\dots+m_n=n-1}} a_1^{m_1-1}\dots a_n^{m_n-1} d_{m_1}\left(\frac{\varphi_1}{\bar{a}_1}; L\right)\dots d_{m_n}\left(\frac{\varphi_n}{\bar{a}_n}; L\right)$$

This completes the proof of the corollary 2.2.3. □

4.2 Proof of Theorem 2.1.1:

For the proof of the theorem 2.1.1 we consider four different cases

$$a_j z_k - a_k z_j \in L, \ \forall\ 1 \leqslant j \neq k \leqslant 3 \tag{4.2.7}$$

$$a_j z_k - a_k z_j \notin L, \ \forall\ 1 \leqslant j \neq k \leqslant 3 \tag{4.2.8}$$

$$a_1 z_2 - a_2 z_1 \in L \ \text{ and } a_1 z_3 - a_3 z_1, a_2 z_3 - a_3 z_2 \notin L. \tag{4.2.9}$$

$$a_1 z_2 - a_2 z_1, a_1 z_3 - a_3 z_1 \in L \ \text{ and } a_2 z_3 - a_3 z_2 \notin L. \tag{4.2.10}$$

Case 1: We assume that

$$a_j z_k - a_k z_j \notin a_j L + a_k L, \ \forall\ 1 \leqslant j \neq k \leqslant 3$$

From the Laurent expansion of Jacobi form $D_L(z;\varphi)$ we have

$$D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k+t_k}{a_k} - z_j\right) = \frac{\bar{a}_j}{\varphi_j} + d_1\left(a_j \frac{z_k+t_k}{a_k} - z_j; L\right) + d_2\left(a_j \frac{z_k+t_k}{a_k} - z_j; L\right)\frac{\varphi_j}{\bar{a}_j} + ..., \ \text{for } j=1,2,3.$$

we use this expansion to express the product

$$\prod_{1\leqslant j\neq k\leqslant 3} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k+t_k}{a_k} - z_j\right)$$

as a serie in terms of variables φ_1, φ_2 and φ_3. To do that, let j_1, j_2 such that : $\{j_1, j_2, k\} = \{1,2,3\}$. Then we obtain

$$\begin{aligned}\prod_{1\leqslant j\neq k\leqslant 3} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k+t_k}{a_k} - z_j\right) &= \frac{\bar{a}_{j_1}\bar{a}_{j_2}}{\varphi_{j_1}\varphi_{j_2}} + \frac{\bar{a}_{j_1}}{\varphi_{j_1}} d_1\left(a_{j_2}\frac{z_k+t_k}{a_k} - z_{j_2}; L\right) + \frac{\bar{a}_{j_2}}{\varphi_{j_2}} d_1\left(a_{j_1}\frac{z_k+t_k}{a_k} - z_{j_1}; L\right)\\ &+ \prod_{1\leqslant j\neq k\leqslant} d_1\left(a_j \frac{z_k+t_k}{a_k} - z_j; L\right) + \frac{\bar{a}_{j_1}}{\bar{a}_{j_2}}\frac{\varphi_{j_2}}{\varphi_{j_1}} d_2\left(a_{j_2}\frac{z_k+t_k}{a_k} - z_{j_2}; L\right) + \frac{\bar{a}_{j_2}}{\bar{a}_{j_1}}\frac{\varphi_{j_1}}{\varphi_{j_2}} d_2\left(a_{j_1}\frac{z_k+t_k}{a_k} - z_{j_1}; L\right)\\ &+ \Big(\text{summation of monomials terms of total degree greater than 1 in terms of } \varphi_{j_1}, \varphi_{j_2}\Big)\end{aligned}$$

Thus, by using the elliptic Raabe formulas and the homogeneity property for $d_j(\varphi; L)$, $j=1,2$, we obtain

$$\begin{aligned}\frac{1}{a_k}\sum_{\bar{t}_k\in L/a_k L}\prod_{1\leqslant j\neq k\leqslant 3} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k+t_k}{a_k} - z_j\right) &= \frac{\bar{a}_1\bar{a}_2\bar{a}_3}{\varphi_1\varphi_2\varphi_3}\varphi_k + \frac{\bar{a}_{j_1}}{\varphi_{j_1}} d_1\left(a_{j_2}z_k - a_k z_{j_2}; L\right)\\ &+ \frac{\bar{a}_{j_2}}{\varphi_{j_2}} d_1\left(a_{j_1}z_k - a_k z_{j_1}; L\right) + \frac{1}{a_k}\sum_{\bar{t}_k\in L/a_k L}\prod_{1\leqslant j\neq k\leqslant 3} d_1\left(a_j \frac{z_k+t_k}{a_k} - z_j; L\right)\\ &+ \frac{\bar{a}_{j_1}}{\bar{a}_k\bar{a}_{j_2}}\frac{\varphi_{j_2}}{\varphi_{j_1}} d_2\left(a_{j_2}z_k - a_k z_{j_2}; L\right) + \frac{\bar{a}_{j_2}}{\bar{a}_k\bar{a}_{j_1}}\frac{\varphi_{j_1}}{\varphi_{j_2}} d_2\left(a_{j_1}z_k - a_k z_{j_1}; L\right) \quad +\\ &\Big(\text{summation of monomial terms of total degree greater than 1 in terms of } \varphi_{j_1}, \varphi_{j_2}\Big)\end{aligned}$$

The function $d_2(\varphi; L)$ is an even function, then

$$\sum_{k=1}^{3} \frac{1}{a_k} \sum_{\bar{t}_k \in L/a_k L} \prod_{1 \leqslant j \neq k \leqslant 3} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k + t_k}{a_k} - z_j\right) = \frac{\bar{a}_1 \bar{a}_2 \bar{a}_3}{\varphi_1 \varphi_2 \varphi_3}(\varphi_1 + \varphi_2 + \varphi_3) +$$

$$\frac{\bar{a}_1}{\varphi_1}\Big(d_1(a_2 z_3 - a_3 z_2; L) + d_1(a_3 z_2 - a_2 z_3; L)\Big) + \frac{\bar{a}_2}{\varphi_2}\Big(d_1(a_1 z_3 - a_3 z_1; L) + d_1(a_3 z_1 - a_1 z_3; L)\Big) +$$

$$\frac{\bar{a}_3}{\varphi_3}\Big(d_1(a_2 z_1 - a_1 z_2; L) + d_1(a_1 z_2 - a_2 z_1; L)\Big) + \frac{1}{a_k} \sum_{\bar{t}_k \in L/a_k L} \prod_{1 \leqslant j \neq k \leqslant 3} d_1\left(a_j \frac{z_k + t_k}{a_k} - z_j; L\right)$$

$$\frac{\bar{a}_1}{\bar{a}_2 \bar{a}_3} \frac{\varphi_2 + \varphi_3}{\varphi_1} d_2(a_2 z_3 - a_3 z_2; L) + \frac{\bar{a}_2}{\bar{a}_1 \bar{a}_3} \frac{\varphi_1 + \varphi_3}{\varphi_2} d_2(a_1 z_3 - a_3 z_1; L) + \frac{\bar{a}_3}{\bar{a}_1 \bar{a}_2} \frac{\varphi_1 + \varphi_2}{\varphi_3} d_2(a_1 z_2 - a_2 z_1; L)$$

$$+\Big(\text{summation of monomial terms of total degree greater than 1 in terms of } \varphi_1, \varphi_2, \varphi_3\Big)$$

Now, we use that the sum $\varphi_1 + \varphi_2 + \varphi_3 = 0$ and $d_1(\varphi; L)$ is odd function , we deduce that

$$\sum_{k=1}^{3} \frac{1}{a_k} \sum_{\bar{t}_k \in L/a_k L} \prod_{1 \leqslant j \neq k \leqslant 3} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k + t_k}{a_k} - z_j\right) = \tag{4.2.11}$$

$$\sum_{k=1}^{3} \frac{1}{a_k} \sum_{\bar{t}_k \in L/a_k L} \prod_{1 \leqslant j \neq k \leqslant 3} d_1\left(a_j \frac{z_k + t_k}{a_k} - z_j; L\right) - \frac{\bar{a}_1}{\bar{a}_2 \bar{a}_3} d_2(a_2 z_3 - a_3 z_2; L)$$

$$-\frac{\bar{a}_2}{\bar{a}_1 \bar{a}_3} d_2(a_1 z_3 - a_3 z_1; L) - \frac{\bar{a}_3}{\bar{a}_2 \bar{a}_1} d_2(a_2 z_1 - a_1 z_2; L) +$$

$$\Big(\text{summation of monomials terms of total degree greater than 1 in terms of } \varphi_1, \varphi_2, \varphi_3\Big)$$

Now we know, according to the theorem 2.2.1, that

$$\sum_{k=1}^{3} \frac{1}{a_k} \sum_{\bar{t}_k \in L/a_k L} \prod_{1 \leqslant j \neq k \leqslant 3} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j \frac{z_k + t_k}{a_k} - z_j\right) = 0$$

Then the coefficients, of our serie in 4.2.11, are all equal to zero. Now, the first coefficient here is zero implies that

$$\sum_{k=1}^{n=3} \frac{1}{a_k} \sum_{\bar{t}_k \in L/a_k L} \prod_{1 \leqslant j \neq k \leqslant n} d_1\left(a_j \frac{z_k + t_k}{a_k} - z_j; L\right) =$$

$$\frac{\bar{a}_1}{\bar{a}_2 \bar{a}_3} d_2(a_2 z_3 - a_3 z_2; L) + \frac{\bar{a}_2}{\bar{a}_1 \bar{a}_3} d_2(a_1 z_3 - a_3 z_1; L) + \frac{\bar{a}_3}{\bar{a}_2 \bar{a}_1} d_2(a_2 z_1 - a_1 z_2; L)$$

Then, we obtain our theorem 2.1.1 in this case 1.

Case 2: We assume that

$$a_j z_k - a_k z_j \in L, \ \forall \ 1 \leqslant j \neq k \leqslant 3$$

We have the following elementary lemma (because the a_1, a_2, a_3 are pairewise coprime) :

Lemma 4.2.1

$$a_j z_k - a_k z_j \in L,\ \forall\ 1 \leqslant j \neq k \leqslant 3 \iff (z_1, z_2, z_3) \in \mathbb{C}(a_1, a_2, a_3) + L^3$$

Then, there exists $(t'_1, t'_2, t'_3) \in L^3$ such that

$$\frac{z_1 + t'_1}{a_1} = \frac{z_2 + t'_2}{a_2} = \frac{z_3 + t'_3}{a_3}$$

Hence, the function $z \to F(z, \overrightarrow{\Phi}, \overrightarrow{A}) = \prod_{j=1}^{n=3} D_L\left(a_j z - z_j; \frac{\varphi_j}{\bar{a}_j}\right)$ has a triple pole at $Z_{1,2,3} = \frac{z_1 + t'_1}{a_1}$.
Now, we compute the residue of $z \to F(z, \overrightarrow{\Phi}, \overrightarrow{A})$ at $z = Z_{1,2,3}$, where $\overrightarrow{A} = (a_1, a_2, a_3)$, $\overrightarrow{\Phi} = (\varphi_1, \varphi_2, \varphi_3)$.
We have

$$\text{(4.2.12)} \quad \operatorname{Res}\left(F(z, \overrightarrow{\Phi}, \overrightarrow{A})dz; z = Z_{1,2,3}\right) = \operatorname{Res}\left(\prod_{j=1}^{n=3} D_L\left(a_j z; \frac{\varphi_j}{\bar{a}_j}\right) z = 0\right) =$$

$$\frac{a_3}{a_1 a_2} d_2\left(\frac{\varphi_3}{\bar{a}_3}; L\right) + \frac{a_2}{a_1 a_3} d_2\left(\frac{\varphi_2}{\bar{a}_2}; L\right) + \frac{a_1}{a_1 a_3} d_2\left(\frac{\varphi_1}{\bar{a}_1}; L\right) +$$

$$\frac{1}{a_1} d_1\left(\frac{\varphi_2}{\bar{a}_2}; L\right) d_1\left(\frac{\varphi_3}{\bar{a}_3}; L\right) + \frac{1}{a_2} d_1\left(\frac{\varphi_1}{\bar{a}_1}; L\right) d_1\left(\frac{\varphi_3}{\bar{a}_3}; L\right) + \frac{1}{a_3} d_1\left(\frac{\varphi_2}{\bar{a}_2}; L\right) d_1\left(\frac{\varphi_1}{\bar{a}_1}; L\right).$$

Now, by using the following expansions near zero

$$d_1(z, L) = \frac{1}{z} - E_2(0; L)z - \frac{\pi}{a(L)}\bar{z} + O(z^2).$$

and

$$d_2(z, L) = -E_2(0; L) - \frac{\pi}{a(L)}\frac{\bar{z}}{z} + \frac{\pi E_2(0; L)}{a(L)}\bar{z}z + o(z).$$

We obtain, the part of monomials of total degree zero (in terms of $\varphi_1, \varphi_2, \varphi_3$ and $\bar{\varphi_1}, \bar{\varphi_2}, \bar{\varphi_3}$) for the following quantity

$$\text{(4.2.13)} \qquad - \operatorname{Res}\left(\prod_{j=1}^{n=3} D_L\left(a_j z; \frac{\varphi_j}{\bar{a}_j}\right)\right)$$

is equal to

$$E_2(0; L)\left(\frac{a_1}{a_2 a_3} + \frac{a_2}{a_1 a_3} + \frac{a_3}{a_1 a_2}\right) + \frac{\pi}{a(L)}\left(\frac{\bar{a}_3}{a_1 a_2}\frac{\bar{\varphi_3}}{\varphi_3} + \frac{\bar{a}_2}{a_1 a_3}\frac{\bar{\varphi_2}}{\varphi_2} + \frac{\bar{a}_1}{a_2 a_3}\frac{\bar{\varphi_1}}{\varphi_1}\right) +$$

$$E_2(0; L)\left(\frac{\bar{a}_2}{a_1 \bar{a}_3}\frac{\varphi_3}{\varphi_2} + \frac{\bar{a}_3}{a_1 \bar{a}_2}\frac{\varphi_2}{\varphi_3} + \frac{\bar{a}_1}{a_2 \bar{a}_3}\frac{\varphi_3}{\varphi_1} + \frac{\bar{a}_3}{a_2 \bar{a}_1}\frac{\varphi_1}{\varphi_3} + \frac{\bar{a}_1}{a_3 \bar{a}_2}\frac{\varphi_2}{\varphi_1} + \frac{\bar{a}_2}{a_3 \bar{a}_1}\frac{\varphi_1}{\varphi_2}\right) +$$

$$\frac{\pi}{a(L)}\left(\frac{\bar a_2}{a_1a_3}\frac{\bar\varphi_3}{\varphi_2}+\frac{\bar a_3}{a_1a_2}\frac{\bar\varphi_2}{\varphi_3}+\frac{\bar a_1}{a_2a_3}\frac{\bar\varphi_3}{\varphi_1}+\frac{\bar a_3}{a_2a_1}\frac{\bar\varphi_1}{\varphi_3}+\frac{\bar a_1}{a_3a_2}\frac{\bar\varphi_2}{\varphi_1}+\frac{\bar a_2}{a_3a_1}\frac{\bar\varphi_1}{\varphi_2}\right)$$

Let us now to use that

$$\varphi_1+\varphi_2+\varphi_3=0$$

we obtain that the part of monomials of total degree zero, of

$$-\operatorname{Res}\left(\prod_{j=1}^{n=3} D_L\left(a_jz;\frac{\varphi_j}{\bar a_j}\right)\right)\tag{4.2.14}$$

is exactly

$$E_2(0;L)\left(\frac{a_1}{a_2a_3}+\frac{a_2}{a_1a_3}+\frac{a_3}{a_1a_2}+\frac{\bar a_2}{a_1\bar a_3}\frac{\varphi_3}{\varphi_2}+\frac{\bar a_3}{a_1\bar a_2}\frac{\varphi_2}{\varphi_3}+\frac{\bar a_1}{a_2\bar a_3}\frac{\varphi_3}{\varphi_1}+\frac{\bar a_3}{a_2\bar a_1}\frac{\varphi_1}{\varphi_3}+\frac{\bar a_1}{a_3\bar a_2}\frac{\varphi_2}{\varphi_1}+\frac{\bar a_2}{a_3\bar a_1}\frac{\varphi_1}{\varphi_2}\right)$$

To finish our computation, we use the Laurent expansion of the Jacobi form $D_L(z;\varphi)$. In fact,

$$D_L\left(\frac{\varphi_j}{\bar a_j};a_j\frac{z_k+t}{a_k}-z_j\right)=\frac{\bar a_j}{\varphi_j}+d_1\left(a_j\frac{z_k+t}{a_k}-z_j;L\right)+d_2\left(a_j\frac{z_k+t}{a_k}-z_j;L\right)\frac{\varphi_j}{\bar a_j}+...,\ \text{for } j=1,2,3.$$

Hence, we obtain the part of monomials of total degree zero

$$\sum_{k=1}^{n}\frac{1}{a_k}\sum_t{}' \prod_{1\leqslant j\neq k\leqslant n} D_L\left(\frac{\varphi_j}{\bar a_j};a_j\frac{z_k+t}{a_k}-z_j\right)\tag{4.2.15}$$

it's correspond to the following quantity

$$\sum_{k=1}^{n}\frac{1}{a_k}\sum_{t\in L/a_kL\setminus\{\bar 0\}}\prod_{1\leqslant j\neq k\leqslant n} d_1\left(a_j\frac{t}{a_k};L\right)+$$

$$E_2(0;L)\left(\frac{\bar a_2}{a_1\bar a_3}\frac{\varphi_3}{\varphi_2}+\frac{\bar a_3}{a_1\bar a_2}\frac{\varphi_2}{\varphi_3}+\frac{\bar a_1}{a_2\bar a_3}\frac{\varphi_3}{\varphi_1}+\frac{\bar a_3}{a_2\bar a_1}\frac{\varphi_1}{\varphi_3}+\frac{\bar a_1}{a_3\bar a_2}\frac{\varphi_2}{\varphi_1}+\frac{\bar a_2}{a_3\bar a_1}\frac{\varphi_1}{\varphi_2}\right)$$

$$-E_2(0;L)\left(\frac{\bar a_2}{\bar a_1\bar a_3}\frac{\varphi_3}{\varphi_2}+\frac{\bar a_3}{\bar a_1\bar a_2}\frac{\varphi_2}{\varphi_3}+\frac{\bar a_1}{\bar a_2\bar a_3}\frac{\varphi_3}{\varphi_1}+\frac{\bar a_3}{\bar a_2\bar a_1}\frac{\varphi_1}{\varphi_3}+\frac{\bar a_1}{\bar a_3\bar a_2}\frac{\varphi_2}{\varphi_1}+\frac{\bar a_2}{\bar a_3\bar a_1}\frac{\varphi_1}{\varphi_2}\right)$$

Finally, we now by theorem 2.2.1, that the quantities 4.2.14 and 4.2.14 are the same. Then, there parts of total degre zero are also the same. The above compuation implies that

$$\sum_{k=1}^{n=3}\frac{1}{a_k}\sum_{t\mathrm{mod}(a_kL)}\prod_{1\leqslant j\neq k\leqslant n} d_1\left(a_j\frac{t}{a_k};L\right)=$$

$$E_2(0;L)\left(\frac{a_1}{a_2a_3}+\frac{a_2}{a_1a_3}+\frac{a_3}{a_1a_2}-\frac{\bar a_1}{\bar a_2\bar a_3}-\frac{\bar a_2}{\bar a_1\bar a_3}-\frac{\bar a_3}{\bar a_1\bar a_2}\right)$$

Thus, we deduce our desired result in this case 2.

Case 3: We assume that

$$a_1z_2 - a_2z_1 \in L \ \text{ and } a_1z_3 - a_3z_1, a_2z_3 - a_3z_2 \notin L.$$

We have $a_1z_2 - a_2z_1 \in L \iff \exists t_1', t_2' \in L : \frac{z_1+t_1'}{a_1} = \frac{z_2+t_2'}{a_2}$. Thanks to $a_1z_3 - a_3z_1, a_2z_3 - a_3z_2 \notin L$, we have $Z_{1,2} = \frac{z_1+t_1'}{a_1}$ is a unique pole of order 2 of the function $F(z, \overrightarrow{\Phi}, \overrightarrow{A}) = \prod_{j=1}^{n=3} D_L\left(a_jz - z_j; \frac{\varphi_j}{\bar{a}_j}\right)$ and all the other poles of F are simple.

Let us to compute the residue of $z \to F(z, \overrightarrow{\Phi}, \overrightarrow{A})$ at $Z_{1,2} = \frac{z_1+t_1'}{a_1}$:
From the Laurent expansion of Jacobi forms

$$D_L\left(a_1z; \frac{\varphi_1}{\bar{a}_1}\right), \quad D_L\left(a_2z; \frac{\varphi_2}{\bar{a}_2}\right)$$

and the Taylor expansion of $D_L\left(a_3z + a_3Z_{1,2} - z_3; \frac{\varphi_3}{\bar{a}_3}\right)$ at $z = 0$, we obtain

$$\text{Res}\left(F(z, \overrightarrow{\Phi}, \overrightarrow{A})dz; z = Z_{1,2}\right) = \tag{4.2.16}$$

$$\text{Res}\left(D_L\left(a_1z; \frac{\varphi_1}{\bar{a}_1}\right) D_L\left(a_2z; \frac{\varphi_2}{\bar{a}_2}\right) D_L\left(a_3z + a_3Z_{1,2} - z_3; \frac{\varphi_3}{\bar{a}_3}\right) dz; z = 0\right) =$$

$$D_L\left(a_3Z_{1,2} - z_3; \frac{\varphi_3}{\bar{a}_3}\right)\left(\frac{d_1\left(\frac{\varphi_2}{\bar{a}_2}; L\right)}{a_1} + \frac{d_1\left(\frac{\varphi_1}{\bar{a}_1}; L\right)}{a_2}\right) + \frac{a_3}{a_1a_2}D_L'\left(a_3Z_{1,2} - z_3; \frac{\varphi_3}{\bar{a}_3}\right)$$

Then

$$\sum_{k=1}^{3}\frac{1}{a_k}\sum_{\bar{t}_k}{}' \prod_{1\leqslant j\neq k\leqslant n=3} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j\frac{z_k+t_k}{a_k} - z_j\right) = -D_L\left(\frac{\varphi_3}{\bar{a}_3}; a_3Z_{1,2} - z_3\right)\left(\frac{d_1\left(\frac{\varphi_2}{\bar{a}_2}; L\right)}{a_1} + \frac{d_1\left(\frac{\varphi_1}{\bar{a}_1}; L\right)}{a_2}\right)$$
$$-\frac{a_3}{a_1a_2}D_L'\left(a_3Z_{1,2} - z_3; \frac{\varphi_3}{\bar{a}_3}\right) e\left(-E_L\left(\left(a_3Z_{1,2} - z_3; \frac{\varphi_3}{\bar{a}_3}\right)\right)\right)$$

where

$$\sum_{\bar{t}_k}{}' = \begin{cases} \sum_{\bar{t}_k \in L/a_kL\setminus\{\overline{t_k'}\}} & \text{if } k = 1, 2 \\ \sum_{\bar{t}_k \in L/a_kL} & \text{For } k = 3 \end{cases}$$

We use Laurent expansion of Jacobi forms $D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j\frac{z_k+t_k}{a_k} - z_j\right)$:

$$D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j\frac{z_k+t_k}{a_k} - z_j\right) = \frac{\bar{a}_j}{\varphi_j} + d_1\left(a_j\frac{z_k+t_k}{a_k} - z_j; L\right) + d_2\left(a_j\frac{z_k+t_k}{a_k} - z_j; L\right)\frac{\varphi_j}{\bar{a}_j} + ..., \text{ for } j = 1, 2, 3$$

to compute the product

$$\prod_{1\leqslant j\neq k\leqslant 3} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j\frac{z_k+t_k}{a_k}-z_j\right)$$

as a serie in terms of variables φ_1, φ_2 and φ_3.
Then, the term corresponding to part of total degree zero of

$$\sum_{k=1}^{n=3}\frac{1}{a_k}\sum_{\bar{t}_k}{}' \prod_{1\leqslant j\neq k\leqslant n=3} D_L\left(\frac{\varphi_j}{\bar{a}_j}; a_j\frac{z_k+t_k}{a_k}-z_j\right) \tag{4.2.17}$$

Hence, this term is exactly the quantity

$$\sum_{k=1}^{n=3}\frac{1}{a_k}\sum_{\bar{t}_k}{}' \prod_{1\leqslant j\neq k\leqslant n} d_1\left(a_j\frac{z_k+t_k}{a_k}-z_j; L\right) - \frac{\bar{a}_1}{\bar{a}_2\bar{a}_3}d_2(a_2z_3-a_3z_2;L) - \frac{\bar{a}_2}{\bar{a}_1\bar{a}_3}d_2(a_1z_3-a_3z_1;L)$$

$$-\frac{\bar{a}_2}{a_1\bar{a}_3}\frac{\varphi_3}{\varphi_2}d_2(a_3Z_{1,2}-z_3;L) - \frac{\bar{a}_1}{a_2\bar{a}_3}\frac{\varphi_3}{\varphi_1}d_2(a_3Z_{1,2}-z_3;L)$$
$$+\frac{\bar{a}_3}{a_1\bar{a}_2}\frac{\varphi_2}{\varphi_3}\left(1-\frac{a_1}{\bar{a}_1}\right)E_2(0;L) + \frac{\bar{a}_3}{a_2\bar{a}_1}\frac{\varphi_1}{\varphi_3}\left(1-\frac{a_2}{\bar{a}_2}\right)E_2(0;L)$$

From the Laurent expansion of $D_L(z;\varphi)$ we obtain

Lemma 4.2.2

$$D_L'(z;\varphi)\, e\left(-E_L(z,\varphi)\right) = -E_2(z;L) - \frac{\pi}{a(L)}\frac{\bar{\varphi}}{\varphi} + \Big(\textit{ serie in terms of } \varphi, \bar{\varphi} \textit{ with power greather than } 1\Big).$$

Now, the term corresponding to the part of total degree zero of

$$-D_L\left(\frac{\varphi_3}{\bar{a}_3}; a_3Z_{1,2}-z_3\right)\left(\frac{d_1\left(\frac{\varphi_2}{\bar{a}_2};L\right)}{a_1}+\frac{d_1\left(\frac{\varphi_1}{\bar{a}_1};L\right)}{a_2}\right)-$$
$$\frac{a_3}{a_1a_2}D_L'\left(a_3Z_{1,2}-z_3;\frac{\varphi_3}{\bar{a}_3}\right)e\left(-E_L\left(\left(a_3Z_{1,2}-z_3;\frac{\varphi_3}{\bar{a}_3}\right)\right)\right)$$

is exactly equal to

$$-\frac{\bar{a}_2}{a_1\bar{a}_3}\frac{\varphi_3}{\varphi_2}d_2(a_3Z_{1,2}-z_3;L) - \frac{\bar{a}_1}{a_2\bar{a}_3}\frac{\varphi_3}{\varphi_1}d_2(a_3Z_{1,2}-z_3;L)$$
$$+\frac{\bar{a}_3}{a_1\bar{a}_2}\frac{\varphi_2}{\varphi_3}E_2(0;L) + \frac{\bar{a}_3}{a_2\bar{a}_1}\frac{\varphi_1}{\varphi_3}E_2(0;L) + \frac{a_3}{a_2a_1}E_2(a_3Z_{1,2}-z_3;L)$$

Finally, we apply our theorem 2.2.1, we obtain

$$\sum_{k=1}^{n=3}\frac{1}{a_k}\sum_{\bar{t}_k\in L/a_kL}\prod_{1\leqslant j\neq k\leqslant n} E_1\left(a_j\frac{z_k+t_k}{a_k}-z_j;L\right) =$$

$$= \frac{\bar{a}_1}{\bar{a}_2\bar{a}_3} d_2(a_2z_3 - a_3z_2; L) + \frac{\bar{a}_2}{\bar{a}_1\bar{a}_3} d_2(a_1z_3 - a_3z_1; L) + \frac{\bar{a}_3}{\bar{a}_2\bar{a}_1} d_2(0; L) + \frac{a_3}{a_1a_2} E_2(a_3Z_{1,2} - z_3; L).$$

This proves the third case of our theorem 2.1.1.

Case 4: We assume that

$$a_1z_2 - a_2z_1, a_1z_3 - a_3z_1 \in L \text{ and } a_2z_3 - a_3z_2 \notin L.$$

In the present case 4 the proof is similar to the case 3. For that reason we omit it. For instance, it is enough to add the term for the contribution coming from the second double pole $Z_{1,2}$ of $F(z, \overrightarrow{\Phi}, \overrightarrow{A})$, which is

$$\frac{a_2}{a_1a_3} E_2\left(z_2 - a_2Z_{1,3}; L\right).$$

Thus we obtain our theorem 2.1.1. □

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