Jacobi forms in two variables: Multiple elliptic Dedekind sums, The Kummer-von Staudt Clausen Congruences for elliptic Bernoulli functions and values of Hecke L-functions

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In this paper we introduce elliptic Bernoulli functions and numbers, which are related to special Jacobi forms of two variables, and study their properties .
More importantly, we state and prove elliptic analogues to the following important theorems:

i) The Dedekind reciprocity Law for Dedekind classical sums, here we introduce enhanced Multiple elliptic Dedekind sums and study their reciprocity law.

ii) The Congruence of Clausen-von-Staudt and Kummer for Bernoulli numbers, here we state and prove it for elliptic Bernoulli numbers.

iii) We obtain Damerell's type result concerning the algebraicity of the special values of the Hecke L-function related to our Jacobi forms.

iv) As a corollary, we connect these elliptic Bernoulli numbers (explicitly computed) to the special values of Hecke L-functions of imaginary quadratic number field and associated to some Grössencharacter.

Contents

1 Introduction

It is well known that the Jacobi forms in one variable are a cross between elliptic functions and modular forms in one variable. They have several applications in differents areas in mathematics, especially in number theory and arithmetical geometry [28].
In this paper, we will study some Jacobi forms in two variables. These forms are arises to Galois module structure of rings of integers of numbers fields [44], construction of classgoups annihilators [12] and stark's units [10], and periods theory [53].
Now, the reasons for being still interested in these forms are multiple. First for all we precise their analytical properties (meromorphy, ellipticity, modularity, functional equation, Laurent expansion, Fourier q-expansion, Eisenstein-Kronecker expansion, distributions formulas) of our Jacobi forms which are of arithmetical nature. The second ground for studying Jacobi forms of

two variables is the following: they define elliptic Bernoulli functions (comes from Coefficients of Laurent expansion of our Jacobi forms, we will state and prove their essential properties: symmetry, periodicity, Raabe's formulas, modular properties). The special values of these elliptic Bernoulli functions give elliptic Bernoulli numbers which are studied in this paper. The third important reason for studying our Jacobi forms, elliptic Bernoulli functions and Bernoulli numbers: we will establish the elliptic analogues of Von-Staudt Clausen theorem and Kummer congruence for elliptic Bernoulli numbers; we computed special values of Hecke L-functions associated to certain Grössencharacter of type $(m, n) \in \mathbb{N}^2$ and expressed them using elliptic Bernoulli numbers. The fourth important reason is the study of the "Enhanced" multiple elliptic analogues of Dedekind sums which are defined here by two different ways: the first one in terms of our Jacobi forms in two variables and the second by using elliptic Bernoulli functions.
Basically, the Bernoulli polynomials $B_n(x)$ are defined by the following identity in the ring $\mathbb{Q}[x][[t]]$.

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n. \tag{1.0.1}$$

Thus, $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$,...
$B_n := B_n(0)$ is the n-th Bernoulli number. Let $\{x\}$ be a fractional part of the real number x. Then the Bernoulli functions $\bar{B}_n(x)$ are defined by

$$\bar{B}_n(x) = \begin{cases} 0, & \text{If } n = 1, x \in \mathbb{Z} \\ B_n(\{x\}) & \text{Otherwise} \end{cases}$$

Equivalently,

$$\sum_{n=0}^{\infty} \frac{\bar{B}_n(x)}{n!} t^n = \frac{te^{tx}}{e^t - 1} + \frac{t}{2} \delta_{0,\{x\}} \tag{1.0.2}$$

where

$$\delta_{0,x} = \begin{cases} 1, & \text{If } x = 0 \\ 0 & \text{Otherwise} \end{cases}$$

is the Kronecker delta function.

In the next section, we introduce an elliptic analogue of these Bernoulli functions $\bar{B}_n(x)$. Our elliptic Bernoulli functions are defined as coefficients of the Laurent expansion of the Jacobi forms in two variables $D_L(z; \varphi)$.

More precisely, this paper is organized as follows.
In the second section we introduce and study modular Jacobi forms of two variables $D_L(z; \varphi)$ and deduce the properties of elliptic Bernoulli functions.
In section three we connected our Jacobi form to Eisenstein series and we study first and second elliptic Bernoulli functions in details.
In section four we define, in two equivalent ways, the elliptic Multiple Dedekind sums in terms of singular values of $D_L(z; \varphi)$ and also in terms of singular values of our elliptic Bernoulli functions.

We will precise the relationship between these two points of view. Our main result on Multiple elliptic Dedekind sums recover and unify all known results concerning reciprocity laws (and also gives a generalization of them) proved by Zagier [51], Beck [13], Rademacher [41], Ito[30], Sczech [42].
In section five we prove algebraicity and Damerell type result for our Jacobi forms and elliptic Bernoulli numbers.
In sections six and seven we would like to establish the analogues of Clausen-von Staudt and Kummer Congruences for singular values of our elliptic Bernoulli functions. The singular values of our elliptic Bernoulli functions are called the "elliptic Bernoulli numbers". Our result recovers and generalizes the Congruences of Clausen-von Staudt and Kummer for Bernoulli-Hurwitz numbers proved by Katz [32].
Section eight contains the study of Hecke L-functions associated to our elliptic Bernoulli functions. The main purpose of this section is to interpolate the values at non negative integers of these L-functions by using The theory of our elliptic Bernoulli functions. As an application, special values of some Hecke L-function will be expressed by elliptic Bernoulli numbers. We deduce new Damerell's type result.

2 Presentation and study of Jacobi forms of two variables: $D_L(z;\varphi)$

2.1 Notations and definitions.

For $\tau \in \mathcal{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ the upper half plane, we consider the following Jacobi's Theta function

$$\theta_\tau(z) = \sum_{n\in\mathbb{Z}} e\left(\frac{1}{2}(n+\frac{1}{2})^2\tau + (n+\frac{1}{2})(z+\frac{1}{2})\right)$$

Or by Jacobi Triple product formula

$$\theta_\tau(z) = iq_\tau^{1/8}\left(e(z/2) - e(-z/2)\right)\prod_{n=1}^{\infty}\left(1-q_\tau^n\right)\left(1-q_\tau^n e(z)\right)\left(1-q_\tau^n e(-z)\right)$$

Where

$$e(z) = e^{2\pi i z}, q_\tau = e(\tau).$$

We shall use the following notation

$$\varphi = \varphi_1\tau + \varphi_2, (\varphi_1, \varphi_2) \in \mathbb{R}^2, \forall \varphi \in \mathbb{C},$$

because $\{\tau, 1\}$ is an $\mathbb{R}$-basis of $\mathbb{C}$.
Now, for each complex lattice L, we fix $\{\omega_1, \omega_2\}$ an $\mathbb{Z}$-oriented basis of L i.e

$$Im\left(\frac{\omega_1}{\omega_2}\right) > 0, L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

We define the following $\mathbb{R}$-alternating bilinear form

$$E_L(z,\varphi) = \frac{\bar{z}\varphi - z\bar{\varphi}}{\omega_1\bar{\omega_2} - \bar{\omega_1}\omega_2} = \frac{\bar{z}\varphi - z\bar{\varphi}}{2i|\omega_2|^2 Im\left(\frac{\omega_1}{\omega_2}\right)}$$

which is the symplectic form on $\mathbb{C}$ associated to the oriented complex lattice L. Note that for two complex lattices $L \subset \Lambda$ we have

$$E_\Lambda = [\Lambda : L]E_L$$

where $[\Lambda : L]$ indicates the number of elements of Λ/L.
Here for $\varphi \in \mathbb{C}$, we can write

$$\varphi = \varphi_1\omega_1 + \varphi_2\omega_2, (\varphi_1, \varphi_2) \in \mathbb{R}^2.$$

Now, we can associate to L a Jacobi form of two variables

$$D_L(z;\varphi) = \frac{1}{\omega_2} e\left(\frac{z}{\omega_2}\varphi_1\right) \frac{\theta'_\tau(0)\theta_\tau\left(\frac{z+\varphi}{\omega_2}\right)}{\theta_\tau\left(\frac{z}{\omega_2}\right)\theta_\tau\left(\frac{\varphi}{\omega_2}\right)}$$

where $\tau = \frac{\omega_1}{\omega_2}$.

2.2 Properties of $D_L(z;\varphi)$.

We regroup in the following the main interesting properties of $D_L(z;\varphi)$. These properties show that our Jacobi form $D_L(z;\varphi)$ is of arithmetical nature. Their proofs are omitted here. For proofs see [5, 8, 9]. Only the properties xiii) , xiv) and xv) are new, we will prove them.

Theorem 2.2.1 *(***Jacobi forms***)*

i) D_L *is meromorphic in the first variable* z*, and only real analytic on the second variable* φ*.*

ii) D_L *is homogenous of degre* -1

$$D_{\lambda L}(\lambda z;\lambda\varphi) = \lambda^{-1}D_L(z;\varphi), \forall\lambda \in \mathbb{C}\backslash\{0\}.$$

In particular, we have the following symmetry

$$D_L(-z;-\varphi) = -D_L(z;\varphi).$$

*iii) (***Periodicity** *of* $D_L(z;\varphi)$*):*

$$\begin{cases} D_L(z;\varphi+\rho) = D_L(z;\varphi) \\ D_L(z+\rho;\varphi) = e(E_L(\rho,\varphi))D_L(z;\varphi) \end{cases}, \forall\rho \in L$$

Where $E_L(u,v) = \frac{1}{2i}\frac{\bar{u}v - \bar{v}u}{Im(\tau)}$.

iv) (**Functional Equation**): $D_L(z;\varphi)e(-E_L(z,\varphi)) = D_L(\varphi;z)$.

v) (**Modularity**): *D_L is a Jacobi modular form for $SL_2(\mathbb{Z})$, with index 0 and weight 1 i.e*

$$D_{\frac{a\tau+b}{c\tau+d}}\left(\frac{z}{c\tau+d};\frac{\varphi}{c\tau+d}\right) = (c\tau+d)D_\tau(z;\varphi), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

where

$$D_\tau(z;\varphi) := D_{L_\tau}(z;\varphi), L_\tau = \mathbb{Z}\tau + \mathbb{Z}, \tau \in \mathcal{H}.$$

vi) *For any $\mathcal{D} = \sum_{i=1}^{r} n_i(a_i)$ principal divisor modulo L. there exists an L-elliptic function having $\mathcal{D}$ as divisor, which is equal to*

$$g_{\mathcal{D}}(z;L) = \prod_{a_i \notin L} D_L(z;-a_i)^{n_i}$$

up to multiplicative constant,

- Weiestrass $\wp'$-function:

$$\wp'_L(z) = -2 \prod_{\bar{t} \in \frac{1}{2}L/L \setminus \{\bar{0}\}} D_L(z;t),$$

vii) (**Twisted square root**):

$$D_L(z,\varphi)D_L(z,-\varphi) = \wp_L(z) - \wp_L(\varphi), \text{ where } \wp_L(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in L \\ \omega \neq 0}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right].$$

viii) (**Infinite Product**) : $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \tau = \frac{\omega_1}{\omega_2}$.

$$D_L(z,\varphi) = \frac{2\pi i}{w_2} q_{\frac{z}{w_2}}^{\frac{\mathrm{Im}\left(\frac{\varphi}{w_2}\right)}{\mathrm{Im}\,\tau}} \times \frac{\left(q_{\frac{z+\varphi}{w_2}}^{\frac{1}{2}} - q_{\frac{z+\varphi}{w_2}}^{-\frac{1}{2}}\right)}{\left(q_{\frac{z}{w_2}}^{\frac{1}{2}} - q_{\frac{z}{w_2}}^{-\frac{1}{2}}\right)\left(q_{\frac{\varphi}{w_2}}^{\frac{1}{2}} - q_{\frac{\varphi}{w_2}}^{-\frac{1}{2}}\right)}$$

$$\prod_{n \geqslant 1} \frac{(1-q_\tau^n)^2 \left(1 - q_\tau^n q_{\frac{z+\varphi}{w_2}}\right)\left(1 - q_\tau^n q_{\frac{z+\varphi}{w_2}}^{-1}\right)}{\left(1 - q_\tau^n q_{\frac{z}{w_2}}\right)\left(1 - q_\tau^n q_{\frac{z}{w_2}}^{-1}\right)\left(1 - q_\tau^n q_{\frac{\varphi}{w_2}}\right)\left(1 - q_\tau^n q_{\frac{\varphi}{w_2}}^{-1}\right)}$$

ix) **Laurent expansion of $D_L(z,\varphi)$**

We have

$$D_\tau(z;\varphi) = \sum_{k \geqslant 0} d_k(\varphi; \mathbb{Z}\tau + \mathbb{Z}) z^{k-1}$$

with

$$d_0(\varphi; \mathbb{Z}\tau + \mathbb{Z}) = 1.$$

For all $k \geqslant 1$, *and* $\varphi = \varphi_1\tau + \varphi_2, (\varphi_1, \varphi_2) \in \mathbb{R}^2$, *we let* $\{\varphi\} = \{\varphi_1\}\tau + \{\varphi_2\}$ *and denoted by* $\{\varphi_1\}\tau, \{\varphi_2\}$ *the fractional parts of the real numbers* z_1, z_2. *Then, we have*

$$\frac{k!}{(2\pi i)^k} d_k(\varphi; \mathbb{Z}\tau + \mathbb{Z}) = \tag{2.2.3}$$

$$B_k\Big(\{\varphi_1\}\Big) + k\{\varphi_1\}^{k-1}\frac{q_{\{\varphi\}}}{q_{\{\varphi\}-1}} + k\sum_{m\geqslant 1}\left((\{\varphi_1\}-m)^{k-1}\frac{q_{\{\varphi\}}^{-1}q^m}{1-q_{\{\varphi\}}^{-1}q^m} - (\{\varphi_1\}+m)^{k-1}\frac{q_{\{\varphi\}}q^m}{1-q_{\{\varphi\}}q^m}\right)$$

where $B_j(X)$ *is the* j*-th Bernoulli Polynomial.*
These coefficients $d_k(\varphi; L)$ *satisfy the following recursive formula*

$$\begin{cases} d_2(\varphi; L) = \frac{1}{2}d_1(\varphi, L)^2 - \frac{1}{2}\wp_L(\varphi), \\ d_{2n}(\varphi; L) = \frac{(2n-1)}{2}G_{2n}(L) - \frac{1}{2}\sum_{i=1}^{2n-1}(-1)^i d_i(\varphi; L)d_{2n-i}(\varphi; L), \\ \text{where } G_{2n}(L) = \sum_{\omega\in L\setminus\{o\}}\frac{1}{|\omega|^{2n}}, \forall n \geqslant 2, \end{cases} \tag{2.2.4}$$

$G_{2n}(L)$ *are the classical Eisenstein series. Here* $L = \mathbb{Z}\tau + \mathbb{Z}$.

*x) (***Cusp at** ∞*) For each* $z, \varphi \in \mathbb{C}\setminus\mathbb{Z}\tau + \mathbb{Z}$, *we have*

(2.2.5)

$$\lim_{\mathrm{Im}(\tau)\to\infty} D_\tau(z,\varphi) = \begin{cases} \pi\Big(cot(\pi\{\varphi\}) + cot(\pi\{z\})\Big)e^{-2i\pi[z_1]\{\varphi_2\}} & \text{If} \quad (z_1,\varphi_1)\in\mathbb{Z}^2 \\ \pi\Big(cot(\pi\{\varphi\}) - i\Big)e^{-2i\pi[z_1]\{\varphi_2\}} & \text{If} \quad \varphi_1\in\mathbb{Z}, z_1\notin\mathbb{Z} \\ \pi\Big(cot(\pi\{z\}) - i\Big)e^{2i\pi\{\varphi_1\}z_2-2i\pi[z_1]\{\varphi_2\}} & \text{If} \quad z_1\in\mathbb{Z}, \varphi_1\notin\mathbb{Z} \\ 0 & \text{If} \quad z_1\notin\mathbb{Z}, \varphi_1\notin\mathbb{Z} \end{cases}$$

Here we denote by $\{z\} = \{z_1\}\tau + \{z_2\}$ *and* $[z] = [z_1]\tau + [z_2]$, *where* $\{z_1\}$, $\{z_2\}$ *are the fractional parts of real numbers* z_1, z_2 *(resp.* $[z_1], [z_2]$ *integer parts of real numbers* z_1, z_2*).*

In the case $z \in \mathbb{R}\setminus\mathbb{Z}$ *and* $\varphi \in \mathbb{C}\setminus\mathbb{Z}\tau + \mathbb{Z}$ *we obtain*

$$\lim_{\mathrm{Im}(\tau)\to\infty} D_\tau(z,\varphi) = \frac{2\pi i e(z\{\varphi_1\})}{e(z)-1} + \frac{2\pi i e(\{\varphi\})}{e(\{\varphi_1\})-1}\delta_{0,\{\varphi_1\}} \tag{2.2.6}$$

xi) For all $\varphi \in \mathbb{C}\backslash\mathbb{Z}\tau+\mathbb{Z}$, *we have*

$$\lim_{\mathrm{Im}(\tau)\to\infty} d_j\Big(\varphi;\mathbb{Z}\tau+\mathbb{Z}\Big) = \begin{cases} \pi cot(\pi\{\varphi\}) & \textit{If} \quad j=1, \varphi_1\in\mathbb{Z}, \\ B_j(\{\varphi_1\})\frac{(2\pi i)^j}{j!} & \textit{Otherwise} \end{cases}$$

*xii) (***Fourier** q**-expansion***)*

$$D_L(z;\varphi) = \frac{\pi}{w_2} q_{\frac{z}{w_2}}^{\frac{\mathrm{Im}\left(\frac{\varphi}{w_2}\right)}{\mathrm{Im}\,\tau}} \left(cot\left(\frac{\pi z}{w_2}\right) + cot\left(\frac{\pi\varphi}{w_2}\right) + 4\sum_{n=1}^{\infty}\sum_{d|n} \; sin\left(2d\frac{\pi z}{w_2} + \frac{2n}{d}\frac{\pi\varphi}{w_2}\right) q_\tau^n \right).$$

xiii) **Kronecker doubles series I:** *One has the following identity*

$$D_L(z;\varphi) = \sum_{\omega\in L}^{(e)} \frac{e(-E_L(\omega,\varphi))}{z+\omega}. \tag{2.2.7}$$

where $\sum_{\omega\in L}^{(e)}$ *is the Eisenstein summation equal to*

$$\sum_{\omega\in L}^{(e)} = \lim_{M,N\to\infty} \sum_{m=-M}^{m=M}\sum_{m=-N}^{m=N}, \quad \textit{Where } \omega = m\omega_1 + n\omega_2.$$

xiv) **Kronecker doubles series II:** *we have the identity*

$$D_{L_\tau}(z;\varphi) = \tag{2.2.8}$$

$$\sum_{\omega\in L_\tau} \frac{e\left(-\frac{1}{2iIm(\tau)}|z+\omega|^2 - E_{L_\tau}(\omega,\varphi)\right)}{z+\omega} + e\left(E_{L_\tau}(z,\varphi)\right) \sum_{\omega\in L_\tau} \frac{e\left(-\frac{1}{2iIm(\tau)}|\varphi+\omega|^2 - E_{L_\tau}(\omega,z)\right)}{\varphi+\omega}$$

xv) **"Eisenstein-Kronecker series of Weight** m **=Bernoulli Functions"**
We set

$$\bar{B}_m(\varphi,L) = \frac{m!}{(2\pi i)^m} d_m(\varphi,L).$$

We have

$$D_L(z;\varphi) = \sum_{m\geqslant 0} \bar{B}_m(\varphi,L)\frac{(2\pi i)^m}{m!} z^{m-1}$$

with

$$\bar{B}_m(\varphi;L) = \begin{cases} -\frac{m!}{(2\pi i)^m} \sum_{\substack{\omega\in L\\ \omega\neq 0}}^{(e)} \frac{e(E_L(\omega,\varphi))}{\omega^m} & \textit{If } m\geqslant 1 \\ 1 & \textit{If } m=0 \end{cases}$$

xvi) (**Coefficients of Laurent expansion at cusp ∞**)
For $\varphi = \varphi_1\tau + \varphi_2 \in \mathbb{C}\backslash\mathbb{Z}\tau + \mathbb{Z}$, we have

$$\lim_{Im(\tau)\to\infty} \bar{B}_m\Big(\varphi, \mathbb{Z}\tau + \mathbb{Z}\Big) = B_m(\{\varphi_1\}) + \frac{e(\{\varphi\})}{e(\{\varphi\}) - 1}\delta_{1,m}\delta_{0,\{\varphi_1\}}$$

Equivalently,

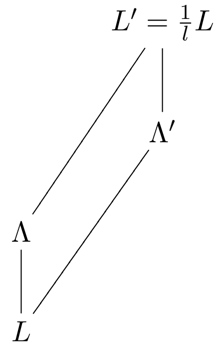
$$\lim_{Im(\tau)\to\infty} \bar{B}_m\Big(\varphi, \mathbb{Z}\tau + \mathbb{Z}\Big) = \begin{cases} \frac{1}{2i} cot(\pi\{\varphi\}) & \textit{If } m = 1 \textit{ and } \{\varphi_1\} = 0 \\ B_m(\{\varphi_1\}) & \textit{Otherwise} \end{cases}$$

xvii) **(Distribution Formulas for $D_L(z;\varphi)$) :**
For L, Λ complex lattices such that : $L \subset \Lambda$, $[\Lambda : L] = l$. We have:

$$\sum_{\bar{t}\in\Lambda/L} D_L(lz; \varphi + t) = D_\Lambda(z;\varphi)$$

xviii) **(Inverse Distribution Formulas for $D_L(z;\varphi)$) :**
Let L, Λ et Λ' complex lattices and l positive integer such that:

$$[\Lambda : L] = [\Lambda' : L] = l \text{ et } \Lambda \cap \Lambda' = L.$$



We have the following inverse distribution formulas

$$D_L(z;\varphi) = \frac{1}{l}\sum_{\bar{t}\in\Lambda'/L} D_\Lambda(\frac{z}{l}; \frac{\varphi}{l} + t).$$

Proof of the property xiii) of Theorem 2.2.1:

We want to prove the equalities 2.2.7 and 2.2.8. We begin with the first one

$$D_L(z;\varphi) = \sum_{\omega\in L}^{(e)} \frac{e(-E_L(\omega,\varphi))}{z+\omega}.$$

Where $\sum_{\omega\in L}^{(e)}$ is the Eisenstein summation equal to

$$\sum_{\omega\in L}^{(e)} = \lim_{M,N\to\infty} \sum_{m=-M}^{m=M} \sum_{m=-N}^{m=N}, \text{ Where } \omega = m\omega_1 + n\omega_2.$$

The functions $z \to D_L(z;\varphi)$ and $F_\varphi : z \to \sum_{\omega\in L}^{(e)} \frac{e(-E_L(\omega,\varphi))}{z+\omega}$ are meromorphic functions with only simple poles in $\omega \in L$. Moreover,

$$\begin{cases} D_L(z+\rho;\varphi) = e(E_L(\rho,\varphi))D_L(z;\varphi) \\ F_\varphi(z+\rho) = e(E_L(\rho,\varphi))F_\varphi(z) \end{cases} \quad \forall \rho \in L.$$

and finally, they have the same residue at $z = \omega, \omega \in L$:

$$\text{Res}\Big(F_\varphi(z)dz, z=\omega\Big) = \text{Res}\Big(D_L(z;\varphi)dz, z=\omega\Big) = e\Big(E_L(\omega,\varphi)\Big)$$

Now, we can conclude in two differents ways:
Firstly, we consider the quotient function $z \to \frac{F_\varphi(z)}{D_L(z;\varphi)}$. It is meromorphic, periodic with periods the lattice L and modulo L it has only one pole (a simple one) at $z = -\varphi$. Then, from Liouville's Theorem, the functions $z \to D_L(z;\varphi)$ and $F_\varphi(z)$ are proportional. But, we know that they have the same residue at $z = \omega, \omega \in L$. Thus we get our desired equality 2.2.7.
Secondly, we can consider the difference function $z \to F_\varphi(z) - D_L(z;\varphi)$ which is meromorphic, periodic with set of periods containing the lattice L and modulo L it has priori a pole at $z = 0$. But, we know that the functions F_φ and $z \to D_L(z;\varphi)$ have the same residue at $z = \omega, \omega \in L$. Then, the function difference $z \to F_\varphi(z) - D_L(z;\varphi)$ is holomorphic on whole complex plan. Then, using again Liouville's Theorem we get our equality 2.2.7.

Proof of the property xiv) of Theorem 2.2.1:

Here, we prove the equality 2.2.8:

$$D_{L_\tau}(z;\varphi) =$$

$$\sum_{\omega\in L_\tau} \frac{e\left(-\frac{1}{2i\text{Im}(\tau)}|z+\omega|^2 - E_{L_\tau}(\omega,\varphi)\right)}{z+\omega} + e\left(E_{L_\tau}(z,\varphi)\right) \sum_{\omega\in L_\tau} \frac{e\left(-\frac{1}{2i\text{Im}(\tau)}|\varphi+\omega|^2 - E_{L_\tau}(\omega,z)\right)}{\varphi+\omega}$$

We set

$$F(z,\varphi) = \sum_{\omega\in L_\tau} \frac{e\left(-\frac{1}{2i\text{Im}(\tau)}|z+\omega|^2 - E_{L_\tau}(\omega,\varphi)\right)}{z+\omega} + e\left(E_{L_\tau}(z,\varphi)\right) \sum_{\omega\in L_\tau} \frac{e\left(-\frac{1}{2i\text{Im}(\tau)}|\varphi+\omega|^2 - E_{L_\tau}(\omega,z)\right)}{\varphi+\omega}$$

This function has the following properties:

i) We remark that the form $E_{L\tau}$ takes integer values on L_τ then after a simple calculation one derives that

$$F(z+\rho,\varphi)=e(E_L(\rho,\varphi))F(z,\varphi);\forall\rho\in L.$$

ii) We claim that the function $z\to F(z,\varphi)$ is meromorphic. We prove that

$$\frac{\partial F}{\partial\bar{z}}(z,\varphi)=0.$$

Precisely,

$$\frac{\partial F}{\partial\bar{z}}(z,\varphi)=\sum_{\omega\in L_\tau}e\left(-\frac{1}{2i\mathrm{Im}(\tau)}|z+\omega|^2-E_{L_\tau}(\omega,\varphi)\right)-\sum_{\omega\in L_\tau}e\left(-\frac{1}{2i\mathrm{Im}(\tau)}|\varphi+\omega|^2-E_{L_\tau}(\omega+\varphi,z)\right)$$

But the function $e\left(-\frac{1}{2i\mathrm{Im}(\tau)}|z|^2\right)=\exp\left(-\frac{\pi}{\mathrm{Im}(\tau)}|z|^2\right)$ is Fourier self-dual. The Fourier transform associated to the symplectic form E_L is defined (for any f in the Schwarz space on $\mathbb{C}$) by the formula

$$\hat{f}(z)=\int_{x\in\mathbb{C}}f(x)e\left(E_L(z,x)\right)d\mu_L(x)$$

where $d\mu_L(x)$ is the Haar measure on $\mathbb{C}$ normalized by the condition $\int_{\mathbb{C}/L}d\mu_L(x)=1$. By Poisson summation formula the distribution $\delta_L=\sum_{\omega\in L}\delta_\omega$ (δ_ω is the Dirac function) is Fourier self-dual. Furthermore, the translation by x goes under Fourier transform to the multiplication by $e\left(-E_L(.,x)\right)$. Then, we obtain

$$\sum_{\omega\in L}f(\omega+x)e\left(-E_L(\omega+x,y)\right)=\sum_{\omega\in L}\hat{f}(\omega+y)e\left(-E_L(\omega,x)\right)$$

Now, we take $f(z)=\exp\left(-\frac{\pi}{\mathrm{Im}(\tau)}|z|^2\right)$. Then

$$\frac{\partial F}{\partial\bar{z}}(z,\varphi)=0.$$

Then, the functions $z\to F(z,\varphi)$ and $z\to D_{L_\tau}(z;\varphi)$ are meromorphic with only simple poles in $\omega\in L$. Moreover,

$$\begin{cases}D_L(z+\rho;\varphi)=e(E_L(\rho,\varphi))D_L(z;\varphi)\\ F(z+\rho,\varphi)=e(E_L(\rho,\varphi))F(z,\varphi)\end{cases}\quad\forall\rho\in L.$$

and finally,

$$\mathrm{Res}\Big(F(z,\varphi)dz,z=\omega\Big)=\mathrm{Res}\Big(D_L(z;\varphi)dz,z=\omega\Big)=e\Big(E_L(\omega,\varphi)\Big)$$

The desired equality 2.2.8 follows.

Proof of the property xv) of Theorem 2.2.1:

To prove this property we use the result 2.2.7. But, we need the Laurent expansion of $z \to \frac{1}{z+\omega}$. For $\omega \in L\backslash\{0\}$ we have

$$\frac{1}{z+\omega} = \sum_{k\geqslant 0}(-1)^k\left(\frac{z}{\omega}\right)^k$$

For our convenance we use $-\omega$ instead ω. Then,

$$D_{L_\tau}(z;\varphi) = \frac{1}{z} + \sum_{m\geqslant 1}\left(\sum_{\substack{\omega\in L\\ \omega\neq 0}}^{(e)} \frac{e(E_L(\omega,\varphi))}{\omega^m}\right) z^{m-1}$$

Then, we obtain

$$\bar{B}_m(\varphi, L) = -\frac{m!}{(2\pi i)^m}\sum_{\substack{\omega\in L\\ \omega\neq 0}}^{(e)}\frac{e(E_L(\omega,\varphi))}{\omega^m}, \quad \forall m \geqslant 1, \text{ and } \bar{B}_0(\varphi, L) = 1.$$

Example 2.2.2 (Jacobi forms of weight 2) *:*
For complex parameter of the non zero 2-division point , $\varphi \in \frac{1}{2}L/L\backslash\{\bar{0}\}$, we have the following Jacobi form of weight 2

$$D_\tau(z;\frac{1}{2}) = \frac{1}{z} + 2\sum_{k\geqslant 0}\frac{(2\pi i)^{2k+2}}{(2k+1)!}\left(\frac{B_{2k+2}}{4k+4} + \sum_{m\geqslant 1}\frac{m^{2k+1}q^m}{1+q^m}\right) z^{2k+1}$$

$$D_\tau(z;\frac{\tau}{2}) = \frac{1}{z} + 2\sum_{k\geqslant 0}\frac{(2\pi i)^{2k+2}}{(2k+1)!}\left(\frac{B_{2k+2}(1/2)}{4k+4} + \sum_{m\geqslant 1}(m+1/2)^{2k+1}\frac{q^{2m+1}+q^{m+\frac{1}{2}}}{\left(1-q^{m+\frac{1}{2}}\right)^2}\right) z^{2k+1}$$

$$D_\tau(z;\frac{\tau+1}{2}) = \frac{1}{z} + 2\sum_{k\geqslant 0}\frac{(2\pi i)^{2k+2}}{(2k+1)!}\left(\frac{B_{2k+2}(1/2)}{4k+4} + \sum_{m\geqslant 1}(m+\frac{1}{2})^{2k+1}\frac{q^{2m+1}-q^{m+\frac{1}{2}}}{\left(1+q^{m+1/2}\right)^2}\right) z^{2k+1}$$

In the following theorem we precise the most important properties of elliptic Bernoulli numbers and functions.

Theorem 2.2.3 *(***Elliptic Bernoulli functions***)*

*i) (***Homogeneity***) For each $m \in \mathbb{N}^*$, $d_m(\varphi, L_\tau)$ is homogenous of degre $-m$ i.e*

$$d_m(\lambda\varphi, \lambda L) = \lambda^{-m} d_m(\varphi, L), \forall\lambda \in \mathbb{C}\backslash\{0\}.$$

ii)

iii) (**Modularity**): *We let* $d_m(\varphi, \tau) := d_m(\varphi, L_\tau)$.
Then, $d_m(\varphi, \tau)$ *is a modular form for* $SL_2(\mathbb{Z})$, *with index 0 and weight m i.e*

$$d_m\left(\frac{\varphi}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^m d_m(\varphi; \tau), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

In other way,

$$d_m\left((d\varphi_1 - c\varphi_2)\frac{a\tau + b}{c\tau + d} + (-b\varphi_1 + a\varphi_2); \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^m d_m(\varphi_1\tau + \varphi_2; \tau),$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), (\varphi_1, \varphi_2) \in \mathbb{R}^2.$$

iv) (**Periodicity**):

$$d_m(\varphi + \rho; \tau) = d_m(\varphi; \tau), \forall \rho \in \mathbb{Z}\tau + \mathbb{Z} \tag{2.2.9}$$

v) *(Symmetry):*

$$d_m(-\varphi; \tau) = (-1)^{m-1} d_m(\varphi; \tau)$$

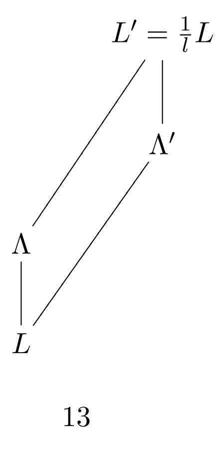
vi)

vii) **(Distribution Formula for** $d_m(\varphi, L)$**):**
For L, Λ *lattices such that :* $L \subset \Lambda$, $[\Lambda : L] = l$, *we have:*

$$\sum_{\bar{t} \in \Lambda / L} d_m(\varphi + t; L) = [\Lambda : L]^{1-m} d_m(\varphi; \Lambda), m \geqslant 1$$

viii) **(Inverse Distribution Formula for** $d_m(\varphi, L)$**):**
Let L, Λ *et* Λ' *complex lattices and* l *a postive integer such that:*

$$[\Lambda : L] = [\Lambda' : L] = l \text{ et } \Lambda \cap \Lambda' = L.$$



We have the following inverse distribution formula

$$d_m(\varphi; L) = \frac{1}{l} \sum_{\bar{t} \in \Lambda'/L} d_m(\frac{\varphi}{l} + t; \Lambda),$$

which comes from the homogeneity and direct distribution properties of $d_m(\varphi, L)$.

ix)

$$\sum_{\bar{t} \in \Lambda/L \setminus \{0\}} d_m(t; L) = \left(1 - \frac{|\omega|^2}{\bar{\omega}^m}\right) E_m(0, L), \forall m \geqslant 2$$

3 Jacobi forms $D_L(z; \varphi)$ and Eisenstein series $E_n(z, L)$

In the following we will connect, in severals ways, Jacobi forms $D_L(z; \varphi)$, Eisenstein series $E_n(z, L)$, Elliptic Bernoulli numbers $d_m(\varphi, L)$ and values of $E_m(z, L)$.

3.1 Connection between $D_L(z; \varphi)$ and Eisenstein series

In the following we state our second main result, giving the relationship between Eisenstein series $E_m(z, L)$, our Jacobi form $D_L(z, \varphi)$ and their coefficients $d_m(\varphi, L)$.
We recall here the definition of Eisenstein series

$$E_m(z, L) = \lim_{s \to 0+} \sum_{w \in L}^{(e)} (w + z)^{-m} |w + z|^{-s}, m = 1, ... \tag{3.1.10}$$

the sum being over $\omega \in L$ if $z \notin L$ an $\omega \in L \setminus \{-z\}$ if $z \in L$.

The following proposition, containing standard distribution formulas satisfied by Eisenstein series, is proved in an almost identical manner in Weil [49], its proof is omitted here.

Proposition 3.1.1 *Let* $\omega \in \mathbb{C} \setminus \{0\}$ *such that:* $\omega L \subset L$. *Then*

i)

$$\sum_{\bar{t} \in L/\omega L} E_n\left(\frac{z+t}{\omega}, L\right) = \omega^n E_n(z, L) = E_n\left(\frac{z}{\omega}, \frac{1}{\omega} L\right)$$

Equivalently,

$$E_n\left(z, \frac{1}{\omega} L\right) = \sum_{\bar{t} \in \frac{1}{\omega} L/L} E_n(z + t, L)$$

ii)

$$\sum_{\bar{t} \in L/\omega L \setminus \{\bar{0}\}} E_n\left(\frac{t}{\omega}, L\right) = (\omega^n - 1) E_n(0, L)$$

Now, we can state the connection between $D_L(z; \varphi)$, their special values and Eisenstein series $E_n(z, L)$.

Theorem 3.1.2

i) Let $\omega \in \mathbb{C}\backslash\{0\}$, $\varphi \in \mathbb{C}\backslash L$ such that: $\omega L \subset L$ and $\bar{\omega}\varphi \in L$. Then

$$D_L(z,\varphi) = \frac{1}{\omega} \sum_{t \in \frac{1}{\omega}L/L} e\Big(- E_L(t, \bar{\omega}\varphi)\Big) E_1\left(\frac{z}{\omega} + t, L\right),$$

In other way,

$$D_L(z,\varphi) = \frac{1}{\omega} \sum_{t \in L/\omega L} e\Big(- E_L(t, \varphi)\Big) E_1\left(\frac{z+t}{\omega}, L\right)$$

ii)

$$d_n(\varphi; L) = \frac{-1}{\omega^n} \sum_{t \in \frac{1}{\omega}L/L} e\Big(E_L(t, \bar{\omega}\varphi)\Big) E_n\left(\frac{t}{\omega}, L\right); \forall n \geqslant 3.$$

Equivalently

$$d_n(\varphi; L) = \frac{-1}{\omega^n} \sum_{t \in L/\omega L} e\Big(E_L(t, \varphi)\Big) E_n\left(\frac{t}{\omega}, L\right), \forall n \geqslant 3$$

*iii) (**Inversion Formula**): For all $\varphi \in \mathbb{C}\backslash L$ we have*

$$E_n(\bar{\varphi}; L) = -\frac{\omega^{n-1}}{\bar{\omega}} \sum_{t \in \frac{1}{\omega}L/L} e\Big(E_L(t, \omega\bar{\varphi})\Big) d_n(t, L)$$

*iv) (**Recursion formula**) Let f be a positive integer, φ primitive parameter of order f. Then*

$$d_3(\varphi; L) = \frac{1}{2f^3} \sum_{t \in L/fL\backslash\{0\}} e\Big(E_L(t, \varphi)\Big) \wp'_L\left(\frac{t}{f}\right)$$

$$d_4(\varphi; L) = \frac{-1}{f^4} \sum_{t \in L/fL\backslash\{0\}} e\Big(E_L(t, \varphi)\Big) \wp_L\left(\frac{t}{f}\right)^2 - \frac{6}{f^4} E_4(0, L);$$

$$d_5(\varphi; L) = \frac{-1}{f^5} \sum_{t \in L/fL\backslash\{0\}} e\Big(E_L(t, \varphi)\Big) \wp_L\left(\frac{t}{f}\right) E_3\left(\frac{t}{f}, L\right)$$

and for $k \geqslant 4$ we have

$$-f^{k+2}(k+1)k(k-1)d_{k+2}(\varphi, L) = 6 \sum_{\substack{p+q=k-2 \\ p\geqslant 1, q\geqslant 1}} \sum_{t \in L/fL} (p+1)(q+1) e(E_L(t,\varphi)) E_{p+2}\left(\frac{t}{f}, L\right) E_{q+2}\left(\frac{t}{f}, L\right) +$$

$$12(k-1) \sum_{t \in L/fL} e\Big(E_L(t, \varphi)\Big) \wp_L\left(\frac{t}{f}\right) E_k\left(\frac{t}{f}, L\right)$$

Proof of theorem 3.1.2:
We prove the **Recursion formula**. We consider the function

$$z \to \wp_L\left(\frac{z-t}{f}\right) - \wp_L\left(\frac{t}{f}\right).$$

$$\begin{aligned}\wp_L\left(\tfrac{z-t}{f}\right) - \wp_L\left(\tfrac{t}{f}\right) = & \sum_{\omega\in L}\left(\frac{1}{\left(\frac{z-t}{f}-\omega\right)^2} - \frac{1}{\left(\frac{-t}{f}-\omega\right)^2}\right)\\ &= f^2\sum_{\omega\in L}\left(\frac{1}{(z-t-f\omega)^2} - \frac{1}{(t+f\omega)^2}\right)\\ &= f^2 \sum_{\rho=t \pmod f}\left(\frac{1}{(z-\rho)^2} - \frac{1}{\rho^2}\right)\end{aligned}$$

Then the expression

$$F(z) = \sum_{\rho=t \pmod f}\left(\frac{e\left(E_L(\rho,\varphi)\right)}{(z-\rho)^2} - \frac{e\left(E_L(\rho,\varphi)\right)}{\rho^2}\right)$$

is equal to

$$f^{-2}e\left(E_L(t,\varphi)\right)\left(\wp_L\left(\frac{z-t}{f}\right) - \wp_L\left(\frac{t}{f}\right)\right)$$

Then

$$F(z) = f^{-2}e\left(E_L(t,\varphi)\right)\left(\wp_L\left(\frac{z-t}{f}\right) - \wp_L\left(\frac{t}{f}\right)\right)$$

we use the relation for $\wp_L$ Weierstrass function:

$$\wp_L''(z) = 6\wp_L(z)^2 - 30E_4(0,L)$$

Then applying this relation to $\wp_L\left(\frac{z-t}{f}\right)$ yields

$$f^4 e\left(-E_L(t,\varphi)\right)F''(z) = 6\left(f^2 e\left(-E_L(t,\varphi)\right)F(z) + \wp_L\left(\frac{t}{f}\right)\right)^2 - 30E_4(0,L)$$

On the other hand we take into account the Laurent expansion

$$F(z) = \sum_{k\geqslant 1}\frac{k+1}{f^{k+2}}e(E_L(t,\varphi))E_{k+2}\left(\frac{t}{f},L\right)z^k$$

We get

$$f^4 e(-E_L(t,\varphi)\sum_{k\geqslant 2}\frac{(k+1)k(k-1)}{f^{k+2}}e(E_L(t,\varphi))E_{k+2}\left(\frac{t}{f},L\right)z^{k-2} =$$

$$6\left(f^2 e(-E_L(t,\varphi))\sum_{k\geqslant 1}\frac{k+1}{f^{k+2}}e(E_L(t,\varphi))E_{k+2}\left(\frac{t}{f},L\right)z^k + \wp_L\left(\frac{t}{f}\right)\right)^2 - 30E_4(0,L)$$

We compare the coefficients of z^k of both sides to get the following **recursion formula** for Eisenstein series $E_k\left(\frac{t}{f}, L\right)$:

$$(k+1)k(k-1)E_{k+2}\left(\frac{t}{f}, L\right) = 6\sum_{\substack{p+q=k-2\\ p\geqslant 1, q\geqslant 1}} (p+1)(q+1)E_{p+2}\left(\frac{t}{f}, L\right)E_{q+2}\left(\frac{t}{f}, L\right) +$$

$$12(k-1)\wp_L\left(\frac{t}{f}\right)E_k\left(\frac{t}{f}, L\right)$$

Finally we multiply the above quantity by $e\Big(E_L(t,\varphi)\Big)$ and make the summation $\sum\limits_{t\in L/fL}$ of both sides to get our desired **recursive formula** for $d_k(\varphi, L)$.

3.2 Weierstrass functions and Eisenstein Series $E_i(z, L)$.

We state the following proposition whose proof is omitted here. For more details you can see [42] p.526 and [21] p.188 proposition 1.5.

Proposition 3.2.1

i)

$$E_0(z, L) = \begin{cases} -1 & \textit{If } z \in L \\ 0 & \textit{Otherwise} \end{cases}$$

ii)

$$E_1(z, L) = \zeta(z, L) - \eta(z, L)$$

where

$$\eta(z, L) = G_2(L)z + \frac{\pi}{a(L)}\bar{z}.$$

iii)

$$\begin{cases} E_2(z, L) = & \wp_L(z) + G_2(L) \\ E_n(z, L) = & \frac{(-1)^n}{(n-1)!}\wp_L{}^{(n-2)}(z), \forall n \geqslant 3 \end{cases}$$

3.3 First and second Elliptic Bernoulli functions $d_i(z, L)$ and $E_i(z, L), i = 1, 2$.

In this subsection we study the first and second Elliptic Bernoulli functions and write them in terms of Eisenstein series.As application, Elliptic Dedekind-Sczech reciprocity Law will be derived from our Multiple Dedekind Sums.

We recall from Lang [38] p.248, that

$$\zeta(z,L) = z\eta_2 + \pi i \frac{e(z)+1}{e(z)-1} + 2\pi i \sum_{m\geqslant 1}\left(\frac{e(-z)q^m}{1-e(-z)q^m} - \frac{e(z)q^m}{1-e(z)q^m}\right) \tag{3.3.11}$$

Where $\eta_2 = \eta(1,L)$, $\eta_1 = \eta(\tau, L)$ and note that

$$\eta(z,L) = \eta(z_1\tau + z_2, L) = z_1\eta(\tau,L) + z_2\eta(1,L)$$

Then

$$\eta(z,L) = z_1\eta_1 + z_2\eta_2, \text{ where } z = z_1\tau + z_2, (z_1,z_2) \in \mathbb{R}^2.$$

On the other hand

$$\eta(z,L) = G_2(L)z + \frac{\pi}{a(L)}\bar{z}.$$

From these equalities and Legendre relation, we obtain

$$\eta(z,L) = G_2(L)(z_1\tau + z_2) + \frac{\pi}{a(L)}(z_1\bar{\tau} + z_2)$$

$$\eta(z,L) = z_1\left(G_2(L)\tau + \frac{\pi}{a(L)}\bar{\tau}\right) + z_2\left(G_2(L) + \frac{\pi}{a(L)}\right)$$

Then, by identification, we have

$$\begin{cases} \eta_1 = G_2(L)\tau + \frac{\pi}{a(L)}\bar{\tau}, \\ \eta_2 = G_2(L) + \frac{\pi}{a(L)} \end{cases}$$

The result of the following theorem will be used, in section 4.4 , to give a new and elementary proof of the reciprocity Theorem 4.4.1 of Sczech.

Theorem 3.3.1 *For all $z \notin L$ we have*

i)

$$E_1(z,L) = 2\pi i B_1(z_1) + \pi i + \pi i \frac{e(z)+1}{e(z)-1} + 2\pi i \sum_{m\geqslant 1}\left(\frac{e(-z)q^m}{1-e(-z)q^m} - \frac{e(z)q^m}{1-e(z)q^m}\right)$$

In an identical way,

$$E_1(z,L) = 2\pi i B_1(z_1) + 2\pi i \frac{e(z)}{e(z)-1} + 2\pi i \sum_{m\geqslant 1}\left(\frac{e(-z)q^m}{1-e(-z)q^m} - \frac{e(z)q^m}{1-e(z)q^m}\right)$$

ii)

$$d_1(z,L) = E_1(z,L)$$

iii)

$$d_2(z,L) = \frac{1}{2}E_1(z,L)^2 - \frac{1}{2}\wp_L(z)$$

Proof : The property i) comes from the equality (3.3.11). The property ii) is a consequence of the equality (2.2.3). Again, from the formula (2.2.3) and the periodicity property (2.2.9) we obtain

$$d_1(z,L) = 2\pi i B_1(z_1) + 2\pi i \frac{e(z)}{e(z)-1} + 2\pi i \sum_{m\geqslant 1}\left(\frac{e(-z)q^m}{1-e(-z)q^m} - \frac{e(z)q^m}{1-e(z)q^m}\right)$$

Hence

$$d_1(z,L) = E_1(z,L).$$

Finally, the property iii) comes from the properties ii) and (2.2.4).

4 "Enhanced" multiple elliptic Dedekind Sums and their applications

The purpose of our study here is not so much a rederivation of old theorems but rather to show a common thread and give a natural generalisation of them to elliptic situation.

These **Multiple elliptic Dedekind sums** generalize and unify various arithmetic sums introduced by Dedekind, Rademacher, Apostol, Carlitz, Zagier, Berndt, Meyer, Sczech, and Dieter. We prove reciprocity laws which are elliptic analogues to The Dedekind reciprocity law satisfied by the classical Dedekind sums [5]. Moreover, from the main result of this section (Multiple elliptic Dedekind reciprocity law) we generalize, recover and unify all classical and elliptic known reciprocity laws on Dedekind-Apostol-Rademacher-Beck-Dieter-Berndt-Sczech-Zagier and others sums [1, 8, 9, 13, 23, 22, 41, 42, 51].
Generalized Dedekind sums appear in various areas such as analytic and algebraic number theory, topology, algebraic and combinatorial geometry, and algorithmic complexity. See [2, 26, 27, 28]

Our theory of **enhanced** Multiple elliptic Dedekind sums developped here can be explored to study the Eisenstein Cohomology of groups as $\mathrm{SL}_2(O_K)$, $\mathrm{GL}_n(O_K)$, where K is an algebraic number field, see [42, 50] and is also closely related to the study elliptic genera, the index theorem of Atiyah-Singer associated to complex manifold [20] and to Meyer's, Euler's and Maslov cocycles [42, 33, 4].

4.1 Statement of multiple elliptic Dedekind reciprocity Laws

Let $a_1, \ldots, a_n, a'_1, \ldots, a'_n$ be elements in $\mathbb{N}^*$, the a_i being pairwise coprime and the a'_i being also pairwise coprime , $m_1, \ldots, m_n, r_1, \ldots, r_n \in \mathbb{N}$, $x_1, \ldots, x_n, x'_1, \ldots, x'_n \in \mathbb{C}$ and $\varphi_1, \cdots, \varphi_n$ be complex variables. A usual $\check{x}$ means that we omit the term x.
We let

$$z_k = -x'_k + x_k \frac{a'_k}{a_k}\tau, \tilde{t}_k = -t'_k + t_k \frac{a'_k}{a_k}\tau,$$

$$\overrightarrow{M_k} = (m_1, \ldots, \check{m}_k, \ldots, m_n), \overrightarrow{R_k} = (r_1, \ldots, \check{r}_k, \ldots, r_n),$$

$$\overrightarrow{Z_k} = (z_1, \ldots, \check{z}_k, \ldots, z_n), \overrightarrow{\phi_k} = (\varphi_1, \ldots, \check{\varphi}_k, \ldots, \varphi_n),$$
$$\overrightarrow{A_k} = (a_1, \ldots, \check{a}_k, \ldots, a_n), \overrightarrow{A'_k} = (a'_1, \ldots, \check{a}'_k, \ldots, a'_n).$$

and finally

$$L = [\tau, 1] := \mathbb{Z}\tau + \mathbb{Z}, \quad L_k = \left[\frac{a'_k}{a_k}\tau, 1\right]$$

We define the multiple elliptic Dedekind Sums

$$d(\overrightarrow{A_k}, \overrightarrow{A'_k}, \overrightarrow{Z_k}, \overrightarrow{\phi_k}, \overrightarrow{M_k}, \overrightarrow{R_k}, \tau) :=$$

$$\frac{(-1)^{m_k} m_k!}{{a'_k}^{m_k+1}} \left(\prod_{1 \leqslant j \neq k \leqslant n} \frac{{a'_j}^{r_j}}{r_j!} \right) \sum_{\tilde{t}_k \in L_k / a'_k L} e\left(E_{L_k}\left(\tilde{t}_k, \frac{\varphi_k}{a_k} \right) \right) \prod_{1 \leqslant j \neq k \leqslant n} D_{L_j}^{(m_j + r_j)} \left(a'_j \frac{z_k + \tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j} \right)$$

Notice that the case

$$\overrightarrow{M_k} = \overrightarrow{R_k} = (0, \ldots, 0)$$

corresponds to

$$d(\overrightarrow{A_k}, \overrightarrow{A'_k}, \overrightarrow{Z_k}, \overrightarrow{\phi_k}, \overrightarrow{M_k}, \overrightarrow{R_k}, \tau) =$$

$$\frac{1}{a'_k} \sum_{\tilde{t}_k \in L_k / a'_k L} e\left(E_{L_k}\left(\tilde{t}_k, \frac{\varphi_k}{a_k} \right) \right) \prod_{1 \leqslant j \neq k \leqslant n} D_{L_j} \left(a'_j \frac{z_k + \tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j} \right)$$

this quantity is also called elliptic multiple Dedekind sums studied in [5, 8, 10].

For arbitrary values of $\overrightarrow{M_k}$, $\overrightarrow{R_k}$ we get here multiple elliptic Apostol-Dedekind-Rademacher version of all considered so-called (in literature) “Apostol, Dedekind, Rademacher and others” sums.

We now formulate the corresponding reciprocity Law for these sums

Theorem 4.1.1 *(***Elliptic reciprocity law in terms of Jacobi forms***)*
For all $j, k, 1 \leqslant j \neq k \leqslant n$, we assume that

$$(a_j x_k - a_k x_j, a'_j x'_k - a'_k x'_j) \notin \mathbb{Z}a_j + \mathbb{Z}a_k \times \mathbb{Z}a'_j + \mathbb{Z}a'_k$$

and

$$\sum_{j=1}^{n} \varphi_j \in L$$

Then

$$\sum_{k=1}^{n} \sum_{\substack{\overrightarrow{R_k} \geqslant \overrightarrow{0} \\ r_1 + \ldots + \check{r}_k + \ldots + r_n = m_k}} d(\overrightarrow{A_k}, \overrightarrow{A'_k}, \overrightarrow{Z_k}, \overrightarrow{\phi_k}, \overrightarrow{M_k}, \overrightarrow{R_k}, \tau) = 0$$

The summation is over $\overrightarrow{R_k} = (r_1, \ldots, \check{r}_k, \ldots, r_n) \in \mathbb{N}^{n-1}$.

Theorem 4.1.2 *i) For all $j, k, 1 \leqslant j \neq k \leqslant n$, we assume that*

$$(a_j x_k - a_k x_j, a'_j x'_k - a'_k x'_j) \notin \mathbb{Z}a_j + \mathbb{Z}a_k \times \mathbb{Z}a'_j + \mathbb{Z}a'_k$$

and

$$\sum_{j=1}^{n} \varphi_j = 0$$

Then

$$\sum_{k=1}^{n} \frac{1}{a'_k} \sum_{\tilde{t_k} \in L_k/a'_k L} \prod_{1 \leqslant j \neq k \leqslant n} D_{L_j} \left(\frac{\varphi_j}{a_j}; a'_j \frac{z_k + \tilde{t}_k}{a'_k} - z_j \right) = 0$$

*ii) (***Complement Formula***) We assume only that*

$$\sum_{j=1}^{n} \varphi_j = 0$$

then

$$\sum_{k=1}^{n} \frac{1}{a'_k} \sum_{\tilde{t_k} \in L_k/a'_k L \backslash \{\bar{0}\}} \prod_{1 \leqslant j \neq k \leqslant n} D_{L_j} \left(\frac{\varphi_j}{a_j}; a'_j \frac{\tilde{t}_k}{a'_k} \right) = -Res \left(\prod_{1 \leqslant j \leqslant n} D_{L_j} \left(a_j z; \frac{\varphi_j}{a_j} \right) dz; z = 0 \right)$$

We will establish in the following that the theorem 4.1.2 contains and generalized the Sczech's result [Scz], in other hand it recovers the Hall-Wilson-Zagier result [HWZ].

Proof of theorem 4.1.2 :
This theorem 4.1.2 comes from theorem 4.1.1. In fact, we consider only the case when

$$m_1 = \cdots = m_n = 0$$

and we obtain our theorem 4.1.2 with using the functional equation of the Jacobi form:

$$D_L(z; \varphi) = e(E_L(z, \varphi)) D_L(\varphi; z)$$

and the condition

$$\sum_{j=1}^{n} \varphi_j = 0. \qquad \square$$

In the following we define an other elliptic analogue of the generalized Dedekind Sums in terms of the elliptic analogous of the Classical Bernoulli functions, as follows:
We recall that

$$D_L(z, \varphi) = \sum_{m \geqslant 0} d_m(\varphi, L) z^{m-1},$$

and

$$\bar{B}_m(\varphi, L) := \frac{m!}{(2\pi i)^m} d_m(\varphi, L)$$

$\bar{B}_m(\varphi, L)$ are our elliptic Bernoulli functions.

Hence

$$D_{L_j}\left(\frac{\varphi_k}{a_j}; a'_j \frac{z_k + \tilde{t}_k}{a'_k} - z_j\right) = \sum_{r_j \geqslant 0} \bar{B}_{r_j}(a'_j \frac{z_k + \tilde{t}_k}{a'_k} - z_j, L_j) \frac{(2\pi i)^{r_j}}{r_j!} \left(\frac{\varphi_j}{a_j}\right)^{r_j - 1}$$

Furthermore,

$$\prod_{1 \leqslant j \neq k \leqslant n} D_{L_j}\left(\frac{\varphi_k}{a_j}; a'_j \frac{z_k + \tilde{t}_k}{a'_k} - z_j\right) = (2\pi i)^{n-1} \sum_{\overrightarrow{R_k} \geqslant \overrightarrow{0}} \prod_{1 \leqslant j \neq k \leqslant n} \bar{B}_{r_j}(a'_j \frac{z_k + \tilde{t}_k}{a'_k} - z_j, L_j) \frac{1}{r_j!} \left(2\pi i \frac{\varphi_j}{a_j}\right)^{r_j - 1}$$

Where $\overrightarrow{R_k} = (r_1, \dots, \check{r}_k, \dots, r_n) \in \mathbb{N}^{n-1}$.

We now introduce the second main object of this section, the multiple elliptic Dedekind sums in terms of elliptic Bernoulli functions:

$$S(\overrightarrow{A_k}, \overrightarrow{A'_k}, \overrightarrow{R_k}, \overrightarrow{Z_k}; \tau) := \frac{1}{a'_k} \sum_{\tilde{t}_k \in L_k / a'_k L} \prod_{1 \leqslant j \neq k \leqslant n} \bar{B}_{r_j}(a'_j \frac{z_k + \tilde{t}_k}{a'_k} - z_j, L_j)$$

Theorem 4.1.3 *(***Elliptic reciprocity law in terms of Elliptic Bernoulli functions***) For all $j, k, 1 \leqslant j \neq k \leqslant n$, we assume that*

$$(a_j x_k - a_k x_j, a'_j x'_k - a'_k x'_j) \notin \mathbb{Z}a_j + \mathbb{Z}a_k \times \mathbb{Z}a'_j + \mathbb{Z}a'_k$$

and

$$\sum_{j=1}^{n} \varphi_j = 0$$

Then

$$\sum_{k=1}^{n} \sum_{\overrightarrow{R_k} \in \mathbb{N}^{n-1}} \frac{S(\overrightarrow{A_k}, \overrightarrow{A'_k}, \overrightarrow{R_k}, \overrightarrow{Z_k}; \tau)}{r_1! \dots \check{r}_j! \dots r_n!} \prod_{1 \leqslant j \neq k \leqslant n} \left(\frac{2\pi i \varphi_j}{a_j}\right)^{r_j - 1} = 0$$

The summation is over $\overrightarrow{R_k} = (r_1, \dots, \check{r}_k, \dots, r_n) \in \mathbb{N}^{n-1}$

Proof of Theorem 4.1.3:

Remark that generating function of the second multiple elliptic Dedekind sum $S(\overrightarrow{A_k}, \overrightarrow{A'_k}, \overrightarrow{R_k}, \overrightarrow{Z_k}; \tau)$ is exactly equal to

$$\frac{1}{(2\pi i)^{n-1}} \times \frac{1}{a'_k} \prod_{1 \leqslant j \neq k \leqslant n} D_{L_j}\left(\frac{\varphi_k}{a_j}; a'_j \frac{z_k + \tilde{t}_k}{a'_k} - z_j\right)$$

After using the functional equation satisfied by Jacobi form

$$D_L(z,\varphi) = e\left(E_L(z,\varphi)\right) D_L(\varphi, z)$$

we obtain

$$\prod_{1\leqslant j\neq k\leqslant n} D_{L_j}\left(\frac{\varphi_k}{a_j}; a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j\right) = e\left(-\sum_{1\leqslant j\neq k\leqslant n} E_{L_j}(a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j, \frac{\varphi_j}{a_j})\right)$$
$$\prod_{1\leqslant j\neq k\leqslant n} D_{L_j}\left(a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j}\right)$$

Hence

$$\prod_{1\leqslant j\neq k\leqslant n} D_{L_j}\left(\frac{\varphi_j}{a_j}; a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j\right) = e\left(\sum_{1\leqslant j\leqslant n} E_{L_j}(z_j, \frac{\varphi_j}{a_j})\right) e\left(-\sum_{1\leqslant j\leqslant n} E_{L_j}(a'_j\frac{z_k+\tilde{t}_k}{a'_k}, \frac{\varphi_j}{a_j})\right)$$
$$e\left(E_{L_k}(\tilde{t}_k, \tfrac{\varphi_k}{a_k})\right) \prod_{1\leqslant j\neq k\leqslant n} D_{L_j}\left(a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j}\right)$$

Now, we use the fact that:

$$\sum_{j=1}^{n}\varphi_j = 0 \text{ and } E_{L_j}(z,\varphi) = \frac{a_j}{a'_j}E_L(z,\varphi)$$

we conclude that

$$\prod_{1\leqslant j\neq k\leqslant n} D_{L_j}\left(\frac{\varphi_j}{a_j}; a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j\right) = e\left(\sum_{1\leqslant j\leqslant n} E_{L_j}(z_j, \frac{\varphi_j}{a_j})\right)$$
$$e\left(E_{L_k}(\tilde{t}_k, \tfrac{\varphi_k}{a_k})\right) \prod_{1\leqslant j\neq k\leqslant n} D_{L_j}\left(a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j}\right)$$

Finally

$$\sum_{\overrightarrow{R_k}\in\mathbb{N}^{n-1}} \frac{S(\overrightarrow{A_k}, \overrightarrow{A'_k}, \overrightarrow{R_k}, \overrightarrow{Z_k}; \tau)}{r_1!\ldots\check{r}_k!\ldots r_n!} \prod_{1\leqslant j\neq k\leqslant n}\left(\frac{2\pi i\varphi_j}{a_j}\right)^{r_j-1} =$$

$$\frac{1}{(2\pi i)^{n-1}} e\left(\sum_{1\leqslant j\leqslant n} E_{L_j}(z_j, \frac{\varphi_j}{a_j})\right) d(\overrightarrow{A_k}, \overrightarrow{A'_k}, \overrightarrow{Z_k}, \overrightarrow{\phi_k}, \overrightarrow{M_k} = (0,\ldots,0), \tau)$$

We conclude, up a multiplicative non zero constant, that the first multiple elliptic Dedekind sums in terms of Jacobi forms corresponding to $\overrightarrow{M_k} = (0,\ldots,0)$ is the generating function of the second one in terms of elliptic Bernoulli functions. Then theorem 4.1.3 is equivalent to theorem 4.1.2. □

4.2 Application 1: Enhanced classical sums of " Dedekind-Apostol-Rademacher-Beck-Berndt-Dieter-Zagier"

In order to state the first corollary of our main theorems of this section, we denote by $\cot^{(m)}$ the m'th derivative of the cotangent function and let $a'_1, \ldots, a'_n \in \mathbb{N}$, $m_1, \ldots, m_n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{C}$. We will study the following sums(the **Enhanced Dedekind cotangent sums**):

$$\mathfrak{C}\begin{pmatrix} a'_1 & a'_2 & \cdots & a'_n \\ m_1 & m_2 & \cdots & m_n \\ x_1 & x_2 & \cdots & x_d \\ \varphi_1 & \varphi_2 & \cdots & \varphi_n \end{pmatrix} := \frac{1}{{a'_1}^{m_1+1}} \sum_{k \bmod a_1} \prod_{j=2}^{n} \left(\cot^{(m_j)} \pi \left(a'_j \frac{k + x'_1}{a'_1} - x'_j \right) - \delta_{0,m_j} \cot^{(m_j)} \pi \left(\frac{\varphi_j}{a_j} \right) \right)$$

and the sum is taken over all k mod a'_1 for which the summand is not singular.Our notation of this sum is similar of Beck's notation in [13]. Indeed, our "**Enhanced**" Dedekind cotangent sums include the so-called "Dedekind cotangent sums" introduced by Beck [13], and various generalized Dedekind sums introduced by Rademacher [22, 41], Apostol [1], Carlitz [17], Zagier [51], Berndt [14], Meyer [39, 40], Sczech [42], and Dieter [19].

In the following, we state the strongest theorem concerning classical reciprocity law

Theorem 4.2.1 *(***Generalisation of Beck's result***)*
For all $j, k, 1 \leqslant j \neq k \leqslant n$, we assume that

$$a'_j x'_k - a'_k x'_j \notin \mathbb{Z}a'_j + \mathbb{Z}a'_k$$

and $\varphi_1, \cdots, \varphi_n$ are real numbers such that:

$$\frac{\varphi_j}{a_j} \notin \mathbb{Z}, \forall j \in \{1, \cdots, \}.$$

Then

$$\sum_{k=1}^{n} (-1)^{m_k} m_k! \sum_{\substack{l_1,\ldots,\widehat{l_k},\ldots,l_n \geqslant 0 \\ l_1+\ldots+\widehat{l_k}+\ldots+l_n = m_k}} \frac{{a'_1}^{l_1} \cdots \widehat{{a'_k}^{l_k}} \cdots {a'_n}^{l_n}}{l_1! \cdots \widehat{l_k!} \cdots l_n!} \mathfrak{C}\begin{pmatrix} a'_k & a'_1 & \cdots & \widehat{a'_k} & \cdots & a'_n \\ m_k & m_1 + l_1 & \cdots & \widehat{m_k + l_k} & \cdots & m_n + l_n \\ x'_k & x'_1 & \cdots & \widehat{x'_k} & \cdots & x'_n \\ \varphi_1 & \varphi_2 & \cdots & \widehat{\varphi_k} & \cdots & \varphi_n \end{pmatrix}$$

$$= \begin{cases} (-1)^{n-1} \dfrac{sin\left(\pi \displaystyle\sum_{j=1}^{n} \frac{\varphi_j}{a_j}\right)}{\displaystyle\prod_{j=1}^{n} sin\left(\pi \frac{\varphi_j}{a_j}\right)} & \textit{if all } m_k = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 4.2.2 (Division points)
Let $a_1, \cdots, a_n \geqslant 2$ be nonnegative integers pairwise coprime, whith $n \geqslant 1$ and $\varphi_1, \cdots, \varphi_n$ real numbers such that:

$$\sum_{j=1}^{n} \varphi_j \in \mathbb{Z} \textit{ and } \frac{\varphi_j}{a_j} \notin \mathbb{Z}, \forall j = 1, \cdots, n.$$

We obtain

$$\sum_{k=1}^{n}\sum_{t_k=1}^{a_k-1}\prod_{j=1}^{n}\left(e^{2\pi i\frac{\varphi_j}{a_j}}\right)^{[a_j\frac{t_k}{a_k}]}\prod_{j\neq k}\frac{e^{\frac{i\pi\varphi_j}{a_j}}}{sin\frac{\pi\varphi_j}{a_j}}=(-1)^n\frac{sin\left(\pi\sum_{j=1}^{n}\frac{\varphi_j}{a_j}\right)}{\prod_{j=1}^{n}sin\left(\pi\frac{\varphi_j}{a_j}\right)}$$

Equivalently

$$\sum_{k=1}^{n}\sum_{t_k=1}^{a_k-1}\prod_{j=1}^{n}\left(e^{-2\pi i\frac{\varphi_j}{a_j}}\right)^{[a_j\frac{t_k}{a_k}]}\prod_{j\neq k}\frac{e^{-\frac{i\pi\varphi_j}{a_j}}}{sin\frac{\pi\varphi_j}{a_j}}=-\frac{sin\left(\pi\sum_{j=1}^{n}\frac{\varphi_j}{a_j}\right)}{\prod_{j=1}^{n}sin\left(\pi\frac{\varphi_j}{a_j}\right)}$$

Remark 4.2.3 In particular, for $\frac{\varphi_j}{a_j}=\frac{1}{2}$, we get

$$\sum_{k=1}^{n}\sum_{t_k=1}^{a_k-1}\prod_{j=1}^{n}(-1)^{[a_j\frac{t_k}{a_k}]}=\begin{cases}-(-1)^{\frac{n-1}{2}} & \text{if n is odd,}\\ 0 & \text{otherwise.}\end{cases}$$

More generally, for $\frac{\varphi_j}{a_j}=\varphi=\frac{k}{N},(k,N)=1$, we have

$$\sum_{l=0}^{n}\sum_{t=1}^{a_l-1}\prod_{k=0}^{n}\left(e^{-2\pi i\varphi}\right)^{[\frac{ta_k}{a_l}]}=-\frac{\sin(\pi n\varphi)}{\sin(\pi\varphi)}e^{i\pi(n-1)\varphi}.$$

Then, theorem 4.2.2 can be seen as an extension of Berndt and Dieter result, theorem 2.1 [15], for complex valued functions.

Question:

i) Can you extend the theorem 2.1 [15] of Berndt-Dieter to complex valued functions? I think that is possible.

ii) If it is, can you give application in terms of signature theory of complex manifolds?

Proof of theorem 4.2.1 :
In order to prove our theorem 4.1.1, in section 4.5, we considered the auxillary function

$$F(z,\overrightarrow{\Phi},\overrightarrow{A},\overrightarrow{M})=\prod_{j=1}^{j=n}D_{L_j}^{(m_j)}\left(a'_jz-z_j;\frac{\varphi_j}{a_j}\right)$$

Where

$$\overrightarrow{A}=\left((a_1\dots,a_n);(a'_1,\dots,a'_n)\right),\overrightarrow{M}=(m_1,\dots,m_n),\overrightarrow{\Phi}=(\varphi_1,\dots,\varphi_n),z_k=-x'_k+x_k\frac{a'_k}{a_k}\tau.$$

Here, we introduce the function

$$f(z,\overrightarrow{\Phi},,\overrightarrow{A},\overrightarrow{M})=\prod_{j=1}^{n}\left(\cot^{(m_j)}\pi\left(a'_jz-x'_j\right)-\delta_{0,m_j}\cot^{(m_j)}\pi\left(\frac{\varphi_j}{a_j}\right)\right)$$

This function f comes essentially from the limit of $F(z,\overrightarrow{\Phi},\overrightarrow{A},\overrightarrow{M})$ when $\mathrm{Im}(\tau)\to\infty$, (for more details we can refer to the property xi) of theorem 2.2.1).
Now, like in the proof of our theorem 4.1.1, in section 4.5, we apply the residue theorem to this function $f(z,\overrightarrow{\Phi},\overrightarrow{A},\overrightarrow{M})$. We integrate f along the simple rectangular path

$$\gamma=[x+iy,x-iy,x+1-iy,x+1+iy,x+iy]\ ,$$

where x and y are chosen such that γ does not pass through any pole of f, and all poles z_p of f have imaginary part $|\mathrm{Im}(z_p)|<y$. By the periodicity of the cotangent function, the contributions of the two vertical segments of γ cancel each other. By definition of the cotangent function,

$$\lim_{y\to\infty}\cot(x\pm iy)=\mp i\ ,$$

and therefore also

$$\lim_{y\to\infty}\cot^{(m)}(x\pm iy)=0$$

for $m>0$. Hence if any of the $m_j>0$,

$$\int_{\gamma}f(z,\overrightarrow{\Phi},,\overrightarrow{A},\overrightarrow{M})\ dz=0\ .$$

If all $m_j=0$, we obtain

$$\int_{\gamma}f(z,\overrightarrow{\Phi},,\overrightarrow{A},\overrightarrow{M})\ dz=\prod_{j=1}^{n}\left(i-\cot(\pi\frac{\varphi_j}{a_j})\right)-\prod_{j=1}^{n}\left(-i-\cot(\pi\frac{\varphi_j}{a_j})\right).$$

This can be rewritten as follows

$$\frac{1}{2i}\int_{\gamma}f(z,\overrightarrow{\Phi},,\overrightarrow{A},\overrightarrow{M})\ dz=\begin{cases}(-1)^{n-1}\dfrac{\sin\pi\left(\displaystyle\sum_{j=1}^{n}\frac{\varphi_j}{a_j}\right)}{\displaystyle\prod_{j=1}^{n}\sin\pi\left(\frac{\varphi_j}{a_j}\right)} & \text{if all } m_j=0,\\ 0 & \text{otherwise.}\end{cases}$$

or, by means of the residue theorem,we obtain

$$(4.2.12)\quad \pi \sum_{z_p} \mathrm{Res}\left(f(z,\overrightarrow{\Phi},\overrightarrow{A},\overrightarrow{M})dz, z_p\right) = \begin{cases} (-1)^{n-1} \dfrac{\sin\pi\left(\displaystyle\sum_{j=1}^{n} \frac{\varphi_j}{a_j}\right)}{\displaystyle\prod_{j=1}^{n} \sin\pi\left(\frac{\varphi_j}{a_j}\right)} & \text{if all } m_j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here the sum ranges over all poles z_p inside γ. It remains to compute their residues. By assumption, f has only simple poles. We will compute the residue at $z_p = \frac{k+z_1}{a_1'}$, $k \in \mathbb{Z}$, the other residues being completely equivalent. We use the Laurent expansion of the cotangent

$$(4.2.13)\qquad \cot\pi\left(a_1'z - z_1\right) = \frac{1}{\pi a_1'}\left(z - \frac{k+z_1}{a_1'}\right)^{-1} + \text{ analytic part } ,$$

and, more generally,

$$\cot^{(m_1)}\pi\left(a_1'z - z_1\right) - \delta_{0,m_1}\cot^{(m_1)}\pi\left(\frac{\varphi_1}{a_1}\right) = \frac{(-1)^{m_1}m_1!}{(\pi a_1')^{m_1+1}}\left(z - \frac{k+z_1}{a_1'}\right)^{-(m_1+1)} + \text{ analytic part } .$$

The other cotangents are analytic at this pole: for $j > 1$,

$$(4.2.14)\qquad \cot^{(m_j)}\pi\left(a_j'z - z_j\right) - \delta_{0,m_1}\cot^{(m_j)}\pi\left(\frac{\varphi_j}{a_j}\right) =$$

$$\sum_{l_j \geqslant 0} \frac{(\pi a_j')^{l_j}}{l_j!}\left(\cot^{(m_j+l_j)}\pi\left(a_j'\frac{k+z_1}{a_1'} - z_j\right) - \delta_{0,m_j+l_j}\cot^{(m_j+l_j)}\pi\left(\frac{\varphi_j}{a_j}\right)\right)\left(z - \frac{k+z_1}{a_1'}\right)^{l_j} .$$

Hence

$$\mathrm{Res}\left(f(z,\overrightarrow{\Phi},\overrightarrow{A},\overrightarrow{M})dz, z = \frac{k+z_1}{a_1'}\right) =$$

$$\frac{(-1)^{m_1}m_1!}{\pi {a_1'}^{m_1+1}} \sum_{\substack{l_1,\ldots,l_d \geqslant 0 \\ l_1+\ldots+l_d = m_1}} \prod_{j=1}^{d} \frac{{a_j'}^{l_j}}{l_j!}\left(\cot^{(m_j+l_j)}\pi\left(a_j'\frac{k+z_1}{a_1'} - z_j\right) - \delta_{0,m_j+l_j}\cot^{(m_j+l_j)}\pi\left(\frac{\varphi_j}{a_j}\right)\right) .$$

Since γ has horizontal width 1, we have a_1' poles of the form $\frac{k+z_1}{a_1'}$ inside γ, where k runs through a complete set of residues modulo a_1'. This gives, by definition of the **generalized Dedekind cotangent sum**,

$$\sum_{k \bmod a_1'} \mathrm{Res}\left(f(z,\overrightarrow{\Phi},\overrightarrow{A},\overrightarrow{M})dz, z = \frac{k+z_1}{a_1'}\right) =$$

$$\frac{1}{\pi}(-1)^{m_1}m_1! \sum_{\substack{l_2,\ldots,\ldots,l_n \geqslant 0 \\ l_2+\ldots+\ldots+l_n = m_1}} \frac{{a_2'}^{l_2}\cdot\cdots\cdot{a_n'}^{l_n}}{l_2!\cdot\cdots\cdot l_n!}\,\mathfrak{C}\left(\begin{array}{c|cccc} a_1' & a_2' & \cdots & \cdots & a_n' \\ m_1 & m_2+l_2 & \cdots & \cdots & m_n+l_n \\ x_1' & x_2' & \cdots & \cdots & x_n' \\ \varphi_1 & \varphi_2 & \cdots & \cdots & \varphi_n \end{array}\right)$$

The other residues are computed in the same way, and give with (4.2.12) the statement:

$$\sum_{k=1}^{n}(-1)^{m_k} m_k! \sum_{\substack{l_1,\dots,\widehat{l_k},\dots,l_n \geqslant 0 \\ l_1+\dots+\widehat{l_k}+\dots+l_n=m_k}} \frac{{a'_1}^{l_1}\cdots\widehat{{a'_k}^{l_k}}\cdots{a'_n}^{l_n}}{l_1!\cdots\widehat{l_k!}\cdots l_n!}\,\mathfrak{C}\begin{pmatrix} a'_k & a'_1 & \cdots & \widehat{a'_k} & \cdots & a'_n \\ m_k & m_1+l_1 & \cdots & \widehat{m_k+l_k} & \cdots & m_n+l_n \\ x'_k & x'_1 & \cdots & \widehat{x'_k} & \cdots & x'_n \\ \varphi_1 & \varphi_2 & \cdots & \widehat{\varphi_k} & \cdots & \varphi_n \end{pmatrix}$$

$$= \begin{cases} (-1)^{n-1}\dfrac{\sin\pi\left(\displaystyle\sum_{j=1}^{n}\frac{\varphi_j}{a_j}\right)}{\displaystyle\prod_{j=1}^{n}\sin\pi\left(\frac{\varphi_j}{a_j}\right)} & \text{if all } m_k=0 \\ 0 & \text{otherwise.} \end{cases}$$

□

To deduce the following result of Beck 4.2.4, from our theorem 4.2.1, we simply take the following values

$$\frac{\varphi_j}{a_j}=\frac{1}{2}, \forall 1\leqslant j\leqslant n$$

. Hence

Corollary 4.2.4 (Beck's original result)
Under the same hypothesis of theorem 4.2.1, we have

$$\sum_{k=1}^{n}(-1)^{m_k} m_k! \sum_{\substack{l_1,\dots,\widehat{l_k},\dots,l_n \geqslant 0 \\ l_1+\dots+\widehat{l_k}+\dots+l_n=m_k}} \frac{a_1^{l_1}\cdots\widehat{a_k^{l_k}}\cdots a_n^{l_n}}{l_1!\cdots\widehat{l_k!}\cdots l_n!}\,\mathfrak{C}\begin{pmatrix} a_k & a_1 & \cdots & \widehat{a_k} & \cdots & a_n \\ m_k & m_1+l_1 & \cdots & \widehat{m_k+l_k} & \cdots & m_n+l_n \\ x_k & x_1 & \cdots & \widehat{x_k} & \cdots & x_n \\ \varphi_1=\frac{a_1}{2} & \varphi_2=\frac{a_2}{2} & \cdots & \widehat{\varphi_k=\frac{a_k}{2}} & \cdots & \varphi_n=\frac{a_n}{2} \end{pmatrix}$$

$$= \begin{cases} (-1)^{\frac{n-1}{2}} & \textit{if all } m_k=0 \textit{ and } n \textit{ odd} \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof of theorem 4.2.2 (***Division points***) :*

To prove this theorem 4.2.2 we need some preliminaries. Let $z=z_1\tau+z_2\in\mathbb{C}$ and $z_1,z_2\in\mathbb{R}$, we denote by $\{z\}=\{z_1\}\tau+\{z_2\}$ and $[z]=[z_1]\tau+[z_2]$, with $\{z_1\}$, $\{z_2\}$ the fractional parts of the real numbers z_1,z_2 (resp. $[z_1],[z_2]$ integer parts of real numbers z_1,z_2). For each $z,\varphi\in\mathbb{C}\backslash\mathbb{Z}\tau+\mathbb{Z}$, using the property x) in theorem 2.2.1, we obtain

$$\lim_{\mathrm{Im}(\tau)\to\infty} D_\tau(z,\varphi) = \begin{cases} \pi\Big(cot(\pi\{\varphi\})+cot(\pi\{z\})\Big)e^{-2i\pi[z_1]\{\varphi_2\}} & \text{If} \quad (z_1,\varphi_1)\in\mathbb{Z}^2 \\ \pi\Big(cot(\pi\{\varphi\})-i\Big)e^{-2i\pi[z_1]\{\varphi_2\}} & \text{If} \quad \varphi_1\in\mathbb{Z}, z_1\notin\mathbb{Z} \\ \pi\Big(cot(\pi\{z\})-i\Big)e^{2i\pi\{\varphi_1\}z_2-2i\pi[z_1]\{\varphi_2\}} & \text{If} \quad z_1\in\mathbb{Z},\varphi_1\notin\mathbb{Z} \\ 0 & \text{If} \quad z_1\notin\mathbb{Z},\varphi_1\notin\mathbb{Z} \end{cases}$$

From the periodicity of $D_\tau(z,\varphi)$, iii) in theorem 2.2.1, we obtain

$$D_L^{(m)}(z+\rho;\varphi) := \frac{\partial^m}{\partial z^m} D_L(z+\rho;\varphi) = e(E_L(\rho,\varphi)) D_L^{(m)}(z;\varphi), \forall \rho \in L$$

For the rest of this proof we assume that $\varphi \in \mathbb{R}\backslash\mathbb{Z}$, then

$$\lim_{\mathrm{Im}(\tau)\to\infty} D_\tau^{(m)}(z,\varphi) = \tag{4.2.15}$$

$$\begin{cases} \pi^{m+1}\Big(\delta_{0,m} cot^{(m)}(\pi\{\varphi\}) + cot^{(m)}(\pi\{z\})\Big) e^{-2i\pi[z_1]\{\varphi_2\}} & \text{If } \{z_1\}=0 \\ \pi^{m+1}\delta_{0,m}\Big(cot(\pi\{\varphi\}) - i\Big) e^{-2i\pi[z_1]\{\varphi_2\}} & \text{If } \{z_1\}\neq 0 \end{cases}$$

Lemma 4.2.5 *We assume $\varphi_j \in \mathbb{R}\backslash\mathbb{Z}, \forall j = 1,\cdots,n$ and let*

$$z_k = -x'_k + x_k\frac{a'_k}{a_k}\tau, \tilde{t}_k = -t'_k + t_k\frac{a'_k}{a_k}\tau, 0 \leqslant t_k < a_k, 0 \leqslant t'_k < a'_k$$

Then, we have

$$\lim_{\mathrm{Im}(\tau)\to\infty} D_{L_j}^{(m_j+r_j)}\left(a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j}\right) = \pi^{m_j+r_j+1}\times \tag{4.2.16}$$

$$\begin{cases} \delta_{0,m_j+r_j}\Big(cot^{(m_j+r_j)}(\pi\frac{\varphi_j}{a_j}) - cot^{(m_j+r_j)}(\pi(a'_j\frac{x'_k+t'_k}{a'_k} - x'_j))\Big) e\Big(-\Big[a_j\frac{x_k+t_k}{a_k} - x_j\Big]\frac{\varphi_j}{a_j}\Big) & \text{If } \{a_j\frac{x_k+t_k}{a_k} - x_j\} = 0 \\ \delta_{0,m_j+r_j}\Big(cot(\pi\frac{\varphi_j}{a_j}) - i\Big) e\Big(-\Big[a_j\frac{x_k+t_k}{a_k} - x_j\Big]\frac{\varphi_j}{a_j}\Big) & \text{If } \{a_j\frac{x_k+t_k}{a_k} - x_j\} \neq 0 \end{cases}$$

Now, for the rest of the proof we take $x_j = 0, \forall j = 1,\cdots,n$. Then

$$\lim_{\mathrm{Im}(\tau)\to\infty} D_{L_j}^{(m_j+r_j)}\left(a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j}\right) = \pi^{m_j+r_j+1}\times \tag{4.2.17}$$

$$\begin{cases} \Big(\delta_{0,m_j+r_j} cot^{(m_j+r_j)}(\pi\frac{\varphi_j}{a_j}) - cot^{(m_j+r_j)}(\pi(a'_j\frac{x'_k+t'_k}{a'_k} - x'_j))\Big) & \text{If } t_k = 0 \\ \delta_{0,m_j+r_j}\Big(cot(\pi\frac{\varphi_j}{a_j}) - i\Big) e\Big(-\Big[a_j\frac{t_k}{a_k}\Big]\frac{\varphi_j}{a_j}\Big) & \text{If } t_k \neq 0 \end{cases}$$

Hence,

$$\lim_{\mathrm{Im}(\tau)\to\infty} \prod_{1\leqslant j\neq k\leqslant n} D_{L_j}^{(m_j+r_j)}\left(a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j}\right) = \pi^{n-1+\sum_{1\leqslant j\neq k\leqslant n}(m_j+r_j)} \times \tag{4.2.18}$$

$$\begin{cases} \prod_{1\leqslant j\neq k\leqslant n} \Big(\delta_{0,m_j+r_j} cot^{(m_j+r_j)}(\pi\frac{\varphi_j}{a_j}) - cot^{(m_j+r_j)}(\pi(a'_j\frac{x'_k+t'_k}{a'_k} - x'_j))\Big) & \text{If } t_k=0 \\ \prod_{1\leqslant j\neq k\leqslant n} \delta_{0,m_j+r_j}\Big(cot(\pi\frac{\varphi_j}{a_j}) - i\Big) e\Big(-\Big[a_j\frac{t_k}{a_k}\Big]\frac{\varphi_j}{a_j}\Big) & \text{If } t_k\neq 0 \end{cases}$$

Then

$$(4.2.19)\quad \lim_{\mathrm{Im}(\tau)\to\infty} \sum_{\substack{0\leqslant t_k<a_k\\ 0\leqslant t'_k<a'_k}} e\left(\mathrm{E}_{L_k}(\tilde{t}_k,\frac{\varphi_k}{a_k})\right) \prod_{1\leqslant j\neq k\leqslant n} D_{L_j}^{(m_j+r_j)}\left(a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j;\frac{\varphi_j}{a_j}\right) =$$

$$\pi^{n-1+\sum_{1\leqslant j\neq k\leqslant n}(m_j+r_j)} \times$$

$$\left(\begin{array}{c} \sum_{t'_k=0}^{a'_k-1} \prod_{1\leqslant j\neq k\leqslant n} \Big(\delta_{0,m_j+r_j} cot^{(m_j+r_j)}(\pi\frac{\varphi_j}{a_j}) - cot^{(m_j+r_j)}(\pi(a'_j\frac{x'_k+t'_k}{a'_k} - x'_j))\Big) \\ + \\ a'_k\sum_{t_k=0}^{a_k-1} e\Big(-\frac{\varphi_k}{a_k}t_k\Big) \prod_{1\leqslant j\neq k\leqslant n} \delta_{0,m_j+r_j}\Big(cot(\pi\frac{\varphi_j}{a_j}) - i\Big) e\Big(-\Big[a_j\frac{t_k}{a_k}\Big]\frac{\varphi_j}{a_j}\Big) \end{array}\right)$$

Finally

$$(4.2.20)\quad \lim_{\mathrm{Im}(\tau)\to\infty} d(\overrightarrow{A_k},\overrightarrow{A'_k},\overrightarrow{Z_k},\overrightarrow{\phi_k},\overrightarrow{M_k},\overrightarrow{R_k},\tau) = \pi^{n-1+\sum_{1\leqslant j\neq k\leqslant n}(m_j+r_j)} \times$$

$$\frac{(-1)^{m_k} m_k!}{{a'_k}^{m_k+1}} \left(\prod_{1\leqslant j\neq k\leqslant n} \frac{{a'_j}^{r_j}}{r_j!}\right) \left(\begin{array}{c} \sum_{t'_k=0}^{a'_k-1} \prod_{1\leqslant j\neq k\leqslant n} \Big(\delta_{0,m_j+r_j} cot^{(m_j+r_j)}(\pi\frac{\varphi_j}{a_j}) - cot^{(m_j+r_j)}(\pi(a'_j\frac{x'_k+t'_k}{a'_k} - x'_j))\Big) \\ + \\ a'_k\sum_{t_k=0}^{a_k-1} e\Big(-\frac{\varphi_k}{a_k}t_k\Big) \prod_{1\leqslant j\neq k\leqslant n} \delta_{0,m_j+r_j}\Big(cot(\pi\frac{\varphi_j}{a_j}) - i\Big) e\Big(-\Big[a_j\frac{t_k}{a_k}\Big]\frac{\varphi_j}{a_j}\Big) \end{array}\right)$$

and $\sum_{1\leqslant j\neq k\leqslant n} r_j = m_k$, then

$$(4.2.21)\quad \lim_{\mathrm{Im}(\tau)\to\infty} \frac{d(\overrightarrow{A_k},\overrightarrow{A'_k},\overrightarrow{Z_k},\overrightarrow{\phi_k},\overrightarrow{M_k},\overrightarrow{R_k},\tau)}{\pi^{n-1+\sum_{1\leqslant j\leqslant n} m_j}} =$$

$$\frac{(-1)^{m_k} m_k!}{{a'_k}^{m_k+1}} \left(\prod_{1\leqslant j\neq k\leqslant n} \frac{{a'_j}^{r_j}}{r_j!} \right) \left(\begin{array}{c} \sum_{t'_k=0}^{a'_k-1} \prod_{1\leqslant j\neq k\leqslant n} \Big(\delta_{0,m_j+r_j} cot^{(m_j+r_j)}(\pi \frac{\varphi_j}{a_j}) - cot^{(m_j+r_j)}(\pi(a'_j \frac{x'_k+t'_k}{a'_k} - x'_j)) \Big) \\ + \\ a'_k \sum_{t_k=0}^{a_k-1} e\Big(-\frac{\varphi_k}{a_k} t_k\Big) \prod_{1\leqslant j\neq k\leqslant n} \delta_{0,m_j+r_j} \Big(cot(\pi \frac{\varphi_j}{a_j}) - i \Big) e\Big(-\Big[a_j \frac{t_k}{a_k}\Big] \frac{\varphi_j}{a_j}\Big) \end{array} \right)$$

This gives the limit

$$(4.2.22) \quad \frac{(-1)^{n-1}}{\pi^{n-1+\sum_{1\leqslant j\leqslant n} m_j}} \lim_{\mathrm{Im}(\tau)\to\infty} \sum_{k=1}^{n} \sum_{\substack{\overrightarrow{R_k}\geqslant \overrightarrow{0} \\ r_1+\ldots+\check{r}_k+\ldots+r_n=m_k}} d(\overrightarrow{A_k}, \overrightarrow{A'_k}, \overrightarrow{Z_k}, \overrightarrow{\phi_k}, \overrightarrow{M_k}, \overrightarrow{R_k}, \tau) =$$

$$\sum_{k=1}^{n} \sum_{\substack{\overrightarrow{R_k}\geqslant \overrightarrow{0} \\ r_1+\ldots+\check{r}_k+\ldots+r_n=m_k}} \frac{(-1)^{m_k} m_k!}{{a'_k}^{m_k+1}} \left(\prod_{1\leqslant j\neq k\leqslant n} \frac{{a'_j}^{r_j}}{r_j!} \right) \times$$

$$\left(\begin{array}{c} \sum_{t'_k=0}^{a'_k-1} \prod_{1\leqslant j\neq k\leqslant n} \Big(cot^{(m_j+r_j)}(\pi(a'_j \frac{x'_k+t'_k}{a'_k} - x'_j)) - \delta_{0,m_j+r_j} cot^{(m_j+r_j)}(\pi \frac{\varphi_j}{a_j}) \Big) \\ + \\ a'_k \sum_{t_k=0}^{a_k-1} \prod_{1\leqslant j\neq k\leqslant n} \delta_{0,m_j+r_j} \Big(i - cot(\pi \frac{\varphi_j}{a_j}) \Big) \prod_{1\leqslant j\leqslant n} e\Big(-\Big[a_j \frac{t_k}{a_k}\Big] \frac{\varphi_j}{a_j}\Big) \end{array} \right)$$

Now, we can give a final formulation

$$\sum_{k=1}^{n} \sum_{\substack{\overrightarrow{R_k}\geqslant \overrightarrow{0} \\ r_1+\ldots+\check{r}_k+\ldots+r_n=m_k}} \frac{(-1)^{m_k} m_k!}{{a'_k}^{m_k+1}} \left(\prod_{1\leqslant j\neq k\leqslant n} \frac{{a'_j}^{r_j}}{r_j!} \right)$$

$$\sum_{t'_k=0}^{a'_k-1} \prod_{1\leqslant j\neq k\leqslant n} \Big(cot^{(m_j+r_j)}(\pi(a'_j \frac{x'_k+t'_k}{a'_k} - x'_j)) - \delta_{0,m_j+r_j} cot^{(m_j+r_j)}(\pi \frac{\varphi_j}{a_j}) \Big)$$

$$= \begin{cases} -(-1)^n \sum_{t_k=0}^{a_k-1} \prod_{1\leqslant j\neq k\leqslant n} \frac{e^{-\pi i \frac{\varphi_j}{a_j}}}{\sin\Big(\pi \frac{\varphi_j}{a_j}\Big)} \prod_{1\leqslant j\leqslant n} e\Big(-\Big[a_j \frac{t_k}{a_k}\Big] \frac{\varphi_j}{a_j}\Big) & \text{if all } m_k = 0 \\ 0 & \text{Otherwise} \end{cases}$$

Or equivalently,

$$\sum_{k=1}^{n}(-1)^{m_k}m_k! \sum_{\substack{l_1,\ldots,\widehat{l_k},\ldots,l_n\geqslant 0\\ l_1+\ldots+\widehat{l_k}+\ldots+l_n=m_k}} \frac{{a'_1}^{l_1}\cdots\widehat{{a'_k}^{l_k}}\cdots{a'_n}^{l_n}}{l_1!\cdots\widehat{l_k!}\cdots l_n!}\,\mathfrak{C}\begin{pmatrix} a'_k & a'_1 & \cdots & \widehat{a'_k} & \cdots & a'_n\\ m_k & m_1+l_1 & \cdots & \widehat{m_k+l_k} & \cdots & m_n+l_n\\ x'_k & x'_1 & \cdots & \widehat{x'_k} & \cdots & x'_n\\ \varphi_1 & \varphi_2 & \cdots & \widehat{\varphi_k} & \cdots & \varphi_n\end{pmatrix}$$

$$= \begin{cases} -(-1)^n\displaystyle\sum_{t_k=0}^{a_k-1}\prod_{1\leqslant j\neq k\leqslant n}\frac{e^{-\pi i\frac{\varphi_j}{a_j}}}{\sin\left(\pi\frac{\varphi_j}{a_j}\right)}\prod_{1\leqslant j\leqslant n} e\Big(-\Big[a_j\frac{t_k}{a_k}\Big]\frac{\varphi_j}{a_j}\Big) & \text{if all } m_k=0\\ \\ 0 & \text{Otherwise .}\end{cases}$$

Now, from theorem 4.2.1 we deduce the statement of theorem 4.2.2:

$$\sum_{k=1}^{n}\sum_{t_k=1}^{a_k-1}\prod_{j=1}^{n}\left(e^{-2\pi i\frac{\varphi_j}{a_j}}\right)^{[a_j\frac{t_k}{a_k}]}\prod_{j\neq k}\frac{e^{-\frac{i\pi\varphi_j}{a_j}}}{\sin\frac{\pi\varphi_j}{a_j}} = -\frac{\sin\left(\pi\displaystyle\sum_{j=1}^{n}\frac{\varphi_j}{a_j}\right)}{\displaystyle\prod_{j=1}^{n}\sin\left(\pi\frac{\varphi_j}{a_j}\right)}.$$

4.3 Application 2: Enhanced sums of " Dedekind-Rademacher-Hall-Wilson-Zagier" in terms of Bernoulli functions.

Let $n\in\mathbb{N}$ be an arbitrary integer $\geqslant 3$ and $1\leqslant k\leqslant n$.
Our second "Enhanced" Multiple elliptic Dedekind-Rademacher sums are

$$S(\overrightarrow{A_k},\overrightarrow{A'_k},\overrightarrow{R_k},\overrightarrow{Z_k};\tau) := \frac{1}{a'_k}\sum_{\tilde{t}_k\in L_k/a'_kL}\prod_{1\leqslant j\neq k\leqslant n}\bar{B}_{r_j}(a'_j\frac{z_k+\tilde{t}_k}{a'_k}-z_j,L_j)$$

The generating function of these sums is exactly equal to

$$\mathfrak{S}(\overrightarrow{A_k},\overrightarrow{A'_k},\overrightarrow{Z_k},\overrightarrow{\Phi_k};\tau) := \frac{1}{(2\pi i)^{n-1}}\times\frac{1}{a'_k}\sum_{\tilde{t}_k\in L_k/a'_kL}\prod_{1\leqslant j\neq k\leqslant n} D_{L_j}\left(\frac{\varphi_k}{a_j};a'_j\frac{z_k+\tilde{t}_k}{a'_k}-z_j\right)$$

Theorem 4.1.3 gives us the reciprocity laws for these "**"Enhanced"** Multiple elliptic Dedekind-Rademacher sums. It can be stated in terms of the generating function as follows (Under the hypothesis of theorem 4.1.3)

$$\sum_{k=1}^{n}\mathfrak{S}(\overrightarrow{A_k},\overrightarrow{A'_k},\overrightarrow{Z_k},,\overrightarrow{\Phi_k};\tau)=0$$

Where $\varphi_1,\cdots,\varphi_n$ are non-zero variables and $\varphi_1+\cdots+\varphi_n=0$.

Let me now state the classical version of our Theorem 4.1.3. More precisely, when $\mathrm{Im}(\tau) \to \infty$ from our result Theorem 4.1.3 we will obtain a formula (generalizing a formula of Hall-Wilson-Zagier in [23] for $n = 3$) which is the Multiple version of Dedekind reciprocity laws in terms of classical Bernoulli functions.

The classical multiple Dedekind sums are defined by the expression

$$S(\overrightarrow{A_k}, \overrightarrow{X_k}, \overrightarrow{R_k}; \tau) := \sum_{t=0}^{a_k-1} \prod_{1 \leqslant j \neq k \leqslant n} \bar{B}_{r_j} \left(a_j \frac{x_k + t}{a_k} - x_j \right)$$

Now, we consider the generating function of these sums

$$\mathfrak{S}(\overrightarrow{A_k}, \overrightarrow{X_k}, \overrightarrow{\Phi_k}) := \sum_{\overrightarrow{R_k} \in \mathbb{N}^{n-1}} \frac{S(\overrightarrow{A_k}, \overrightarrow{X_k}, \overrightarrow{R_k}; \tau)}{r_1! \dots \check{r}_j! \dots r_n!} \prod_{1 \leqslant j \neq k \leqslant n} \left(\frac{\varphi_j}{a_j} \right)^{r_j - 1}$$

Now, we state our main result of this subsection

Theorem 4.3.1 *For all $j, k, 1 \leqslant j \neq k \leqslant n$, we assume that*

$$a'_j x'_k - a'_k x'_j \notin \mathbb{Z} a'_j + \mathbb{Z} a'_k \text{ and } \sum_{j=1}^{n} \varphi_j = 0$$

Then

(4.3.23)

$$\sum_{k=1}^{n} \mathfrak{S}(\overrightarrow{A_k}, \overrightarrow{X_k}, 2\pi i \overrightarrow{\Phi_k}) = \sum_{k=1}^{n} \sum_{\substack{0 \leqslant t < a_k \\ I(t,k) \neq \emptyset}} \frac{1}{(2i)^{\ Card\ I(t,k)}} \prod_{j \notin I(t,k) \cup \{k\}} \frac{e\left(\frac{\varphi_j}{a_j} \{ a_j \frac{x_k + t}{a_k} - x_j \} \right)}{e\left(\frac{\varphi_j}{a_j} \right) - 1} \times$$

$$\left(\prod_{j \in I(t,k)} cot \left(\frac{\pi \varphi_j}{a_j} \right) - \frac{1}{a'_k} \sum_{t'=0}^{a'_k - 1} \prod_{j \in I(t,k)} \left(cot \left(\frac{\pi \varphi_j}{a_j} \right) - cot \pi \left(a'_j \frac{x'_k + t'}{a'_k} - x'_j \right) \right) \right).$$

Equivalently,

(4.3.24)

$$\sum_{k=1}^{n} \mathfrak{S}(\overrightarrow{A_k}, \overrightarrow{X_k}, \overrightarrow{\Phi_k}) = \sum_{k=1}^{n} \sum_{\substack{0 \leqslant t < a_k \\ I(t,k) \neq \emptyset}} \frac{1}{2^{\ Card\ I(t,k)}} \prod_{j \notin I(t,k) \cup \{k\}} \frac{e^{\frac{\varphi_j}{a_j} \{ a_j \frac{x_k + t}{a_k} - x_j \}}}{e^{\frac{\varphi_j}{a_j}} - 1} \times$$

$$\left(\prod_{j \in I(t,k)} coth \left(\frac{\varphi_j}{2a_j} \right) - \frac{1}{a'_k} \sum_{t'=0}^{a'_k - 1} \prod_{j \in I(t,k)} \left(coth \left(\frac{\varphi_j}{2a_j} \right) + i\, cot \pi \left(a'_j \frac{x'_k + t'}{a'_k} - x'_j \right) \right) \right).$$

Where

$$I(t, k) = \left\{ 1 \leqslant j \neq k \leqslant n : \left\{ a_j \frac{x_k + t}{a_k} - x_j \right\} = 0 \right\}.$$

Proof of theorem 4.3.1:

From the equality (2.2.6) we quote

(4.3.25)

$$\lim_{\mathrm{Im}(\tau)\to\infty} D_\tau\left(\frac{\varphi_j}{a_j}, a'_j\frac{z_k+\tilde{t}_k}{a'_k}-z_j\right)=\frac{e\left(\frac{\varphi_j}{a_j}\{a_j\frac{x_k+t}{a_k}-x_j\}\right)}{e\left(\frac{\varphi_j}{a_j}\right)-1}+\frac{2\pi ie\left(-(a'_j\frac{x'_k+t'}{a'_k}-x'_j)\right)}{e\left(-(a'_j\frac{x'_k+t'}{a'_k}-x'_j)\right)-1}\delta_{0,\left\{a_j\frac{x_k+t}{a_k}-x_j\right\}}$$

Then we deduce a **cotangent version of the theorem 4.1.2**

(4.3.26)

$$\sum_{k=1}^{n}\frac{1}{a'_k}\sum_{t'=0}^{a'_k-1}\sum_{t=0}^{a_k-1}\prod_{j\neq k}\left(\frac{e\left(\frac{\varphi_j}{a_j}\{a_j\frac{x_k+t}{a_k}-x_j\}\right)}{e\left(\frac{\varphi_j}{a_j}\right)-1}+\frac{2\pi ie\left(-(a'_j\frac{x'_k+t'}{a'_k}-x'_j)\right)}{e\left(-(a'_j\frac{x'_k+t'}{a'_k}-x'_j)\right)-1}\delta_{0,\left\{a_j\frac{x_k+t}{a_k}-x_j\right\}}\right)=0$$

Now, we set

$$\overrightarrow{R_k}!=\prod_{j\neq k} r_j!$$

we can reformulate this result in terms of Bernoulli functions, from the equality (1.0.2) we deduce

(4.3.27)

$$\sum_{k=1}^{n}\sum_{\overrightarrow{R_k}\geqslant\overrightarrow{0}}\frac{1}{\overrightarrow{R_k}!}.\frac{1}{a'_k}\sum_{t'=0}^{a'_k-1}\sum_{t=0}^{a_k-1}\prod_{j\neq k}$$
$$\left(B_{r_j}\left(\{a_j\tfrac{x_k+t}{a_k}-x_j\}\right)+\left(-\tfrac{1}{2i}\cot\pi\left(a'_j\tfrac{x'_k+t'}{a'_k}-x'_j\right)\right)\delta_{1,r_j}\delta_{0,\left\{a_j\frac{x_k+t}{a_k}-x_j\right\}}\right)\left(\tfrac{2\pi i\varphi_j}{a_j}\right)^{r_j-1}=0$$

Then

(4.3.28)

$$\sum_{k=1}^{n}\sum_{\overrightarrow{R_k}\geqslant\overrightarrow{0}}\frac{1}{\overrightarrow{R_k}!}.\frac{1}{a'_k}\sum_{t'=0}^{a'_k-1}\sum_{t=0}^{a_k-1}\prod_{j\in I(t,k)}\left(B_{r_j}(0)+\left(-\frac{1}{2i}\cot\pi\left(a'_j\frac{x'_k+t'}{a'_k}-x'_j\right)+\frac{1}{2}\right)\delta_{1,r_j}\right)$$
$$\prod_{j\notin I(t,k)\cup\{k\}}B_{r_j}\left(\{a_j\frac{x_k+t}{a_k}-x_j\}\right)\left(\frac{2\pi i\varphi_j}{a_j}\right)^{r_j-1}=0$$

Now, we distinguish two cases for our summation over $I(t,k)=\emptyset$ or $\delta_{1,r_j}=1$ and $I(t,k)\neq\emptyset$ or $\delta_{1,r_j}=0$. We remark that the case $I(t,k)=\emptyset$ or $\delta_{1,r_j}=1$ gives us our quantity

$$\sum_{k=1}^{n}\mathfrak{S}(\overrightarrow{A_k},\overrightarrow{X_k},2\pi i\overrightarrow{\Phi_k})$$

that corresponds to the first member of our equality 4.3.23) and the rest corresponds to the second member of (4.3.23).
Hence, we obtain our desired theorem 4.3.1. □

In particular, we have the following corollaries

Corollary 4.3.2 *(***Generic case***)*

For all $j, k, 1 \leqslant j \neq k \leqslant n$, we assume that

$$a'_j x'_k - a'_k x'_j \notin \mathbb{Z}a'_j + \mathbb{Z}a'_k \text{ and } \sum_{j=1}^{n} \varphi_j = 0$$

Indeed, if we assume that

$$\overrightarrow{X} \in \mathbb{R}\overrightarrow{A} + \mathbb{Z}^n$$

Then

$$\sum_{k=1}^{n} \mathfrak{S}(\overrightarrow{A_k}, \overrightarrow{X_k}, 2\pi i\overrightarrow{\Phi_k}) = \frac{1}{(2i)^{n-1}} \left(\prod_{1 \leqslant j \neq k \leqslant n} cot\left(\frac{\pi\varphi_j}{a_j}\right) - Im\left(\prod_{j=1}^{n}\left(cot\left(\frac{\pi\varphi_j}{a_j}\right) + i\right)\right)\right)$$

Equivalently,

$$\sum_{k=1}^{n} \mathfrak{S}(\overrightarrow{A_k}, \overrightarrow{X_k}, \overrightarrow{\Phi_k}) = \frac{1}{2^{n-1}} \prod_{1 \leqslant j \neq k \leqslant n} coth\left(\frac{\varphi_j}{2a_j}\right) - \frac{1}{2^n}\left(\prod_{j=1}^{n}\left(coth\left(\frac{\varphi_j}{2a_j}\right) + 1\right) - \prod_{j=1}^{n}\left(coth\left(\frac{\varphi_j}{2a_j}\right) - 1\right)\right)$$

Where $\overrightarrow{A} = (a_1, \cdots, a_n), \overrightarrow{X} = (x_1, \cdots, x_n)$

Proof of the corollary 4.3.2:
It's comes from the following fact

$$\overrightarrow{X} \in \mathbb{R}\overrightarrow{A} + \mathbb{Z}^n \iff \exists \text{ a unique } 0 \leqslant t_k \leqslant a_k - 1 \text{ such that } I(t_k, k) = \{1, \cdots, n\}\backslash\{k\}$$

Then for $0 \leqslant t \neq t_k \leqslant a_k - 1$ we have $I(t, k) = \emptyset$. The corollary 4.3.2 comes from the theorem 4.3.1. □

Now, for $n = 3$ we can formulate our Theorem 4.3.1 as follows

Corollary 4.3.3 (Hall-Wilson-Zagier: $n = 3$**)** *Let a_1, a_2, a_3 be three positive integers which no common factor, $\overrightarrow{A} = (a_1, a_2, a_3)$, x_1, x_2, x_3 three real numbers, and $\varphi_1, \varphi_2, \varphi_3$ three variables with sum zero. Then*

$$\sum_{k=1}^{3} \mathfrak{S}(\overrightarrow{A_k}, \overrightarrow{X_k}, \overrightarrow{\Phi_k}) = \begin{cases} -\frac{1}{4} & \textit{If } \overrightarrow{X} \in \mathbb{R}\overrightarrow{A} + \mathbb{Z}^3 \\ 0 & \textit{otherwise.} \end{cases}$$

Where $\overrightarrow{A} = (a_1, a_2, a_3), \overrightarrow{X} = (x_1, x_2, x_3)$

Proof of the corollary 4.3.3:
Since $n = 3$ then $0 \leqslant \text{card}I(t, k) \leqslant 2$. If $\text{card}I(t, k) = 2, \forall 1 \leqslant k \leqslant 3$ then $\overrightarrow{X} \in \mathbb{R}\overrightarrow{A} + \mathbb{Z}^3$ (remark that t is unique for each k). Hence we use the result of corollary 4.3.2 to obtain the first part of the corollary 4.3.3.

Otherwise, we take the real part of the two members of the equality (4.3.24) of the theorem 4.3.1 to obtain the rest of our desired corollary 4.3.3. □

4.4 Application 3: Dedekind-Sczech reciprocity result.

In this section we give a new and elementary proof of the reciprocity Theorem 4.4.1 of Sczech. Our proof is clearly different to Sczech one.
We recall here Sczech's result [42].
For $a, c \in O_L := \{z \in \mathbb{C} : zL \subset L\}$, we define

$$D(a,c) := \frac{1}{c} \sum_{\bar{t} \in L/cL} E_1\left(\frac{t}{c}; L\right) E_1\left(\frac{at}{c}; L\right)$$

Theorem 4.4.1 *(R.Sczech). For $a, c \in O_L \backslash \{0\}$ pairwise coprime. Then*

$$D(a,c) + D(c,a) = 2iE_2(0; L) Im\left(\frac{a}{c} + \frac{c}{a} + \frac{1}{ac}\right).$$

In the following, we attempt to deduce the result of Sczech above from our general Dedekind Multiple elliptic reciprocity Laws satisfied by elliptic Multiple Dedekind Sums. We reformulate a weak version of our reciprocity Law in Theorem 4.1.2 for Three variables φ_1, φ_2 and φ_3.

Theorem 4.4.2 *For $\varphi_1, \varphi_2, \varphi_3$ Three variables, $a_1, a_2, a_3 \in O_L$, pairwise coprime such that:*

$$\varphi_1 + \varphi_2 + \varphi_3 = 0.$$

Then, we have

$$\frac{1}{a_1} \sum_{\bar{t} \in L/cL} D_L\left(\frac{\varphi_2}{\bar{a_2}}, \frac{a_2 t}{a_1}\right) D_L\left(\frac{\varphi_3}{\bar{a_3}}, \frac{a_3 t}{a_1}\right) + \frac{1}{a_2} \sum_{\bar{t} \in L/cL} D_L\left(\frac{\varphi_1}{\bar{a_1}}, \frac{a_1 t}{a_2}\right) D_L\left(\frac{\varphi_3}{\bar{a_3}}, \frac{a_3 t}{a_2}\right) +$$

$$\frac{1}{a_3} \sum_{\bar{t} \in L/cL} D_L\left(\frac{\varphi_1}{\bar{a_1}}, \frac{a_1 t}{a_3}\right) D_L\left(\frac{\varphi_2}{\bar{a_2}}, \frac{a_2 t}{a_3}\right) =$$

$$-\left(\frac{d_1(\frac{\varphi_2}{\bar{a_2}}, L) d_1(\frac{\varphi_3}{\bar{a_3}}, L)}{a_1} + \frac{d_1(\frac{\varphi_1}{\bar{a_1}}, L) d_1(\frac{\varphi_3}{\bar{a_3}}, L)}{a_2} + \frac{d_1(\frac{\varphi_1}{\bar{a_1}}, L) d_1(\frac{\varphi_2}{\bar{a_2}}, L)}{a_3}\right)$$

$$-\left(\frac{d_2(\frac{\varphi_1}{\bar{a_1}}, L) a_1}{a_2 a_3} + \frac{d_2(\frac{\varphi_2}{\bar{a_2}}, L) a_2}{a_1 a_3} + \frac{d_2(\frac{\varphi_3}{\bar{a_3}}, L) a_3}{a_1 a_2}\right)$$

Proof of Theorem 4.4.2: We consider the function

$$F(z) = \prod_{1 \leqslant j \leqslant 3} D_{L_j}\left(a_j z; \frac{\varphi_j}{\bar{a_j}}\right).$$

This function is meromorphic, and periodic with periods the Lattice L, because $\varphi_1 + \varphi_2 + \varphi_3 = 0$, and have poles (modulo L) at $z = \frac{t}{a_j}, \bar{t} \in L/a_j L$ all its poles have a simple multiplicity except at $z = 0$ modulo L which is of order 3. Then, by residue Theorem, we obtain

$$\sum_{k=1}^{3} \frac{1}{a_k} \sum_{\bar{t_k} \in L_k / a_k L \backslash \{\bar{0}\}} \prod_{1 \leqslant j \neq k \leqslant 3} D_L\left(\frac{\varphi_j}{\bar{a_j}}; a_j \frac{\tilde{t}_k}{a_k}\right) = -\text{Res}\left(\prod_{1 \leqslant j \leqslant 3} D_L\left(a_j z; \frac{\varphi_j}{\bar{a_j}}\right) dz; z = 0\right)$$

Now, we use the Laurent expansion of the form

$$D_L(z,\varphi) = \sum_{k\geqslant 0} d_k(\varphi, L) z^{k-1}$$

with

$$d_0(\varphi, L) = 1,$$

we obtain

$$\operatorname{Res}\left(\prod_{1\leqslant j\leqslant 3} D_L\left(a_j z; \frac{\varphi_j}{\bar{a_j}}\right) dz; z=0\right) = \frac{1}{a_1} d_1\left(\frac{\varphi_2}{\bar{a_2}}, L\right) d_1\left(\frac{\varphi_3}{\bar{a_3}}, L\right) + \frac{1}{a_2} d_1\left(\frac{\varphi_1}{\bar{a_1}}, L\right) d_1\left(\frac{\varphi_3}{\bar{a_3}}, L\right) +$$

$$\frac{1}{a_3} d_1\left(\frac{\varphi_1}{\bar{a_1}}, L\right) d_1\left(\frac{\varphi_2}{\bar{a_2}}, L\right) + \quad \frac{a_1}{a_2 a_3} d_2\left(\frac{\varphi_1}{\bar{a_1}}, L\right) + \frac{a_2}{a_1 a_3} d_2\left(\frac{\varphi_2}{\bar{a_2}}, L\right) + \frac{a_3}{a_1 a_2} d_2\left(\frac{\varphi_3}{\bar{a_3}}, L\right)$$

which is the desired theorem 4.4.2.

As a consequence of our Theorem 4.4.2 we obtain the result of Sczech 4.4.1. In fact, in the folllowing we prove the implication: **Theorem 4.4.2** $\Longrightarrow$ **Theorem 4.4.1.**

Proof of theorem 4.4.2 $\Longrightarrow$ *Theorem 4.4.1:*

Without loss of generality we can assume

$$\left|\frac{\varphi_1}{\varphi_2}\right| < 1$$

1) Let us compute the constant term of the quantity

$$A(\varphi_1, \varphi_2, \varphi_3) = \frac{d_1(\frac{\varphi_2}{\bar{a_2}}, L) d_1(\frac{\varphi_3}{\bar{a_3}}, L)}{a_1} + \frac{d_1(\frac{\varphi_1}{\bar{a_1}}, L) d_1(\frac{\varphi_3}{\bar{a_3}}, L)}{a_2} + \frac{d_1(\frac{\varphi_1}{\bar{a_1}}, L) d_1(\frac{\varphi_2}{\bar{a_2}}, L)}{a_3}$$

We use the following Lemma

Lemma 4.4.3 *We have*

$$d_1(z, L) = \frac{1}{z} - G_2(L) z - \frac{\pi}{a(L)} \bar{z} - G_4(L) z^3 + o(z^3).$$

The lemma comes from the proposition 3.2.1, theorem 3.3.1 ($d_1(z,L) = E_1(z,L)$) and the power series of $\zeta(z,L)$ at the origin,

$$\zeta(z, L) = \frac{1}{z} - \sum_{n\geqslant 2} G_{2n}(L) z^{2n+1}.$$

Using $\varphi_3 = -\varphi_1 - \varphi_2$ and $\left|\frac{\varphi_1}{\varphi_2}\right| < 1$, we can get the Laurent expansions

$$\begin{aligned} \tfrac{\varphi_2}{\varphi_3} &= -1 + \tfrac{\varphi_1}{\varphi_2} + ..., \\ \tfrac{\varphi_1}{\varphi_3} &= -\tfrac{\varphi_1}{\varphi_2} + ... \end{aligned}$$

Then we obtain, from these equalities and lemma 4.4.3, that the constant coefficient of

$$\frac{d_1(\frac{\varphi_2}{\bar{a_2}}, L) d_1(\frac{\varphi_3}{\bar{a_3}}, L)}{a_1}$$

is equal to

$$\left(\frac{\bar{a_2}}{a_1 \bar{a_3}} + \frac{\bar{a_3}}{a_1 \bar{a_2}} \right) G_2(L)$$

and the constant coefficient of

$$\frac{d_1(\frac{\varphi_1}{\bar{a_1}}, L) d_1(\frac{\varphi_3}{\bar{a_3}}, L)}{a_2}$$

is equal to

$$\frac{\bar{a_1}}{a_2 \bar{a_3}} G_2(L)$$

and the constant coefficient of

$$\frac{d_1(\frac{\varphi_1}{\bar{a_1}}, L) d_1(\frac{\varphi_2}{\bar{a_2}}, L)}{a_3}$$

is zero.
Hence, the constant term of

$$A(\varphi_1, \varphi_2, \varphi_3)$$

is equal to

$$\left(\frac{\bar{a_2}}{a_1 \bar{a_3}} + \frac{\bar{a_3}}{a_1 \bar{a_2}} + \frac{\bar{a_1}}{a_2 \bar{a_3}} \right) G_2(L)$$

2) Secondly, we compute the constant term of the quantity

$$B(\varphi_1, \varphi_2, \varphi_3) = \frac{d_2(\frac{\varphi_1}{\bar{a_1}}, L) a_1}{a_2 a_3} + \frac{d_2(\frac{\varphi_2}{\bar{a_2}}, L) a_2}{a_1 a_3} + \frac{d_2(\frac{\varphi_3}{\bar{a_3}}, L) a_3}{a_1 a_2}$$

to do that we use the following lemma

Lemma 4.4.4 *We have*

$$d_2(z, L) = -G_2(L) - \frac{\pi}{a(L)} \frac{\bar{z}}{z} + \frac{\pi G_2(L)}{a(L)} \bar{z} z + o(z).$$

The lemma comes from the proposition 3.2.1, theorem 3.1.2 and the power series of $\wp(z, L)$ at the origin,

$$\wp(z, L) = \frac{1}{z^2} + \sum_{n \geqslant 1} (2n+1) G_{2n+2}(L) z^{2n}.$$

Then, the constant term of $B(\varphi_1, \varphi_2, \varphi_3)$ is equal to

$$-G_2(L) \left(\frac{a_1}{a_2 a_3} + \frac{a_2}{a_1 a_3} + \frac{a_3}{a_1 a_2} \right)$$

Finally, the constant term in the Laurent expansion of

$$-A(\varphi_1, \varphi_2, \varphi_3) - B(\varphi_1, \varphi_2, \varphi_3)$$

is equal to

$$G_2(L) \left(\frac{a_1}{a_2 a_3} + \frac{a_2}{a_1 a_3} + \frac{a_3}{a_1 a_2} - \frac{\bar{a_2}}{a_1 \bar{a_3}} - \frac{\bar{a_3}}{a_1 \bar{a_2}} - \frac{\bar{a_1}}{a_2 \bar{a_3}} \right)$$

that is the constant term of Laurent expansion of the right member of the equality in Theorem 4.4.2.

3) Now we compute the constant term of Laurent expansion of the left member of the equality in Theorem 4.4.2:
In the following we compute successively the constant terms of laurent expansions of the expressions

$$\frac{1}{a_1} \sum_{\bar{t} \in L/cL} D_L\left(\frac{\varphi_2}{\bar{a_2}}, \frac{a_2 t}{a_1}\right) D_L\left(\frac{\varphi_3}{\bar{a_3}}, \frac{a_3 t}{a_1}\right), \frac{1}{a_2} \sum_{\bar{t} \in L/cL} D_L\left(\frac{\varphi_1}{\bar{a_1}}, \frac{a_1 t}{a_2}\right) D_L\left(\frac{\varphi_3}{\bar{a_3}}, \frac{a_3 t}{a_2}\right)$$

and

$$\frac{1}{a_3} \sum_{\bar{t} \in L/cL} D_L\left(\frac{\varphi_1}{\bar{a_1}}, \frac{a_1 t}{a_3}\right) D_L\left(\frac{\varphi_2}{\bar{a_2}}, \frac{a_2 t}{a_3}\right)$$

To do that we use the Laurent expansion of The Jacobi form

$$D_\tau(z; \varphi) = \sum_{k \geqslant 0} d_k(\varphi, L) z^{k-1}, d_0(\varphi, L) = 1,$$

and the fact that $\varphi_3 = -\varphi_1 - \varphi_2$ and $\left|\frac{\varphi_1}{\varphi_2}\right| < 1$. Hence, we obtain

$$D_L\left(\frac{\varphi_2}{\bar{a_2}}, \frac{a_2 t}{a_1}\right) = \frac{\bar{a_2}}{\varphi_2} + d_1\left(\frac{a_2}{a_1} t, L\right) + \frac{\varphi_2}{\bar{a_2}} d_2\left(\frac{a_2}{a_1} t, L\right) + \dots$$

$$D_L\left(\frac{\varphi_3}{\bar{a_3}}, \frac{a_3 t}{a_1}\right) = \frac{\bar{a_3}}{\varphi_3} + d_1\left(\frac{a_3}{a_1} t, L\right) + \frac{\varphi_3}{\bar{a_3}} d_2\left(\frac{a_3}{a_1} t, L\right) + \dots$$

and

$$D_L\left(\frac{\varphi_2}{\bar{a_2}},\frac{a_2t}{a_1}\right)D_L\left(\frac{\varphi_3}{\bar{a_3}},\frac{a_3t}{a_1}\right)=d_1\left(\frac{a_3t}{a_1},L\right)d_1\left(\frac{a_2t}{a_1},L\right)+\frac{\bar{a_2}\bar{a_3}}{\varphi_2\varphi_3}+\frac{\bar{a_2}}{\varphi_2}d_1\left(\frac{a_3t}{a_1},L\right)+$$

$$\frac{\bar{a_3}}{\varphi_3}d_1\left(\frac{a_2t}{a_1},L\right)+\frac{\bar{a_2}}{\varphi_2}\frac{\varphi_3}{\bar{a_3}}d_2\left(\frac{a_3t}{a_1},L\right)+\frac{\varphi_2}{\bar{a_2}}\frac{\bar{a_3}}{\varphi_3}d_2\left(\frac{a_2t}{a_1},L\right)+\text{ powers series in terms of }\varphi_1\text{ and }\varphi_2.$$

Then, for fixed $t\in\in L/cL$, the constant term of Laurent expansion of

$$\frac{1}{a_1}D_L\left(\frac{\varphi_2}{\bar{a_2}},\frac{a_2t}{a_1}\right)D_L\left(\frac{\varphi_3}{\bar{a_3}},\frac{a_3t}{a_1}\right)$$

is equal to

$$d_1\left(\frac{a_3t}{a_1},L\right)d_1\left(\frac{a_2t}{a_1},L\right)-\frac{\bar{a_2}}{a_1\bar{a_3}}d_2\left(\frac{a_3t}{a_1},L\right)-\frac{\bar{a_3}}{a_1\bar{a_2}}d_2\left(\frac{a_2t}{a_1},L\right).$$

In a similar way, we obtain

$$D_L\left(\frac{\varphi_1}{\bar{a_1}},\frac{a_1t}{a_2}\right)=\frac{\bar{a_1}}{\varphi_1}+d_1\left(\frac{a_1}{a_2}t,L\right)+\frac{\varphi_1}{\bar{a_1}}d_2\left(\frac{a_1}{a_2}t,L\right)+\dots$$

$$D_L\left(\frac{\varphi_3}{\bar{a_3}},\frac{a_3t}{a_2}\right)=\frac{\bar{a_3}}{\varphi_3}+d_1\left(\frac{a_3}{a_2}t,L\right)+\frac{\varphi_3}{\bar{a_3}}d_2\left(\frac{a_3}{a_2}t,L\right)+\dots$$

and

$$D_L\left(\frac{\varphi_1}{\bar{a_1}},\frac{a_1t}{a_2}\right)D_L\left(\frac{\varphi_3}{\bar{a_3}},\frac{a_3t}{a_2}\right)=d_1\left(\frac{a_3t}{a_2},L\right)d_1\left(\frac{a_1t}{a_2},L\right)+\frac{\bar{a_1}\bar{a_3}}{\varphi_1\varphi_3}+\frac{\bar{a_1}}{\varphi_1}d_1\left(\frac{a_3t}{a_2},L\right)+$$

$$\frac{\bar{a_3}}{\varphi_3}d_1\left(\frac{a_1t}{a_2},L\right)+\frac{\bar{a_1}}{\varphi_1}\frac{\varphi_3}{\bar{a_3}}d_2\left(\frac{a_3t}{a_2},L\right)+\frac{\varphi_1}{\bar{a_1}}\frac{\bar{a_3}}{\varphi_3}d_2\left(\frac{a_1t}{a_2},L\right)+\text{ power series in terms of }\varphi_1\text{ and }\varphi_2.$$

Then, the constant term in the Laurent expansion of

$$D_L\left(\frac{\varphi_1}{\bar{a_1}},\frac{a_1t}{a_2}\right)D_L\left(\frac{\varphi_3}{\bar{a_3}},\frac{a_3t}{a_2}\right)$$

is equal to

$$d_1\left(\frac{a_3t}{a_2},L\right)d_1\left(\frac{a_1t}{a_2},L\right)-\frac{\bar{a_1}}{a_2\bar{a_3}}d_2\left(\frac{a_3t}{a_2},L\right)$$

Finally,

$$D_L\left(\frac{\varphi_1}{\bar{a_1}},\frac{a_1t}{a_2}\right)=\frac{\bar{a_1}}{\varphi_1}+d_1\left(\frac{a_1}{a_2}t,L\right)+\frac{\varphi_1}{\bar{a_1}}d_2\left(\frac{a_1}{a_2}t,L\right)+\dots$$

$$D_L\left(\frac{\varphi_3}{\bar{a_3}},\frac{a_3t}{a_2}\right)=\frac{\bar{a_3}}{\varphi_3}+d_1\left(\frac{a_3}{a_2}t,L\right)+\frac{\varphi_3}{\bar{a_3}}d_2\left(\frac{a_3}{a_2}t,L\right)+\dots$$

and

$$D_L\left(\frac{\varphi_1}{\bar{a_1}},\frac{a_1t}{a_3}\right)D_L\left(\frac{\varphi_2}{\bar{a_2}},\frac{a_2t}{a_3}\right)=d_1\left(\frac{a_1t}{a_3},L\right)d_1\left(\frac{a_2t}{a_3},L\right)+\frac{\bar{a_1}\bar{a_2}}{\varphi_1\varphi_2}+$$

$$\frac{\bar{a_1}}{\varphi_1} d_1\left(\frac{a_2 t}{a_3}, L\right) + \frac{\bar{a_2}}{\varphi_2} d_1\left(\frac{a_1 t}{a_3}, L\right) + \text{ powers series in terms of } \varphi_1 \text{ and } \varphi_2.$$

Then, the constant term in the Laurent expansion of

$$D_L\left(\frac{\varphi_1}{\bar{a_1}}, \frac{a_1 t}{a_3}\right) D_L\left(\frac{\varphi_2}{\bar{a_2}}, \frac{a_2 t}{a_3}\right)$$

is equal to

$$d_1\left(\frac{a_1 t}{a_3}, L\right) d_1\left(\frac{a_2 t}{a_3}, L\right)$$

Now from our calculation **1), 2), 3)** and Theorem 3.3.1, we obtain

$$\sum_{i=1}^{3} \frac{1}{a_i} \sum_{\bar{t} \in L/a_i L \setminus \{0\}} \prod_{1 \leqslant j \neq k \leqslant 3} E_1\left(\frac{a_j t}{a_i}; L\right) = G_2(L)\left(\frac{a_1}{a_2 a_3} + \frac{a_2}{a_1 a_3} + \frac{a_3}{a_1 a_2} - \frac{\bar{a_1}}{a_2 \bar{a_3}} - \frac{\bar{a_2}}{a_1 \bar{a_3}} - \frac{\bar{a_3}}{a_1 \bar{a_2}}\right) +$$

$$\frac{\bar{a_2}}{a_1 \bar{a_3}} \sum_{\bar{t} \in L/a_1 L \setminus \{0\}} d_2\left(\frac{a_3 t}{a_1}, L\right) + \frac{\bar{a_3}}{a_1 \bar{a_2}} \sum_{\bar{t} \in L/a_1 L \setminus \{0\}} d_2\left(\frac{a_2 t}{a_1}, L\right) + \frac{\bar{a_1}}{a_2 \bar{a_3}} \sum_{\bar{t} \in L/a_2 L \setminus \{0\}} d_2\left(\frac{a_3 t}{a_2}, L\right)$$

Now using the distribution formula

$$\sum_{\bar{t} \in L/cL \setminus \{0\}} d_2\left(\frac{t}{c}, L\right) = G_2(0)\left(1 - \frac{c}{\bar{c}}\right), \forall c \in O_L \setminus \{0\}.$$

Then, we obtain

$$\sum_{i=1}^{3} \frac{1}{a_i} \sum_{\bar{t} \in L/a_i L \setminus \{0\}} \prod_{1 \leqslant j \neq k \leqslant 3} E_1\left(\frac{a_j t}{a_i}; L\right) = G_2(L)\mathbf{I}\left(\frac{a_1}{a_2 a_3} + \frac{a_2}{a_1 a_3} + \frac{a_3}{a_1 a_2}\right)$$

where $\mathbf{I}(z) = z - \bar{z}$.
This result is an Homogenization of the Theorem 4.4.1. In particular, for $a_1 = a, a_2 = c, a_3 = 1$, we get our implication : **Theorem 4.4.2 $\Longrightarrow$ Theorem 4.4.1.** $\square$

4.5 Proof of Theorems 4.1.1 and 4.1.3

We consider the function

$$F(z, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M}) = \prod_{j=1}^{j=n} D_{L_j}^{(m_j)}\left(a'_j z - z_j; \frac{\varphi_j}{a_j}\right)$$

Where

$$\overrightarrow{A} = \left((a_1 \dots, a_n); (a'_1, \dots, a'_n)\right), \overrightarrow{M} = (m_1, \dots, m_n),$$

$$\overrightarrow{\Phi} = (\varphi_1, \dots, \varphi_n), z_k = -x'_k + x_k \frac{a'_k}{a_k} \tau.$$

We assume that:

i) $\sum_{i=1}^{n} \varphi_i \in L = \mathbb{Z}\tau + \mathbb{Z}$,

ii) $\left(a'_i x'_j - a'_j x'_i, a_i x_j - a_j x_i\right) \notin \left\langle a'_i, a'_j \right\rangle \mathbb{Z} \times \langle a_i, a_j \rangle \mathbb{Z}$

The function

$$F : z \to F(z, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M})$$

has the following properties

i) F is meromorphic, has poles on

$$z = \frac{z_i}{a'_i} - \frac{t'_i}{a'_i}\tau + \frac{t_i}{a_i}, (t_i, t'_i) \in \mathbb{Z}^2$$

this pole has order equal to $m_i + 1$.

ii) F is periodic with periods containing the lattice L i.e

$$F(z + \rho, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M}) = F(z, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M}), \forall \rho \in L$$

this comes from

$$\begin{aligned} D_{L_i}(z + \rho, \varphi) &= e\left(E_{L_i}(\rho, \varphi)\right) D_{L_i}(z, \varphi) \\ &= e\left(\tfrac{a'_i}{a_i} E_L(\rho, \varphi)\right) D_{L_i}(z, \varphi) \end{aligned}$$

Then, $\forall \rho \in L$

$$\begin{aligned} F(z + \rho, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M}) &= e\left(E_L(\rho, \sum_{i=1}^{n} \varphi_i)\right) F(z, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M}), \\ &= F(z, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M}), \\ &\quad \text{because } \sum_{i=1}^{n} \varphi_i \in L \end{aligned}$$

Then, using Liouville residue theorem for an elliptic function with periods containing lattice L, we obtain

$$\sum_{i=1}^{n} \sum_{\substack{(t_i, t'_i) \in \mathbb{Z}^2 \\ 0 \leqslant t_i < a_i, 0 \leqslant, t'_i < a'_i}} \operatorname{Res}\left(F(z, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M}) dz; z = \frac{z_i}{a'_i} - \frac{t'_i}{a'_i}\tau + \frac{t_i}{a_i}\right) = 0$$

It remains to compute their residues. The function

$$F : z \to F(z, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M})$$

has poles at $z = \frac{z_i}{a'_i} - \frac{t'_i}{a'_i}\tau + \frac{t_i}{a_i}, (t_i, t'_i) \in \mathbb{Z}^2$ with order $m_i + 1$.
We will compute the residue at $\frac{z_i}{a'_i} - \frac{t'_i}{a'_i}\tau + \frac{t_i}{a_i}, (t_i, t'_i) \in \mathbb{Z}^2$, the other residues being completely equivalent. We use the Laurent expansion of the Jacobi form

$$D_{L_k}^{(m_k)}\left(a'_k z - z_k; \frac{\varphi_k}{a_k}\right) = \frac{(-1)^{m_k} m_k!}{{a'_k}^{m_k+1}\left(z - \frac{z_k+\tilde{t}_k}{a'_k}\right)^{m_k+1}} + \text{analytic part.}$$

where $\tilde{t}_k = -t'_k + t_k \frac{a'_k}{a_k}\tau, 0 \leqslant t_k < a_k, 0 \leqslant t'_k < a'_k$
and, more generally,

$$D_{L_j}^{(m_j)}\left(a'_j z - z_j; \frac{\varphi_j}{a_j}\right) = \sum_{r_j \geqslant 0} \frac{{a'_j}^{r_j}}{r_j!} D_{L_j}^{(m_j+r_j)}\left(a'_j \frac{z_k+\tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j}\right)\left(z - \frac{z_k+\tilde{t}_k}{a'_k}\right)^{r_j}$$

Hence

$$\operatorname{Res}\left(F(z, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M})dz; z = \frac{z_k+\tilde{t}_k}{a'_k}\right) =$$

$$e\left(E_{L_k}(\tilde{t}_k, \frac{\varphi_k}{a_k})\right) \frac{(-1)^{m_k} m_k!}{{a'_k}^{m_k+1}} \sum_{\substack{r_1,\ldots,\check{r}_k,\ldots,r_n \geqslant 0 \\ r_1+\ldots+\check{r}_k+\ldots+r_n = m_k}} \prod_{1 \leqslant j \neq k \leqslant n} \frac{{a'_j}^{r_j}}{r_j!} D_{L_j}^{(m_j+r_j)}\left(a'_j \frac{z_k+\tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j}\right)$$

Let us give some details on this fact. We begin with the change of variable $z \to z + \frac{\tilde{t}_k}{a_k}$, $\tilde{t}_k \in L_k$. Then, we obtain

$$\operatorname{Res}\left(F(z, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M})dz; z = \tfrac{z_k+\tilde{t}_k}{a'_k}\right) = \operatorname{Res}\left(F(z + \tfrac{\tilde{t}_k}{a'_k}, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M})dz; z = \tfrac{z_k}{a'_k}\right) =$$

$$\operatorname{Res}\left(D_{L_k}^{(m_k)}\left(a'_k z + \tilde{t}_k - z_k; \tfrac{\varphi_k}{a_k}\right) \prod_{1 \leqslant j \neq k \leqslant n} D_{L_j}^{(m_j)}\left(a'_j z + a'_j \frac{\tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j}\right) dz; z = \frac{z_k}{a'_k}\right) =$$

$$e\left(E_{L_k}(\tilde{t}_k, \tfrac{\varphi_k}{a_k})\right) \operatorname{Res}\left(D_{L_k}^{(m_k)}\left(a'_k z - z_k; \tfrac{\varphi_k}{a_k}\right) \prod_{1 \leqslant j \neq k \leqslant n} D_{L_j}^{(m_j)}\left(a'_j z + a'_j \frac{\tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j}\right) dz; z = \frac{z_k}{a'_k}\right)$$

Now, we know that

$$D_{L_k}^{(m_k)}\left(a'_k z - z_k; \frac{\varphi_k}{a_k}\right) = \frac{(-1)^{m_k} m_k!}{{a'_k}^{m_k+1}\left(z - \frac{z_k+\tilde{t}_k}{a'_k}\right)^{m_k+1}} + \text{analytic part.}$$

and

$$D_{L_j}^{(m_j)}\left(a'_j z - z_j; \frac{\varphi_j}{a_j}\right) = \sum_{r_j \geqslant 0} \frac{{a'_j}^{r_j}}{r_j!} D_{L_j}^{(m_j+r_j)}\left(a'_j \frac{z_k+\tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j}\right)\left(z - \frac{z_k+\tilde{t}_k}{a'_k}\right)^{r_j}$$

Hence,

$$\operatorname{Res}\left(F(z, \overrightarrow{\Phi}, \overrightarrow{A}, \overrightarrow{M})dz; z = \tfrac{z_k+\tilde{t}_k}{a'_k}\right) = \tfrac{(-1)^{m_k} m_k!}{{a'_k}^{m_k+1}} e\left(E_{L_k}(\tilde{t}_k, \tfrac{\varphi_k}{a_k})\right)$$
$$\times \sum_{\substack{r_1,\ldots,\check{r}_k,\ldots,r_n \geqslant 0 \\ r_1+\ldots+\check{r}_k+\ldots+r_n = m_k}} \prod_{1 \leqslant j \neq k \leqslant n} \frac{{a'_j}^{r_j}}{r_j!} D_{L_j}^{(m_j+r_j)}\left(a'_j \frac{z_k+\tilde{t}_k}{a'_k} - z_j; \frac{\varphi_j}{a_j}\right)$$

After using the functional equation satisfied by the Jacobi form

$$D_L(z,\varphi) = e\left(E_L(z,\varphi)\right) D_L(\varphi, z)$$

we obtain

$$\mathrm{Res}\left(F(z,\overrightarrow{\Phi},\overrightarrow{A},\overrightarrow{M})dz; z=\tfrac{z_k+\tilde{t}_k}{a'_k}\right) = \tfrac{(-1)^{m_k} m_k!}{{a'_k}^{m_k+1}} \times e\left(\sum_{1\leqslant j\neq k\leqslant n} E_{L_j}(a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j, \frac{\varphi_j}{a_j})\right) e\left(E_{L_k}(\tilde{t}_k, \tfrac{\varphi_k}{a_k})\right)$$

$$\times \sum_{\substack{r_1,\ldots,\check{r}_k,\ldots,r_n\geqslant 0\\ r_1+\ldots+\check{r}_k+\ldots+r_n=m_k}} \prod_{1\leqslant j\neq k\leqslant n} \frac{{a'_j}^{r_j}}{r_j!} D_{L_j}^{(m_j+r_j)}\left(\frac{\varphi_j}{a_j}; a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j\right)$$

Now, we use the following :

$$E_{L_j}(z,\varphi) = \frac{a_j}{a'_j} E_L(z,\varphi)$$

and the fact that:

$$\sum_{j=1}^{n} \varphi_j \in L$$

we conclude that

$$\mathrm{Res}\left(F(z,\overrightarrow{\Phi},\overrightarrow{A},\overrightarrow{M})dz; z=\tfrac{z_k+\tilde{t}_k}{a'_k}\right) = \tfrac{(-1)^{m_k} m_k!}{{a'_k}^{m_k+1}} \times e\left(-\sum_{1\leqslant j\leqslant n} E_{L_j}(z_j, \frac{\varphi_j}{a_j})\right)$$
$$\times \sum_{\substack{r_1,\ldots,\check{r}_k,\ldots,r_n\geqslant 0\\ r_1+\ldots+\check{r}_k+\ldots+r_n=m_k}} \prod_{1\leqslant j\neq k\leqslant n} \frac{{a'_j}^{r_j}}{r_j!} D_{L_j}^{(m_j+r_j)}\left(\frac{\varphi_j}{a_j}; a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j\right)$$

The other residues are computed in the same way.
Finally, we have

$$e\left(-\sum_{1\leqslant j\leqslant n} E_{L_j}(z_j, \frac{\varphi_j}{a_j})\right)$$
$$\sum_{k=1}^{n} \frac{(-1)^{m_k} m_k!}{{a'_k}^{m_k+1}} \sum_{\substack{0\leqslant t_k\leqslant a_k-1; 0\leqslant t'_k\leqslant a'_k-1\\ \tilde{t}_k = t_k\frac{a_k}{a'_k}\tau - t'_k}} \sum_{\substack{r_1,\ldots,\check{r}_k,\ldots,r_n\geqslant 0\\ r_1+\ldots+\check{r}_k+\ldots+r_n=m_k}} \prod_{1\leqslant j\neq k\leqslant n} \frac{{a'_j}^{r_j}}{r_j!} D_{L_j}^{(m_j+r_j)}\left(\frac{\varphi_j}{a_j}; a'_j\frac{z_k+\tilde{t}_k}{a'_k} - z_j\right)$$
$$= 0$$

this gives our statement

$$\sum_{k=1}^{n} (-1)^{m_k} m_k! \sum_{\substack{\overrightarrow{R_k}\geqslant \overrightarrow{0}\\ r_1+\ldots+\check{r}_k+\ldots+r_n=m_k}} \prod_{1\leqslant j\neq k\leqslant n} \frac{{a'_j}^{r_j}}{r_j!} d(\overrightarrow{A_k}, \overrightarrow{A'_k}, \overrightarrow{Z_k}, \overrightarrow{\phi_k}, \overrightarrow{M_k}+\overrightarrow{R_k}, \tau) = 0$$

□

5 Algebraicity and Damerell's type result for Jacobi forms $D_\tau(z;\varphi)$ and elliptic Bernoulli numbers $B_m(\varphi;\tau)$.

5.1 Algebraicity of Jacobi forms $D_\tau(z;\varphi)$

We fix a Weierstrass Model

$$\left(E, \frac{dx}{y}\right) : \begin{cases} y^2 = 4x^3 - g_2(\Omega)x - g_3(\Omega), \\ g_k(\Omega) = d_k \sum\limits_{\rho\in\Omega,\rho\neq 0} \rho^{-2k}, \quad k \in \{2,3\}, \\ d_2 = 60, d_3 = 140 \end{cases}$$

for an elliptic curve E, where Ω is the complex Lattice which is formed by the complex periods of $\left(E, \frac{dx}{y}\right)$.

We note $F = \mathbb{Q}\left(g_2(\Omega), g_3(\Omega)\right)$ the definition field of the Weierstrass Model $\left(E, \frac{dx}{y}\right)$. For any $\sigma \in \mathrm{Aut}(\mathbb{C}/F)$ we consider

$$\rho \longmapsto \rho^{[\sigma]}$$

application of: $\mathbb{C}/\Omega \to \mathbb{C}/\Omega$ via the action of σ over the coordinates $(\mathcal{P}_\Omega(\rho), \mathcal{P}'_\Omega(\rho))$ the image of the point $\rho \in \mathbb{C}/\Omega$ in Weierstrass Model $\left(E, \frac{dx}{y}\right)$.

Let $L \supset \Omega$ be a complex Lattice with finite index. We associate to L the Jacobi form $D_L(z,\varphi)$:

$$D_L(z;\varphi) = \frac{1}{\omega_2} e\left(\frac{z}{\omega_2}\varphi_1\right) \frac{\theta'_\tau(0)\theta_\tau\left(\frac{z+\varphi}{\omega_2}\right)}{\theta_\tau\left(\frac{z}{\omega_2}\right)\theta_\tau\left(\frac{\varphi}{\omega_2}\right)}$$

where $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \tau = \frac{\omega_1}{\omega_2} \in \mathcal{H}$, $\varphi = \varphi_1\omega_1 + \varphi_2\omega_2, (\varphi_1, \varphi_2) \in \mathbb{R}^2$.

We have the following properties of $D_L(z;\varphi)$:

Proposition 5.1.1 *Let p be an integer > 1.*

i) The fonction $z \longmapsto D_\Omega(z;\varphi)$ is defined over the field $F(E[p])$, extension of the field F obtained by adjonction of the coordinates of the p-torsion points of E.

ii) For all $\sigma \in Aut(\mathbb{C}/F)$, we have

$$D_L(z;\varphi)^\sigma = D_{L^\sigma}\left(z^{[\sigma]};\varphi^{[\sigma]}\right) .$$

*iii) In particular, the function $z \mapsto D_L(z;\varphi)$ is defined over $F(\varphi \pmod L), L/\Omega)$. Where $F(\varphi \pmod L), L/\Omega)$ is the smallest subextension of $F(E[p])/F$ such that :
the point φ modulo L and the subgroup L/Ω are defined.*

5.2 Algebraicity of elliptic Bernoulli numbers $B_m(\varphi;\tau)$.

Let η be the eta Dedekind function defined by:
For $\tau \in \mathcal{H}$

$$\eta(\tau) = q_\tau^{\frac{1}{24}} \prod_{n\geqslant 1} (1 - q_\tau^n), q_\tau = e^{2\pi i\tau}$$

The main result of this subsection is the following

Theorem 5.2.1 *Let τ belonging to an imaginary quadratic field with $Im(\tau) > 0$. For positive integers f, k with $k \geqslant 3$ and φ complex parameter of a primitive f-division point i.e φ complex parameter of f-division of order f in $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$. Then*

$$\frac{f^k B_k(\varphi;\tau)}{\eta(\tau)^{2k}}$$

is an algebraic integer.

For this result we give **two differents proofs.** In these proofs we use the q-expansion principle and the complex multiplication theory.
For N a positive integer we define

$$\Gamma(N) := \left\{ M \in \mathrm{SL}_2(\mathbb{Z}) : M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we define the function

$$\tilde{M} : \mathcal{H} \to \mathcal{H}, \tilde{M}(\tau) = \frac{a\tau + b}{c\tau + d}$$

Definition 5.2.2 *By a modular function of level N, we mean a function f defined on $\mathcal{H}$ such that:*

i) f *is meromorphic on* $\mathcal{H}$

ii) $fo\tilde{M} = f, \forall M \in \Gamma(N)$

iii) $\forall M \in SL_2(\mathbb{Z}), \exists B > 0$ *such that for all* $\omega \in \mathcal{H}$ *with* $Im(\omega) > B$, *we have*

$$fo\tilde{M}(\omega) = \sum_{n\geqslant N_0} a_n q^{\frac{n}{N}}, N_0 \in \mathbb{Z}, a_n\mathbb{C}, \text{ and } q^{\frac{1}{N}} = e^{\frac{2\pi i\omega}{N}}.$$

This expansion is called the Fourier expansion or q-expansion of $fo\tilde{M}$ and the a_n are called their coefficients.

As usual we denote the absolute modular invariant by $j(\omega)$. We state the following propositions whose proofs are omitted here.

Proposition 5.2.3 *(q-expansion principle):*
a) Let f be a modular function of level one, with Fourier expansion given by

$$f(\omega) = \sum_{n \geqslant N_0} a_n q^{\frac{n}{N}}$$

- *i) If all the a_n are contained in a subfield K of $\mathbb{C}$, then f belongs to $K(j)$ i.e f is a rational function of j with coefficients in K.*
- *ii) If f is holomorphic on $\mathcal{H}$ and all the a_n are contained in some additive subgroup A of $\mathbb{C}$, then f belongs to $A[j]$ i.e f is a polynomial function of j with coefficients in A.*

b) Let f be a modular function of level N on $\mathcal{H}$. Suppose further that for each $M \in SL_2(\mathbb{Z})$, the Fourier Coefficients of $fo\tilde{M}$ are contained in some additive subgroup A of $\mathbb{C}$. Then f is integral over $A[j]$.

We know that every elliptic curve with complex multiplication is defined over an algebraic extension of $\mathbb{Q}$. Precisely, if the complex Lattice L' admits a complex multiplier ω, then it is of the form αL, where L is a Lattice contained in $K = \mathbb{Q}(\omega)$ be an imgainary quadratic field. Next, we recall the main result concerning the integrality of j.

Proposition 5.2.4 *Let L a complex Lattice with complex multiplier ω, take $\{1, \tau\}$ a basis for O_K, $K = \mathbb{Q}(\omega)$ imaginary quadratic field, with $Im(\tau) > 0$. Then*

$$j(O_K) = \frac{1728 g_2(O_K)^3}{\Delta(O_K)}$$

is algebraic over $\mathbb{Q}$, where

$$\Delta(O_K) = \Delta(1, \tau) = g_2(O_K)^3 - 27 g_3(O_K)^2 = \Big(2\pi i \eta(\tau)^2\Big)^{12},$$

Remark that $q_\tau = e^{2\pi i \tau}$ is a real number (because τ is a quadratic imaginary complex). Then $\eta(\tau)$ is a real number.
We have

$$\Delta(O_K) = \pm\tilde{\omega}^{12}, \tilde{\omega} = 2\pi|\eta(\tau)|^2.$$

Then $g_2(\tilde{\omega}O_K)$ and $g_3(\tilde{\omega}O_K)$ are algebraic over $\mathbb{Q}$.
Indeed, if L is a Lattice contained in K. As it is commensurable with O_K, $\tilde{\omega}L$ is commensurable with $\tilde{\omega}O_K$.Then for all $m \geqslant 2$, $g_m(\tilde{\omega}O_K)$ is algebraic over $\mathbb{Q}$, see [49] pp.38-39. Then , by the homogeneity of g_m, for all $m \geqslant 2$

$$\frac{g_m(O_K)}{\tilde{\omega}^{2m}} \text{ is algebraic over } \mathbb{Q}.$$

Theorem 5.2.5 *Let K a quadratic imaginary number field and L a complex lattice in K. Then, we have for $m \geqslant 2$*

$$\frac{g_m(O_K)}{\tilde{\omega}^{2m}} \text{ and } \frac{E_m(\alpha, O_K)}{\tilde{\omega}^m}$$

are algebraic over $\mathbb{Q}$, for all $\alpha \in \mathbb{Q}L \backslash L$.

First Proof of theorem 5.2.1:
We use the following distribution formulas:

$$\sum_{\bar{t}\in L/NL\backslash\{\bar{0}\}} \left(E_2\left(\frac{t}{N}, L\right) - E_2(0,L) \right) = 0,$$

More generally, for any $\omega \in \mathbb{C}\backslash\{0\}$ such that: $\omega L \subset L$, we have

$$\sum_{\bar{t}\in L/\omega L\backslash\{\bar{0}\}} \left(E_n\left(\frac{t}{\omega}, L\right) - E_n(0,L) \right) = \begin{cases} 0 & \text{If } n \text{ odd} \\ \omega(\omega^{n-1} - \bar{\omega})E_n(0,L) & \text{Otherwise} \end{cases}$$

Or equivalently,

$$\sum_{\bar{t}\in L/\omega L\backslash\{\bar{0}\}} E_n\left(\frac{t}{\omega}, L\right) = \begin{cases} 0 & \text{If } n \text{ is odd} \\ (\omega^n - 1)E_n(0,L) & \text{Otherwise} \end{cases}$$

Now, we take in particular $\omega = N$, Then for $1 \leqslant k \leqslant N^2$, we have by induction on N:

$$\sum_{\bar{t}\in L/NL\backslash\{\bar{0}\}} E_N\left(\frac{t}{N}, L\right)^k \in \mathbb{Q}\Big[E_n(0,L), n \geqslant 4\Big].$$

We obtain a linear system, $1 \leqslant k \leqslant N^2$, we deduce that:
$E_N\left(\frac{t}{N}, L\right)$ are integral over the ring $\mathbb{Q}[E_n(0,L), n \geqslant 4]$ for all $\bar{t} \in L/NL\backslash\{\bar{0}\}$.
We know, that for $L = \tilde{\omega}O_K$ and $N \geqslant 2$, $g_n(\tilde{\omega}O_K)$ is algebraic over $\mathbb{Q}$. Then $E_N\left(\frac{t}{N}, L\right)$ is algebraic over $\mathbb{Q}$. Hence,

$$\frac{E_N\left(\frac{t}{N}, O_K\right)}{\tilde{\omega}^N}$$

is algebraic over $\mathbb{Q}$.
From theorem 3.1.2 we get, that for $f \in \mathbb{C}\backslash\{0\}$,

$$d_k(\varphi; L) = -\frac{1}{f^k} \sum_{t\in L/fL} e(-E_L(t,\varphi))E_k\left(\frac{t}{f}, L\right), \quad \forall k \geqslant 2.$$

or equivalently

$$f^k B_k(\varphi, L) = -\frac{k!}{(2\pi i)^k} \sum_{t\in L/fL} e(-E_L(t,\varphi))E_k\left(\frac{t}{f}, L\right), \quad \forall k \geqslant 2.$$

Now, our *theorem 5.2.1* comes from theorem 5.2.5. □

Second Proof of theorem 5.2.1:
By the q-expansion principle, we claim that

$$\tau \to f^k \frac{\bar{B}_k(\varphi, L = \mathbb{Z}\tau + \mathbb{Z})}{\eta(\tau)^{2k}}$$

is holomorphic modular function for $\Gamma(12f)$.
We prove this fact, we know that

$$B_m\left((d\varphi_1-c\varphi_2)\frac{a\tau+b}{c\tau+d}+(-b\varphi_1+a\varphi_2);\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^m B_m(\varphi_1\tau+\varphi_2;\tau),$$

and

$$\eta^{2k}\left(\frac{a\tau+b}{c\tau+d}\right)=\zeta_{a,b,c,d}^{-k}(c\tau+d)^k\eta^{2k}(\tau),$$

$$\forall\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}),(\varphi_1,\varphi_2)\in\mathbb{R}^2$$

where $\zeta_{a,b,c,d}$ is a root of unity and

$$\zeta_{a,b,c,d}=1,\forall\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma(12).$$

Then by the q-expansion principle $\tau\to f^k\frac{\bar{B}_k(\varphi,L=\mathbb{Z}\tau+\mathbb{Z})}{\eta(\tau)^{2k}}$ is algebraic over $A[j]$ where $A=$ ring generated by Fourier coefficients of q-expansion of this function. Explicitely, we know that

$$f^kB_k(\varphi,L=\mathbb{Z}\tau+\mathbb{Z})=f^kB_k(\varphi_1)+(-1)^{k-1}k\sum_{0\leqslant\mu,\nu<f}e\left(E_L(t,\varphi)\right)\sum_{\substack{mm_2>0\\ m_2=\nu \pmod f}}m^{k-1}(sign(m))e\left(\frac{m\mu}{f}\right)q^{\frac{mm_2}{f}}$$

Then

$$A\subset\mathbb{Z}\left[e^{\frac{2\pi i}{f}}\right].$$

An other point of view, we can deduce this formula by using the following Hecke result

$$E_k\left(\frac{t}{f},L=\mathbb{Z}\tau+\mathbb{Z}\right)=\delta_{0,\{\frac{t_1}{f}\}}\sum_{n=t_2 \pmod f}{}^{'}\frac{1}{n^k}+\frac{1}{f^k}\frac{(-2\pi i)^k}{(k-1)!}\sum_{\substack{mn>0\\ m=t_1 \pmod f}}n^{k-1}e\left(\frac{nt_2}{f}\right)e\left(\frac{mn}{f}\tau\right)$$

Then, for $\varphi=\varphi_1\tau+\varphi_2,(\varphi_1,\varphi_2)\in\mathbb{R}^2$, we have

$$E_k\left(\varphi,L=\mathbb{Z}\tau+\mathbb{Z}\right)=\delta_{0,\{\varphi_1\}}\sum_{n\in\mathbb{Z}}{}^{'}\frac{1}{(\varphi_2+n)^k}+\frac{1}{f^k}\frac{(-2\pi i)^k}{(k-1)!}\sum_{\substack{mn>0\\ m=\varphi_1 \pmod f}}n^{k-1}e\left(\frac{n\varphi_2}{f}\right)q^{\frac{mn}{f}}$$

Note that

$$\sum_{n\in\mathbb{Z}}{}^{'}\frac{1}{(\varphi_2+n)^k}=\begin{cases}\left(1+(-1)^k\right)\zeta(k) & \text{If } \varphi_2=0\\ \zeta(\varphi_2,k)+(-1)^k\zeta(-\varphi_2,k) & \text{If } 0<\varphi_2<1\end{cases}$$

Using this result of Hecke and the following formula

$$f^kB_k(\varphi,L)=-\frac{k!}{(2\pi i)^k}\sum_{t\in L/fL}e(-E_L(t,\varphi))E_k\left(\frac{t}{f},L\right),\quad\forall k\geqslant 2$$

Again, we deduce

$$f^k B_k(\varphi, L) = f^k B_k(\varphi_1) + (-1)^{k-1} k \sum_{0 \leqslant \mu, \nu < f} e\left(E_L(t, \varphi)\right) \sum_{\substack{m m_2 > 0 \\ m_2 = \nu \pmod f}} m^{k-1} (sign(m)) e\left(\frac{m\mu}{f}\right) q^{\frac{m m_2}{f}}$$

Then, by complex multiplication theory (proposition 5.2.4) we deduce that

$$\frac{f^k B_k(\varphi; \tau)}{\eta(\tau)^{2k}}$$

is an algebraic integer. □

6 The Congruence of Clausen-von Staudt for elliptic Bernoulli functions

In the degnerate case " cups at ∞", we obtain the Bernoulli numbers. In the case of $L = \mathbb{Z}i + \mathbb{Z}$ (case of the lemniscatic curve $y^2 = 4x^3 - 4x, \omega = \frac{dx}{y}$), we obtain a generalization of the so-called Hurwitz numbers. Our elliptic Bernoulli numbers can be regarded as a generalization of the so-called Bernoulli-Hurwitz stutied by Katz [32].
The main purpose of this section is to settle the theorem of von staudt-Clausen for elliptic Bernoulli numbers $B_k(\varphi, \mathbb{Z}\tau + \mathbb{Z})$ in the case τ is imaginary quadratic and φ is the rational parameter of torsion point on the elliptic curve $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$.

We begin this section by an overview of the classical Von Staudt Clausen congruence and introducing the Weber functions. Indeed , we establish a recursion formula satisfied by Eisenstein series. These functions are particulary well suited to question of analogous von Staudt clausen congruence type. Another fondamental step in our study, to obtain the main result of this section, is principally based on the systematic use of the results of Hasse [24] and Herglotz [25]. In particular, we obtain arithmetic information for so-called singular values of Bernoulli ellitpic functions and Eisenstein series, that is to say when the lattice L admits complex multiplications.
Throught this section K denotes a quadratic imaginary field.

Let us make an overview of Von Staudt Clausen congruence for ordinary Bernoulli numbers. For now, let n be an even positive integer. An elementary property of Bernoulli numbers is the following discovered independently by T. Clausen [18] and K. G. C. von Staudt [46] in 1840. The von Staudt-Clausen theorem states that

- The structure of the denominator of B_n is given by

  (6.0.29) $$B_n + \sum_{p-1 \mid n} \frac{1}{p} \in \mathbb{Z}$$

  Equivalently

- (6.0.30) $$\text{denominator of } (B_n) = \prod_{p-1|n} p\,.$$

Sometimes the theorem is stated in this alternative form:

For an even integer $k \geqslant 2$ and any prime p the product pB_k is p-integral, that is, pB_k is a rational number t/s such that p does not divide s. Moreover:

(6.0.31) $$pB_k \equiv \begin{cases} -1 \pmod p & \text{if } (p-1) \text{ divides } k \\ 0 \pmod p & \text{if } (p-1) \text{ does not divide } k. \end{cases}$$

6.1 Weber's functions and Eisenstein series.

Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ complex lattice i.e $\tau = \frac{\omega_1}{\omega_2} \in \mathcal{H}$.
We define the Weber function $h_L(z)$ by

$$h_L(z) = \begin{cases} -2^7 3^5 \frac{g_2(L) g_3(L)}{\Delta(L)} \wp_L(z) & \text{if} \quad d_K < -4 \\ 2^8 3^4 \frac{g_2(L)^2}{\Delta(L)} \wp_L(z)^2 & \text{if} \quad d_K = -4 \\ -2^9 3^6 \frac{g_3(L)}{\Delta(L)} \wp_L(z)^3 & \text{if} \quad d_K = -3 \end{cases}$$

Let j, γ_2, γ_3 be functions as in Weber [48] by

(6.1.32) $$\begin{cases} j = 2^6 3^3 \frac{g_2(L)^3}{\Delta(L)}, \\ \Delta(L) = g_2(L)^3 - 27 g_3(L)^2 = (2\pi i)^{12} \eta(\tau)^{24}, \\ \gamma_2 = \sqrt[3]{j} = 2^2 3 \frac{g_2(L)}{\sqrt[3]{\Delta}}, \\ \gamma_3 = \sqrt{j - 1728} = 2^3 3^3 \frac{g_3(L)}{\sqrt{\Delta}}, \end{cases}$$

Let K be an imaginary quadratic number field. We let d_K denote the discriminant of K and fix a non-zero fractional O_K-ideal L. Let $\mathfrak{f}$ be an integral ideal in K and f be the smallest positive integer divisible by $\mathfrak{f}$. Let ω_1, ω_2 be a basis of the ideal L with $\operatorname{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0$. Thus $K = \mathbb{Q}(\tau), \tau = \frac{\omega_1}{\omega_2}$, $j(\tau)$ is algebraic integer.

We put
$$h_t := h_L\left(\frac{t}{f}\right), t = t_1\omega_1 + t_2\omega_2, 0 \leqslant t_1, t_2 < f.$$

Then it is known, by Hasse [24], that these f-division values of $h_L(z)$ are algebraic numbers whose denominators are at most divisible by prime factors of f.
Now, we state the following result about singular values of Eisenstein series

Theorem 6.1.1 *$\forall t \in L/fL\backslash\{0\}$, we have*

$$(k-1)!E_k\left(\frac{t}{f},L\right) = \sum_{\substack{2a+3b+4c=k,\\ a,b,c\geqslant 0}} A_{a,b,c}(t)\wp_L\left(\frac{t}{f}\right)^a \wp_L^{'}\left(\frac{t}{f}\right)^b g_2(L)^c$$

with $A_{a,b,c}(t) \in \mathbb{Q}(\xi_f)$, ξ_f is a primitive root of unity, in particular the numerator of $A_{a,b,c}(t)$ is an integer in $\mathbb{Q}(\xi_f)$ and the denominator of $A_{a,b,c}(t)$ is at most powers of 2.

Proof : We proced by induction on k.
We examine the cases: $k = 4, 5$:

$$(4-1)!E_4\left(\frac{t}{f},L\right) = 6\wp_L\left(\frac{t}{f}\right)^2 - \frac{1}{2}g_2(L)$$

$$(5-1)!E_5\left(\frac{t}{f},L\right) = -4!\wp_L\left(\frac{t}{f}\right)\wp_L^{'}\left(\frac{t}{f}\right)$$

Then the Theorem holds for $k = 4, 5$.
Now assume that the Theorem holds for $k \leqslant n+1$. Then by recursive formula for $E_k\left(\frac{t}{f},L\right)$, we have

$$(k+1)!E_{k+2}\left(\frac{t}{f},L\right) = 6\sum_{\substack{p+q=k-2\\ p\geqslant 1,q\geqslant 1}} \frac{(p+1)(q+1)(k+1)!}{(k+1)k(k-1)}E_{p+2}\left(\frac{t}{f},L\right)E_{q+2}\left(\frac{t}{f},L\right)+$$

$$12\frac{(k+1)!(k-1)}{(k+1)k(k-1)}\wp_L\left(\frac{t}{f}\right)E_k\left(\frac{t}{f},L\right)$$

Hence

$$(k+1)!E_{k+2}\left(\frac{t}{f},L\right) = 6\sum_{\substack{p+q=k-2\\ p\geqslant 1,q\geqslant 1}} C_{p+q}^{p}(p+1)!E_{p+2}\left(\frac{t}{f},L\right)(q+1)!E_{q+2}\left(\frac{t}{f},L\right)+$$

$$12(k-1)!\wp_L\left(\frac{t}{f}\right)E_k\left(\frac{t}{f},L\right).$$

Then by induction we get the Theorem. □

6.2 Statement and proof of elliptic Von-Staudt Clausen Congruence.

We state our elliptic analogue of the von-Staudt Clausen congruence. Let τ belong to an imaginary quadratic field K with $\mathrm{Im}(\tau) > 0$. In this subsection we set $L = \mathbb{Z}\tau + \mathbb{Z}$.

Theorem 6.2.1 *(***Even index***)*
Let $\mathfrak{f}$ be an integral ideal in $K \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$ and f be the smallest positive integer divisible by $\mathfrak{f}$. Then for φ a complex parameter of a primitive f-division point of $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ and for $k \geqslant 3$, we have

$$f^{2k} \frac{B_{2k}(\varphi;\tau)}{\eta(\tau)^{4k}} = C_{2k} + D_{2k}$$

where

$$C_{2k} = (-1)^{k+1} \frac{\gamma_2^u \gamma_3^v j^w}{6} + \sum_{\substack{p-1 | 2k \\ pprime \geqslant 5}} \frac{A_p(\omega_1,\omega_2)^{\frac{2k}{p-1}}}{p} + \gamma_2^u \gamma_3^v P_k(j)$$

$$D_{2k} = 2k \sum_{t \in L/fL \backslash \{0\}} e(E_L(t,\varphi)) \sum_{\substack{a+3b+2c=k \\ a,b,c \geqslant 0}} A_{a,2b,c}(t)(-1)^a 2^{2b+2c-2k} 3^{c-k} \gamma_2^{3c-k} \gamma_3^{2c-k} h_t^a \left(-h_t^3 + 3\gamma_2^3\gamma_3^2 h_t - 2\gamma_2^3\gamma_3^4\right)^b.$$

$$k = 6w + 2u + 3v, 0 \leqslant u \leqslant 2, 0 \leqslant v \leqslant 1, P_k(j) \in \mathbb{Z}[j]$$

and $A_p(\omega_1,\omega_2)$ denote the penultimate coefficient in the multiplicative equation

$$x^{p+1} - A_1(\omega_1,\omega_2)x^p + \ldots - A_p(\omega_1,\omega_2)x + (-1)^{\frac{p-1}{2}} = 0$$

satisfied by

$$x = p\frac{\eta^2(p\tau)}{\eta^2(\tau)} = \sqrt[12]{\frac{\Delta\left(\frac{\omega_1}{p},\omega_2\right)}{\Delta(\omega_1,\omega_2)}}$$

and $A_{a,2b,c}(t) \in \mathbb{Q}(\xi_f)$ and the denominator of $A_{a,2b,c}(t)$ is at most powers of 2. Finally,

$$A_p(\omega_1,\omega_2) = \gamma_2^{u_p} \gamma_3^{v_p} f_p(j),$$

with

$$\frac{p-1}{2} = 6t_p + 2u_p + 3v_p, 0 \leqslant u_p \leqslant 2, 0 \leqslant v_p \leqslant 1, f_p(j) \in \mathbb{Z}[j], d^\circ f_p = t_p.$$

Corollary 6.2.2 *In the* **even index** *case the denominator of the number*

$$f^{2k} \frac{B_{2k}(\varphi;\tau)}{\eta(\tau)^{4k}}$$

is divisible by at most prime factors of 2, 3, f, γ_2,γ_3 or prime $p \geqslant 5$ such that $p-1$ divides $2k$.

Theorem 6.2.3 *(***Odd index***)*
Let $\mathfrak{f}$ be an integral ideal in $K \neq \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$ and f be the smallest positive integer divisible by $\mathfrak{f}$. Then for φ a complex parameter of a primitive f-division point of $\mathbb{C}/L$ and for $k \geqslant 3$, we have

$$f^{2k+1} \frac{B_{2k+1}(\varphi;\tau)}{\eta(\tau)^{4k+2}} = C_{2k+1} + D_{2k+1}$$

where

$$C_{2k+1} = 0$$

and

$$D_{2k+1} = \frac{2k+1}{(2\pi i)^3\eta^6(\tau)} \times \sum_{t \in L/fL\setminus\{0\}} e(E_L(t,\varphi))\wp'_L\left(\frac{t}{f}\right) \times$$

$$\sum_{\substack{a+3b+2c=k-1\\ a,b,c\geqslant 0}} A_{a,2b+1,c}(t)(-1)^a 2^{2b+2c-2k+2}3^{c-k+1}\gamma_2^{3c-k+1}\gamma_3^{2c-k+1}h_t^a\Big(-h_t^3+3\gamma_2^3\gamma_3^2h_t-2\gamma_2^3\gamma_3^4\Big)^b.$$

Corollary 6.2.4 *In the* **odd index** *case the denominator of the number*

$$f^{2k+1}\frac{B_{2k+1}(\varphi;\tau)}{\eta(\tau)^{4k+2}}$$

is only divisible by at most prime factors of 2, 3, f, $\gamma_2(L)$ and $\gamma_3(L)$.

Theorem 6.2.5 $K = \mathbb{Q}(i)$
Let $\mathfrak{f}$ be an integral ideal in $K = \mathbb{Q}(i)$ and f be the smallest positive integer divisible by $\mathfrak{f}$. Then for φ a complex parameter of a primitive f-division point of $\mathbb{C}/O_K$ and for $k \geqslant 3$, we have

i) The number

$$f^{4k}\frac{B_{4k}(\varphi;\tau)}{\eta(\tau)^{8k}}$$

is divisible by at most prime factors of 2,3, f and prime $p \geqslant 5$ such that p divides $4k$.

ii) For $k \not\equiv 0 \pmod 4$, we have the number

$$f^{k}\frac{B_{k}(\varphi;\tau)}{\eta(\tau)^{2k}}$$

is divisible by at most prime factors of 2,3, f.

Theorem 6.2.6 $K = \mathbb{Q}(\sqrt{-3})$
Let $\mathfrak{f}$ be an integral ideal in $K = \mathbb{Q}(\sqrt{-3})$ and f be the smallest positive integer divisible by $\mathfrak{f}$. Then for φ a complex parameter of a primitive f-division point of $\mathbb{C}/O_K$ and for $k \geqslant 3$,

i) The number

$$f^{6k}\frac{B_{6k}(\varphi;\tau)}{\eta(\tau)^{12k}}$$

is divisible by at most prime factors of 2,3, f and prime $p \geqslant 5$ such that p divides $6k$.

ii) For $k \not\equiv 0 \pmod 6$, we have the number

$$f^{k}\frac{B_{k}(\varphi;\tau)}{\eta(\tau)^{2k}}$$

is divisible by at most prime factors of 2,3, f.

Remark 6.2.7 *The first part C_k of our elliptic Bernoulli number $f^k\frac{B_k(\varphi;\tau)}{\eta(\tau)^{2k}}$ is already studied by G. Herglotz [25] and H.Lang [36, 37]. Consequently our attention is to study in details the second object D_k.*

Proof of theorems 6.2.1, 6.2.3 and 6.2.5:

We get from Theorem 3.1.2

$$\begin{aligned} -f^{2k}\frac{B_{2k}(\varphi;\tau)}{\eta(\tau)^{4k}} &= \frac{(2k)!}{\eta^{4k}(\tau)}\sum_{t\in L/fL} e(E_L(t,\varphi))E_{2k}\left(\frac{t}{f},L\right) \\ &= \frac{(2k)!}{\eta^{4k}(\tau)}E_{2k}\left(0,L\right)+\frac{(2k)!}{\eta^{4k}(\tau)}\sum_{t\in L/fL\backslash\{0\}} e(E_L(t,\varphi))E_{2k}\left(\frac{t}{f},L\right)\end{aligned}$$

Then in G. Herglotz [25] and H.Lang [36, 37] it is showed, that

$$\frac{(2k)!}{\eta^{4k}(\tau)}E_{2k}\left(0,L\right)=(-1)^{k+1}\frac{\gamma_2^u\gamma_3^v j^w}{6}+\sum_{\substack{p-1|2k\\ p\text{prime}}}\frac{A_p(\omega_1,\omega_2)^{\frac{2k}{p-1}}}{p}+\gamma_2^u\gamma_3^v G_{2k}(j),$$

where $G_{2k}(j)\in\mathbb{Z}[j]$, p prime $\geqslant 5$, and

$$\begin{aligned} D_{2k} &= \frac{(2k)!}{\eta^{4k}(\tau)}\sum_{t\in L/fL\backslash\{0\}} e(E_L(t,\varphi))E_{2k}\left(\frac{t}{f},L\right) \\ &= \frac{2k}{(2\pi i)^{2k}\eta^{4k}(\tau)}\sum_{t\in L/fL\backslash\{0\}} e(E_L(t,\varphi))(2k-1)!E_{2k}\left(\frac{t}{f},L\right) \\ &= \frac{2k}{(2\pi i)^{2k}\eta^{4k}(\tau)}\sum_{t\in L/fL\backslash\{0\}} e(E_L(t,\varphi))\sum_{\substack{a+3b+2c=k\\ a,b,c\geqslant 0}} A_{a,2b,c}(t)\wp_L\left(\frac{t}{f}\right)^a\wp_L^{'}\left(\frac{t}{f}\right)^{2b}g_2(L)^c\end{aligned}$$

Now, using the Weierstrass $\wp$-function model,

$$\wp_L^{'}(z)^2=4\wp_L(z)^3-g_2(L)\wp_L(z)-g_3(L)$$

Here $d_K<-4$**,** we use the Weber's functions and notations to obtain

$$(6.2.33)\qquad \begin{cases}\Delta(L)^{\frac{1}{6}}=(2\pi i)^2\eta(\tau)^4,\\ g_2(L)=2^{-2}3^{-1}\Delta^{\frac{1}{3}}\gamma_2,\\ g_3(L)=2^{-3}3^{-3}\Delta^{\frac{1}{2}}\gamma_3,\\ \wp_L(z)=-2^{-2}3^{-1}\Delta^{\frac{1}{6}}\gamma_2^{-1}\gamma_3^{-1}h_L(z),\\ \wp_L^{'}(z)^2=2^{-4}3^{-3}\Delta^{\frac{1}{2}}\gamma_2^{-3}\gamma_3^{-3}\Big(-h_t^3+3\gamma_2^3\gamma_3^2h_t-2\gamma_2^3\gamma_3^4\Big)\end{cases}$$

Then

$$D_{2k}=\frac{2k}{(2\pi i)^{2k}\eta^{4k}(\tau)}\sum_{t\in L/fL\setminus\{0\}}e(E_L(t,\varphi))\sum_{\substack{a+3b+2c=k\\ a,b,c\geqslant 0}}A_{a,2b,c}(t)\wp_L\left(\frac{t}{f}\right)^a\Big(4\wp_L(z)^3-g_2(L)\wp_L(z)-g_3(L)\Big)^b g_2(L)^c$$

Hence

$$D_{2k}=2k\sum_{t\in L/fL\setminus\{0\}}e(E_L(t,\varphi))\sum_{\substack{a+3b+2c=k\\ a,b,c\geqslant 0}}A_{a,2b,c}(t)(-1)^a 2^{2b+2c-2k}3^{c-k}\gamma_2^{3c-k}\gamma_3^{2c-k}h_t^a\Big(-h_t^3+3\gamma_2^3\gamma_3^2h_t-2\gamma_2^3\gamma_3^4\Big)^b.$$

By theorem 6.1.1, we know that the denominator of $A_{a,2b,c}(t)$ is at most powers of 2 in the other hand the f-division values of Weber's function h_t are algebraic numbers whose denominators are at most divisible by prime factors of f cf. Hasse [24]. This completes the proof of theorem 6.2.1.

Using the relation (6.2.33) the *proof of theorem 6.2.3* is similar as the theorem 6.2.1. Now, to prove the *theorem 6.2.5 and theorem 6.2.6* you use same techniques as in the proof of the theorem 6.2.1, we must only remark that:

$$\text{(6.2.34)}\qquad \begin{cases} j=1728,\gamma_2=2^23,\gamma_3=0 & \text{If } K=\mathbb{Q}(i)\\ j=0,\gamma_2=0,\gamma_3=2^33\sqrt{-3} & \text{If } K=\mathbb{Q}(\sqrt{-3})\end{cases}$$

This complete the proof of our result of elliptic analogue to Von Staudt Clausen over an imaginary quadratic number field $K=\mathbb{Q}(\tau)$ for the elliptic Bernoulli numbers $B_{2k}(\varphi,\tau)$. □

Theorem 6.2.8 *(2-division points)*
Let L be an arbitrary complex lattice with $L=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2,\tau=\frac{\omega_1}{\omega_2}\in\mathcal{H}$. Then for φ a complex parameter of a non zero 2-division point of $\mathbb{C}/L$, we have

$$B_{2k+1}(\varphi,\tau)=0,\forall k\geqslant 0$$

and

$$f^{2k}\frac{B_{2k}(\varphi,\tau)}{\eta^{4k}(\tau)}=C_{2k}+D_{2k},\forall k\geqslant 2$$

where C_{2k} and D_{2k} are the same as in the Theorem 6.2.1

In this case the coefficients $B_k(\varphi,\tau)$ are explicitely given by example 2.2.2.

7 The Kummer Congruence for elliptic Bernoulli functions

It should be noted that our results in this paragraph are to be brought closer to those obtained by Villegas, in terms of special values of certain function L, in its paper [47].

In this section, it will be shown that the singular values of elliptic Bernoulli numbers satisfy a Kummer congruence.

7.1 An overview of the classical Kummer Congruence

We begin with an overview of classical Kummer congruences :
Let k,, and r with $k > r \geqslant 1$ and p be prime with $p-1 \not| k$. The classical Kummer congruence [35] states that the Bernoulli numbers satisfy

$$\sum_{s=0}^{r}(-1)^s C_r^s \frac{B_{k+s(p-1)}}{k+s(p-1)} \equiv 0 \pmod{p^r}.$$

In the study of p-adic L-series Iwasawa introduced generalized Bernoulli numbers B_{χ_f}, associated to characters χ_f of conductor f. Carlitz [16] proved a Kummer congruence for these generalized Bernoulli numbers. He showed that with k, p, and r as before and with $p \not| f$, then

$$\sum_{s=0}^{r}(-1)^s C_r^s \frac{B_{k+s(p-1)}}{k+s(p-1)} \equiv 0 \pmod{p^r}.$$

In our main result of this section, we are insterested by a related Kummer congruence which was established by Vandiver [45] who demonstrated that if k, and r with $k > r \geqslant 1$ and p be prime with $p-1 \not| k$, and a and f positive integers, then

$$\sum_{s=0}^{r}(-1)^s C_r^s f^{k+s(p-1)} \frac{B_{k+s(p-1)}\left(\frac{a}{f}\right)}{k+s(p-1)} \equiv 0 \pmod{p^r}.$$

7.2 Statement and proof of the elliptic Kummer congruence.

We state now our Kummer congruence type satisfied by the elliptic Bernoulli numbers $B_k(\varphi, L)$.

Theorem 7.2.1 *let $p \geqslant 5$ be prime and let k, r be integers with $k > r \geqslant 1$, $f > 1$ and $p-1 \not| k$, $\varphi \in \mathbb{C}\backslash\{0\}$ of order f i.e $f = [L + \mathbb{Z}\varphi : L]$, then if τ is an imaginary quadratic complex number with $Im\tau > 0$, We have*

$$\sum_{s=0}^{r}(-1)^{s(1+\frac{p-1}{2})} C_r^s f^{k+s(p-1)} \frac{B_k(\varphi, L)}{k+s(p-1)} \equiv 0 \pmod{p^r}.$$

Siegel in [43] has shown that the values at positive integers of $L(s, \chi)$, where K is an imaginary quadratic field, $\mathfrak{f}$ is an integral ideal of K, $\chi = \epsilon\psi$ ϵ is a primitive ray-class character mod$\mathfrak{f}$ and ψ is a Grössencharacter, is expressed in terms of the elliptic Bernoulli numbers. Consequently, our theorem 7.2.1 also give a Kummer congruence for the values of this Hecke L-series.

Proof of theorem 7.2.1:
In the proof of this theorem we need the q-expansion of $B_k(\varphi, L)$ and the standard Kummer congruence. We have the following q-expansion formula

$$f^k B_k(\varphi, L) = f^k B_k(\varphi_1) + (-1)^{k-1} k \sum_{0 \leqslant \mu, \nu < f} e\left(E_L(t, \varphi)\right) \sum_{\substack{mm_2 > 0 \\ m_2 = \nu \pmod{f}}} m^{k-1}(sign(m)) e\left(\frac{m\mu}{f}\right) q^{\frac{mm_2}{f}}$$

Hence

$$B_k(\varphi, L) \in \mathbb{Z}\left[\frac{1}{f}, \xi_f\right][[q]]$$

Now using classical Kummer congruence and Vandiver result. Then, we write the sum

$$\sum_{s=0}^{r}(-1)^{s(1+\frac{p-1}{2})}C_r^s f^{k+s(p-1)}\frac{B_k(\varphi, L)}{k+s(p-1)} = -\sum_{s=0}^{r}(-1)^s C_r^s f^{k+s(p-1)}\frac{B_{k+s(p-1)}(\varphi)}{k+s(p-1)}+$$

$$\sum_{0\leqslant\mu,\nu<f} e\left(E_L(t,\varphi)\right) \sum_{\substack{mm_2>0\\ m_2=\nu \pmod f}} m^{k-1}(sign(m))e\left(\frac{m\mu}{f}\right) q^{\frac{mm_2}{f}} \times \sum_{s=0}^{r}(-1)^{k++s(p-1)}C_r^s m^{k+s(p-1)}$$

We remark , after Newton's binomial, that

$$\sum_{s=0}^{r}(-1)^{k++s(p-1)}C_r^s m^{k+s(p-1)} = m^{k-1}\left((-m)^{p-1}-1\right)^r \equiv 0 \pmod{p^r}.$$

Let $R(p,q)$ denote the ring of q-expansions of the form

$$\sum a_m q^{\frac{m}{12f}}$$

with algebraic and p-integral coefficients, with only finite number of nonzero terms with $m<0$. Hence one can use the last equalities to write

$$\sum_{s=0}^{r}(-1)^{s(1+\frac{p-1}{2})}C_r^s f^{k+s(p-1)}\frac{B_k(\varphi, L)}{k+s(p-1)} \equiv 0 \pmod p R(p,q). \tag{7.2.35}$$

Hence, we can conclude with the result of Theorem 5.2.1 and the congruence 7.2.35, that the Kummer congruence for elliptic Bernoulli numbers $B_k(\varphi,\tau)$ is valid.

8 Hecke L-functions associated to Elliptic Bernoulli functions

The main purpose of this section is to study special values of Hecke L-Functions and their connection with elliptic Bernoulli numbers.

8.1 Statement of the main result

Let K be a number Field, $\zeta_K(s)$ the Dedekind zeta Function of K. We can break up ζ_K into a finite sum

$$\zeta_K(s) = \sum_A \zeta_K(s,A),$$

where A runs over the ideal classgroup of K and

$$\zeta_K(s,A) = \sum_{\mathfrak{a}\in A}\frac{1}{N(\mathfrak{a})^s}, \quad \mathrm{Re}(s)>1.$$

Then $\zeta_K(s, A)$ is, after analytic continuation, a meromorphic function of s with a simple pole at $s = 1$ as its only singularity. Moreover, the residue of $\zeta_K(s, A)$ at $s = 1$ is independent of the ideal class A chosen; this fact, discovered by Dirichlet(for quadratic fields) is at the basis of the analytic determination of class number of K. If we consider the Laurent expansion of $\zeta_K(s, A)$ at $s = 1$, however, say

$$\zeta_K(s, A) = \frac{c_K}{s-1} + \rho_0(A) + \rho_1(A)(s-1) + \dots$$

More generally, to a character χ of the ideal class group of K, we associate the L-series

$$L(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}, \text{ for } \operatorname{Re}(s) > 1$$

The sum runs over all ideals $\mathfrak{a}$ of K.
Now, χ being an ideal class character, we may write

$$\begin{aligned} L(s, \chi) &= \sum_A \chi(A) \sum_{\mathfrak{a} \in A} N(\mathfrak{a})^{-s} \\ &= \sum_A \chi(A) \zeta_K(s, A) \\ &= \\ &= \sum_A \chi(A) \left(\frac{c_K}{s-1} + \rho_0(A) + \rho_1(A)(s-1) + \dots \right) \end{aligned}$$

Then

$$L(s, \chi) = \sum_A \chi(A) \rho_0(A) + o(s-1)$$

since, χ being $\neq \chi_0 = 1, \sum_A \chi(A) = 0$.
Taking the limit as $s \to 1$, we get

$$L(s = 1, \chi) = \sum_A \chi(A) \rho_0(A)$$

Then it transpires that the constant term $\rho_0(A)$ is no longer independent of the choice of A. When K is an imaginary quadratic field, the evaluation of $\rho_0(A)$ accomplished by Kronecker[34] (the so-called "First Kronecker limit Formula"), when K is real quadratic field $\rho_0(A)$ evaluated by D. Zagier [52].
Our interest here is to connect $L(s = m, \chi), m \geqslant 1$, to elliptic Bernoulli numbers is as follows. One may express the special values of Hecke L-functions of K using the coefficients of Laurent expression of Jacobi forms $D_L(z, \varphi)$.
Let K be an imaginary quadratic field with ring of integers O_K. We consider O_K and any ideal $\mathcal{F}$ of O_K to be lattices in $\mathbb{C}$ through a fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Le χ be an algebraic Hecke character of ideals in K, with conductor $\mathfrak{f}$ and type (m, n) i.e

$$\chi : I_K(\mathfrak{f}) \to \bar{\mathbb{Q}}^\times$$

homomorphism of the form

$$\chi((\beta)) = \epsilon(\beta)\beta^m\bar{\beta}^n$$

for some finite character

$$\epsilon : (O_K/\mathfrak{f})^\times \to \bar{\mathbb{Q}}^\times,$$

we set $\tilde{\chi}(\beta) = \beta^m\bar{\beta}^n$ where $\beta \in O_K$ is prime to $\mathfrak{f}$

$$\epsilon((\beta)) = \epsilon(\beta), (\beta, \mathfrak{f}) = 1$$

we extend ϵ so that $\epsilon(\beta) = 0$ for $(\beta, \mathfrak{f}) \neq 1$ Then the Hecke L-function associated to χ is defined by

$$L_{\mathfrak{f}}(s, \chi) = \sum_{(\mathfrak{a},\mathfrak{f})=1} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}$$

where the sum is over integral ideals of O_K prime to $\mathfrak{f}$, and this series is absolutely convergent if $\mathrm{Re}(s) > \frac{m+n}{2} + 1$. The Hecke L-function is known to have a meromorphic continuation to the whole complex s-plane and a functional equation.
The function $L_{\mathfrak{f}}(s, \chi)$ has a pole at $s = s_0$ if and only if the conductor of χ is trivial, $m = n$ and $s_0 = \frac{m+n}{2} + 1$.
Now, we can state the main result of this section. The Hecke L-function may be expressed in terms of Elliptic Bernoulli numbers as follows.

Theorem 8.1.1 *(***Interpolation***)*
Let $\mathfrak{f}$ be an ideal of K. Let $I_K(\mathfrak{f})$ be the subgroup of the ideal groups of K consisting of ideal prime to $\mathfrak{f}$, and $P_K(\mathfrak{f})$ be the subgroup of $I(\mathfrak{f})$ of principal ideals (α) such that $\alpha \equiv 1 \pmod f$. Let χ be an algebraic Hecke Character of K of conductor $\mathfrak{f}$ and type $(-m, m)$. Then we have

$$L_{\mathfrak{f}}(s = m, \chi) = \frac{(2\pi i)^{2s}}{(2m)!\omega_{\mathfrak{f}} G_\chi} \sum_A \bar{\chi}(\mathfrak{b}_A) \left(\frac{y_{\mathfrak{b}_A} N(\omega_1(\mathfrak{b}_A))}{\sqrt{|D|}} \right)^s B_{2s}(\varphi(A); \mathfrak{b}_A)$$

Or equivalently

$$L_{\mathfrak{f}}(s = m, \chi) = \frac{1}{\omega_{\mathfrak{f}} G_\chi} \sum_A \bar{\chi}(\mathfrak{b}_A) \left(\frac{y_{\mathfrak{b}_A} N(\omega_1(\mathfrak{b}_A))}{\sqrt{|D|}} \right)^s d_{2s}(\varphi(A); \mathfrak{b}_A)$$

we have $\omega_{\mathfrak{f}}$ is the number of roots of 1 congruent to $1 \pmod{\mathfrak{f}}$, A runs over all ray-classes mod$\mathfrak{f}$, is the imaginary part of $\frac{\omega_2(\mathfrak{b}_A)}{\omega_1(\mathfrak{b}_A)}$, D is the discriminant of K. G_χ is the Gaussian sum associated to χ, defined as follows.
Let $\mathfrak{D}_K$ be the different of K. Let $\mathfrak{a}$ be an ideal such that $(\mathfrak{a}, \mathfrak{f}) = 1$, belonging to the inverse class of $\mathfrak{f}^{-1}{\mathfrak{D}_K}^{-1}$. Then there exists an element $\gamma \in K$ such that

$$(\gamma) = \mathfrak{a}\mathfrak{f}^{-1}{\mathfrak{D}_K}^{-1}.$$

We fix γ one for all. The Gaussian sum is defined by

$$G_\chi = \sum_{\alpha \pmod f} \bar{\chi}(\alpha) e^{2\pi i S(\alpha\gamma)}$$

where α runs over the complete set of representatives $\pmod{\mathfrak{f}}$ and S denotes the trace from K to $\mathbb{Q}$.

Let

$$\mathfrak{a}^* = \{\lambda \in K : S(\lambda\alpha) \in \mathbb{Z}, \forall \alpha \in \mathfrak{a}\}$$

ideal of K. It is known that $\mathfrak{a}\mathfrak{a}^*$ is independent of $\mathfrak{a}$ and in fact:

$$\mathfrak{a}\mathfrak{a}^* = (1)^* = \mathfrak{D}_K{}^{-1}.$$

Furthermore $(\mathfrak{a}^*)^* = \mathfrak{a}$ and $N(\mathfrak{D}_K) = |D_K|$, D_K: the discriminant of K. We make explicit calculation if K is a quadratic field over $\mathbb{Q}$, with discriminant D_K. Let $\{\omega_1, \omega_2\}$ be an integral basis of $\mathfrak{a}$:

$$\mathfrak{a} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \mathfrak{a}^* = \{\lambda \in K : S(\lambda\omega_1), S(\lambda\omega_2) \in \mathbb{Z}\}$$

Then

$$\begin{aligned} S(\lambda\omega_1) &= m \\ S(\lambda\omega_2) &= n \end{aligned}$$

where $m, n \in \mathbb{Z}$

i.e

$$\begin{aligned} \lambda\omega_1 + \lambda\bar{\omega}_1 &= m \\ \lambda\omega_2 + \bar{\lambda}\bar{\omega}_1 &= n \end{aligned}$$

Then

$$\lambda = m\frac{\bar{\omega}_2}{\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1} - n\frac{\bar{\omega}_1}{\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1}$$

Hence

$$\mathfrak{a}^* = \mathbb{Z}\frac{\bar{\omega}_2}{\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1} + \mathbb{Z}\frac{-\bar{\omega}_1}{\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1}$$

Now,

$$\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1 = \pm N(\mathfrak{a})\sqrt{|D_K|}$$

and this means that

$$\mathfrak{a}^* = \mathfrak{a}^{-1}\left(\frac{1}{\sqrt{|D_K|}}\right), \text{ i.e } \mathfrak{a}\mathfrak{a}^* = \left(\frac{1}{\sqrt{|D_K|}}\right) = \mathfrak{D}_K{}^{-1}.$$

Moreover $N(\mathfrak{D}_K) = |D_K|$.

We have

a) consider the ideal $\mathfrak{a} = \mathfrak{f}^{-1}\mathfrak{D}_K{}^{-1}$ in K; clearly $\mathfrak{a}^* = \mathfrak{f}$.

b) Let us choose in the class of $\mathfrak{f}\mathfrak{D}_K$, an integral ideal $\mathfrak{a}$ coprime to $\mathfrak{f}$. Then

$$\mathfrak{a}\mathfrak{f}^{-1}\mathfrak{D}_K{}^{-1} = (\gamma), \text{ for } \gamma \in K$$

i.e

$\mathfrak{a}\mathfrak{f}^{-1} = (\gamma)\mathfrak{D}_K$ has exact denominator $\mathfrak{f}$.

Lemma 8.1.2

i)

$$|G_\chi|^2 = N(\mathfrak{f}) \ i.e \ |G_\chi| = \sqrt{N(\mathfrak{f})}.$$

ii)

$$\chi(\beta) = G_\chi^{-1} \sum_{\lambda \pmod{\mathfrak{f}}} \bar{\chi}(\lambda) e^{2\pi i S(\lambda\beta\gamma)},$$

Proof :

$$\begin{aligned} G_\chi \bar{G_\chi} &= G_\chi \sum_{\alpha \pmod{\mathfrak{f}}} \chi(\alpha) e^{-2\pi i S(\alpha\gamma)} \\ &= \sum_{\lambda \pmod{\mathfrak{f}}} \bar{\chi}(\lambda) \sum_{\alpha \pmod{\mathfrak{f}}} \chi(\alpha) e^{-2\pi i S(\alpha(\lambda-1)\gamma)} \end{aligned}$$

Because

$$G_\chi = \sum_{\alpha \pmod{)f}} \bar{\chi}(\alpha) e^{2\pi i S(\alpha\gamma)}$$

α be an integer in K coprime to $\mathfrak{f}$, then $\alpha\lambda$ runs over a complete set of prime residue classes modulo $\mathfrak{f}$ when λ does so. Thus

$$G_\chi = \bar{\chi}(\alpha) \sum_{\lambda \pmod{)f}} \bar{\chi}(\lambda) e^{2\pi i S(\alpha\lambda\gamma)}$$

Thus

$$\sum_{\alpha \pmod{)f}} e^{2\pi i S(\mu\alpha\gamma)} = \begin{cases} 0 & \text{If } \mu \notin \mathfrak{f} \\ N(\mathfrak{f}) & \text{If } \mu \in \mathfrak{f} \end{cases}$$

From the equalities above, we have then

$$|G_\chi|^2 = N(\mathfrak{f}).$$

From the last equality above, we obtain directly the second assertion of our lemma:

$$\chi(\alpha) = G_\chi^{-1} \sum_{\lambda \pmod{\mathfrak{f}}} \bar{\chi}(\lambda) e^{2\pi i S(\lambda\alpha\gamma)}.$$

8.2 Proof of the Interpolation Theorem 8.1.1: Computation of $L_{\mathfrak{f}}(s,\chi)$

There are three key lemmas for our proof of the main result of this section.
First of all, the Hecke L-function associated to χ is defined by

$$L_{\mathfrak{f}}(s,\chi) = \sum_{(\mathfrak{a},\mathfrak{f})=1} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}$$

where the sum is over all integral of O_K coprime to $\mathfrak{f}$,

$$L_{\mathfrak{f}}(s,\chi) = \sum_{A}\sum_{\mathfrak{a}\in A} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s},$$

where A runs over all the ideal classes in the wide sense and $\mathfrak{a}$ over all the non-zero integrals ideals in A, which are coprime to $\mathfrak{f}$.
In the class A^{-1}, we can choose an integral ideal $\mathfrak{b}_A$ coprime to $\mathfrak{f}$ and

$$\mathfrak{a}\mathfrak{b}_A = (\beta)$$

where β is an integer divisible by $\mathfrak{b}_A$ and coprime to $\mathfrak{f}$. Conversely, any principal ideal (β) divisible by $\mathfrak{b}_A$ and coprime to $\mathfrak{f}$ is of the form $\mathfrak{a}\mathfrak{b}_A$, where $\mathfrak{a}$ is an integral ideal in A coprime to $\mathfrak{f}$.
Moreover,

$$\chi(\mathfrak{b}_A)\chi(\mathfrak{a}) = \chi((\beta)), \text{ and } N(\mathfrak{b}_A)N(\mathfrak{a}) = |N(\beta)|.$$

where $N(\beta)$ is the norm of β. Then

$$L_{\mathfrak{f}}(s,\chi) = \sum_{A} \bar{\chi}(\mathfrak{b}_A)N(\mathfrak{b}_A)^s \sum_{\mathfrak{b}_A|(\beta)} \frac{\chi((\beta))}{N(\beta)^s}.$$

Where the inner summation is over all principal ideal (β) divisible by $\mathfrak{b}_A$ and coprime to $\mathfrak{f}$. Since A^{-1} runs over all classes in the wide sense when A does so, we may assume that $\mathfrak{b}_A$ is an integral ideal in A coprime to $\mathfrak{f}$. Then

$$L_{\mathfrak{f}}(s,\chi) = \sum_{A} \bar{\chi}(\mathfrak{b}_A)N(\mathfrak{b}_A)^s \sum_{\mathfrak{b}_A|(\beta)} \frac{v(\beta)\chi(\beta)}{N(\beta)^s}.$$

Lemma 8.2.1 *Any character $\chi(\beta)$ of the ray class group modulo $\mathfrak{f}$ may be written in the form*

$$\chi((\beta)) = v(\beta)\chi(\beta)$$

where $v(\beta)$ is a character "of signature" and $\chi(\beta)$ is a character of the group $(O_K/\mathfrak{f})^*$. We may extend $\chi(\beta)$ to all residue classes modulo $\mathfrak{f}$ by setting

$$\chi(\beta) = 0 \textit{ for } \beta \textit{ not coprime to } \mathfrak{f}.$$

Next step, we have to replace $\chi(\beta)$ by an exponential of the form $e^{2\pi i(mu+nv)}$, comes from lemma 8.1.2. Hence

Proposition 8.2.2 *For any character χ with conductor $\mathfrak{f}$,*

$$L_{\mathfrak{f}}(s,\chi) = \frac{1}{G_\chi} \sum_{\lambda \pmod{\mathfrak{f}}} \sum_{A} \bar{\chi}(\mathfrak{b}_A)N(\mathfrak{b}_A)^s \sum_{\mathfrak{b}_A|(\beta)} \frac{v(\beta)e^{2\pi i S(\lambda\beta\gamma)}}{N(\beta)^s}.$$

In other hand

$$L_{\mathfrak{f}}(s,\chi) = \frac{1}{\omega_K G_\chi} \sum_{\lambda \pmod{\mathfrak{f}}} \sum_{A} \bar{\chi}(\mathfrak{b}_A)N(\mathfrak{b}_A)^s \sum_{\mathfrak{b}_A|\beta\neq 0} \frac{v(\beta)e^{2\pi i S(\lambda\beta\gamma)}}{N(\beta)^s}.$$

where the summation is over all $\beta \neq 0$ in $\mathfrak{b}_A$.

Note that λ runs over all a full system of representatives of the prime residue classes modulo $\mathfrak{f}$ and $\mathfrak{b}_A$ over a complete set of representatives integral and coprime to $\mathfrak{f}$ of the classes in the wide sense, then $(\lambda)\mathfrak{b}_A$ covers exactly $\frac{\omega_K}{\omega_{\mathfrak{b}_A}}$ times, a complete system of representatives of the ray classes modulo $\mathfrak{f}$. Then

$$L_{\mathfrak{f}}(s,\chi) = \frac{1}{\omega_{\mathfrak{f}} G_\chi} \sum_A \bar{\chi}(\mathfrak{b}_A) N(\mathfrak{b}_A)^s \sum_{\beta\in\mathfrak{b}_A,\beta\neq 0} \frac{v(\beta)e^{2\pi i S(\beta\gamma)}}{N(\beta)^s}.$$

Where A runs over the ray classes modulo $\mathfrak{f}$ and $\mathfrak{b}_A$ is a fixed integral ideal in A coprime to $\mathfrak{f}$.

Lemma 8.2.3 *Let $\{\omega_1(A), \omega_2(A)\}$ be an integral basis of $\mathfrak{b}_A$; we can assume, without loss of generality that $\frac{\omega_1(A)}{\omega_2(A)} = \tau_A = x_A + iy_A$ with $y_A > 0$, $y_A = Im\left(\frac{\omega_1(A)}{\omega_2(A)}\right)$.*
Then if $\beta \in \mathfrak{b}_A$, $\beta = m\omega_1(A) + n\omega_2(A)$ for rational m, n and

$$N(\beta) = N(\omega_2(A))|m + n\tau_A|^2 = |\omega_2(A)|^2|m + n\tau_A|^2.$$

Moreover,

$$N(\mathfrak{b}_A)\sqrt{|D_K|} = |\omega_1(A)\bar{\omega}_2(A) - \omega_2(A)\bar{\omega}_1(A)|$$
$$N(\mathfrak{b}_A)\sqrt{|D_K|} = 2y_A|\omega_2(A)|^2$$

$$N(\mathfrak{b}_A) = \frac{2y_A|\omega_2(A)|^2}{\sqrt{|D_K|}}.$$

Thus

$$N(\mathfrak{b}_A)^s \sum_{\beta\in\mathfrak{b}_A,\beta\neq 0} \frac{v(\beta)e^{2\pi i S(\beta\gamma)}}{N(\beta)^s} = \left(\frac{2y_A|\omega_2(A)|^2}{\sqrt{|D_K|}}\right)^s \sum_{\omega\in\mathfrak{b}_A\setminus\{0\}} \frac{v(\omega)e^{2\pi i S(\omega\gamma)}}{N(\omega)^s}.$$

Now for $\omega = m\omega_1(A) + n\omega_2(A)$, let us set

$$u_A = S(\omega_1(A)\gamma), v_A = S(\omega_2(A)\gamma)$$

and let f be the smallest positive integer divisible by $\mathfrak{f}$. Then in view of the fact that $(\gamma)\mathfrak{D}_K$ has exact denominator $\mathfrak{f}$ and $\mathfrak{b}_A$ is coprime to $\mathfrak{f}$, it follows that u_A and v_A are rational numbers with the reduced denominator f. Since $\mathfrak{f} \neq (1)$, u_A and v_A are not simultaneously integral. We then have

$$N(\mathfrak{b}_A)^s \sum_{\omega\in\mathfrak{b}_A,\omega\neq 0} \frac{v(\omega)e^{2\pi i S(\omega\gamma)}}{N(\omega)^s} = \left(\frac{2y_A|\omega_2(A)|^2}{\sqrt{|D_K|}}\right)^s \sum_{\omega\in\mathfrak{b}_A\setminus\{0\}} \frac{v(\omega)e^{2\pi i(mu_A+nv_A)}}{|\omega|^{2s}}.$$

We recall that

$$E_{\mathfrak{b}_A}(\omega == m\omega_1(A) + n\omega_2(A), \varphi = -v_A\omega_1(A) + u_A\omega_2(A)) = mu_A + nv_A$$

Finally, we obtain

Theorem 8.2.4 *Let χ be a Hecke character of conductor $\mathfrak{f}$.*
Then we have

$$L_{\mathfrak{f}}(s,\chi) =$$

$$\frac{1}{\omega_{\mathfrak{f}}G_\chi}\sum_A \bar{\chi}(\mathfrak{b}_A)\left(\frac{2y_A|\omega_2(A)|^2}{\sqrt{|D_K|}}\right)^s \sum_{\omega\in\mathfrak{b}_A\setminus\{0\}} \frac{v(\omega)e\Big(E_{\mathfrak{b}_A}\left(\omega = m\omega_1(A)+n\omega_2(A),\varphi_A\right)\Big)e^{2\pi i(mu_A+nv_A)}}{\omega^s\bar{\omega}^s},$$

where

$$\varphi_A = -v_A\omega_1(A) + u_A\omega_2(A)$$

In particular, for

i)

$$v(\omega) = \left(\frac{\omega}{|\omega|}\right)^{-2m} = \omega^{-m}\bar{\omega}^m, m \geqslant 1$$

We obtain

$$L_{\mathfrak{f}}(s,\chi) = \frac{1}{\omega_{\mathfrak{f}}G_\chi}\sum_A \bar{\chi}(\mathfrak{b}_A)\left(\frac{2y_A|\omega_2(A)|^2}{\sqrt{|D_K|}}\right)^s \sum_{\omega\in\mathfrak{b}_A\setminus\{0\}} \frac{v(\omega)e(E_{\mathfrak{b}_A}(\omega,\varphi_A))e^{2\pi i(mu_A+nv_A)}}{\omega^{m+s}\bar{\omega}^{s-m}}.$$

As consequence, we have

$$L_{\mathfrak{f}}(s=m,\chi) = \frac{1}{\omega_{\mathfrak{f}}G_\chi}\sum_A \bar{\chi}(\mathfrak{b}_A)\left(\frac{2y_A|\omega_2(A)|^2}{\sqrt{|D_K|}}\right)^m d_{2m}(\varphi_A,\mathfrak{b}_A); \forall m \geqslant 1.$$

ii) If

$$v(\omega) = \left(\frac{\omega}{|\omega|}\right)^{-m}, m \geqslant 1$$

We obtain

$$L_{\mathfrak{f}}(s=m,\chi) = \frac{1}{\omega_{\mathfrak{f}}G_\chi}\sum_A \bar{\chi}(\mathfrak{b}_A)\left(\frac{2y_A|\omega_2(A)|^2}{\sqrt{|D_K|}}\right)^{\frac{m}{2}} d_m(\varphi_A,\mathfrak{b}_A); \forall m \geqslant 1.$$

Theorem 8.2.5 *(***Damerell's type result***)*
For each Hecke character χ of conductor $\mathfrak{f}$.

i) For

$$v(\omega) = \omega^{-m}\bar{\omega}^m, m \geqslant 1$$

we have that

$$\frac{L_{\mathfrak{f}}(s=m,\chi)}{(2\pi i)^{2m}\eta(\tau)^{4m}}$$

is algebraic

ii) For

$$v(\omega) = \left(\frac{\omega}{|\omega|}\right)^{-m}, m \geqslant 1,$$

we have that

$$\frac{L_{\mathfrak{f}}(s=m,\chi)}{(2\pi i)^{2m}\eta(\tau)^{4m}}$$

is algebraic

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