Microsimulation and population dynamics

in longevity, credit, HFT modelling

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Plan

Why a course on population dynamics ?

Population dynamics and Longevity

- First motivation, longevity risk
- To take into account the complexity of the gobal population
- General Model for population dynamics in Ecology

Marked Point Process: Renew of interest

Useful tool, under different denominations, for other domains:

- Credit risk Modelling
- Hawkes processes in Hight Frequency Trading
- Brain study
- Data Mining

Why a course on population dynamics II

Probability Theory and Simulation

- Based on useful in probability theory, Poisson Point measure,
- Birth and Death process
- Particular Method for Monte Carlo Simulation

Data and Calibration

- Completely different situations for different domains
- Hard to calibrate
- Hard to simulate

Plan

Demographic Transition in a nutshell

Mortality Transition, Canning (2011)

+30 years for Life Expectancy (LE) in the last century The demographic observation

- Substantial decline in mortality rate, in particular in small ages
- followed by reduction in fertility rate
- Heath transition (physical and cognitive development) and compression of morbidity

Economics aspects

- Economic growth, (income by head) and
- increase in social and political policy (education, democratie..)
- Growth in world population, citer Cohen

Health point of view from Cutler,Deaton,alii (2006)

- Decline in infectious disease (60% of deaths in1848, $< 5\%$ in 1971 in UK)
- Nutritional improvement (debate on the importance)
- Progress in medecine, vaccins, ...

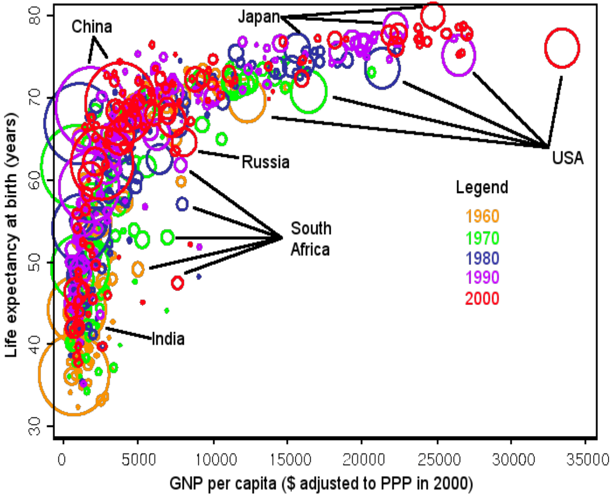
Public policies

- Macro public health: big public work projects (water purifica- tion explain half of mortality reduction in US ($1900 \sim 1930$)
- Reduction in alcoholism, in smoking
- public and private health,..also contribute with complex impacts,
- *large heterogenity* with differences by age, type of sub-population, countries, with reverse or delayed effects.

Economic and Wealth Point of view

- Strong evidence on the links, but only 20% as impact
- Relation non-linear and concave
- Unexplained recent slower pace for LifeExp in US /Europa

Wealth and longevity: complex dependency



Evolutionary theory for aging

Example of Biological views

- Aging is characterized by the decline of physiological capacity
- Explain heterogeneity and randomness in individual patterns
- Nevertheless a robust observation in evolution theory, Gompertz (1825): The log mortality rate between 35-80y is linear in age.
- After 80y, large debate on the rectangularization of the survival curve, the question of "limited human life span"?

Example of data: EU15, 2011, Age-specific mortality rates per 100.000

1. $[0, 1y], 486(382)$
2. $[1y, 10y], 19(15)|$ $[11y, 20y, 41(19)|$ $[21y, 30y], 93(32)$
3. $[31y, 40y], 133(63)|$ $[41y, 50y], 313(163)|$ $[51y, 60y], 750(385)$
4. $[61y, 70y], 1869(953) \mid [71y -$

Aggregate mortality indicators

Life exptectancy at birth

- Lifetime of an individual: τ
- Life expectancy at birth: $\mathbb{E}[\tau]$, at ten $\mathbb{E}[\tau - 10/\tau > 10]$

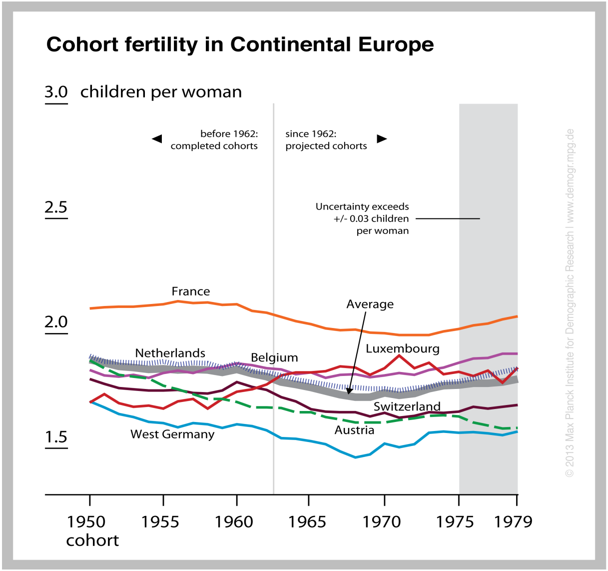
Death rate

- Death rate $d(a)$ such that $\mathbb{P}(\tau > a) = e^{-\int_0^a d(s)ds}$
- In practice annual death probability reduction $q(a) = \mathbb{P}(\tau < a + 1 \mid \tau \geq a)$
- Mortality plateau (old ages)

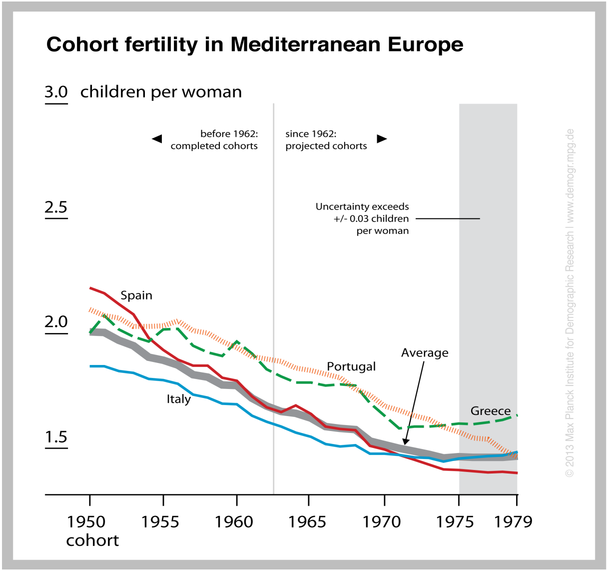
Fertility rate

- Complex notion
- With large political connotation (Fertility, Immigration)

Fertility rate in Continental Europa/1950-80

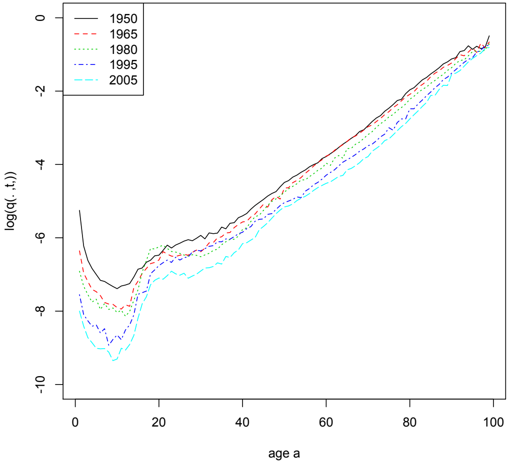


Fertility rate in Mediterranean Europa

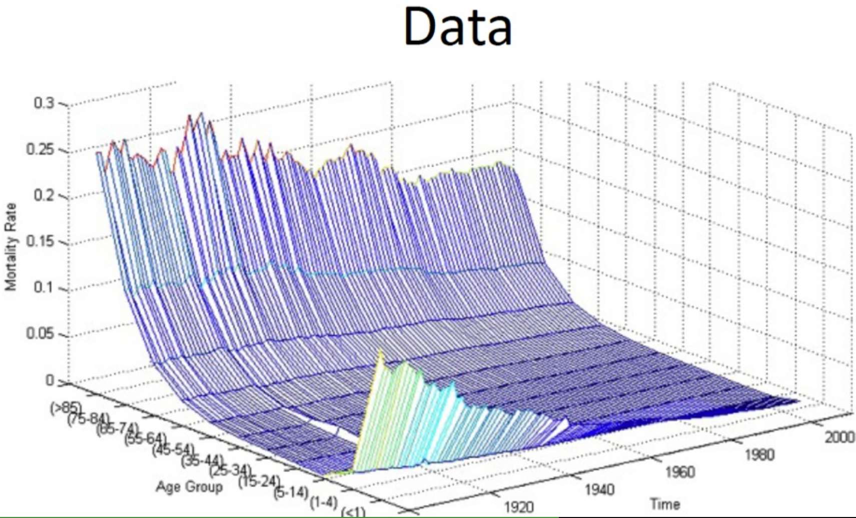


National mortality: log q(a,t)

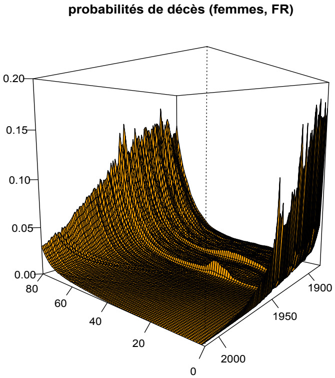
- Looking at $log\ q(a,t)$ age a in [0,100]
- for different years t (1950,1965,1980,1995,2005)

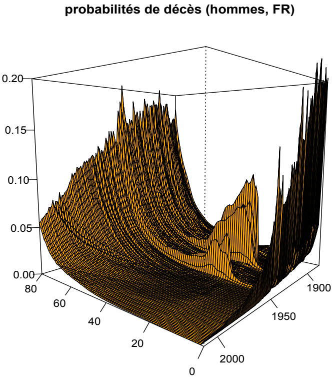


National mortality Surface



National mortality by gender (France)





Femmes

Hommes

New economic and social challenge

Aging populations: new phenomenon, without past historical reference

- viability of shared collective systems, in particular (state or private) pension systems
- new generational equilibrium
- role and place of aging population in the society

Complex phenomenon, multi-causes

- Difficult to model.
- The role of age
- The heterogenity

Complex Estimation

- Coherence of the data
- Age, cohort, period

Plan

Aims of microscopic models

- Provide population evolution at the scale of individual
- allows to understand paterns of aggregate indicators

Two examples in these lessons

- Impact of aging
- How individual birth patterns in heterogeneous population can create artificial mortality changes ("Cohort effect")

Microscopic models in different fields:

Individual-Based models in Public Economy:

- Agent-based models in economics (Orcutt, 1957)
- Microsimulation models of government bodies (Ex : INSEE, model "DESTINIE")
- Individual-Based models in ecology (mathematical framework):

Individual-Based models in ecology:

- Modelling a population with birth, death, and mutation at birth
- Population structured by traits (*i.e.* individual characteristics) (Fournier-Méléard 2004) (Champagnat-Ferrière-Méléard 2006), with age (Tran 2006, Ferrière-Tran 2009)

Microsimulation exercise

First step Define clear specification of the objectives, for determining methods, assumptions and scenarios, in view of constitution of *Data base* storing the information on all individuals on study

- *State space*: state of variables to be projected, as traits, attributes of individuals, (as age, sex, residence, level of schooling, wealth)
- *support variables* (marital statuts, children..) used to predict events.
- *Covariates* Y or factors of type demographic or environmental.

At the macrolevel, the state space is the set of all combinations of individual state variables.

Dynamic simulation over the time

- to predict the future state, be careful on the causality of the events, since demographic events influences population,
- and intensity of demographic events are themselves influenced by the composition of the population.

Analysis of the "artificial" database

The classical point of view in population dynamics

- As a family of individual biographies influenced by the others, or by covariates
- Valid only in the linear case
- Essential assumption in demographic practice

Cross-sectional point of view

- The population is described every date by the characteristics of its individuals
- well- adapted to interacting individuals
- very similar description as for interacting particules system in physics

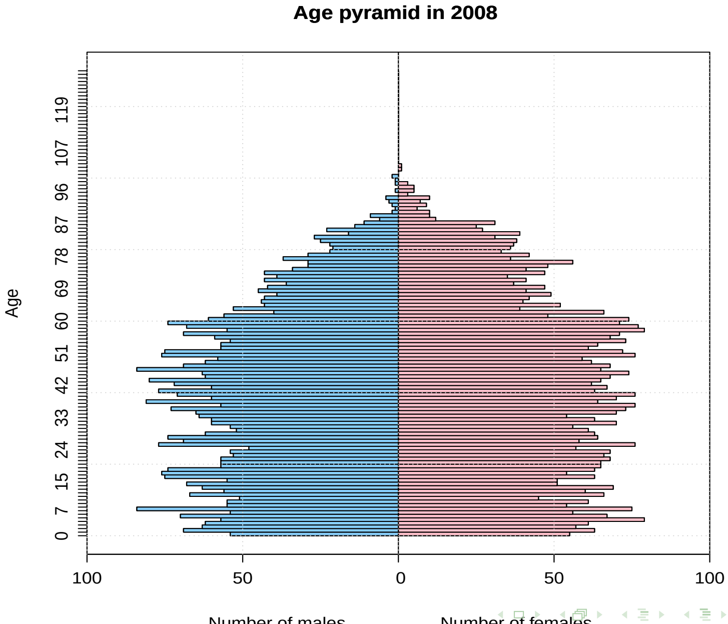
The "macro point of view" is cross-sectional (similar to continuous time Markov chain

Sources of randomness

From Vanimhoff 1998

- inherent randomness due to Monte Carlo Methods, reduced by increase the number of runs, or the size of the database or variance reduction. Not equivalent in general
- starting-population randomness: in general a subsample of the population; be careful that any deviation of the sample distribution impact future projections
- *Specification randomness* Choice of the number of state variables: more variables increase the MC randomness, calibrations errors, implied correlation due to calibration.
- Reduced by *sorting methods*, or alignment methods to respect of some macro properties

Initial population for N=10 000 in 2008



Plan

Classical example: Poisson population

- A *marked population* χ is a finite set of individuals, points, particles, characterized by quantitative attributes with values in E
- Uncertainty concerns $\mathrm{card}(\xi)$=nb of individuals, and the "vector" of attributes $(X_1, ...X_n..)$

Poisson population:

- Let $(X_1, ...X_n..)$ be an iid sample of $\mu(dx)$ on E, stopped randomly at ν, an independent $\sim \mathrm{Pois}(\lambda)$.
- The marked population $\xi = \{X_1, ...X_\nu\}$ is a Poisson population, with mean $\eta(dx) = \lambda\mu(dx)$, iff $N^\xi(B) = \mathrm{card}(\xi \cap B) = \sum_k^\nu \mathbf{1}_{X_n \in B}$ is a Poisson variable $\mathrm{Pois}(\lambda\mu(B))$.

Restricted or thinned Poisson population:

- The population $\xi^B = \{X_1, ...X_\nu\} \cap B$ restricted to B is still a Poisson population with Poisson parameter $\lambda^B = \lambda\mu(B)$ and spatial distribution $\mu^B(dx) = \mu(dx\,|B) = \mathbf{1}_B(x)\mu(dx)/\mu(B)$.

Classical example: Poisson population II

- All standard properties of finite Poisson measure are satisfied:
  - $N(B) = \sum_{n=1}^{\nu} \mathbf{1}_B(X_n) = \text{card} \sim \text{Pois}(\eta(B)$
  - $(N(B_i))$ are independent if $B_i \cap B_j) = \emptyset$

- Assume η is only σ-finite, and finite on an increasing sequence of (finite measure) windows K_k, with reunion E.
  - By assumption, the Poisson set restricted to K_k is a finite Poisson population ξ^k.
  - The "countable " reunion of these sets ξ^k is now countable. The decomposition is not unique.

Random marked population

- A marked population χ is an at most countable set of individual with patterns in E. The space of outcomes is the so-called population (configuration) space Γ(E)
- To study the spatial distribution of ξ, we introduce
  - $N(B)(\xi)$ the number of individuals of ξ in B, that is $N(B)(\xi) = \mathrm{card}(\xi \cap B)$ (additive properties)
  - $\mathrm{V}(B)(\xi)$ the *vacancy indicator* and the vacancy set $V(B) = \{\xi : N(B)(\xi) = 0\} = \{\xi : \text{no points in B}\}$
- $N_{R \cup Y}(B) = N_R(B) + N_Y(B)$ if $\xi_R \cap \xi_Y = \emptyset$ and $\mathrm{V}_{R \cup Y}(B) = \mathrm{V}_R(B)\mathrm{V}_Y(B)$
- The first moment, called *mean measure*, $\eta(B) = \mathbb{E}(N(B))$ is often viewed as the main information on the population
- The remarkable property of $\mathrm{V}(B)$ is that this "minimal" operator characterizes the distribution of the marked population

Dynamic Poisson population

On-line Poisson process = Poisson population on $\mathbb{R}^+$ with mean measure proportional to Lebesgue $\eta(dt) = \lambda \mathrm{Leb}(dt)$

- Restricted to a window $[0, T]$, $N_T = N(]0, T]) \sim \mathrm{Poi}(\lambda T)$
- Conditionally to $N_T = n$, the temporal characteristics (θ_i^T) are uniformly distributed on $[0, T]$
- Process representation as non decreasing function in T, with jumps times T_j and waiting times $\tau_j = T_j - T_{j-1}$ exponential with parameter Λ

Dynamic Poisson population= marks added at the temporal component

- Finite total mass m. Simple extension of the static construction with iid sample (X_n) of $m(dx)/m(E)$, $\xi_t = \{X_1 .... X_{N_t}\}$
- Extension to σ finite case, without difficulty

General population with stochastic intensity

Notation $z \mapsto \langle m, f_z \rangle = \int_{x \in E} \int_{y \in F} f(x, y, z) m(dx, dy)$= function

General filtration and Martingale point of view

- Stochastic intensity $\mu(t, dx)dt$: $N_t(B) - \int_0^t \mu(s, B)ds$ is a $\mathcal{F}_t$ martingale
- $\mathcal{F}_t$-Poisson process: martingale property for $(N_t(B) - \mu(B)t)$ is equqival to the independence of $(N_{t+h} - N_t)$ to $\mathcal{F}_t$
- Extension to $\mathcal{P} \otimes \mathcal{B}(E)$ integrable predictable processes f_t: $(\langle N_t, f \rangle - \int_0^t \langle \mu, f_s \rangle ds)$ is a martingale

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First projection point of view, with digital predictable intensity

- Assume that $\mathbf{1}_D(\omega, t, x)$ is $\mathcal{P} \otimes \mathcal{B}(E)$ mesurable, with $\int_0^t \int_E \mathbf{1}_D(\omega, s, x) \, ds\mu(dx) < \infty, \quad \eta(dt, dx) = dt\mu(dx)$
- The marked population $N^D(dt, dx) = \mathbf{1}_D(\omega, t, x).N(dt, dx)$ has the intensity: $\eta^D(dt, dx) = \mathbf{1}_D(t, x).\eta(dt, dx) = \mu_t^D(dx)dt$

- Let us consider a predictable measure, $\eta^\lambda(\omega, dt, dx) = \lambda(\omega, t, x)\eta(dt, dx)$, where η is a deterministic product measure on $\mathbb{R}^+ \times E$. (Density assumption)
- How the construct a dynamic random population with intensity $\eta^\lambda(dt, dx)$ from the Poisson measure $Q(dt, dx)$ with intensity $\eta(dt, dx)$?

Thinning of extended Poisson measure

- Introduce a *thinning parameter* θ erasing all the points (t, x) such that $\lambda(t, x) < \theta$. Put $D(\omega, t, x, \theta) = \{\theta \leq \lambda(\omega, t, , x)\}$
- On $E \times \mathbb{R}^+ \times \mathbb{R}_+$, with current point (t, x, θ) and product measure $q(dt, dx, d\theta) = dt\, m(dx)\, d\theta$, and Poisson measure $Q(dt, dx, d\theta)$,
- Restricted $Q(dt, dx, d\theta)$, to $Q^D(dt, dx, d\theta)$
- Projected $Q^D(dt, dx, d\theta)$ on $E \times \mathbb{R}^+$ into $N(dt, dx)$
- The new dynamic population has the desired intensity.

Birth Dates in on-line Poisson process

- From a set point of view, put $\xi_t = \{T_1, T_{N_t}\}$, where T_k is the date of birth (of entry time) of the k individual in the population. Obviously, (ξ_t) is not a Poisson population.
- Put $\bar{N}(dt, du) = \sum_{n\geq 1} \delta_{T_n}(dt)\delta_{T_n}(du)$. As point process with two components, the mean measure is proportional to the Lebesgue measure on the diagonal of $\mathbb{R}^2_+$ denoted $\Delta(dt, du) = dt\, \delta_t(du)$. It is not a product measure, so $\bar{N}$ is not a Poisson process

Cohort Market Dynamic population

- With marks, the random set becomes $\xi_t = \{(T_1, X_1), (T_{N_t}, X_{N_t})\}$ with intensity measure $\Delta(dt, du)m(dx)$.
- The population is said to be structured by cohort, with a natural order of enumeration of individuals in the population.

Dynamic population structured by age

- The new population becomes $\tilde{\xi}_t = \{(t - T_1, X_1), (t - T_{N_t}, X_{N_t})\}$.
- The mark is now depending on the point of time t by $r_t(a) = (t - a)^+$,
- $\tilde{N}^a_t(A \times B) = N_t(r_t(A) \times B)$ is no more a measure in t, although if its expectation $\int_0^t \mathbf{1}_A(t-s)dsm(B) = \int_0^t \mathbf{1}_A(s)ds$

Deterministic formula for counting measure with age

- For differentiable f in age, coupled with integration by parts (formula for the online process)
$$z_t(f) = f(0)z_t(1) + \int_0^t z_v(f')dv.$$
- $z_t(f)$ is of finite variation
- Application to Hawkes process

Thinning equations for Birth processes

The thinning construction can be used to define a wide variety of processes as solution to stochastic equations.

Intensity for linear Birth Process

- Generalisation of Poisson process, but pure jump Markov process on $\mathbb{N}$, (N_t) non decreasing in time with jump 1 and intensity λn
- The time between two jumps is exponential of parameter λn, and independent

Representation as solution of SDE

- Stochastic intensity $\lambda_t N_{t-}$
- Equation $dN_t = \int_{\mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_t N_{t-}\}} Q(dt, d\theta), N_0 = x$
- Solution by recursive method starting with the process
  $dX^1_t = \int_{\mathbb{R}_+} \mathbf{1}_{\{\theta \leq \lambda_t x\}} Q(dt, d\theta)$,
  $dX^2_t = \int_{\mathbb{R}_+} \mathbf{1}_{\{X^1_{t-} > x\}} \mathbf{1}_{\{\theta \leq \lambda_t X^1_{t-}\}} Q(dt, d\theta)$ and so on....

Plan

Linear self-exciting processes

- A point process N with jump times (T_n) and path-dependent intensity λ_t, ($N_t = N_0 + \int_{(0,t]} \lambda_s dt\ + \mathcal{F}$-martingale)
- Hawkes (1971): Linear self-excitation

$$\lambda_t = \bar{\mu} + \int_{(0,t)} \phi(t-s) \underset{\cdot}{\mathrm{N}}_s = \bar{\mu} + \sum_{T_n<t} \phi(t-T_n),$$

- ϕ=fertility function

- Simple Hawkes example: Autoregressive point process,

$$\phi(t) = \alpha e^{-\beta t} \qquad \lambda_t = \bar{\mu} + \alpha \int_{(0,t)} e^{-\beta(t-s)} \underset{\cdot}{\mathrm{N}}_s$$

- Credit: (Gieseke, Dufie,....)
  (i) $\lambda_t = \bar{\mu} + \int_{(0,t)} \psi(s) \underset{\cdot}{\mathrm{N}}_s = \bar{\mu} + \sum_{T_n<t} \psi(T_n)$,
  (ii) λ_t is differentiable in time,
- No true in general for Hawkes intensity

Self-exciting models for applications

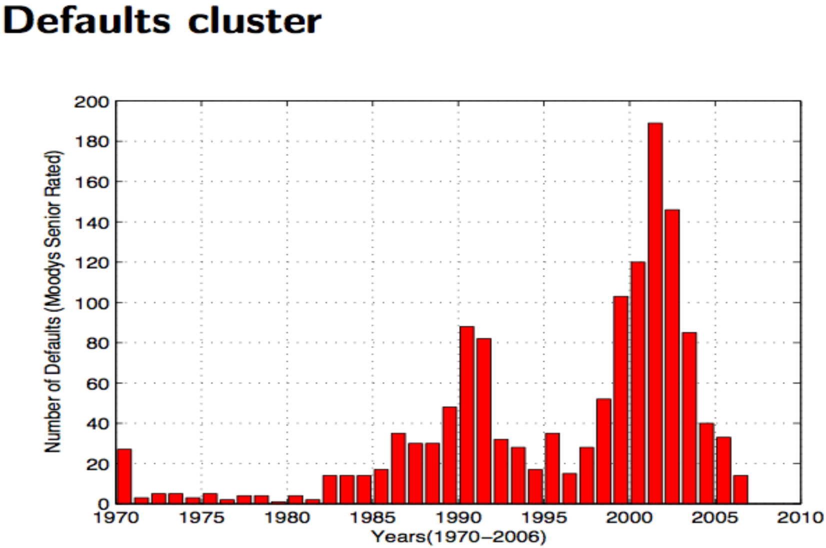
Physics and Biology

- seismology (AfterShoks), epidemiology
- ecology
- neuroscience, DNA modelling

Social Sciences

- epidemics in Socio-Economic networks
- finance: credit risk, contagion, mortgage prepayments
- insurance: risk processes, ruin theory, surrender lapse
- High Frequency trading and market microstructure

from Gieseke,1970-2006



Financial applications

"Low frequency"

- Credit risk (e.g. Gieseke,Errais et al. 2010)
- Daily financial data (e.g. Embrechts et al. 2011)
- Financial contagion (e.g. At-Sahalia et al. 2010)

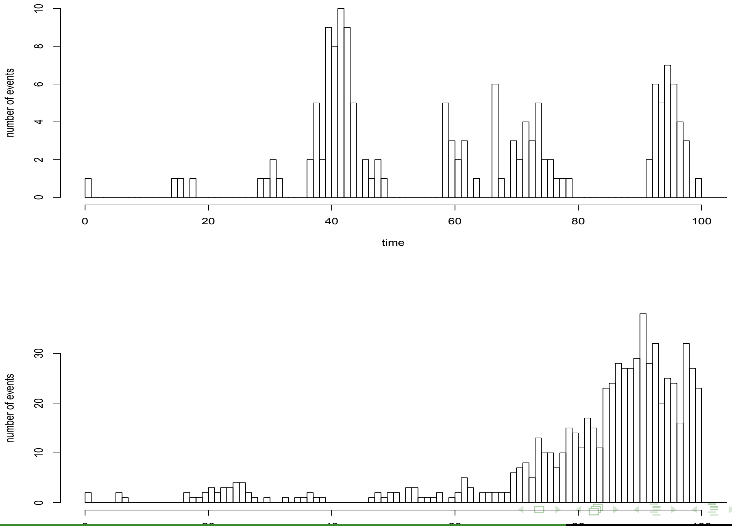
"High frequency "

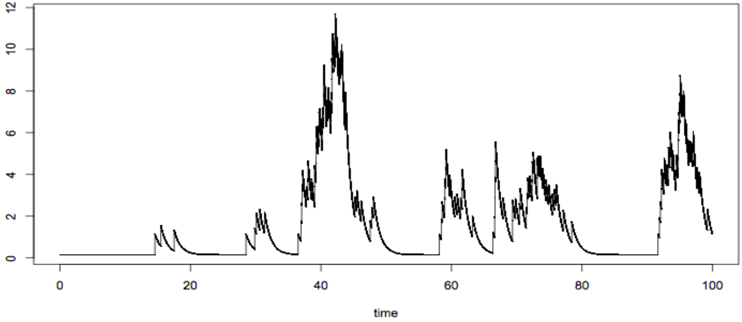
- Midquote and transaction prices, market impact (e.g. Bacry & Muzy, 2013)
- Limit order book (e.g. Large, 2007)
- Scaling limits (e.g. Jaisson & Rosenbaum, 2013)

[Review from Jaisson & Rosenbaum (2013)]

History for two Populations

Human population: $\phi^2(a) = \mu + k \exp -c(t - t_f)^{+,2})$

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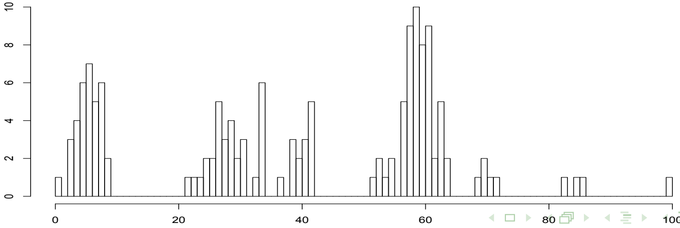
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Population point of view

Birth process with immigration

- Each individual has an age a
- Immigrants arrive according to a Poisson $(\bar{\mu})$
- Any individual aged a gives birth with rate $\phi(a)$

Same definition as in Hawkes process

Age pyramid at time t:

- Fix t. $Z_t([\alpha, \beta))$ is the number of events with age in $[\alpha, \beta)$.
- $N_t = Z_t(\mathbb{R}^+) = \langle Z_t, \mathbf{1} \rangle$
- $\langle Z_t, f \rangle = \int_{\mathbb{R}_+} f(a) Z_t(da) = \sum_n \mathbf{1}_{[0,T_n]}(t) f(t - T_n) = \int_0^t f(t-s) dN_s$
- Intensity process: $\lambda_t = \bar{\mu} + \langle Z_{t-}, \phi \rangle$

Dynamics

Differential property

- What is the dynamics of the age pyramid $Z_t(da)$ over time ?
- Recall that $N_t = \langle Z_t, \mathbf{1} \rangle$

Key (!) lemma

For each differentiable f, $\langle Z_t, f \rangle = f(0)dN_t + \underbrace{\langle Z_t, f' \rangle}_{\text{ageing}} dt$.

Proof Use that $f(t-s) - f(0) = \int_s^t f(t-u)du$ and make an integration by parts.

- Define the Hawkes process as the solution to the stochastic equation

$$N_t = \int_{(0,t)} \int_{\mathbb{R}_+} \mathbf{1}_{[0, \bar{\mu} + \int_{(0,s)} \phi(s-u) dN_u]}(\theta) Q(ds, d\theta),$$

- Existence easy if ϕ is bounded by K, by using $\mathbf{1}_{[0,K]}.d\theta = d\theta^K$ and the sequence (S_n, Θ_n) associated with $dt \otimes d\theta^K$.
- By Picard iteration, starting with $N_t^0 = N_0$ and

$$N_t^K = N_0 + \int_{(0,t)} \int_{\mathbb{R}_+} \mathbf{1}_{[0, \bar{\mu} + \int_{(0,s)} \phi(s-u) dN_u^{K-1}]}(\theta) Q(d, d\theta),$$

Avantage of the SDE representation

- Strong solution even in the non linear case
- Allows comparison theorem since same noise
- Study the sensitivity to the initial condition

Classical Exponential fertility function

$$\langle Z_t, f\rangle = f(0)N_t + \int_0^t \langle Z_s, f'\rangle ds.$$

- If $\phi(a) = \alpha e^{\beta a}$, $\phi' = \beta\phi$ and

$$\langle Z_t, \phi\rangle = \langle Z_u, \phi\rangle + \alpha \int_u^t \int_{\mathbb{R}_+} \mathbf{1}_{[0,\bar{\mu}+\langle Z_{s-},\phi\rangle]}(\theta) Q(ds, d\theta) + \beta \int_u^t \langle Z_s, \phi\rangle ds.$$

- EDS in $\langle Z_t, \phi\rangle$ only, and $\lambda_t = \bar{\mu} + \langle Z_{t-}, \phi\rangle$ is a Markov process

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Distribution properties for $\phi(a) = \alpha e^{-\beta a}$

- Errais, Gieseken et al. (2010)
- At-Sahalia et al. (2010)
- DASSIOS, (2011)
- Da Fonseca and Zaatour (2014)

Fertility function: setting

The map $a \in \mathbb{R}_+ \mapsto \phi(a)$ is of class $\mathcal{C}^n(\mathbb{R}_+)$ and is solution to the equation

$$\phi^{(n)} = c_{-1} + \sum_{k=0}^{n-1} c_k \phi^{(k)},$$

with initial conditions $\phi^{(k)}(0) = m_k$, for $0 \leq k \leq n-1$. For instance exponential functions multiplied by a polynome

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Fertility function: examples

- Approximation of a power law kernel ($\sim \frac{1}{t^{1+\epsilon}}$) with cut-off (Hardiman-Bercot-Bouchaud, 2013)

$$\phi(t) = \frac{n}{Z}\left(\sum_{i=0}^{M-1} \frac{e^{-t/(\tau_0 m^i)}}{(\tau_0 m^i)^{1+\epsilon}} - S e^{-t/(\tau_0 m^i)}\right).$$

- Z is such that $\int_0^\infty \phi = n$ and S such that $\phi(0) = 0$.
- Used to allow tractable likelihood

Dynamics

$$\phi^{(n)} = c_{-1} + \sum_{k=0}^{n-1} c_k \phi^{(k)}, \; \phi^{(k)}(0) = m_k$$

A system differential linear

$$N_t = \int_0^t \int_{\mathbb{R}_+} frm[o]{-}{-}_{[0,\bar{\mu}+\langle Z_{s-},\phi\rangle]}(\theta) Q(ds,\theta)$$

$$\langle Z_t, \phi \rangle = m_0 N_t + \int_0^t \langle Z_s, \phi' \rangle ds$$

$$\vdots$$

$$\langle Z_t, \phi^{(k)} \rangle = m_k N_t + \int_0^t \langle Z_s, \phi^{(k+1)} \rangle ds$$

$$\vdots$$

$$\langle Z_t, \phi^{(n-1)} \rangle = m_{n-1} N_t + \int_0^t \langle Z_s, \phi^{(n)} \rangle ds$$

Dynamics

A linear system of affine type

- The $(n+1)$-dimensional process $X_t := (\langle Z_t, 1\rangle, \langle Z_t, \phi\rangle, ..., \langle Z_t, \phi^{(n-1)}\rangle)$ is solution of the affine differential sytem

$$X_t = N_t \hat{m} + \int_0^t C X_s ds, \quad N_t - \int_0^t X_s^1 ds = \text{martingale}$$

- The Laplace transform has a closed form

$$\mathbb{E}\left[\exp\left(v.X_T\right)\right] = \exp\left(-\bar{\mu}\int_0^T (1 - e^{A_s.\hat{m}})ds\right),$$

- Martingale $= \exp\left(\alpha_t N_t - \int_0^t \alpha'_s N_s ds - \int_0^t (e^{\alpha_s} - 1)(\bar{\mu} + \lambda_s) ds\right)$
- The matrix A is solution of the deterministic equation

$$^tCA_t + A'_t + (e^{A_t.\hat{m}} - 1)\mathbf{1}_{i=2} = 0, \quad A_T = v$$