The regularizing effect of perturbed superlinear gradient terms in elliptic equations

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Joint work with M. Latorre Balado & S. Segura de León

Introduction

Some stuff on superlinear problems

Superlinear gradient...what?

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Example

⇝ If

$$\lambda u - \Delta u = c|\nabla u|^q + f \quad \text{and} \quad u_t - \Delta u = c|\nabla u|^q + f$$

$$\text{when} \quad q \in (0,1],$$

then u is enough as test function because

$$c\int |\nabla u|^q u \le c\left(\int |\nabla u|^2\right)^{\frac{q}{2}} \left(\int u^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \quad \text{and} \quad \frac{2}{2-q} \le 2$$

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⇝ Instead, if

$$\lambda u - \Delta u = c|\nabla u|^q + f \quad \text{and} \quad u_t - \Delta u = |u|^r + |\nabla u|^q + f$$

$$\text{with } q \text{ superlinear},$$

then $\frac{2}{2-q} > 2$ and we cannot close the estimate as before.

We need something more than u!

Superlinear issues

Point 1. The superlinearity influences the choice of the data: unlike coercive problems, we need a **compatibility condition** between the superlinear terms and the data.

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If

$$u_t - \Delta_p u = |\nabla u|^q + f, \quad u(0,x) = u_0(x) \quad \text{with } 1 < q < 2$$

f and u_0 have to be regular enough w.r.t. q in order to have solutions.

✎ M. Ben-Artzi, P. Souplet & F. Weissler, J. Math. Pures Appl. (2002).
Linear Cauchy problem ($p = 2$, semigroup theory).

✎ M. M., Nonlin. Anal. (2018).
Nonlinear Cauchy-Dirichlet problem ($\sim 1 < p < N$).

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✎ N. Grenon, F. Murat & A. Porretta, Ann. Sc. Norm. Super. Pisa (2014).
Nonlinear Dirichlet problem ($\sim 1 < p < N$).

Superlinear issues

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What if...

... we assume a lower regularity in the data when the growth is superlinear?
Nonexistence occurs because we do not respect the compatibility condition!

M. Ben-Artzi, P. Souplet & F. Weissler; N. Grenon, F. Murat & A. Porretta

✎ M. M. & A. Porretta, Proc. London Math. Society (2019).
Nonlinear Cauchy-Dirichlet problem ($p \sim 2$).

Superlinear issues

Point 2. We need to deal with solutions belonging to a certain **well posedness class** which depends on the compatibility condition.

What if...

...the solution is not regular enough?

Uniqueness may fail because solutions no longer belong to the right class!

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✎ T. Leonori & M. M., Comm. Pure Appl. Anal. (2019).

Parabolic problem with superlinear but subnatural growth $(1 < q < 2)$:

$$\exists u > 0 \text{ solution of } u_t - \Delta u = |\nabla u|^q,\ u(0,x) = 0$$

but $\exists! u$ if u belongs to the well posedness class

$$|u|^{\frac{N(q-1)}{2(2-q)}} \in L^2(0,T;H_0^1(\Omega)).$$

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✎ **B. Abdellaoui, A. Dall'Aglio & I. Peral, J. Mat. Pures et Appl. (2008).**

Parabolic problem with natural growth ($q = 2$):

$$\exists u > 0 \text{ solution of } u_t - \Delta u = |\nabla u|^2,\ u(0,x) = 0$$

but $\exists! u$ if u belongs to the well posedness class

$$(e^u - 1) \in L^2(0,T;H_0^1(\Omega)).$$

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✎ N. Grenon, F. Murat & A. Porretta, Ann. Sc. Norm. Super. Pisa (2014).
Elliptic problem with superlinear but subnatural growth ($1 < q < 2$):

$$\exists u > 0 \text{ solution of } -\Delta u = |\nabla u|^q$$

but $\exists! u$ if u belongs to the well posedness class

$$|u|^{\frac{(N-2)(q-1)}{2(2-q)}} \in H_0^1(\Omega).$$

Perturbations and superlinearity

We now focus on gradient term superlinearities in the elliptic setting.
What happens if **we perturb the superlinearity**, i.e.

$$\lambda u - \Delta u = g(u)|\nabla u|^q + f,$$

but **we still want q to be superlinear**?

Wait...**what is** $g(u)$?

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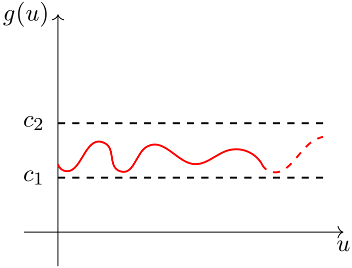
Some typical choices:



$$0 < c_1 \leq g(u) \leq c_2$$

Basically, the case $g(u) =$ const.
(N. Grenon, F. Murat & A. Porretta)

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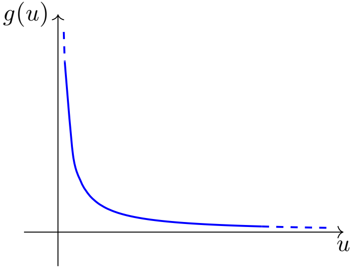

The singular case

D. Giachetti, F. Petitta & S. Segura de León

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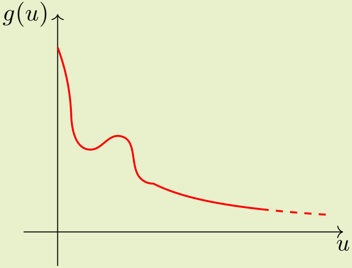
We choose a mashup



$g(u)$ does not explode in 0
and decays to 0 when $u \to \infty$.

Known stuff

D. Arcoya, L. Boccardo, D. Giachetti, T. Leonori, F. Petitta, A. Porretta, S. Segura de León, ...

$$-\Delta_p u = g(u)\,|\nabla u|^p + f$$

✎ A. Porretta & S. Segura de León, J. Math. Pures Appl. (2006).
$g(u)$ not singular in 0 and $g(u) \leq \frac{c}{|u|}$.

✎ D. Arcoya, L. Boccardo, T. Leonori & A. Porretta, J. Diff. Eq. (2010).
$p=2$, $u>0$, $f\geq 0$, $g(u)$ singular in 0 and $g(u) \leq \frac{c}{u}$.

✎ D. Giachetti, F. Petitta & S. Segura de León, Comm. Pure Appl. Anal. (2012).
$p=2$, $g(u)$ singular in 0 and $g(u) \leq \frac{c}{|u|^\alpha}$ for $0<\alpha<1$.

The perturbation term $g(u)$ regularizes: we have the same solutions regularity of the case $g(u)\equiv$ const. for lower regularities of f.

The superlinear elliptic problem

A middle way between the cases $\lambda|\nabla u|^q$ and $\lambda|\nabla u|^p/|u|^\alpha$

The elliptic problem

Let $\Omega \subsetneq \mathbb{R}^N$ to be open and bounded. The problem we consider is

$$\begin{cases} -\Delta u = g(u)|\nabla u|^q + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

being q superlinear, $g : \mathbb{R} \to (0, +\infty)$ continuous s.t.

$$g(u) \le \frac{\lambda}{|u|^\alpha} \quad \text{with} \quad \lambda,\, \alpha > 0,$$

and $f \in L^m(\Omega)$ for a certain $m = m(\alpha, q, N)$.

(We will assume $u \ge 0$, $f \ge 0$ and $\Delta\cdot$. The incoming results hold for nonlinear divergence operators and for sign changing u.)

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Aim: we are interested in the regularizing effect induced by $g(u)$. For this reason, we do not take into account a possible singular behavior in 0.

The superlinear setting

$$-\Delta u = \lambda \frac{|\nabla u|^q}{u^\alpha} + f \quad \text{with} \quad \alpha > 0$$

Case $0 < \alpha < 1$: q is superlinear but subnatural if

$$1 + \alpha < q < 2.$$

(Hint: look at the homogeneity!)
Case $\alpha \geq 1$: no superlinear growth occurs!

Consequences

If $0 < \alpha < 1$ and $1 + \alpha < q < 2$ we need $f \in L^m(\Omega)$ with $m \geq \overline{m} = \overline{m}(q, \alpha)$.
Having $\alpha \geq 1$ makes the problem easier and $f \in L^1(\Omega)$ is enough.

Perturbed VS no perturbed case

If $\alpha = 0$, then $1 < q < 2$. Having $\alpha > 0$ improves the superlinear threshold!

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Focusing on m

We suppose that $1+\alpha < q < 2$, $0 < \alpha < 1$.

$$-\Delta u = \lambda \frac{|\nabla u|^q}{u^\alpha}$$

We take a test function which behaves as $u^{\sigma-1}$ and the estimate is closed if

$$\sigma \geq \frac{(N-2)(q-(1+\alpha))}{2-q}.$$

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$$-\Delta u=f\in L^m(\Omega)$$

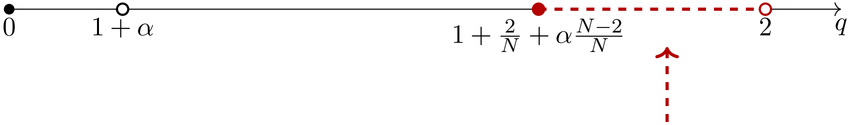
Let $\sigma=(N-2)(q-(1+\alpha))/(2-q)$. We need

$$m\geq\frac{N\sigma}{N+\sigma-2}=\frac{N(q-(1+\alpha))}{q-2\alpha}.$$

On σ and m

$$0 < \alpha < 1, \qquad 1 + \alpha < q < 2$$

$$\sigma = \frac{(N-2)(q-(1+\alpha))}{2-q}, \quad m = \frac{N(q-(1+\alpha))}{q-2\alpha}$$



$$\sigma \geq 2$$

$$\frac{2N}{N+2} \leq m < \frac{N}{2}$$

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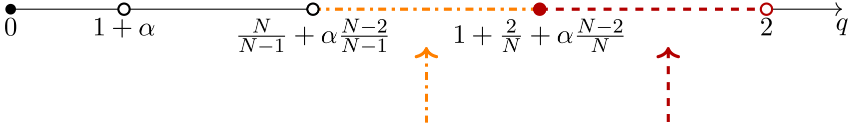

$1 < \sigma < 2$ $\qquad$ $\sigma \geq 2$

$$1 < m < \frac{2N}{N+2} \qquad \frac{2N}{N+2} \leq m < \frac{N}{2}$$

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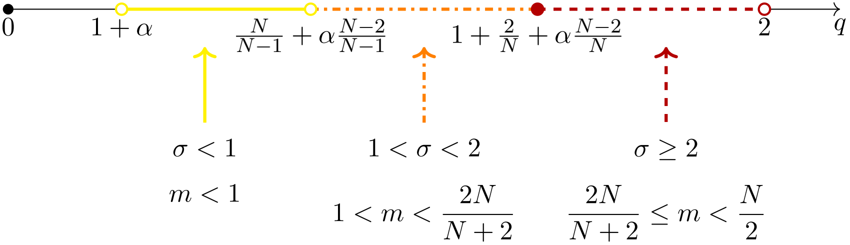

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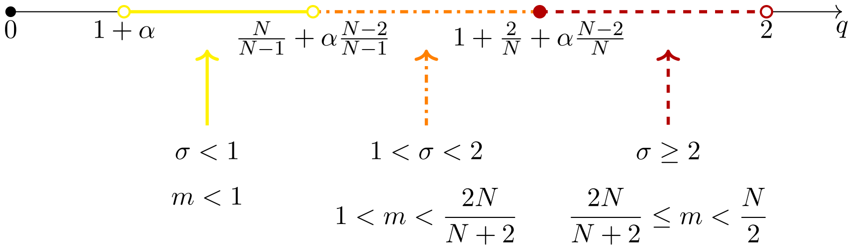

The case $q=\frac{N}{N-1}+\alpha\frac{N-2}{N-1}$: something more than L^1-data is needed.

The regularity class

$$-\Delta u = \lambda \frac{|\nabla u|^q}{u^\alpha} + f \quad \text{with } \alpha + 1 < q < 2$$

In order to have the problem well-posed we need to work with

$$\left\{ u : \quad u^{\frac{\sigma}{2}} \in H_0^1(\Omega) \right\} \quad \text{when } \sigma \geq 2,$$

$$\left\{ u : \quad (1+u)^{\frac{\sigma}{2}-1} u \in H_0^1(\Omega) \right\} \quad \text{when } 1 < \sigma < 2.$$

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What about the case $\sigma < 1$?

Existence results

To the a priori estimates and beyond

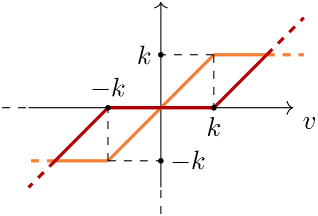
Tools and guideline

Tools

$$T_k(v) = \max\{-k, \min\{k, v\}\}$$

$$G_k(v) = (|v| - k)_+ \mathsf{sign}(v)$$

$$v = T_k(v) + G_k(v)$$



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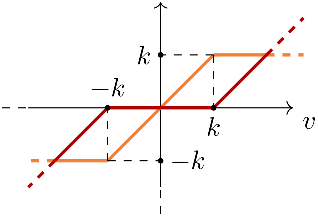


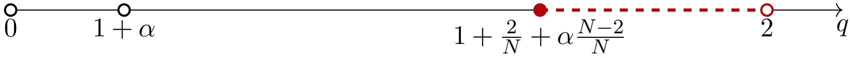
$G_k(v)$ and $T_k(v)$

Guideline

We reason through approximation techniques. Hence, we need

- suitable uniform bounds;
- the strong convergence in $L^1(\Omega)$ of the r.h.s..

The case $\sigma \geq 2$



$$\sigma \geq 2, \qquad \frac{2N}{N+2} \leq m < \frac{N}{2}$$

Definition - Finite Energy Solutions (FES)

A function $u \in H_0^1(\Omega)$ is a finite energy solution if $g(u)|\nabla u|^q \in L^1(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega g(u)|\nabla u|^q \varphi + \int_\Omega f(x)\varphi$$

holds for all $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

The a priori estimate

Proposition

Let $f \in L^m(\Omega)$ with $m = \frac{N(q-(1+\alpha))}{q-2\alpha}$, $\|f\|_{L^m}$ small and $u^{\frac{\sigma}{2}} \in H_0^1(\Omega)$. Then

$$\left\|u^{\frac{\sigma}{2}}\right\|_{H_0^1} \le M$$

with $M > 0$ depending on the parameters of the problem and $\|f\|_{L^m}$.

Sketch of the proof

We take $G_k(u)^{\sigma-1}$ as test function so that

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$$\frac{4(\sigma-1)}{\sigma^2} \left\| \left| \nabla G_k(u)^{\frac{\sigma}{2}} \right| \right\|_{H_0^1}^2 = \int_\Omega g(u) \left|\nabla u\right|^q G_k(u)^{\sigma-1} + \int_\Omega f G_k(u)^{\sigma-1}$$

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since $(\sigma-1)m' = 2^* \frac{\sigma}{2}$ and by Sobolev's inequality.

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$$\frac{4(\sigma-1)}{\sigma^2} \left\| \left| \nabla G_k(u)^{\frac{\sigma}{2}} \right| \right\|_{H_0^1}^2 \leq \lambda \int_\Omega |\nabla G_k(u)|^q \, G_k(u)^{\sigma-1-\alpha} + c_S \|f\|_{L^m} \left\| \left| \nabla G_k(u)^{\frac{\sigma}{2}} \right| \right\|_{H_0^1}^{2\frac{\sigma-1}{\sigma}}$$

since $(\sigma-1)m' = 2^* \frac{\sigma}{2}$ and by Sobolev's inequality.

$$\lambda \int |\nabla G_k(u)|^q \, G_k(u)^{\sigma-1-\alpha} \leq \lambda \frac{2^q}{\sigma^q} \int \left| \nabla G_k(u)^{\frac{\sigma}{2}} \right|^q G_k(u)^{\sigma-1-\alpha-q\left(\frac{\sigma}{2}-1\right)}$$

$$\leq \lambda \frac{2^q}{\sigma^q} \left\| \left| \nabla G_k(u)^{\frac{\sigma}{2}} \right| \right\|_{H_0^1}^q \left(\int G_k(u)^{\left(\sigma-1-\alpha-q\left(\frac{\sigma}{2}-1\right)\right)\frac{2}{2-q}} \right)$$

$$\leq \lambda \frac{2^q}{\sigma^q} c_S \left\| \left| \nabla G_k(u)^{\frac{\sigma}{2}} \right| \right\|_{H_0^1}^{2\frac{N-q}{N-2}}$$

since $\left(\sigma-1-\alpha-q\left(\frac{\sigma}{2}-1\right)\right)\frac{2}{2-q} = 2^*\frac{\sigma}{2}$ and by Sobolev's inequality.

So far

for $Y_k = \left\| \left| \nabla G_k(u)^{\frac{\sigma}{2}} \right| \right\|_{H_0^1}^2$ and $C_i = C_i(N,q,\gamma)$.

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$$C_1 Y_k \le C_2 Y_k^{\frac{N-q}{N-2}} + C_3 \|f\|_{L^m} Y_k^{\frac{\sigma-1}{\sigma}}$$

for $Y_k = \left\|\left|\nabla G_k(u)^{\frac{\sigma}{2}}\right|\right\|_{H_0^1}^2$ and $C_i = C_i(N, q, \gamma)$.

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Let

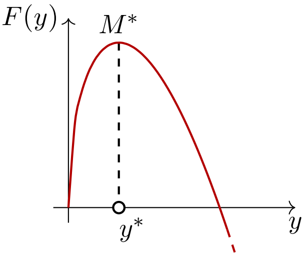
$$F(y) = C_1 y^{\frac{1}{\sigma}} - C_2 y^{\frac{N-q}{N-2}-\frac{\sigma-1}{\sigma}}.$$

We have

$$F(Y_k) \le C_3 \|f\|_{L^m} \qquad \forall k \ge 0.$$

Properties of $F(y) = C_1 y^{\frac{1}{\sigma}} - C_2 y^{\frac{N-q}{N-2} - \frac{\sigma-1}{\sigma}}$

- $\frac{1}{\sigma} < \frac{N-q}{N-2} - \frac{\sigma-1}{\sigma}$;
- $F(0) = 0$;
- $\lim_{y\to\infty} F(y) = -\infty$;
- $\exists y^* : \quad M^* = F(y^*) = \max F(y)$.



We take f such that

$$F(Y_k) \le C_3 \|f\|_{L^m} < M^*.$$

Note that $F(y) = C_3 \|f\|_{L^m}$ has two roots $y_-(\|f\|_{L^m}) < y^* < y_+(\|f\|_{L^m})$.

Key point: since $u^{\frac{\sigma}{2}} \in H_0^1(\Omega)$ then

$$k \longrightarrow Y_k = \left\| \left| \nabla G_k(u)^{\frac{\sigma}{2}} \right| \right\|_{H_0^1}^2$$

is continuous and $Y_k \to 0$ when $k \to \infty$,hence

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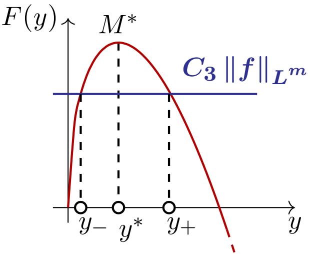

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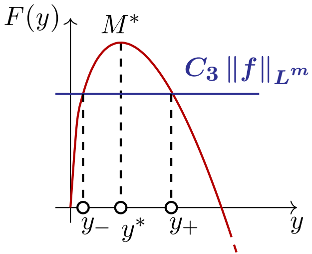

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The approximating problem

✎ A. Porretta & S. Segura de León, J. Math. Pures Appl. (2006).

We consider the approximating problem

$$\begin{cases} -\Delta u_n = T_n\left(g(u_n)|\nabla u_n|^q + f(x)\right) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

which admits solutions $\{u_n\}_n \in L^\infty(\Omega) \cap H_0^1(\Omega)$ verifying

$$\int_\Omega \nabla u_n \cdot \nabla \varphi = \int_\Omega T_n\left(g(u_n)|\nabla u_n|^q + f(x)\right)\varphi \qquad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Convergence results

- The previous Proposition ensures that $\left\{u_n^{\frac{\sigma}{2}}\right\}_n$ is uniformly bounded in $H_0^1(\Omega)$. Then, the same holds for $\{u_n\}_n$ (test u_n). We thus deduce

$$u_n \longrightarrow u \quad \text{a.e. in } \Omega,$$

and $u_n \rightharpoonup u$ in $H_0^1(\Omega)$.

- The r.h.s. is equi-integrable in $L^1(\Omega)$, indeed

$$\sup_n \int_E g(u_n)|\nabla u_n|^q \leq c \sup_n \int_E |\nabla u_n|^q < \varepsilon \quad \text{for } |E| < \delta_\varepsilon.$$

Then, L. Boccardo & F. Murat applies and

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega.$$

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$$\text{with } S \in W^{1,\infty}(\mathbb{R}) \quad \text{s.t.} \quad \text{supp}(S(u)) \subseteq [-L, L]$$

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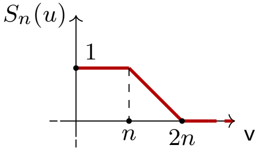
$$S_n(u) = \begin{cases} 1 & 0 < u \le n, \\ \dfrac{2n-u}{n} & n < u \le 2n, \\ 0 & u > 2n. \end{cases}$$



$$-\frac{1}{n}\int_{\{n<u\le 2n\}} |\nabla u|^2\varphi + \int_\Omega S_n(u)\nabla u\cdot\nabla\varphi = \int_\Omega F\varphi S_n(u) \qquad \varphi \quad \text{smooth.}$$

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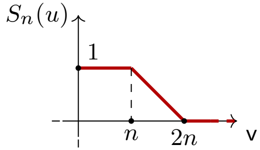

$$\frac{1}{n}\int_{\{n<u\le 2n\}} |\nabla u|^2\varphi = \int_\Omega S_n(u)\nabla u\cdot\nabla\varphi - \int_\Omega F\varphi S_n(u) \qquad \varphi \quad \text{smooth.}$$

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Advantages

We read the equation for the bounded part of u.

But... what about large value of u?

We need to ask for the **asymptotic condition**

$$\lim_{n\to\infty} \frac{1}{n} \int_{\{n<u\leq 2n\}} |\nabla u|^2 \varphi = 0$$

in order to recover the usual weak formulation.

The case $1 < \sigma < 2$

$$1 < \sigma < 2, \quad 1 < m < \frac{2N}{N+2}$$

Definition - Renormalized Solutions (RS) with Lebesgue data

A function

$$u \in \mathcal{T}_0^{1,2}(\Omega) = \Big\{ v \text{ a.e. finite s.t. } \quad T_k(v) \in H_0^1(\Omega) \quad \forall k > 0 \Big\}$$

is a renormalized solution if

- $g(u)\,|\nabla u|^q \in L^1(\Omega)$;
- $\forall S \in W^{1,\infty}(\mathbb{R})$ with compact support, $\forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ s.t. $S(u)\varphi \in H_0^1(\Omega)$ it holds

$$\int_\Omega S'(u)\varphi|\nabla u|^2 + \int_\Omega S(u)\nabla u \cdot \nabla\varphi = \int_\Omega g(u)|\nabla u|^q S(u)\varphi + \int_\Omega f(x)S(u)\varphi.$$

Some comments

Stranger things...

...but not too much: the asymptotic condition is not needed because it is naturally induced by the regularity class, i.e. if

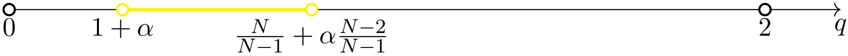
$$(1+u)^{\frac{\sigma}{2}-1}u \in H_0^1(\Omega)$$

then

$$\lim_{n\to\infty} \frac{1}{n} \int_{\{n<u\leq 2n\}} |\nabla u|^2 \varphi = 0.$$

The proofs in this case are very (very) similar to the previous one.
Again, the regularity class is **crucial**, both in the a priori estimate and in the convergences proofs.

The case $\sigma < 1$



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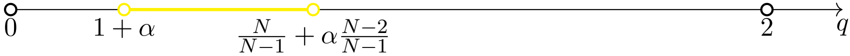

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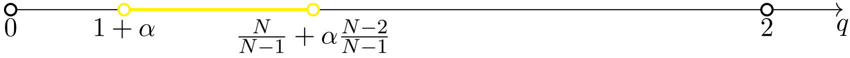

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Properties of $\mu \in \mathcal{M}_b(\Omega)$

$$\forall \mu \in \mathcal{M}_b(\Omega) \quad \exists! \mu_0,\, \mu_s : \quad \mu = \mu_0 + \mu_s$$

$$\mu_0 \in L^1(\Omega) + H^{-1}(\Omega), \quad \mu_s \lfloor E \quad (\mu_s(B) = \mu_s(B \cap E)\ \forall B)$$

✎ L. Boccardo, T. Galloüet & L. Orsina, Ann. Inst. Henri Poincaré, Non Lin. Anal. (1996).

✎ G. Dal Maso, F. Murat, L. Orsina & A. Prignet, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (1999).

Renormalized Solutions

$$\mathcal{M}_b(\Omega) \ni \mu = \mu_0 + \mu_s$$

Definition - RS with measure data

A function $u \in \mathcal{T}_0^{1,2}(\Omega)$ is a renormalized solution if

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$$\int_\Omega S'(u)\varphi|\nabla u|^2 + \int_\Omega S(u)\nabla u \cdot \nabla\varphi = \int_\Omega g(u)|\nabla u|^q S(u)\varphi + \int_\Omega S(u)\varphi\, d\mu_0;$$

- the asymptotic condition holds:

$$\lim_{n\to\infty} \frac{1}{n} \int_{\{n\le u\le 2n\}} |\nabla u|^2\, \varphi = \int_\Omega \varphi\, d\mu_s \qquad \forall\varphi \in C_b(\Omega).$$

A preliminary result

Marcinkievicz spaces M^p

$$M^p(\Omega) = \left\{ u \text{ meas. s.t. } [\,u\,]_{M^p} = \sup_{k>0} \{k^p |\{x \in \Omega \,:\, u(x) > k\}|\}^{\frac{1}{p}} < \infty \right\}$$

$$L^p(\Omega) \hookrightarrow M^p(\Omega) \hookrightarrow L^{p-\omega}(\Omega) \quad \forall 0 < \omega < p-1$$

Lemma

If

$$\int_\Omega |\nabla T_\ell(u)|^2 \le \ell^r M$$

then

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This Lemma is an extension of the one contained in

✎ P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre & J. L. Vazquez, Ann. Scuola Norm. Sup. Pisa (1995).

The a priori estimate

Proposition

If $\|\mu\|_{\mathcal{M}_b}$ is small enough, we have

$$\int_\Omega g(u)\,|\nabla u|^q \le c \quad \text{and} \quad \int_{\{u>k\}} g(u)\,|\nabla u|^q \to 0 \quad \text{for } k \to \infty.$$

Sketch of the proof

Let $s > q$ and $0 < \theta < 1$ to be fixed later.

We test with $\psi(u) = T_j\left((\varepsilon + G_k(u))^\theta - \varepsilon^\theta\right)$ and compute

$$\int_\Omega \nabla G_k(u) \cdot \nabla \psi(u) \le \int_\Omega \frac{|G_k(u)|^q}{G_k(u)^\alpha}\psi(u) + \int_\Omega \psi(u)\,d\mu$$

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$$c\int_\Omega \left|\nabla T_{j^{\frac{\theta+1}{2\theta}}}\,(\varepsilon + G_k(u))^{\frac{\theta+1}{2}}\right|^2 \le \int_\Omega \frac{|G_k(u)|^q}{G_k(u)^\alpha}\psi(u) + \int_\Omega \psi(u)\,d\mu$$

by definition of $\psi(\cdot)$.

The a priori estimate

Proposition

If $\|\mu\|_{\mathcal{M}_b}$ is small enough, we have

$$\int_\Omega g(u)\,|\nabla u|^q \le c \quad \text{and} \quad \int_{\{u>k\}} g(u)\,|\nabla u|^q \to 0 \quad \text{for } k\to\infty.$$

Sketch of the proof

Let $s>q$ and $0<\theta<1$ to be fixed later.

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by definition of $\psi(\cdot)$.

We set $r = \dfrac{2\theta}{\theta+1}$, $\quad \ell = j^{\frac{1}{r}}$, $\quad \varphi_{\varepsilon,k}(u) = (\varepsilon + G_k(u))^{\frac{\theta+1}{2}}$ and rewrite

$$\int_\Omega \left|\nabla T_\ell\left(\varphi_{\varepsilon,k}(u)\right)\right|^2 \le \ell^r \left[\int_\Omega \frac{|\nabla G_k(u)|^q}{G_k(u)^\alpha} + \|\mu\|_{\mathcal{M}_b}\right] = \ell^r M.$$

We apply the Lemma:

$$\left[|\nabla\varphi_{\varepsilon,k}(u)|\right]_{M^{\frac{N(2-r)}{N-r}}} \le c\left[\int_\Omega \frac{|\nabla G_k(u)|^q}{G_k(u)^\alpha} + \|\mu\|_{\mathcal{M}_b}\right]^{\frac{1}{2-r}}$$

$$\le \dots$$

$$\le \left[\left\||\nabla\varphi_{\varepsilon,k}(u)|\right\|_{L^s}^{s\frac{N-q}{N-s}} + \|\mu\|_{\mathcal{M}_b}\right]^{\frac{1}{2-r}}$$

fixing suitable values of s, θ. In particular, $s < \frac{N(2-r)}{N-r}$. Then

$$\left\||\nabla\varphi_{\varepsilon,k}(u)|\right\|_{L^s} \le c\left[|\nabla\varphi(u)|\right]_{M^{\frac{N(2-r)}{N-r}}} \le c\left[\left\||\nabla\varphi_{\varepsilon,k}(u)|\right\|_{L^s}^{s\frac{N-q}{N-s}} + \|\mu\|_{\mathcal{M}_b}\right]^{\frac{1}{2-r}}.$$

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Let $Y_{k,\varepsilon} = \left\|\left|\nabla \varphi_{\varepsilon,k}(u)\right|\right\|_{L^s}$. Then we have

$$Y_{k,\varepsilon} - C_1 Y_{k,\varepsilon}^{s\frac{N-q}{N-s}\frac{1}{2-r}} \leq C_2 \left\|\mu\right\|_{\mathcal{M}_b}^{\frac{1}{2-r}}$$

Properties of $F(y) = y - C_1 y^{s\frac{N-q}{N-s}\frac{1}{2-r}}$

- $s\frac{N-q}{N-s}\frac{1}{2-r} = q - \alpha > 1$;
- $F(0) = 0$;
- $\lim_{y\to\infty} F(y) = -\infty$;
- $\exists y^* : \quad M^* = F(y^*) = \max F(y)$.

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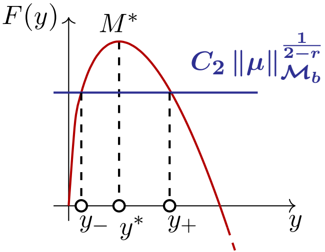

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The approximating problem

✎ A. Porretta, 2002-Fez conference on Partial Differential Equations, Electron. J. Diff. Eqns. Conf. 09 (2002).

$$\begin{cases} -\Delta u_n = T_n\left(g(u_n)|\nabla u_n|^q\right) + \mu_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

where μ_n smooth (e.g. constructed by convolution) satisfies

$$\|\mu_n\|_{L^1} \leq \|\mu\|_{\mathcal{M}_b} \quad \text{and} \quad \lim_{n\to\infty} \int_\Omega \varphi \mu_n = \int_\Omega \varphi \, d\mu \quad \forall \varphi \in C(\overline{\Omega}).$$

There exists $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Convergence results

- The Proposition applies and then $u_n \to u$ in $L^s(\Omega)$ for $1 \le s < \frac{N(2-r)}{N-2}$, $u_n \to u$ a.e., $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ for $1 \le p < \frac{N(2-r)}{N-r}$.
- We have $\nabla u_n \to \nabla u$ a.e. and $\int_\Omega g(u_n)|\nabla u_n|^q \to \int_\Omega g(u)\,|\nabla u|^q$ (L.Boccardo & F. Murat + Vitali's Theorem).
- The strong convergence of

$$T_k(u_n) \to T_k(u) \quad \text{in } H_0^1(\Omega)$$

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Resuming

Theorem - M. Latorre Balado, M.M. & S. Segura de León

Let $0 < \alpha < 1$ and $1 + \alpha < q < 2$. Then

- if $1 + \frac{2}{N} + \alpha\frac{N-2}{N} \leq q < 2$ and $\|f\|_{L^m}$ is small for $m = \frac{N(q-(1+\alpha))}{q-2\alpha}$, we have FES satisfying
$$\left\{u : \quad u^{\frac{\sigma}{2}} \in H^1_0(\Omega)\right\};$$
- if $\frac{N}{N-1} + \alpha\frac{N-2}{N-1} < q < 1 + \frac{2}{N} + \alpha\frac{N-2}{N}$ and $\|f\|_{L^m}$ is small for $m = \frac{N(q-(1+\alpha))}{q-2\alpha}$, we have RS satisfying
$$\left\{u : \quad (1+u)^{\frac{\sigma}{2}-1}u \in H^1_0(\Omega)\right\};$$
- $1 + \alpha < q < \frac{N}{N-1} + \alpha\frac{N-2}{N-1}$ and $\mu \in \mathcal{M}_b(\Omega)$, $\|\mu\|_{\mathcal{M}_b}$ small, we have RS.

The critical case

$$-\Delta u = \frac{\lambda}{u^\alpha}|\nabla u|^{\frac{N}{N-1}+\alpha\frac{N-2}{N-1}} + f$$

In this case

$$m = \left.\frac{(N-2)(q-(1+\alpha))}{q-2\alpha}\right|_{q=\frac{N}{N-1}+\alpha\frac{N-2}{N-1}} = 1$$

but **L^1-data are NOT enough!!**

✎ N. Grenon, F. Murat & A. Porretta, Ann. Sc. Norm. Super. Pisa (2014). Nonlinear Dirichlet problem with $g(u) = $ const..

Metatheorem - M. Latorre Balado, M.M. & S. Segura de León

If $q = \frac{N}{N-1} + \alpha\frac{N-2}{N-1}$ and $\|f\|$ is small then it is enough to take

- $f \in L^{1+\varepsilon}(\Omega)$, $\varepsilon > 0$;
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Theorem - M. Latorre Balado, M.M. & S. Segura de León

If λ is small and $f \in L^m(\Omega)$ for

- $\frac{2N}{N+2} \leq m < \frac{N}{2}$, we have FES satisfying $u^{\frac{m(N-2)}{2(N-2m)}} \in H_0^1(\Omega)$;
- $1 < m < \frac{2N}{N+2}$, we have RS satisfying $(1+u)^{\frac{m(N-2)}{2(N-2m)}-1} u \in H_0^1(\Omega)$;
- $m = 1$, we have RS.

The smallness of λ is needed here:

$$C_1 Y_k^{\frac{1}{\sigma}} \leq \lambda C_2 Y_k^{\frac{1}{\sigma}} + C_3 \|f\|_{L^m} \quad \text{for } Y_k = \left\| \left| \nabla G_k(u)^{\frac{\sigma}{2}} \right| \right\|_{H_0^1}^2 .$$

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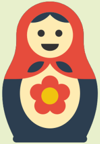
Regularity results



The regularizing^2 effect

$$-\Delta u = \lambda |\nabla u|^q + f \in L^r(\Omega)$$

with $\quad 1 < q < 2$

- If $r > \frac{N}{2} \Rightarrow \quad u$ bounded;
- if $r = \frac{N}{2} \Rightarrow \quad u \in L^d(\Omega)$ for $d < \infty$;
- if $\frac{2N}{N+2} < r < \frac{N}{2}$
  $\Rightarrow \quad u^{\frac{t}{2}} \in H_0^1(\Omega)$ for $t = \frac{r(N-2)}{N-2r}$;
- if $\frac{N(q-1)}{q} < r < \frac{2N}{N+2}$
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- If $\frac{N}{2} < s (< r) \Rightarrow \quad u$ bounded;
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Concluding...

Theorem - M. Latorre Balado, M.M. & S. Segura de León

Our solutions have the same regularity of the case $\alpha = 0$ for lower f regularities and/or greater q growth.

Remark

The regularity class plays an important role in the proof.