# THE REGULARIZING EFFECT OF SUPERLINEAR GRADIENT TERMS IN NONLINEAR PARABOLIC EQUATIONS


MARTINA MAGLIOCCA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROME TOR VERGATA


## 1. THE SUPERLINEAR PROBLEM

We consider the following problem:

$$\begin{cases} u_t - \Delta_p u = |\nabla u|^q & \text{in } Q_T, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{P}$$

assuming $1 < p < N$,

$$u_0 \in L^\sigma(\Omega) \quad \text{with} \quad \sigma \geq \max\left\{1, \frac{N(q-(p-1))}{p-q}\right\} \tag{ID}$$

and superlinear $q$ growths in the gradient term, i.e.

$$\max\left\{\frac{p}{2}, \frac{p(N+1)-N}{N+2}\right\} < q < p. \tag{Q}$$

**AIM:** proving long time decay of (renormalized) solutions of (P).

The incoming results hold in a general nonlinear setting (see [MP, M2]).

## 3. MAIN RESULT

**THEOREM** We assume that (Q), (ID) and (RC)–(ET) are in force.
Then, solutions of (P) are bounded for positive times and they decay as solutions of coercive problems.

## 4. THE KEY POINT

We sketch the case $\sigma > 1$, then (RC) holds. Let

$$G_k(u) = (|u|-k)_+ \text{sign}(u).$$

**A first smallness condition:** if we take $k \gg 1$ s.t.

$$\|G_k(u_0)\|_{L^\sigma(\Omega)} < \delta \qquad k \gg 1, \tag{SC1}$$

$\delta > 0$ small enough, then

$$\|G_k(u(t))\|_{L^\sigma(\Omega)} \leq \delta \qquad k \gg 1, \forall t > 0.$$

Indeed, testing (P) with $\int_0^{G_k(u)} (\varepsilon + |v|)^{\sigma-3}|v|\,dv$, $\varepsilon \geq 0$, and thanks to (RC) and to a continuity argument in time, we find

$$\int_\Omega |G_k(u(t))|^\sigma\,dx - \int_\Omega |G_k(u_0)|^\sigma\,dx + c(1-\delta)\iint_{Q_t}\left|\nabla\left[(1+|G_k(u)|)^{\frac{\sigma-2}{p}}|G_k(u)|\right]\right|^p dx\,dt \leq 0.$$

In particular, we deduce that $u(t) \in L^\sigma(\Omega)$ for $t > 0$.

**CONSEQUENCE:** if $u_0 \in L^\infty(\Omega)$ then we take $k = \|u_0\|_{L^\infty(\Omega)}$ obtaining

$$\|u(t+\tau)\|_{L^\infty(\Omega)} \leq \|u(\tau)\|_{L^\infty(\Omega)}. \tag{1}$$

## 2. SUPERLINEAR STUFF

**RMK 1:** if (ID) is not satisfied then no solution exists.
**RMK 2:** when (ID) verifies $\sigma > 1$, we have to take $u$ s.t.

$$\left\{u \text{ solution s.t.} \quad (1+|u|)^{\frac{\sigma-2}{p}} u \in L^p(0,T;W_0^{1,p}(\Omega))\right\}. \tag{RC}$$

This class makes the problem well posed: on the contrary, we lose uniqueness!
**RMK 3:** when $\sigma = 1$, (RC) is replaced with

$$\lim_{n\to\infty}\frac{1}{n}\iint_{\{n\leq|u|\leq 2n\}} |\nabla u|^p = 0. \tag{ET}$$

The existence of solutions of (P) satisfying (RC)–(ET) is contained in [M1].

**N.B.:** the requests in **RMK 1**, **RMK 2** are common features among superlinear problems. **RMK 3** is related to the renormalized setting.

## 5. REGULARIZING EFFECT

Taking advantage of (SC1), we manage to prove that

- $\|G_k(u)\|_{L^r(\Omega)}$, $r > \sigma$, decays polynomially in $t$ and $u(t) \in L^r(\Omega)$ for $t > 0$;
- $\|G_k(u)\|_{L^\infty(\Omega)}$ decays polynomially in $t$ (exponentially if $p = 2$) and $u(t) \in L^\infty(\Omega)$ for $t > 0$;

for $k \gg 1$ (as in (SC1)) and with the same rates of coercive problems.

**CONCLUSION 1.:** the $G_k(u)$ function decays as solutions of coercive problem for large values of $k$ (see (SC1)).

## 6. THE TURNING POINT

Combining the decay of $\|G_k(u)\|_{L^\infty(\Omega)}$ with (SC1) and (1), we manage to recover the one of the whole solution at an unspecified rate:

$$\|u(t)\|_{L^\infty(\Omega)} \to 0 \quad \text{for} \quad t \to \infty.$$

**A second smallness condition:** we take $\tau \gg 1$ s.t.

$$\|u(\tau)\|_{L^\infty(\Omega)} < \delta \qquad \tau \gg 1, \tag{SC2}$$

$\delta > 0$ small enough, then we use (SC2) instead of (SC1) in the previous proofs: in this way, all the results obtained for $G_k(u)$ hold for the whole $u$.

**CONCLUSION 2.:** the problem (P) decays as coercive problems for large time.

## 7. THE CASE $\sigma = 1$

The condition (ET) takes the place of (RC). Then, letting $k \gg 1$ s.t. (SC1) holds with $\sigma = 1$, we recover the "key point" result.
Once we get $u \in L^\infty(0,T;L^{1+\omega}(\Omega))$, $\omega > 0$, then the proofs follow as before.

## 8. COMPARISON WITH THE COERCIVE CASE AND THE SUPERLINEAR POWER PROBLEM

The results below are contained in [Po]. Consider

$$u_t - \Delta_p u = 0, \quad u_0 \in L^\sigma(\Omega).$$

If $1 < p < \frac{2N}{N+\sigma}$ then

$$\|u(t)\|_{L^r(\Omega)} \leq c\|u_0\|_{L^\sigma(\Omega)}^{\frac{\sigma[2N-p(N+r)]}{r[2N-p(N+\sigma)]}} t^{-\frac{N(\sigma-r)}{r(2N-p(N+\sigma))}} \quad \text{with} \quad r < \sigma.$$

If $\frac{2N}{N+\sigma} < p < N$ then the decay above holds with $r > \sigma$ and

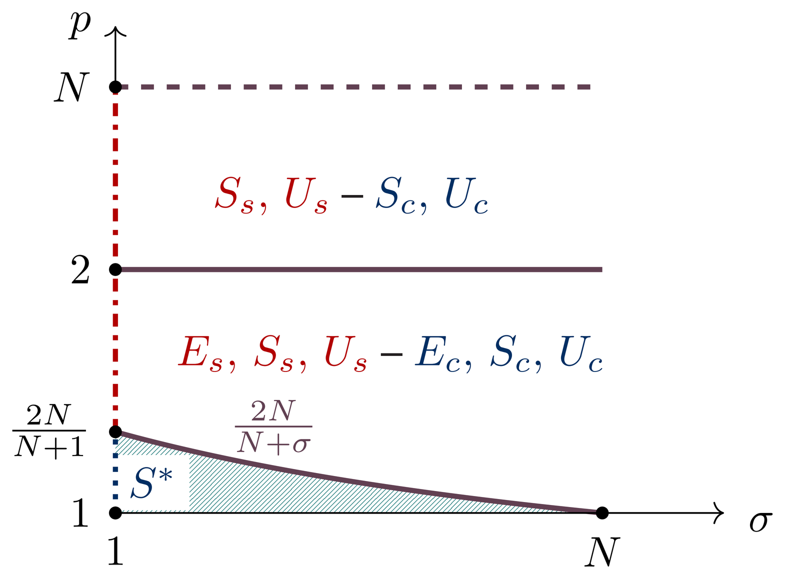
$$\|u(t)\|_{L^\infty(\Omega)} \leq c\|u_0\|_{L^\sigma(\Omega)}^{\frac{p\sigma}{p(N+\sigma)-2N}} t^{-\frac{N}{p(N+\sigma)-2N}}.$$

Having (Q) implies that

$$p > \frac{2N}{N+\sigma}$$

and this means that, if we are in the superlinear setting (see (Q)) and a solution of (P) exists, then such a solution regularizes.

Superlinear problem ($X_s$) Coercive problem ($X_c$)




$S_s, S_c$ = regularizing effect/decay $L^\sigma - L^r$, $r > \sigma$
$U_s, U_c$ = regularizing effect/decay $L^\sigma - L^\infty$
$E_s, E_c$ = extinction in finite time
$S_c^*$ = decay $L^\sigma - L^r$, $r < \sigma$
= nonexistence for superlinear $q$
= $q > \max\left\{\frac{p}{2}, \frac{p(N+1)-N}{N+2}\right\}$
= $q \leq \max\left\{\frac{p}{2}, \frac{p(N+1)-N}{N+2}\right\}$

**RMK 4:** the decay is not obvious in superlinear settings: as shown in [P], the problem

$$u_t - \Delta u = |u|^q \qquad \text{with} \quad q > 1$$

does not admit global solutions and blow up phenomena occur!

## 9. REFERENCES


[M1] M. Magliocca, *Existence results for a Cauchy-Dirichlet parabolic problem with a repulsive gradient term*, Nonlin. Anal., Vol. 166 (2018), pp. 102-143.

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