Global existence and decay to equilibrium for some crystal surface models

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Joint work with R. Granero Belinchón

Crystal...what?

Tools and presentation of the problem

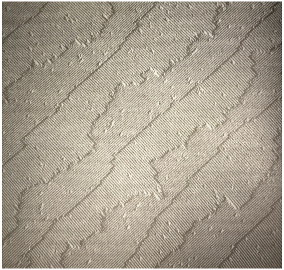
Crystal surfaces

How crystal surfaces are

Crystal surfaces is made up of terraces separated by steps of atomic height. These steps contain straight parts separated by kinks. On the terraces, there are surface vacancies (or advacancies) resulting from missing surface atoms.

How crystal surfaces evolve

Under ultra-high vacuum conditions, atoms are sent onto the surface, and they diffuse until they are incorporated. These process depends on the crystal material, temperature, pressure...



Physics of Crystal Growth
A. Pimpinelli & J. Villain

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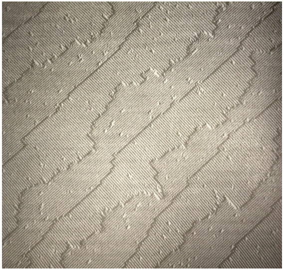

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From micro to macroscopic descriptions

Crystal surfaces are described through EDOs at microscopic level and by PDEs at the macroscopic one. These are obtained as limit of EDOs.

Our mathematical point of view

We will focus on IV order parabolic PDEs describing the evolution of crystal surfaces.

Applications

"It is important to study this evolution [the crystal ones], *because the manufacture of crystal films lies at the heart of modern nanotechnology."*

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- Modern electronic devices (e.g. mobile phone antennae).
- New technologies based on nonlinear optics, plasmonics, photovoltaics and photocalysis.

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  and references therein (*Nature*, *J. Electrochem. Soc.*, *J. Stat. Phys.*,...).
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  ✎ M. Khenner, J. Appl. Phys. (2018)
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Basic tools
The Fourier series

- k-th Fourier coefficient of 2π-periodic $u = u(t,x)$:

$$\widehat{u}(t,k) = \frac{1}{(2\pi)^N}\int_{\mathbb{T}^N} u(t,x)e^{-ix\cdot k}\,dx \quad k\in\mathbb{Z}^N;$$

- the Fourier series:

$$u(t,x) = \sum_{k\in\mathbb{Z}^N}\widehat{u}(t,k)e^{ix\cdot k};$$

- some properties of $\widehat{\cdot}$:

$$\widehat{uv}(t,k) = \sum_{j\in\mathbb{Z}^N}\widehat{u}(t,j)\,\widehat{v}(t,k-j) \quad k\in\mathbb{Z}^N;$$

$$\widehat{(\partial^\alpha u)}(t,k) = (ik)^\alpha\widehat{u}(t,k) \quad k\in\mathbb{Z}^N.$$

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Basic tools

Sobolev and Wiener spaces

- Sobolev spaces $W^{n,p}(\mathbb{T}^N)$, $H^\alpha(\mathbb{T}^N)$:

$$W^{n,p}(\mathbb{T}^N) = \left\{ u \in L^p(\mathbb{T}^N) : \quad \|u\|^p_{W^{n,p}} = \|u\|^p_{L^p} + \|\partial^n_x u\|^p_{L^p} < \infty \right\},$$

$$H^\alpha(\mathbb{T}^N) = \left\{ u \in L^2(\mathbb{T}^N) : \quad \|u\|^2_{H^\alpha} = \sum_{k \in \mathbb{Z}^N} |k|^{2\alpha} |\widehat{u}(k)|^2 < \infty \right\};$$

- Wiener space $A^\alpha(\mathbb{T}^N)$:

$$A^\alpha(\mathbb{T}^N) = \left\{ u \in L^1(\mathbb{T}^N) : \quad \|u\|_{A^\alpha} = \sum_{k \in \mathbb{Z}^N} |k|^{\alpha} |\widehat{u}(k)| < \infty \right\};$$

- interpolation inequality between Wiener spaces

$$\|u\|_{A^s} \leq \|u\|^{1-\theta}_{A^0} \|u\|^{\theta}_{A^r} \quad \forall 0 \leq s \leq r\,, \ \theta = \frac{s}{r}.$$

Why using Wiener spaces?

Motivations through rough examples

- **The heat equation:**
  - $u_t - \Delta u = 0 \quad + \quad u_0 \in L^2(\mathbb{T}^N) \Rightarrow \quad u \in L^2(0,T;H^1(\mathbb{T}^N))$
  - $u_t - \Delta u = 0 \quad + \quad u_0 \in A^0(\mathbb{T}^N) \Rightarrow \quad u \in L^1(0,T;A^2(\mathbb{T}^N))$

  (Fourier series $\rightsquigarrow \quad \partial_t \widehat{u}(t,k) + |k|^2 \widehat{u}(t,k) = 0 \quad \forall k \in \mathbb{Z}^N$, $\partial_t |\widehat{u}(t,k)| = \mathrm{Re}(\overline{\widehat{u}}(t,k)\partial_t \widehat{u}(t,k))/|\widehat{u}(t,k)| \rightsquigarrow \frac{d}{dt}\|u(t)\|_{A^0} + \|u(t)\|_{A^2} = 0.$)

- **The biharmonic heat equation:**
  - $u_t + \Delta^2 u = 0 \quad + \quad u_0 \in L^2(\mathbb{T}^N) \Rightarrow \quad u \in L^2(0,T;H^2(\mathbb{T}^N))$
  - $u_t + \Delta^2 u = 0 \quad + \quad u_0 \in A^0(\mathbb{T}^N) \Rightarrow \quad u \in L^1(0,T;A^4(\mathbb{T}^N))$

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We gain spatial regularity!!

The main problem

$$\begin{cases} \partial_t u = \Delta e^{-\Delta u} & \text{in } (0,T)\times\mathbb{T}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{T}^N, \end{cases} \qquad \text{with} \quad \langle u_0\rangle := \frac{1}{|\mathbb{T}^N|}\int_{\mathbb{T}^N} u_0\,dx = 0,$$

where u represents the crystal height. This model has been introduced by

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Possible ways and results

$$u_t = \Delta e^{-\Delta u} \qquad \text{in } (0,T)\times\Omega, \quad \Omega \subsetneq \mathbb{R}^N, \quad N \geq 1$$

Y. Gao, J.-G. Liu, Y. Lu, X. Xu, ...

- Lyapunov functionals;
- passing from IV order equations to II order elliptic-parabolic systems;
- Sobolev initial data (with/without smallness conditions);
- uniqueness for Δu regular enough.

The change of variable

We choose $v = f(u)$ such that the fully nonlinear problem becomes quasilinear.

The change of variable: $v = \Delta u$

$$\begin{cases} \partial_t v = \Delta^2 e^{-v} & \text{in } (0,T)\times\mathbb{T}^N, \\ v(0,x) = v_0(x) & \text{in } \mathbb{T}^N, \end{cases} \qquad \text{with} \quad \langle v_0\rangle = 0.$$

Definition

A function $v \in L^\infty(0,T;L^\infty(\mathbb{T}^N))$ is a weak solution if

$$\int_{\mathbb{T}^N} \varphi(0,x)v_0(x)\,dx - \iint_{(0,T)\times\mathbb{T}^N} \partial_t\varphi(t,x)v(t,x) + \Delta^2\varphi(t,x)\,e^{-v(t,x)}\,dx\,dt = 0,$$

for all $\varphi \in W^{1,1}(0,T;L^1(\mathbb{T}^N)) \cap L^1(0,T;W^{4,1}(\mathbb{T}^N))$.

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Main results

Existence and Regularity

Main results

Theorem - R. Granero Belinchón & M. M.

Existence: If $v_0 \in A^0(\mathbb{T}^N)$ is a medium size datum*, then there exists at least a global solution such that

$$v \in L^\infty(0,T;L^\infty(\mathbb{T}^N)) \cap L^{\frac{4}{3}}(0,T;W^{3,\infty}(\mathbb{T}^N))$$
$$\cap \mathcal{M}(0,T;W^{4,\infty}(\mathbb{T}^N)) \cap L^2(0,T;H^2(\mathbb{T}^N)),$$

$$\|v(t)\|_{L^\infty} \le \|v_0\|_{A^0} e^{-c(v_0)t} \qquad \forall t \in (0,T).$$

Regularity: If v_0 verifies also $v_0 \in A^0(\mathbb{T}^N) \cap H^2(\mathbb{T}^N)$, then

$$v \in C([0,T];H^2(\mathbb{T}^N)) \cap L^2(0,T;H^4(\mathbb{T}^N)),$$

$$\|v(t)\|^2_{H^2} + \int_0^T \|v(t)\|^2_{H^4}\, ds \le c(v_0),$$

$$\|v(t)\|_{H^r} \le c(v_0) e^{-c(v_0)t} \quad \text{con} \quad 0 \le r < 2.$$

* Its smallness condition is explicit.

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The a priori estimate

$$\begin{cases} \partial_t v_n = \Delta^2 \sum_{j=0}^{n} \frac{(-1)^j}{j!} v_n^j & \text{in } (0,T)\times\mathbb{T}^N, \\ v_n(0,x) = v_0 = \Delta u_0 & \text{in } \mathbb{T}^N. \end{cases}$$

Proposition - A priori estimate

If $v_0 \in A^0(\mathbb{T}^N)$ is a medium size datum, then $\{v_n\}_n$ is uniformly bounded in

$$L^\infty(0,T;A^0(\mathbb{T}^N)) \cap L^1(0,T;A^4(\mathbb{T}^N)).$$

Furthermore

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Remark

We have the same spaces of the biharmonic heat equation.

Sketch of the proof

1. First step: the quasilinear form

$$\partial_t v_n = \sum_{j=0}^{n} \frac{(-1)^j}{j!} (v_n^j)_{,ii\ell\ell}$$

$f_{,j} = \frac{\partial f}{\partial x_j}$ with $j = 1, \ldots, N$.

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N.B. $N_j = N_j(v_n, \dots, (v_n)_{,ii\ell\ell})$.

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2. Second step: the Fourier series

$$\partial_t \widehat{v}_n(t,k) = -|k|^4 \widehat{v}_n(t,k) + \sum_{j=2}^{n} \frac{(-1)^j}{(j-1)!} \widehat{N}_j$$

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$$\frac{d}{dt}\|v_n(t)\|_{A^0} \leq -\|v_n(t)\|_{A^4} + \sum_{j=2}^{n} \frac{1}{(j-1)!}\|N_j(t)\|_{A^0}.$$

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The interpolation inequality leads to

$$\|N_j(t)\|_{A^0} \leq c(j-1, j-2, j-3) \|v_n(t)\|_{A^0}^{j-1} \|v_n(t)\|_{A^4}.$$

$f_{,j} = \frac{\partial f}{\partial x_j}$ with $j = 1, \ldots, N$.

We consider the sum over $j \geq 2$, getting

$$\frac{d}{dt}\|v_n(t)\|_{A^0} \leq \|v_n(t)\|_{A^4}\left(\delta\left(\|v_n(t)\|_{A^0}\right) - 1\right)$$

for $\quad \delta\left(\|v_n(t)\|_{A^0}\right) = \sum_{j\geq 2} c(j-1, j-2, j-3)\frac{\|v_n(t)\|_{A^0}^{j-1}}{(j-1)!}.$

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We need to ask for

$$v_0 \in A^0(\mathbb{T}^N) \text{ such that } \delta(\|v_0\|_{A^0}) - 1 \leq 0.$$

In this way

$$\|v_n(t)\|_{A^0} + (1 - \delta(\|v_0\|_{A^0}))\int_0^t \|v_n(s)\|_{A^4}\, ds \leq \|v_0\|_{A^0},$$

$$\|v_n(t)\|_{A^0} \leq e^{-\left(1-\delta(\|v_0\|_{A^0})\right)t}\|v_0\|_{A^0}.$$

The smallness condition

The inequality

$$\delta\left(\|v_0\|_{A^0}\right) = \sum_{j\geq 2} c(j-1, j-2, j-3)\frac{\|v_0\|_{A^0}^{j-1}}{(j-1)!}$$

$$\Rightarrow$$

$$\boldsymbol{\delta(\|v_0\|_{A^0}) = e^{\|v_0\|_{A^0}}\left(1 + 7\|v_0\|_{A^0} + 6\|v_0\|_{A^0}^2 + \|v_0\|_{A^0}^3\right) - 1} < 0$$

Remark

WolframAlpha $\Rightarrow \quad \|v_0\|_{A^0} < 0.023.$

Compactness results

1. Banach-Alaoglu Theorem: $\{v_n\}_n \overset{*}{\rightharpoonup} v \quad \text{in} \quad L^\infty(0,T;L^\infty(\mathbb{T}^N))$.

2. Interpolation inequality in Wiener spaces:

$$\int_0^T \|v_n(t)\|_{H^2}^2\,dt \le \int_0^T \|v_n(t)\|_{A^2}^2\,dt \le \sup_{t\in(0,T)} \|v_n(t)\|_{A^0} \int_0^T \|v_n(t)\|_{A^4}\,dt < c$$

$$\Rightarrow \quad \{v_n\}_n \rightharpoonup v \quad \text{in} \quad L^2(0,T;H^2(\mathbb{T}^N)).$$

3. Riesz Theorem:

$$\|\partial_t v_n(t)\|_{H^{-2}} = \sup_{\varphi\in H^2,\, \|\varphi\|_{H^2}\le 1} \left|\langle \partial_t v_n(t),\varphi\rangle\right| \overset{\mathbf{2.}+\text{a priori estimate}}{<} c$$

$$\Rightarrow \{\partial_t v_n\}_n \rightharpoonup \partial_t v \quad \text{in} \quad L^2(0,T;H^{-2}(\mathbb{T}^N))$$

4. Gathering **2.** + **3.** + Aubin$\Rightarrow$ $\{v_n\}_n \to v \quad \text{in} \quad L^2(0,T;L^2(\mathbb{T}^N))$.

Conclusion of the existence

Existence: Let us consider

$$\int_{\mathbb{T}^N} \varphi(0,x) v_n(0,x)\, dx - \iint_{(0,T)\times\mathbb{T}^N} \partial_t \varphi v_n + \Delta^2 \varphi\, e^{-v_n}\, dx\, dt = 0.$$

The previous convergences imply

$$\left| \int_{\mathbb{T}^N} \Delta^2 \varphi \left(e^{-v_n} - e^{-v}\right) dx \right| \leq e^{\|v_n(t)\|_{L^\infty} + \|v(t)\|_{L^\infty}} \|\Delta\varphi(t)\|_{L^2} \|v_n(t) - v(t)\|_{L^2},$$

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Exponential decay: Weakly-$\star$ lower semicontinuity of the norm:

$$\|v(t)\|_{L^\infty} \le \liminf_{N\to\infty} \|v_n(t)\|_{A^0} \le e^{-(1-\delta(0))t}\|v_0\|_{A^0}$$

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Regularity

Regularity with $u_0 \in A^0(\mathbb{T}^N)$

- Banach-Alaoglu Theorem ⇝ $\mathcal{M}\left(0,T;W^{4,\infty}(\mathbb{T}^N)\right)$.
- Uniform boundedness + interpolation ⇝ $L^{\frac{4}{3}}\left(0,T;W^{3,\infty}(\mathbb{T}^N)\right)$.
- Sobolev interpolation ⇝ strong convergence in $L^2\left(0,T;H^r(\mathbb{T}^N)\right)$ for $r<2$.

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We prove suitable a priori boundedness and compactness results. It is worth to point out that these results hold with the same smallness condition.

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Remarks

Remarks on Wiener spaces and technique

On the Wiener space $A^0(\mathbb{T}^N)$

The scaling $v_\lambda(t,x) = v(\lambda t, \lambda^4 x)$ is invariant for the $A^0(\mathbb{T}^N)$ norm, i.e.

$$\|v\|_{A^0} = \|v_\lambda\|_{A^0}.$$

In particular

$$\partial_t v_\lambda = \Delta^2 e^{-v_\lambda}.$$

On the technique

- Fourier series $\rightsquigarrow$ derivatives turn into products;
- the smallness condition is explicit and depends only on $\|v_0\|_{A^0}$ $\rightsquigarrow$ good for experiments in lab;
- no restriction on the dimension $N \geq 1$.

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Remarks on solutions and domains

From v to u

If $\Delta u_0 \in A^0(\mathbb{T}^N)$ is medium size, then there exists at least a global weak solution $u \in W^{2,\infty}(\mathbb{T}^N)$.

From $\mathbb{T}^N$ to $\mathbb{R}^N$

$\mathbb{T}^N$

- Fourier series;
- Wiener space defined through Fourier series.

- Fourier transform $\frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{ik\cdot x} \widehat{u}(k)\, dk$;
- Wiener space defined through Fourier transform:

$$A^s(\mathbb{R}^N) = \left\{ v \in L^1(\mathbb{R}^N) : \quad \|v\|_{A^s} = \int_{\mathbb{R}^N} |k|^s |\widehat{v}(k)|\, dk < \infty \right\}$$

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✎ J.G. Liu & R.M. Strain, Interfaces Free Bound. (2019).

More crystal surface models

Another crystal surface problem...

$$\begin{cases} \partial_t u = -u^2 \Delta^2 (u^3) & \text{in } (0,T) \times \mathbb{T}^N, \\ u(0,x) = u_0(x) > 0 & \text{in } \mathbb{T}^N. \end{cases}$$

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✎ **H.A.H. Shehadeh, R.V. Kohn & J. Weare, Phys. D (2011).**

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Theorem - R. Granero Belinchón & M. M.

The same existence and regularity results hold, for a different smallness (but always explicit) condition on $\|v_0\|_{A^0}$.

... and even more!

$$\begin{cases} \partial_t u = \Delta^2(u^{-3}) & \text{in } (0,T)\times\mathbb{T}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{T}^N. \end{cases}$$

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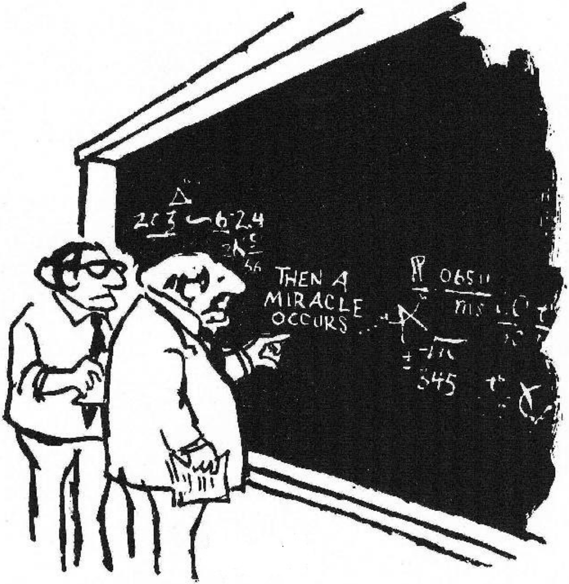

"I think you should be more explicit here in step two."