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Méthodes probabilistes en EDP: applications aux équations dispersives et en mécanique des fluides

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Thèse de mathématiques

MÉTHODES PROBABILISTES EN EDP : APPLICATIONS AUX ÉQUATIONS DISPERSIVES ET EN MÉCANIQUE DES FLUIDES

Mickaël Latocca

Sous la direction de
Nicolas Burq et Isabelle Gallagher

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Chapitre 1

Introduction générale

L'étude des équations aux dérivées partielles dont les données initiales sont considérées aléatoirement constitue l'objet principal du travail présenté dans cette thèse. Les équations étudiées sont d'origine physique diverses, on considère aussi bien l'équation des ondes et l'équation de Schrödinger qui seront qualifiées de *dispersives* mais on s'intéresse également à l'équation d'Euler qui modélise un fluide parfait incompressible.

Cette introduction poursuit un double objectif. Il s'agit à la fois de présenter les méthodes générales qui sont généralement utilisées pour étudier ces équations aux dérivées partielles à données aléatoires, mais aussi de présenter les travaux de cette thèse en les replaçant chacun dans leur contexte.

À cet effet, la section 2 doit être lue comme une longue présentation des deux méthodes du domaine qui me semblent les plus générales et les adaptables, sans doute utiles dans un autre contexte. Les sections suivantes, numérotées de 3 à 6 introduisent chacune un chapitre de la thèse. On replace les travaux dans leur contexte et on met en lumière certaines difficultés qui se manifestent.

Enfin, cette introduction est écrite dans un style peu rigoureux. En particulier on préfère donner des idées de preuve plutôt que des preuves complètes, lesquelles peuvent être lues dans les articles originaux auxquels on réfère.

Les travaux qui sont présentés ont tous donné lieu à une publication ou une pré-publication. Les résultats originaux de la thèse énoncés dans cette introduction sont numérotés par des lettres et ont été publiés ou pré-publiés dans les articles suivants :

- *Almost Sure Existence of Global Weak Solutions for Supercritical Wave Equations*, Journal of Differential Equations, Vol 273, pp. 83-121.
- *Almost Sure Scattering at Mass Regularity for Radial Schrödinger Equations*, pré-publication arXiv:2011.06309.
- *Construction of High Regularity Invariant Measures for the Euler Equations and Remarks on the Growth of the Solutions*, pré-publication arXiv:2002.11086
- *Probabilistic Local Well-Posedness for the Schrödinger Equation Posed for the Grushin Laplacian*, pré-publication arXiv:2103.03560.

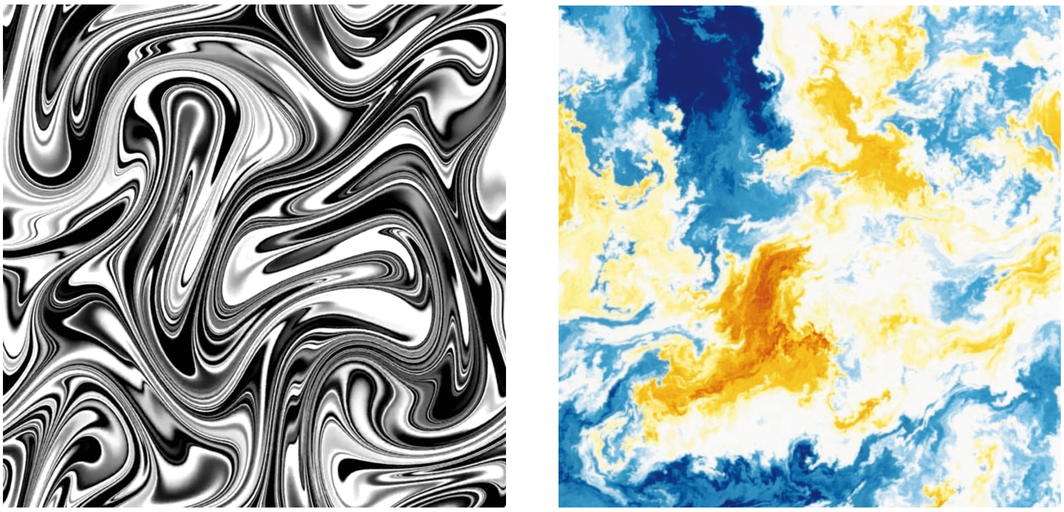
Avant d'introduire les deux méthodes fondamentales de l'étude des équations à données aléatoires, on donne quelques raisons physiques et mathématiques pour lesquelles on s'intéresse à des données *aléatoires*.

1. Pourquoi de l'aléatoire dans l'étude des EDP ?

Dans cette section, on se borne à évoquer quelques raisons qui rendent utile, voire désirable l'introduction des données aléatoires pour l'étude des équations aux dérivées partielles. En particulier, cette section ne prétend à aucune rigueur mathématique.

1.1. Raisons physiques. Une raison principale qui pousse à l'étude de données initiales aléatoires trouve sa source dans le besoin d'une description statistique des solutions de certaines équations. Un exemple particulièrement important est celui des phénomènes de turbulence et de mélange dans les équations de la mécanique des fluides, phénomènes physiques qui sont encore très mal compris à ce jour. Leur description ou l'établissement rigoureux de certaines « lois » empiriques est un enjeu majeur. On consultera par exemple [BBPS19] pour des progrès récents.

Même si chacun peut faire l'expérience des phénomènes de turbulence, par exemple en regardant l'écoulement de l'air autour de l'aile d'un avion, on peut encore plus facilement faire l'expérience du phénomène de mélange : du lait déposé à la surface d'une tasse de café aura tendance à se mélanger au café selon un procédé d'étirement-repliement, formant rapidement des structures de grande complexité. La Figure 1 (a) illustre ce phénomène, la Figure 1 (b) donnant une simulation numérique des équations décrivant le phénomène.



(a) Mélange du lait et du café (b) Simulation numérique

FIGURE 1. Phénomène de mélange[1]

On conçoit ainsi toute la complexité de décrire de tels phénomènes.

Face à une telle complexité des solutions observées on peut essayer de décrire le comportement « en moyenne » de telles solutions, en déduire le comportement *typique* des solutions et caractériser la déviation par rapport à ce comportement moyen, trouver des invariants, des quantités qui caractérisent le comportement de la grande majorité des solutions, d'où l'idée d'étudier ces équations avec des données aléatoires.

Voici une autre raison qui pousse à considérer des données aléatoires. On peut reprocher aux solutions considérées dans les énoncés mathématiques leur manque de sens physique. Toujours en mécanique des fluides, un exemple frappant est celui de la non unicité des solutions faibles de l'équation d'Euler. Sans entrer dans les détails, Sheffer et Shnirelman [Sch93, Shn97] et plus récemment [DLS09, Ise18] montrent que l'on peut produire de telles solutions faibles qui sont nulles entre l'instant initial $t = 0$ et un temps $t = 1$, qui ne sont pas nulles pour $t \in [1, 2]$ et qui redeviennent nulles pour tout $t \geqslant 2$. Concrètement cela signifie qu'un fluide initialement au repos s'agite subitement au temps $t = 1$ sans force extérieure et revient ensuite au repos au temps $t = 2$. On peut s'interroger sur la pertinence d'une telle notion. Les *solutions* présentées sont exclues par la mécanique des fluides et contraires à l'expérimentation. Doit-on en déduire que les équations d'Euler sont un mauvais modèle des phénomènes physiques qu'elles sont supposées modéliser ? Ou bien est-ce la notion de solution considérée par les mathématiciens qui n'est pas la bonne, parce que l'espace des solutions admissibles contient des exemples pathologiques ? On peut ainsi vouloir développer une notion générique de solution, de telle sorte que ces constructions pathologiques ne soient pas le cas typique des solutions considérées.

Remarque 1.1. Il ne faut pas tomber dans l'excès inverse et penser que la classe des fonctions considérées en physique doit se borner aux fonctions très régulières. La résolution des problèmes élémentaires en mécanique quantique nécessite l'emploi de fonctions peu régulières qui appartiennent aux espaces de Sobolev dont on fera usage dès la section 2. Pour ne donner qu'un exemple, la solution du problème du puits semi-infini en mécanique quantique (traité dans [BD02]) demande que les fonctions considérées aient des dérivées discontinues. De façon similaire la notion de charge ponctuelle en un point P considérée en électostatique, se formalise par une distribution de Dirac δ_P, qui n'est même pas une fonction.

1.2. Dépasser les obstructions déterministes. Il existe des raisons purement mathématiques qui plaident en faveur de l'étude des données aléatoires. L'une des premières questions à laquelle doit répondre l'étude d'une équation

[1](a) est emprunté au magazine Quanta à l'adresse, `www.quantamagazine.org/mathematicians-prove-batchelors-law-of-turbulence-20200204` et (b) est issue d'une simulation numérique [Kra96]

aux dérivées partielles est de donner un sens à ses solutions. On rappelle ce que l'on entend d'ordinaire par « résoudre » localement et globalement l'équation d'évolution abstraite :

$$\left\{ \begin{array}{rcl} \partial_t u(t) & = & F(u(t)) \\ u(0) & = & u_0 \in X \,, \end{array} \right. \tag{1.1}$$

où $F : X \to Y$ avec X et Y des espaces de Banach et l'inconnue est $u : \mathbb{R} \to X$, la variable d'évolution t représentant généralement le *temps*. On dit aussi *résoudre le problème de Cauchy*.

Avec les notations précédentes on a la définition suivante.

Definition 1.2. On dit que (1.1) est localement bien posé dans l'espace X si pour tout ensemble borné $B \subset X$ et toute donnée initiale $u_0 \in X$ il existe $T = T(u_0) > 0$ et une unique solution $u \in \mathcal{C}^0([0,T], X)$ à (1.1), et si de plus l'application $u_0 \mapsto u$ définit une fonction continue $B \to \mathcal{C}^0([0,T], X)$. Le problème est également dit *uniformément bien posé* lorsque cette application est uniformément continue sur B.

On réfère à [Tzv06] pour plus de détails. Enfin, le problème est dit globalement bien posé si on peut prendre $T = \infty$ dans cette définition.

On peut voir cette définition comme le cahier des charges de la notion de solution. En plus de l'existence et l'unicité on demande en particulier une propriété de continuité par rapport aux données initiales qui font la robustesse de telles solutions.

Pour certaines équations, de telles théories de Cauchy locales ou globales sont inconnues voire fausses dans des espaces où l'on aimerait pourtant construire des solutions. On prend l'exemple de l'équation de Schrödinger posée sur une variété riemannienne compacte (M, g) de dimension 2 :

$$i\partial_t u + \Delta_g u = |u|^2 u \,, \tag{1.2}$$

avec pour condition initiale $u(0) = u_0 \in H^s(M)$, les espaces de Sobolev usuels associés à la variété M.

Le problème de Cauchy local pour (1.2) est bien posé dans les espaces de Sobolev $H^s(M)$ dès que $s > \frac{1}{2}$, voir [BGT04]. Dans le cas de la sphère ce seuil est même abaissé à $s > \frac{1}{4}$, [BGT05]. Pour le tore, dont les propriétés arithmétiques aident à abaisser ce seuil, on a même $s > 0$, voir [Bou93].

Dans ces deux derniers cas, les résultats sont essentiellement optimaux, dans la mesure où il est connu que pour $s < \frac{1}{4}$ (resp. $s < 0$), le problème de Cauchy n'est pas uniformément bien posé [BGT02, Ban04] (resp. [BGT02, CCT03, Mol09]). De façon plus précise, dans les articles [CCT03] et [CKS+10], les auteurs montrent que pour tous $\varepsilon, \tau \ll 1$ et tout $K \gg 1$, on peut construire des données initiales ε proches qui évoluent en des solutions telles qu'au temps $t = \tau$ les solutions sont à distance au moins K.

La conservation, au moins à un niveau informel, de la quantité $t \mapsto \|u(t)\|^2_{L^2(M)}$ par l'équation (1.2) permettrait de construire des solutions globales, à condition de savoir d'abord produire des solutions locales. Or une théorie de Cauchy locale n'est pas connue à ce niveau de régularité. L'introduction des données aléatoires permet de dépasser ces seuils de régularité dans certaines situations que l'on étudiera. La base d'une telle stratégie sera expliquée dans la section 2.

Dans le cas où le problème de Cauchy est mal posé à niveau de régularité fixé, on peut aussi se demander si ce caractère mal posé vaut pour *toutes* les données initiales, ou si au contraire on peut en identifier une classe pour laquelle on peut construire des solutions de façon unique. Dans ce cas, on remarque qu'une théorie *probabiliste* jouerait au moins le rôle de pourvoyeur d'exemples. En effet, imaginons une équation dont la résolution locale (ou globale) dans H^{s_0} soit inconnue. Supposons aussi que l'on ne connaisse aucune solution explicite dans H^{s_0} hormis les quelques exemples triviaux comme la fonction nulle. Une construction probabiliste du type « pour presque toute donnée initiale dans H^{s_0} il existe une unique solution associée » produit *a fortiori* au moins une solution que l'on ne connaissait pas par les méthodes déterministes.

On fixe un espace probabilisé $(\Omega, \mathcal{F}, \mathbb{P})$ dont la mesure de probabilités est $\mathbb{P}$. On notera toujours ω pour désigner un élément de Ω dans le reste de cette introduction.

2. Le coeur de l'approche probabiliste en deux thèmes

On aborde maintenant l'exposé des deux méthodes probabilistes que l'on peut qualifier de centrales dans l'étude des équations aux données initiales aléatoires. Cette thèse fera usage de l'esprit de ces deux méthodes. La grande quantité d'applications et la robustesse des énoncés font de ces méthodes les piliers de l'étude des équations aux dérivées partielles avec des données aléatoires. Nous les présenterons dans leur ordre chronologique d'apparition, d'abord la méthode des mesures invariantes qui globalise des solutions et remonte au moins à [Bou94] puis la méthode locale utilisée dans [BT08a]. Ces articles peuvent être considérés comme les travaux fondateurs de toute l'étude des équations dispersives à données aléatoires.

La revue de la littérature sera minimale dans cette section, en particulier nous ne présenterons pas les développements très récents de ces méthodes. La présentation des deux méthodes principales se fera à travers l'exemple d'une équation modèle de Schrödinger fractionnaire avec gain de dérivée :

$$\text{(fNLS)} \qquad \begin{cases} i\partial_t u - (-\Delta)^\alpha u &= \langle \nabla \rangle^{-\beta}(|u|^2 u) \\ u(0) &= u_0 \in H^s(\mathbb{T}^d) \,. \end{cases}$$

On s'intéressera au cas $(d, \alpha, \beta) = (3, 2, 1)$, mais les résultats de la section 2.1 et la section 2.2 seront énoncés pour des valeurs génériques de d, $\alpha, \beta > 0$.

Le choix d'une telle équation est motivé par des raisons d'exposition. En particulier, l'emploi d'une équation d'ordre 1 en temps résulte d'une volonté d'écrire des équations compactes, et le gain de dérivé de la non-linéarité encodé par $\beta > 0$ simplifie grandement les arguments présentés. Lorsque $\beta = 0$ les arguments deviennent plus difficiles à présenter, on se réfère par exemple à [Bou96] pour un tel exemple.

Avant d'entrer dans les détails de ces méthodes, on rappelle la méthode usuelle pour obtenir des solutions locales puis globales de façon déterministe.

Pour montrer que le problème de Cauchy pour (fNLS) est uniformément localement bien posé dans H^s pour s assez grand, on applique un argument de point fixe dont l'ingrédient essentiel est l'utilisation de l'injection de Sobolev dont nous ferons usage dans ce chapitre. Ainsi on utilisera les résultats suivants auxquels on réfère en tant que « injections de Sobolev ».

(i) Pour $s > \frac{d}{2}$, l'injection $H^s(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ est continue.

(ii) Pour $s \in [0, \frac{d}{2}[$, l'injection $H^s(\mathbb{T}^d) \hookrightarrow L^p(\mathbb{T}^d)$ est continue, dès que $\frac{1}{p} \geqslant \frac{1}{2} - \frac{s}{d}$.

(iii) Pour $\varepsilon > 0$ et $p \geqslant 2$ tels que $\varepsilon p > d$ on a $W^{\varepsilon,p}(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$.

Lemme 1.3. *Soit $s > \frac{d}{2}$. Alors le problème de Cauchy pour* (fNLS) *est uniformément localement bien posé dans l'espace $H^s(\mathbb{T}^d)$.*

Démonstration. L'équation (fNLS) se reformule de façon intégrale en

$$u(t) = e^{-it(-\Delta)^\alpha} u_0 - i \int_0^t \frac{e^{-i(t-t')(-\Delta)^\alpha}}{\langle \nabla \rangle^\beta} \left(|u(t')|^2 u(t') \right) \mathrm{d}t' \,, \tag{1.3}$$

que l'on appellera la formulation Duhamel (pour un Français il s'agit de la formule de la variation des constantes). Résoudre (1.3) revient donc à résoudre une équation de point fixe $\Phi(u) = u$ pour la fonctionnelle $\Phi : \mathcal{C}([0,T], H^s) \longrightarrow \mathcal{C}([0,T], H^s)$ définie par le membre de droite de (1.3). On va utiliser le théorème de point fixe appliqué à une restriction de $\Phi : B_{R,T} \to B_{R,T}$ où $B_{R,T}$ désigne la boule fermée de centre 0 de rayon R de l'espace $\mathcal{C}([0,T], H^s)$.

Soit $u_0 \in H^s(\mathbb{T}^d)$. Pour une raison qui sera claire dans les estimations, on fixe $R > 2\|u_0\|_{H^s}$. On cherche un temps $T > 0$ tel que Φ stabilise la boule $B_{R,T}$ (ce qui n'a *a priori* rien d'immédiat) et est contractante sur cette boule. Comme souvent, la deuxième partie découle des calculs de la première partie ainsi on se borne à montrer que Φ stabilise la boule $B_{R,T}$ pour T assez petit, c'est à dire qu'il nous faut montrer que pour des u qui satisfont $\|u\|_{L^\infty_T H^s} \leqslant R$, on a par exemple $\|\Phi(u)\|_{L^\infty_T H^s} \leqslant R$.

Pour de tels u et $t \in [0,T]$, on commence par appliquer l'inégalité triangulaire :

$$\|\Phi(u)(t)\|_{H^s} \leqslant \|e^{-it(-\Delta)^\alpha} u_0\|_{H^s} + \int_0^t \left\| e^{-i(t-t')(-\Delta)^\alpha} \left(|u(t')|^2 u(t') \right) \right\|_{H^{s-\beta}} \mathrm{d}t' \,.$$

On utilise maintenant que $e^{-it(-\Delta)^\alpha}$ est une isométrie dans $H^s(\mathbb{T}^d)$, ce qui se voit grâce au théorème de Parseval. On a donc obtenu

$$\begin{aligned}\|\Phi(u)(t)\|_{H^s} &\leqslant \|u_0\|_{H^s} + \int_0^T \||u(t')|^2 u(t')\|_{H^{s-\beta}} \\ &\leqslant \frac{R}{2} + T\||u|^2 u\|_{L^\infty_T H^{s-\beta}}\,.\end{aligned}$$

Pour conclure, on voudrait une inégalité du type

$$\||u|^2 u\|_{H^{s-\beta}} \lesssim \|u\|^3_{H^s}\,.$$

De telles inégalités peuvent se démontrer avec des lois de produits. Donnons en simplement l'idée. Lorsque l'on dérive un terme comme u^3, s fois, on obtient des termes, dont celui qui fait apparaître le plus de dérivées est celui qui porte toutes les dérivées, il s'agit de $u^2\nabla^s u$. On s'attend à ce que ce terme soit le facteur limitant de l'analyse qui va suivre. On ne formalise pas cette heuristique et on se borne à écrire informellement que $\nabla^s(|u|^2 u) \simeq |u|^2 \nabla^s u$, et donc par Hölder on a

$$\||u|^2 u\|_{H^s} \lesssim \|u\|^2_{L^\infty} \|u\|_{H^s}\,.$$

En utilisant $\||u|^2 u\|_{H^{s-\beta}} \leqslant \||u|^2 u\|_{H^s}$ et l'injection de Sobolev qui donne $\|u\|_{L^\infty} \lesssim \|u\|_{H^s}$, car $s > \frac{d}{2}$, on obtient alors

$$\|\Phi(u)\|_{L^\infty_T H^s} \leqslant \frac{R}{2} + TR^3\,,$$

qui est majoré par R dès que $T \leqslant \frac{1}{2}R^2$. La résolution du problème de Cauchy locale en découle. □

La théorie que nous venons d'établir peut être améliorée, en particulier toute la puissance des injections de Sobolev n'a pas été utilisée et on a majoré grossièrement $\||u|^2 u\|_{H^{s-\beta}} \leqslant \||u|^2 u\|_{H^s}$, perdant l'intérêt du gain de dérivées donné par β. On peut alors raffiner les calculs précédents pour obtenir la théorie locale suivante.

Lemme 1.4. *Pour $s > \frac{d-\beta}{2}$, le problème de Cauchy pour* (fNLS) *est uniformément localement bien posé dans $H^s(\mathbb{T}^d)$.*

Démonstration. En reprenant les calculs précédents on repart de

$$\|\Phi(u)\|_{L^\infty H^s} \leqslant \frac{R}{2} + T\||u|^2 u\|_{L^\infty H^{s-\beta}}\,,$$

de sorte qu'il nous suffit d'expliquer que pour $s > \frac{d-\beta}{2}$ et tout $u \in H^s$ on a

$$\||u|^2 u\|_{H^{s-\beta}} \lesssim \|u\|^3_{H^s}\,. \tag{1.4}$$

À cet effet, on utilise encore l'heuristique

$$\nabla^{s-\beta}(|u|^2 u) \simeq \nabla^{s-\beta} u |u|^2 + \{\text{termes d'ordres inférieurs}\} ,$$

ce qui permet d'appliquer une inégalité de Hölder avec p et q finis qui satisfont $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ puis l'injection de Sobolev :

$$\begin{aligned} \| |u|^2 u \|_{H^{s-\beta}} &\lesssim \|\nabla^{s-\beta} u\|_{L^p} \|u\|_{L^{2q}}^2 \\ &\lesssim \|u\|_{H^{s-\beta+d\left(\frac{1}{2}-\frac{1}{p}\right)}} \|u\|^2_{H^{d\left(\frac{1}{2}-\frac{1}{2q}\right)}} . \end{aligned}$$

On obtient alors (1.4) à condition que

$$s-\beta+d\left(\frac{1}{2}-\frac{1}{p}\right) \leqslant s \text{ et } d\left(\frac{1}{2}-\frac{1}{2q}\right) \leqslant s .$$

La première condition impose $\frac{1}{q} \leqslant \frac{\beta}{d}$ et la seconde $\frac{1}{q} \geqslant 1-\frac{2s}{d}$. Ces deux conditions peuvent être simultanément vérifiées dès que $1-\frac{2s}{d} \leqslant \frac{\beta}{d}$, c'est à dire pour $s \geqslant \frac{d-\beta}{2}$, ce qui est vérifié par hypothèse. □

Remarque 1.5. La preuve précédente n'est pas non plus optimale. Aucune propriété spécifique de l'équation n'est utilisée, sinon que le semi-groupe linéaire associé est une isométrie dans L^2 (et dans les espaces H^s). Une théorie plus fine tiendrait compte des effets dispersifs de cette équation. Pour des raisons d'expositions on ne présente que la méthode la plus simple pour obtenir une théorie de Cauchy locale afin de l'améliorer par des méthodes probabilistes.

On a ainsi obtenu une théorie de Cauchy locale qui peut s'écrire de façon compacte de la façon suivante : pour $u_0 \in H^s(\mathbb{T}^d)$ il existe un temps d'existence $\tau \sim \|u_0\|_{H^s}^{-2}$ tel que le problème de Cauchy pour (fNLS) associé à u_0 est bien posé sur l'intervalle de temps $[0,\tau]$. De plus on peut choisir un tel τ tel que pour tout $t \in [0,\tau]$ on a le contrôle $\|u(t)\|_{H^s} \leqslant 2\|u_0\|_{H^s}$.

Une méthode pour globaliser les solutions est d'itérer un tel argument en appliquant à nouveau la théorie locale au nouveau temps initial $t=\tau$. On voit alors qu'une condition suffisante pour atteindre n'importe quel temps $T>0$ est que la quantité $t \mapsto \|u(t)\|_{H^s}$ reste bornée quand t croît. Il faudrait donc disposer d'une estimation *a priori* sur $\|u\|_{H^s}$. En général, une telle information est difficile à obtenir. Par exemple pour (fNLS), en multipliant l'équation par $\langle\nabla\rangle^{\beta}\bar{u}$ et en intégrant formellement on obtient que la quantité

$$E(t) := \frac{1}{2}\|u(t)\|^2_{H^{\alpha+\frac{\beta}{2}}} + \frac{1}{4}\|u\|^4_{L^4} ,$$

est formellement conservée. On en déduit que la norme $H^{\alpha+\frac{\beta}{2}}$ de $u(t)$ est bornée en temps. Ainsi, toute solution locale produite par le lemme 1.4 au niveau $H^{\alpha+\frac{\beta}{2}}$ est globale. Dans le cas $(d,\alpha,\beta)=(3,2,1)$ on a construit une théorie locale dans $H^s(\mathbb{T}^3)$ dès que $s > 1$. Puisque $\alpha + \frac{\beta}{2} = \frac{5}{2}$ on en déduit une théorie globale dans $H^{\frac{5}{2}}$.

Nous allons à présent montrer une autre méthode de globalisation, fondée sur une *conservation statistique*, ce qui aboutira à la construction de solutions globales à un niveau de régularité plus faible que H^2 dans notre exemple.

2.1. Mesures invariantes : l'argument de globalisation. Définissons d'abord ce que l'on entend par mesure invariante. Étant donné une équation différentielle dont le flot est donné par $\phi_t : X \to X$, une mesure μ sur l'espace X est dite invariante par ϕ_t si pour tout ensemble mesurable $A \subset X$ et pour tout $t \geqslant 0$ on a

$$\mu(\phi_{-t}A) = \mu(A)\,.$$

Intuitivement cela signifie que les propriétés statistiques encodées par μ sont conservées par le flot, la statistique de la dynamique restant la même à chaque instant.

L'obtention et l'étude de telles mesures invariantes pour des systèmes dynamiques comme les équations aux dérivées partielles est d'une grande importance en physique. En particulier, lorsqu'un espace X, sur lequel agit un flot ϕ_t est muni d'une mesure de probabilité invariante μ, deux résultats essentiels s'appliquent :

1. Le théorème de récurrence de Poincaré, qui énonce essentiellement que dans ce cas, pour tout ensemble mesurable $A \subset X$ tel que $\mu(A) > 0$, presque tout $u \in A$ est *récurrent*, au sens où $\phi_t u \in A$ pour une infinité de t.
2. Le théorème de Birkhoff, qui fait le lien entre les moyennes temporelles de $f \circ \phi_t$ et les moyennes statistiques de f pour une classe de fonctions $f : X \to X$ appelées fonctions μ-ergodiques.

Il existe une classe d'équations différentielles qui admettent naturellement des mesures invariantes, il s'agit des équation dite hamiltoniennes, c'est à dire qui s'écrivent sous la forme

$$\begin{cases} \dot{q} &= \partial_p H(q,p) \\ \dot{p} &= -\partial_q H(q,p)\,, \end{cases} \tag{1.5}$$

avec des conditions initiales $(q(0), p(0)) = (q_0, p_0) \in \mathbb{R}^N \times \mathbb{R}^N$ où $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$.

Dans ce cas, une mesure de probabilité invariante est donnée par

$$d\mu = Z^{-1} e^{-H(q,p)}\, \mathrm{d}q_0\, \mathrm{d}p_0\,,$$

où $\mathrm{d}q_0\, \mathrm{d}p_0$ désigne la mesure de Lebesgue de $\mathbb{R}^{2N}$ et Z est une constante de normalisation.

Pour montrer l'invariance, on vérifie immédiatement que $t \mapsto H(q(t), p(t))$ est de dérivée nulle, donc $H(q,p)$ est une quantité conservée par le flot de (1.5). L'invariance de μ découle alors du lemme suivant.

Lemme 1.6 (Liouville)**.** *La mesure* $\mathrm{d}q_0\,\mathrm{d}p_0$ *est invariante par le flot de* (1.5)*.*

Démonstration. On note ϕ_t le flot associé à (1.5) et l'on considère $f : (q,p) \mapsto f(q,p)$ une fonction mesurable. A partir de

$$((\phi_t)_*\mathrm{d}q_0\,\mathrm{d}p_0, f) \coloneqq \int f(\phi_t(q_0,p_0))\,\mathrm{d}q_0\,\mathrm{d}p_0 = \int f(q(t),p(t))\,\mathrm{d}q_0\,\mathrm{d}p_0\,,$$

on calcule formellement que

$$\frac{\mathrm{d}}{\mathrm{d}t}((\phi_t)_*\mathrm{d}q_0\,\mathrm{d}p_0, f) = \int \left(\dot{q}\partial_q f(q,p) + \dot{p}\partial_p f(q,p)\right)\,\mathrm{d}q_0\,\mathrm{d}p_0\,.$$

Il suffit alors d'utiliser (1.5) et d'intégrer par parties pour en déduire que $t \mapsto (\phi_t^*\mathrm{d}q_0\,\mathrm{d}p_0, f)$ est constant, d'où l'invariance de la mesure $\mathrm{d}q_0\,\mathrm{d}p_0$. □

Bien que posée en dimension infinie, (fNLS) jouit d'une structure analogue. Pour le voir, on développe $u \in L^2(\mathbb{T}^d)$ en série de Fourier :

$$u(x) = \sum_{n\in\mathbb{Z}^d} u_n e^{in\cdot x}\,,$$

et on écrit $u_n = q_n + ip_n$. Les variables réelles $q = (q_n)_{n\in\mathbb{Z}^d}$ et $p = (p_n)_{n\in\mathbb{Z}^d}$ suffisent à décrire u. On vérifie en effet que $u = q + ip$ résout (fNLS) si et seulement si le couple (p,q) est solution de

$$\begin{cases} \partial_t q &= \partial_p H(q,p) \\ \partial_t p &= -\partial_q H(q,p)\,, \end{cases}$$

muni de la condition initiale $(q(0),p(0)) = (\mathrm{Re}(\hat{u}_0), \mathrm{Im}(\hat{u}_0))$, avec $H(q,p) = H(u)$ définit par

$$H(u) \coloneqq \frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}}u\|_{L^2}^2 + \left\langle \langle\nabla\rangle^{-\beta}(|u|^2u), u\right\rangle_{L^2}\,,$$

dont on vérifie de plus qu'il s'agit d'une quantité conservée par (fNLS) et s'obtient en multipliant (fNLS) par $\bar{u}$ puis en intégrant par parties.

Notons $\mathrm{d}u \coloneqq \mathrm{d}p\,\mathrm{d}q$, mesure qu'il faudra définir de façon rigoureuse par la suite. Alors on s'attend à ce que la mesure

$$\text{(1.6)} \qquad \mathrm{d}\mu(u)\text{“} = \text{”}Z^{-1}\exp\left(-\frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}}u\|_{L^2}^2 - \left\langle\langle\nabla\rangle^{-\beta}(|u|^2u), u\right\rangle\right)\mathrm{d}u\,,$$

soit invariante pour (fNLS), où Z est une constante de renormalisation.

En plus du problème de donner un sens à $\mathrm{d}u$, s'ajoute que le cadre précédent est celui des équations différentielles, donc de la dimension finie, alors que (fNLS) est posée en dimension infinie.

Expliquons rapidement comment on peut déjà construire formellement la mesure $\mathrm{d}u$. Plus précisément, on construit directement la mesure

$$\mathrm{d}\mu_0(u)\text{“} = \text{”}Z^{-1}\exp\left(-\frac{1}{2}\|u\|_{L^2}^2 - \frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}}u\|_{L^2}^2\right)\mathrm{d}u\,,$$

la mesure $\mathrm{d}\mu$ pouvant ensuite être construite comme une mesure absolument continue par rapport à $\mathrm{d}\mu_0$. On réfère par exemple au chapitre 4 de cette thèse pour un tel exemple.

Au moins informellement, on peut s'imaginer $Z^{-1}\mathrm{d}u$ comme la mesure produit $\bigotimes_{n\in\mathbb{Z}^d}\mathrm{d}u_n$. En évrivant u en variables u_n on va construit la mesure

$$\mathrm{d}\mu_0(u) = \bigotimes_{n\in\mathbb{Z}^d}(2\pi)^{-1}e^{-\frac{1}{2}(1+|n|^{2\alpha})|u_n|^2}\,\mathrm{d}u_n\,, \tag{1.7}$$

où l'introduction du $(2\pi)^{-1}$ sert à normaliser car on reconnaît des densités de mesures gaussiennes indépendantes suivant chaque coefficient de Fourier. Plus précisément, la mesure μ_0 s'obtient comme mesure image par la variable aléatoire

$$\sum_{n\in\mathbb{Z}^d}\frac{g_n^\omega}{\sqrt{1+|n|^{2\alpha}}}e^{in\cdot x}\,, \tag{1.8}$$

où les $(g_n)_{n\in\mathbb{Z}^d}$ sont des variables aléatoires gaussiennes complexes centrées de variance 1, identiquement distribuées et indépendantes.

Remarque 1.7. On s'aperçoit que sans l'ajout du terme $\|u\|^2$ dans la définition de la mesure μ_0, un problème de division par $n=0$ aurait pu poser problème dans la définition que l'on donne de μ_0.

Nous allons maintenant présenter l'idée principale de *l'argument de globalisation* utilisé dans [Bou94]. Grâce à sa grande flexibilité, cette méthode a connu un succès conséquent depuis son introduction. Nous énonçons ce résultat sous forme de méta-théorème car il ne s'agit pas à proprement parler d'un résultat rigoureux mais d'un canevas de preuve.

Méta-théorème (Argument de globalisation)**.** *Soient $s\in\mathbb{R}$ et ϕ_t le flot d'une équation d'évolution du type* (1.1)*. On suppose qu'il existe une mesure μ sur H^s telle que :*

- (*i*) *μ est invariante par ϕ_t.*
- (*ii*) *Pour tout $\lambda>0$, on a $\mu(\|u\|_{H^s}>\lambda)\lesssim e^{-c\lambda^2}$.*
- (*iii*) *On dispose d'une bonne théorie locale au niveau H^s, c'est à dire que si $\|u_0\|_{H^s}\leqslant\lambda$ alors il existe une unique solution locale sur $[0,\tau]$ associée à u_0 où l'on peut prendre $\tau\sim\lambda^{-K}$ pour un certain $K>0$ et de plus $\|u(t)\|_{H^s}\leqslant 2\lambda$ pour tout $t\leqslant[0,\tau]$.*

Sous ces hypothèses, pour μ-presque toute donnée initiale $u_0 \in H^s$, il existe une unique solution globale à (1.1) associée à u_0 et de plus on a pour tout $t > 0$,

$$\|u(t)\|_{H^s} \leqslant C(u_0) \log^{\frac{1}{2}}(1+t)\,.$$

Remarque 1.8. Les mesures gaussiennes comme μ_0 définies par (1.7) satisfont naturellement l'hypothèse (ii). En effet, nous verrons plus tard que ceci s'obtient grâce au théorème 1.14. Enfin la condition (iii) est ce qu'une théorie locale par point fixe comme celle du lemme 1.4 nous fournit.

Démonstration du méta-théorème. Plutôt que de chercher à construire directement des solutions globales pour presque toute donnée initiale, on commence plutôt par construire des solutions globales jusqu'à un temps $T > 0$ que l'on fixe, pour un ensemble de données initiales dont la mesure est plus grande que $1-\varepsilon$, où $\varepsilon > 0$ est également fixé.

Pour cela, fixons une taille $\lambda > 0$, que l'on ajustera par la suite. On rappelle que le temps local donné par la théorie locale (iii) est $\tau \sim \lambda^{-K}$ pour des données initiales u_0 qui satisfont $\|u_0\|_{H^s} \leqslant \lambda$. Ainsi, par cette même théorie locale on sait que les données initiales dans l'ensemble défini par

$$G_{\lambda,T} \coloneqq \left\{ u_0, \text{ tels que pour tout } n \leqslant \left\lfloor \frac{T}{\tau} \right\rfloor, \|u(n\tau)\|_{H^s} \leqslant \lambda \right\}\,,$$

produisent des solutions sur $[0,T]$ telles que $\|u(t)\|_{H^s} \leqslant 2\lambda$ pour tout $t \in [0,T]$. La représentation graphique de telles solutions est donnée par la Figure 2.

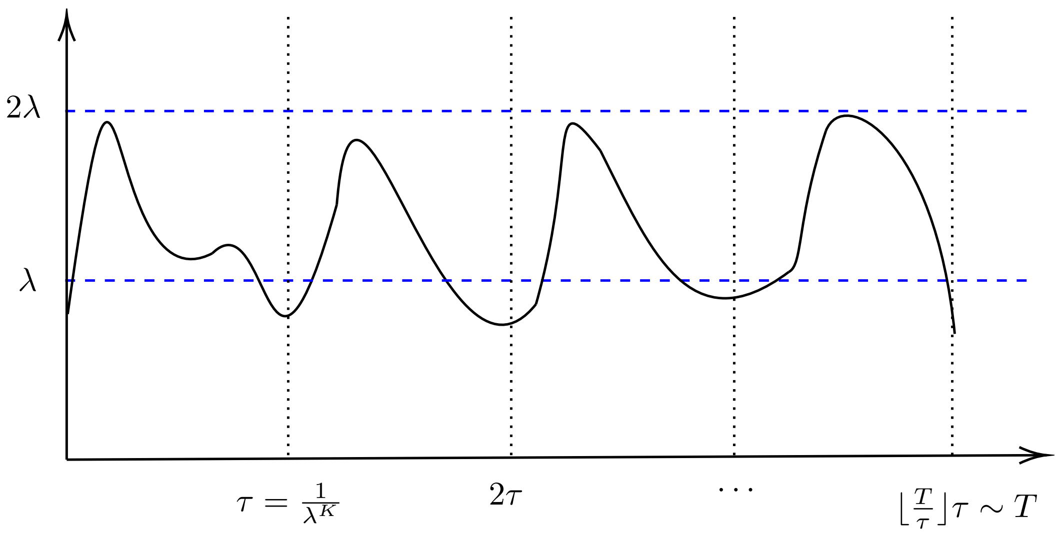


FIGURE 2. Représentation des solutions issues de $G_{\lambda,T}$

On va évaluer la probabilité de l'ensemble complémentaire et montrer que cet ensemble est petit. Plus précisément, le complémentaire, est donné par :

$$B_{\lambda,T} \coloneqq H^s \setminus G_{\lambda,T} = \bigcup_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \{u_0, \|u(n\tau)\|_{H^s} > \lambda\} \ ,$$

de sorte que par majoration brutale de la mesure de l'union s'écrive :

$$\mu(B_{\lambda,T}) \leqslant \sum_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \mu\left(\{u_0, \|\phi_{n\tau} u_0\|_{H^s} > \lambda\}\right) \ .$$

On utilise ensuite ensuite le fait que $u(n\tau) = \phi_{n\tau} u_0$ et l'invariance de la mesure (i) pour écrire

$$\begin{aligned} \mu\left(\{u_0, \|\phi_{n\tau} u_0\|_{H^s} > \lambda\}\right) &= \mu\left(\phi_{-n\tau}\left(\{u_0 \in H^s, \|u_0\|_{H^s} > \lambda\}\right)\right) \\ &= \mu\left(\{u_0 \in H^s, \|u_0\|_{H^s} > \lambda\}\right) \ , \end{aligned}$$

puis en utilisant (ii) on obtient finalement :

$$\mu(B_{\lambda,T}) \leqslant \sum_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \mu\left(\{u_0 \in H^s, \|u_0\|_{H^s} > \lambda\}\right) \leqslant \frac{T}{\tau} e^{-c\lambda^2} \ .$$

Le fait que $\tau \sim \lambda^{-K}$ entraîne alors que

$$\mu(B_{\lambda,T}) \lesssim T\lambda^K e^{-c\lambda^2} \ ,$$

et donc $\mu(B_{\lambda,T}) \leqslant \varepsilon$ dès que $\lambda \sim \log^{1/2}\left(\frac{T}{\varepsilon}\right)$. On fixe un tel λ. On a ainsi obtenu un ensemble $G_{T,\varepsilon}$ de mesure au moins $1 - \varepsilon$ tel que $\|u(t)\| \leqslant 2\log^{\frac{1}{2}}(\frac{T}{\varepsilon})$ pour tout $t \in [0, T]$.

Pour conclure on utilise un argument à la Borel-Cantelli. On introduit l'ensemble

$$G \coloneqq \bigcup_{k \geqslant 0} \bigcap_{n \geqslant k} G_{n, \frac{1}{n^2}} \ .$$

D'une part, pour des données initiales dans G on a bien l'existence d'une unique solution globale associée dont la croissance ne dépasse pas $C(u_0)\log^{\frac{1}{2}}(1+t)$. Enfin, par monotonie on a

$$\mu(H^s \setminus G) \leqslant \limsup_{k \to \infty} \mu\left(\bigcup_{n \geqslant k} H^s \setminus G_{n, \frac{1}{n^2}}\right) \leqslant \limsup_{k \to \infty} \sum_{n \geqslant k} \frac{1}{n^2} \ ,$$

qui vaut 0 car la série des $\frac{1}{n^2}$ converge. □

Remarque 1.9. L'argument présenté est l'analogue de la globalisation à l'aide d'une quantité conservée (ou bornée). En effet, plutôt que d'exploiter une borne sur $t \mapsto \|u(t)\|_H^s$ de façon déterministe, on exploite l'information statistique que $\|u(t)\|_H^s$ est contrôlé *pour tous temps* $t = n\tau$, car contrôlé en $t = 0$, grâce à l'invariance de la mesure. La théorie locale permet alors d'estimer la variation sur sur les intervalles $[n\tau, (n+1)\tau]$.

Remarque 1.10. L'utilisation d'un argument de Borel-Cantelli est assez typique des preuves probabilistes en EDP : un énoncé presque-sûr se ramène à un énoncé avec « grande probabilité ».

Les hypothèses du méta-théorème sont flexibles. En particulier on appliquera une version dont les hypothèses sont modifiées dans les chapitres 4 et 5 de cette thèse. Principalement deux hypothèses peuvent être assouplies. La première est que l'invariance de la mesure μ supposée en (i) peut être remplacée par une information sur $t \mapsto \mu(\phi_t A)$ en fonction de $\mu(A)$, sans demander l'égalité de ces deux quantités, c'est ce qui nous permettra d'exploiter les mesures quasi-invariantes dans le chapitre 4.

De la même façon, les estimations sous-gaussiennes supposées en (ii) peuvent-être affaiblies, par exemple les estimations du type

$$\mu(\{u_0, \|u_0\|_{H^s} > \lambda\}) \lesssim \lambda^{-\gamma}\,,$$

suffiront dans le chapitre 4.

Dans le cas de ces deux modifications, la preuve du méta-théorème suit le même plan mais la conclusion s'en trouve affaiblie, la croissance en temps devenant plus forte qu'un logarithme.

La seule hypothèse dont on ne semble pas pouvoir se passer est celle d'une théorie locale de la forme (iii), c'est à dire une *bonne* théorie locale.

Signalons que le méta-théorème est si flexible que son idée principale a été utilisée récemment dans un contexte apparemment très différent, celui de la théorie des nombres [Tao19].

Ainsi, lorsque l'on dispose d'une mesure invariante, au moins au niveau informel, il faut disposer d'une théorie locale au niveau de régularité de la mesure invariante pour appliquer le méta-théorème. Supposons qu'une équation donnée possède une mesure invariante dont le support est à régularité H^s pour un certain réel s. Trois cas de figure peuvent se présenter.

(1) *Cas très favorable.* Au niveau H^s une théorie locale et même globale est déjà connue. L'argument de globalisation se révèle tout de même utile ! En effet, il permet de construire μ presque sûrement des solutions qui ont une très faible croissance, logarithmique. Afin de comparer, les solutions construites par point fixe le sont de telle sorte que sur un intervalle du type $[0, \tau]$ la solution ne fait pas plus que doubler sa norme, ainsi il est fréquent que de telles constructions donnent lieu à un contrôle exponentiel de la croissance.

(2) *Cas favorable.* Au niveau H^s une bonne théorie locale est connue, mais pas de théorie globale. L'argument de globalisation construit alors

μ presque sûrement des solutions globales à croissance lente. C'est le cas étudié par Bourgain [Bou94] qui démontre qu'en dimension 1, l'équation de Schrödinger quintique est presque sûrement globalement bien posée sur le support de sa mesure de Gibbs, c'est à dire à un niveau de régularité $H^{\frac{1}{2}^-}$ et que de plus on a une estimation du type $\|u(t)\|_{H^s} \lesssim \log^{\frac{1}{2}}(1+t)$ pour tout $t > 0$.

Ce résultat entre dans le cadre où une théorie de Cauchy locale est déjà connue sur le support de la mesure de Gibbs : lorsque Bourgain écrit son article, une théorie locale au niveau H^{0^+} est déjà connue [Bou93], la mesure de Gibbs ayant un support de régularité $H^{\frac{1}{2}^-}$, le méta-théorème s'applique.

(3) *Cas défavorable.* Au niveau H^s, aucune théorie locale n'est connue. Il faut alors essayer de construire une bonne théorie locale, mais de façon probabiliste cette fois, pour des données initiales du type (1.8). Il s'agit du cas de figure étudié dans [Bou96] et qui démontre que l'équation (fNLS) dans le cas $(d, \alpha, \beta) = (2, 1, 0)$ est globalement bien posé sur le support de sa mesure de Gibbs, de régularité H^{0^-}, alors que la meilleure théorie locale connue est de niveau de régularité H^{0^+}. Le tour de force de l'article est la construction d'une théorie locale probabiliste à ce niveau de régularité.

Remarque 1.11. L'énoncé du méta-théorème laisse penser que l'argument de globalisation présenté requiert déjà qu'un flot global existe pour l'équation considérée, ce qui peut paraître comme un argument circulaire. Afin de se sortir de cette impasse, un argument rigoureux est d'approcher (fNLS) par des équations de dimension finie, par troncature en fréquences par exemple. On montre alors que les solutions u_N à ces approximations sont faciles à construire et que de même des mesures invariante μ_N pour les flots ϕ_N des équations approchées sont faciles à produire. On peut alors appliquer le méta-théorème à (μ_N, u_N) qui fournira des solutions globale dont la croissance sera estimée *uniformément* en N. On exploite alors des propriétés de convergence $(\phi_N, \mu_N) \to (\phi, \mu)$ en un sens qui reste à préciser. On en déduit alors l'existence de solutions globales à croissance faible et l'invariance de μ.

Traitons l'exemple de (fNLS) où l'on rappelle que $(d, \alpha, \beta) = (3, 2, 1)$. D'une part la théorie locale du lemme 1.4 se situe au niveau $H^{1^+}(\mathbb{T}^3)$. D'autre part, on dispose d'une mesure formellement invariante μ dont les données du support sont de la forme (1.8) donc de régularité $H^{\alpha-\frac{d}{2}^-} = H^{\frac{1}{2}^-}$. En effet, pour $s \in \mathbb{R}$ on calcule :

$$\mathbb{E}\left[\|u_0^\omega\|_{H^s}^2\right] = \sum_{n\in\mathbb{Z}^d} \frac{\langle n\rangle^{2s}}{1+|n|^{2\alpha}} \mathbb{E}[|g_n|^2],$$

et puisque $\mathbb{E}[|g_n|^2] = 1$, la convergence a lieu dès que $2s-2\alpha < -d$, ce qui donne bien le résultat annoncé. Ainsi le cas de l'équation (fNLS) pour $(d, \alpha, \beta) = (3, 2, 1)$ appartient à la catégorie (3), le méta-théorème ne peut pas s'appliquer directement, il nous faut développer une théorie locale probabiliste.

2.2. Le problème de Cauchy local probabiliste. Nous allons maintenant établir une théorie locale pour des données initiales aléatoires de la forme (1.8), c'est à dire développer une théorie de Cauchy probabiliste à une régularité H^s pour s qui sera suffisamment bas pour appliquer le méta-théorème pour sur (fNLS) et sa mesure invariante.

À partir des années 1970 et surtout dans les années 1990, l'étude du problème de Cauchy pour des équations similaires à (fNLS) a fait une utilisation grandissante des propriétés spécifiques de l'équation, en l'occurence l'existence d'estimations de Strichartz, c'est à dire des estimations dans des normes espace-temps des solutions de certaines équations linéaires.

Pour mieux comprendre l'intérêt des données aléatoires dans l'étude du problème de Cauchy local, et mettre en évidence l'importance des estimations de Strichartz, c'est à dire en norme $L^p_T L^q_x$, déterministes ou probabilistes, on propose d'oublier l'aspect aléatoire pour le moment. On rappelle que le lemme 1.4 construit une théorie de Cauchy locale uniquement à partir de l'injection de Sobolev, ce que l'on souhaite dépasser. Comment peut-on abaisser l'exposant s pour lequel on peut résoudre le problème de Cauchy localement dans H^s ?

On propose l'heuristique suivante fondée sur la formulation Duhamel (1.3) pour (fNLS), que l'on rappelle :

$$u(t) = e^{-it(-\Delta)^\alpha} u_0 - i \int_0^t \frac{e^{-i(t-t')(-\Delta)^\alpha}}{\langle \nabla \rangle^\beta} \Big(|u(t')|^2 u(t') \Big) \,\mathrm{d}t' \,.$$

La partie $u_L(t) := e^{-it(-\Delta)^\alpha} u_0$ est explicite, il s'agit de la solution de l'équation linéaire

$$i\partial_t u - (-\Delta)^\alpha u = 0 \,, \tag{fLS}$$

de condition initiale $u_L(0) = u_0$, résoluble pour toute régularité H^s. Ainsi on peut décomposer

$$u(t) = u_L(t) + v(t) \,, \tag{1.9}$$

avec v qui est donnée par

$$v(t) := -i \int_0^t \frac{e^{-i(t-t')(-\Delta)^\alpha}}{\langle \nabla \rangle^\beta} (|u(t')|^2 u(t')) \mathrm{d}t' \,.$$

Il est important de voir pourquoi cette décomposition peut nous aider : puisque l'on a $u \in H^s$, on a aussi $|u|^2 u \in H^s$ (voir la preuve du lemme 1.4) et donc grâce à l'opérateur $\langle \nabla \rangle^{-\beta}$ on obtient que $v \in H^{s+\beta}$, c'est à dire que ce terme est plus régulier dès que $\beta > 0$, ce que l'on suppose désormais.

Remarque 1.12. Sans la régularisation ad-hoc $\langle \nabla \rangle^{-\beta}$ dans (fNLS), le terme v n'est *a priori* pas plus régulier que H^s et donc cette stratégie est plus difficile à mettre en place.

La décomposition (1.9) possède donc une partie calculable et globale $u_L \in H^s$ ainsi qu'une partie plus régulière $v \in H^{s+\beta}$ qui demande à être construite. Si l'on choisit s tel que $s+\beta$ est supérieur au seuil de résolution locale de (fNLS), donné par $\frac{d-\beta}{2}$ c'est à dire $s > \frac{d}{2} - \frac{3\beta}{2}$, on espère pouvoir construire un tel v en utilisant une théorie locale similaire à celle du lemme 1.4.

L'équation intégrale qui est formellement satisfaite par $v \in H^{s+\beta}$, est de la forme $v = \Psi(v)$ où

$$\Psi(v)(t) := -i \int_0^t \frac{e^{-i(t-t')(-\Delta)^\alpha}}{\langle \nabla \rangle^\beta} \Big(|u_L + v|^2 (u_L + v) \Big) \, \mathrm{d}t' \,, \tag{1.10}$$

qui n'est pas exactement le cadre d'application du lemme 1.4 à cause des termes u_L. On va donc effectuer une preuve similaire. On verra apparaître des conditions contraignantes sur des normes $\|u_L\|_{L^p_T L^q_x}$ qui peuvent empêcher notre programme d'aboutir. Comme dans le lemme 1.4 on cherche une estimation de la forme

$$\|\Psi(v)\|_{L^\infty_T H^{s+\beta}} \leqslant C \|v\|^3_{L^\infty_T H^s} \,,$$

qui se révèle être le point fondamental dans l'élaboration d'une stratégie par point fixe. En raisonnant comme dans le lemme 1.4 on obtient immédiatement

$$\|\Psi(v)\|_{L^\infty_T H^{s+\beta}} \leqslant \| |u_L + v|^2 (u_L + v) \|_{L^1_T H^s} \,,$$

et l'on rappelle que l'on sait que $u_L(t) \in H^s$ et $v(t) \in H^{s+\beta}$.

On doit ensuite développer le terme cubique et l'estimer dans $L^1_T H^s$. Dans le développement on voit apparaître $|u_L|^2 u_L$, $|v|^2 v$ et des termes restants qui seront interpolés entre ces deux termes. Pour $|v|^2 v$ on applique le même raisonnement qu'au lemme 1.4 et pour $|u_L|^2 u_L$ on utilise des lois de produit dans les espaces de Sobolev. Finalement on obtient

$$\|\Psi(v)\|_{L^\infty_T H^{s+\beta}} \lesssim \|u_L\|^2_{L^2_T L^\infty_x} \|u_L\|_{L^\infty_T H^s} + T \|v\|^3_{L^\infty_T H^{s+\beta}} \,.$$

Bien évidemment on a $\|u_L\|_{L^\infty_T H^s} \leqslant \|u_0\|_{H^s}$ qui est fini car $u_0 \in H^s$. La borne la plus difficile à obtenir semble être $\|u_L\|_{L^2_T L^\infty} < \infty$, que l'on doit estimer en fonction de $\|u_0\|_{H^s}$.

C'est ici que les données aléatoires font leur entrée en jeu. En effet, dans le cas déterministe et uniquement à l'aide d'injections de Sobolev, on ne peut conclure que si $s > \frac{d}{2}$. On pose alors la question suivante : pour quelle classe de données $u_0 \in H^s$, et $s < \frac{d-\beta}{2}$, a-t-on $u_L \in L^2_T L^\infty$?

Répondre à une telle question est précisément le but des données aléatoires avec une randomisation assez générale, de type sous-gaussienne et qui sont des randomisations qui ressemblent à celle donnée par les mesures invariantes de la section précédente. On se donne une suite $(u_n)_{n\in\mathbb{Z}^d}$ telle que

$$u_0(x) \coloneqq \sum_{n\in\mathbb{Z}^d} u_n e^{in\cdot x} \in H^s(\mathbb{T}^d)\,.$$

On considère des variables aléatoires indépendantes et identiquement distribuées $(X_n)_{n\in\mathbb{Z}^d}$, telles qu'il existe $c > 0$ tel que pour tout $\gamma > 0$ on ait l'estimation

$$\mathbb{E}[e^{\gamma X_n}] \leqslant e^{c\gamma^2}\,. \tag{1.11}$$

On pourra penser aux X_n comme des gaussiennes. Alors la *randomisation* associée à u_0 est définie par

$$u_0^\omega(x) \coloneqq \sum_{n\in\mathbb{Z}^d} X_n^\omega u_n e^{in\cdot x}\,. \tag{1.12}$$

On peut vérifier facilement que

$$\mathbb{E}\left[\|u_0^\omega\|_{H^s}^2\right] = C\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2s}|u_n|^2 = C\|u_0\|_{H^s}^2 < \infty\,,$$

de sorte que cette randomisation soit de niveau H^s. On montre mieux.

Lemme 1.13 (Randomisation exactement H^s)**.** *On considère la randomisation* (1.12) *où les* $(X_n)_{n\in\mathbb{Z}^d}$ *satisfont de plus* $\mathbb{P}(|X_n| > 0) > 0$*. On suppose que pour tout* $\varepsilon > 0$*, on a* $u_0 \in H^s \setminus H^{s+\varepsilon}$*. Alors pour presque tout* ω *on a :*

$$u_0^\omega \in H^s \setminus \bigcup_{\varepsilon>0} H^{s+\varepsilon}\,.$$

Une démonstration de ce lemme se trouve dans [BT08a], lemme B.1. Ainsi les données randomisées par (1.12) sont bien de même régularité que u_0. En effet, il est important de noter que la randomisation ne régularise pas les données initiales. Si par exemple, pour $u_0 \in H^s$ on avait un résultat du type $u_0^\omega \in H^{\frac{d-\beta}{2}+\varepsilon}$ presque sûrement, alors on pourrait directement appliquer le résultat du lemme 1.4.

L'avantage de telles données aléatoires apparaît dans l'estimation des normes L_x^p de u_0^ω. En particulier pour tout p fini, cette norme est presque sûrement contrôlée par la norme L^2 de u_0. L'ingrédient essentiel de cette démonstration et connu depuis les années 1930, il s'agit du résultat suivant dont une démonstration est donnée en Annexe A.

Théoreme 1.14 (Kolmogorov, Paley, Zygmund)**.** *Soient* $(a_n)_{n\geqslant 0}$ *une suite réelle et* $(X_n)_{n\geqslant 0}$ *une suite de variables aléatoires indépendantes, identiquement* *distribuées vérifiant* (1.11)*. Alors on a*

$$\mathbb{P}\left(\left|\sum_{n\geqslant 0} a_n X_n\right| > \lambda\right) \leqslant \exp\left(-c\frac{\lambda^2}{\|a_n\|_{\ell^2}^2}\right) ,$$

et donc

$$\left\|\sum_{n\geqslant 0} a_n X_n\right\|_{L^r_\omega} \lesssim \sqrt{r}\left(\sum_{n\geqslant 0}|a_n|^2\right)^{\frac{1}{2}} .$$

Ce théorème permet d'étudier les normes $L^q_T L^p_x$ de

$$e^{-it(-\Delta)^\alpha} u_0^\omega = \sum_{n\in\mathbb{Z}^d} e^{it|n|^{2\alpha}} a_n e^{in\cdot x} X_n .$$

Pour cela applique l'inégalité de Minkowski pour $r \geqslant p, q$, le théorème 1.14 et le fait que $|e^{it|n|^{2\alpha}} a_n e^{in\cdot x}| = |a_n|$ pour obtenir

$$\left\|\sum_{n\in\mathbb{Z}^d} e^{it|n|^{2\alpha}} a_n e^{in\cdot x} X_n\right\|_{L^r_\omega L^q_T L^p_x} \lesssim \sqrt{r}\left\|\left(\sum_n |a_n|^2\right)^{\frac{1}{2}}\right\|_{L^q_T L^p_x} .$$

D'autre part, en appliquant Parseval et en utilisant que le tore est de volume fini on obtient

$$\left\|\left(\sum_n |a_n|^2\right)^{\frac{1}{2}}\right\|_{L^q_T L^p_x} \lesssim T^{\frac{1}{q}} \|u_0\|_{L^2_x} .$$

L'inégalité de Markov entraîne alors le résultat suivant.

Corollaire 1.15 (Estimations de « Strichartz » aléatoires)**.** *Soient* $u_0 \in H^s(\mathbb{T}^d)$ *et* u_0^ω *la randomisation définie par* (1.12)*. Soient* $q, p \in [2, \infty[$*. Alors en dehors d'un ensemble de mesure au plus* e^{-cR^2} *on a*

$$\|u_L^\omega\|_{L^q_T W^{s,p}} \leqslant RT^{\frac{1}{q}} \|u_0\|_{H^s} ,$$

où $u_L^\omega(t) = e^{-it(-\Delta)^\alpha} u_0^\omega$*.*

Remarque 1.16. Il convient de noter que ces considérations sont très spécifiques au tore, dans le cas général de fonctions propres $e_n(x)$ du laplacien sur une variété M, on a seulement par application de Minkowski que en dehors d'un ensemble de mesure au plus e^{-cR^2},

$$\left\|e^{it\Delta} u_0\right\|_{L^q_T L^p_x} \leqslant RT^{\frac{1}{q}}\left(\sum_{n\in\mathbb{Z}^d} |a_n|^2 \|e_n\|^2_{L^p(M)}\right)^{\frac{1}{2}} .$$

Remarque 1.17 (Découplage probabiliste)*.* Afin de comprendre l'intérêt de telles estimation, prenons N variables aléatoires indépendantes, qui suivent une loi uniform sur $\{-1,1\}$, notées $(\varepsilon_n)_{1\leqslant n\leqslant N}$. Alors un corollaire du theorème 1.14 est que en dehors d'un ensemble de mesure au plus e^{-cR^2}, on a

$$\left|\sum_{n=1}^{N}\varepsilon_n^{\omega}a_n\right|\leqslant R\left(\sum_{n=0}^{N}|a_n|^2\right)^{\frac{1}{2}}.$$

La comparaison par rapport à une estimation déterministe est frappante. En effet, par application de Cauchy-Schwarz on a seulement

$$\left|\sum_{n=1}^{N}\varepsilon_n^{\omega}a_n\right|\leqslant N^{\frac{1}{2}}\left(\sum_{n=1}^{N}|a_n|^2\right)^{\frac{1}{2}},$$

ainsi l'aspect probabiliste de *découplage* des a_n fait l'économie d'un facteur $N^{\frac{1}{2}}$.

On peut à présent établir une théorie locale probabiliste pour (fNLS).

Proposition 1.18. *Soit $s>\max\{\frac{d}{2}-\frac{3\beta}{2},0\}$ et $u_0\in H^s(\mathbb{T}^d)$. On considère des données initiales u_0^{ω} de la forme (1.12). Pour presque toute donnée u_0^{ω}, il existe $T=T(\omega)>0$ tel que (fNLS) admette une unique solution associée à u_0^{ω} dans l'espace*

$$u_L^{\omega}+\mathcal{C}^0([0,T),H^{s+\beta}),$$

où $u_L^{\omega}(t)=e^{-it(-\Delta)^{\alpha}}u_0^{\omega}$.

Démonstration. On reprend le raisonnement qui nous a conduit à chercher une solution u à (fNLS) sous la forme $u(t)=u_L+v$, où v résout $\Psi(v)=v$, et Ψ est définit par (1.10) on obtient

$$\|\Psi v\|_{L_T^{\infty}H_x^{s+\beta}}\leqslant\|u_L\|_{L_T^4L_x^{\infty}}^2\|u_L\|_{L_T^2H_x^s}+T\|v\|_{L_T^{\infty}H_x^{s+\beta}}^3.$$

Le corollaire 1.15 permet d'affirmer que en dehors d'un ensemble de probabilité au plus e^{-cR^2} on a

$$\|u_L\|_{L_T^2H_x^s}\leqslant RT^{\frac{1}{2}}\|u_0\|_{H^s},$$

mais en revanche on ne peut pas directement l'appliquer à la norme $L_T^4L_x^{\infty}$. Pour cela on choisit un $\varepsilon\in]0,s]$ tel que $u_0\in H^{\varepsilon}$, et alors par injection de Sobolev,

$$\|u_L\|_{L_T^4L_x^{\infty}}\lesssim\|u_L\|_{L_T^4W_x^{\varepsilon,p(\varepsilon)}},$$

pour un certain $p(\varepsilon)\in[2,\infty[$ tel que $\varepsilon p(\varepsilon)>d$. On peut à présent appliquer le corollaire 1.15. Plus précisément, on fixe un $R>0$, et on introduit l'ensemble

$$\Omega_R\coloneqq\{\|u_L\|_{L_T^4L_x^{\infty}}\lesssim RT^{\frac{1}{4}}\|u_0\|_{H^s}\}\cap\{\|u_L\|_{L_T^2H^s}\leqslant RT^{\frac{1}{2}}\|u_0\|_{H^s}\},$$

qui est tel que $\mathbb{P}(\Omega\setminus\Omega_R)\leqslant 2e^{-cR^2}$. De plus pour tout $\omega\in\Omega_R$ on a

$$\|\Psi(v)\|_{L^{\infty}H^{s+\beta}}\lesssim T\left(R^3+\|v\|_{H^{s+\beta}}^3\right),$$

donc pour tout $\|v\|_{L^\infty_T H^{s+\beta}_x} \leqslant R$ et $T = \frac{1}{CR^2}$ on a

$$\|\Psi(v)\|_{L^\infty_T H^{s+\beta}_x} \leqslant \frac{1}{2}\|v\|_{L^\infty_T H^{s+\beta}_x}\,,$$

ainsi pour $\omega \in \Omega_R$ on résout localement (fNLS) en temps $T \sim \frac{1}{R^2}$. On passe à un résultat d'existence presque-sûr par un argument de Borel-Cantelli similaire à celui appliqué pour prouver le méta-théorème. □

On a donc réussi à produire des solutions de façon probabiliste pour des valeurs de s inférieures à $\frac{d}{2} - \frac{\beta}{2}$. La méthode probabiliste nous a fait gagner β dérivées et l'ingrédient essentiel s'est révélé être le fait que pour $u_0 \in H^\varepsilon$, on a presque sûrement $u_L^\omega \in L^q_T L^\infty$, là où l'utilisation de l'injection de Sobolev aurait coûté $\frac{d}{2}$ dérivées et non ε. Ce résultat est donc la conséquence de la structure des données aléatoires et non de la structure de l'équation. On peut obtenir des estimations déterministes $u_L \in L^q_T L^p_x$, qui sont appelées estimations de Strichartz, qui ne sont vérifiées que pour certains couples (p, q). Dans le cas du tore $\mathbb{T}^d$ ces estimations sont difficiles à obtenir et reposent sur des propriétés arithmétiques du tore $\mathbb{T}^d$. Pour ne citer qu'un exemple, donnons l'estimation de Strichartz sur $\mathbb{T}^2$.

Théoreme 1.19 (Zygmund, [Zyg74])**.** *Soit $T > 0$, alors $\|e^{it\Delta}u_0\|_{L^4_{T,x}} \lesssim \|u_0\|_{L^2_x}$.*

En combinant les résultats de cette section et de la section précédente on en déduit le résultat suivant.

Théoreme 1.20 (Solutions globales probabilistes pour (fNLS))**.** *On considère (fNLS) pour $(d, \alpha, \beta) = (3, 2, 1)$. Alors pour presque toute donnée initiale de la forme u_0^ω définie par (1.8) il existe une unique solution globale à (fNLS) telle que pour tout $s < \frac{1}{2}$ et tout $t > 0$ on a*

$$\|u(t)\|_{H^s} \lesssim \log^{\frac{1}{2}}(1 + t)\,.$$

Esquisse de démonstration. Il s'agit essentiellement de vérifier les hypothèses du méta-théorème. On utilise la mesure μ, construite formellement par (1.6) et invariante par la dynamique de (fNLS), ainsi les hypothèses (i) et (ii) sont vérifiées. Enfin, pour les données initiales sur le support de la mesure μ, c'est à dire de la forme (1.8) de régularité $H^{\alpha - \frac{d}{2}^-} = H^{\frac{1}{2}^-}$ la théorie locale de la proposition 1.18 s'applique et on en déduit que l'hypothèse (iii) du méta-théorème est vérifiée. □

2.3. Inconvénient des mesures de Gibbs. Dans la section 2.1, on a réussi à construire des mesures invariantes en suivant la méthode hamiltonienne. À partir d'une telle mesure invariante, le méta-théorème est en mesure de fournir des solutions globales à croissance très lente. Il faut garder à l'esprit que cette méthode souffre d'une limitation essentielle : on ne choisit ni cette mesure ni sa régularité. En effet, les mesures de Gibbs sont assez rares dans la pratique car elles proviennent d'équations hamiltoniennes. De plus, leur niveau de régularité est fixé. Par exemple dans le cas de (fNLS), on a montré que cette régularité est de niveau $H^{\alpha-\frac{d}{2}^-}(\mathbb{T}^d)$. On voit ainsi que cette régularité se détériore avec la dimension, rendant toute théorie de Cauchy probabiliste fondée sur l'exploitation de telles mesures très difficile lorsque d est grand.

Voici une obstruction générale quant à l'existence d'une mesure invariante, il est souvent nécessaire d'exploiter un phénomène de compacité pour pouvoir construire de telles mesures. Dans la section 2.1 on a donné une construction sur le tore, domaine que nous avons abondamment utilisé grâce aux séries de Fourier. Qu'en est-il pour (fNLS) posée sur $\mathbb{R}^d$? Dans ce cas, c'est la dispersion qui fait obstruction à l'existence des mesures invariantes.

Proposition 1.21 (Proposition 3.2 de [BT20])**.** *Soit μ une mesure de probabilité sur $L^2(\mathbb{R})$, invariante par le flot ϕ_t de l'équation*

$$i\partial_t u + \partial_x^2 u = |u|^4 u\,.$$

Alors $\mu = \delta_0$.

Démonstration. Cette équation est globalement bien posée dans L^2 et ses solutions font *scattering*, c'est à dire qu'il existe $u^+ \in L^2$ tel que l'on ait $\|u(t) - e^{it\Delta}u^+\|_{L^2} \underset{t\to\infty}{\longrightarrow} 0$. Ceci est démontré dans [Dod16a]. Pour une fonction χ, lisse et à support compact, on écrit que pour tout t,

$$\int_{L^2} \frac{\|\chi u\|_{L^1}}{1+\|u\|_{L^2}}\,\mathrm{d}\mu = \int_{L^2} \frac{\|\chi \phi_t u\|_{L^1}}{1+\|u\|_{L^2}}\,\mathrm{d}\mu(u)\,,$$

où on a utilisé l'invariance de μ et noté ϕ_t le flot de l'équation. On observe par Cauchy-Schwarz et conservation de la norme L^2 que

$$\frac{\|\chi \phi_t u\|_{L^1}}{1+\|u\|_{L^2}} \leqslant \frac{\|\chi\|_{L^2}\|\phi_t u\|_{L^2}}{1+\|u\|_{L^2}} \lesssim 1\,.$$

Avec l'information $\|\chi\phi_t u\|_{L^1} \leqslant \|\chi e^{it\Delta}u^+\|_{L^1} + \|\chi\|_{L^2}\|\phi_t u - e^{it\Delta}u^+\|_{L^2} \to 0$ donnée par le scattering et la dispersion, on en déduit par convergence dominée que

$$\int_{L^2} \frac{\|\chi u\|_{L^1}}{1+\|u\|_{L^2}}\,\mathrm{d}\mu = 0\,,$$

et donc $u = 0$, μ-presque partout. $\square$

On vient de voir que la construction d'un mesures invariante pour une équations posée sur $\mathbb{R}^d$ n'est pas toujours possible, ce qui rend la tâche de construire des solutions globales plus difficiles. On peut toutefois chercher à construire des solutions locales probabilistes par la méthode de la section 2.2.

Pour cela on introduit des données aléatoires sur l'espace $\mathbb{R}^d$. La série aléatoire introduite pour le cas du tore (1.12) randomise les fonctions à l'échelle unité en fréquences, on va donc imiter ce procédé sur l'espace euclidien. Soit $u_0 \in H^s(\mathbb{R}^d)$. On introduit une partition de l'unité $\sum_{n\in\mathbb{Z}^d} \chi(\xi - n) = 1$ où χ est une fonction lisse à support compact. On fixe aussi une suite $(X_n)_{n\in\mathbb{Z}^d}$ de variables sous-gaussiennes qui satisfont (1.11) indépendantes et identiquement distribuées. On définit alors la randomisation u_0^ω par sa transformée de Fourier :

$$\mathcal{F}_{x\to\xi}(u_0^\omega)(\xi) = \sum_{n\in\mathbb{Z}^d} X_n^\omega \chi(\xi - n)\hat{u}_0(\xi)\,,$$

ce que l'on écrit dans l'espace physique comme

$$u_0^\omega = \sum_{n\in\mathbb{Z}^d} X_n^\omega \chi(D - n)u_0\,. \tag{1.13}$$

Une telle randomisation est appelée *randomisation de Wiener*, elle est par exemple utilisée dans [BOP15b]. On en déduit, comme dans le cas des tores, des estimations de Strichartz probabilistes. En effet, l'inégalité de Bernstein appliquée aux couronnes de largeur unité $\{n \leqslant |\xi| \leqslant n+1\}$ assure que pour $p \geqslant 2$,

$$\|\chi(D-n)u\|_{L^p(\mathbb{R}^d)} \lesssim \|u\|_{L^2_x(\mathbb{R}^d)}\,.$$

En combinant avec la preuve du théorème 1.14 on obtient les estimations de Strichartz probabilistes suivantes.

Théoreme 1.22. *Soit $u_0 \in H^s(\mathbb{R}^d)$ et $p, q \in [2,\infty[$. Soit u_0^ω sa randomisation donnée par (1.13). Alors en dehors d'un ensemble de mesure au plus e^{-cR^2} on a*

(i) $\|u_0^\omega\|_{L^p_x} \leqslant R\|u_0\|_{L^2}$.

(ii) $\|e^{it\Delta}u_0^\omega\|_{L^q_T L^p_x} \leqslant RT^{\frac{1}{q}}\|u_0\|^2_{L^2_x}$.

Signalons que dans le cas $\mathbb{R}^d$ le phénomène de dispersion permet de prouver des estimations de Strichartz déterministes bien plus générales que sur le tore. Pour écrire ces estimations nous introduisons l'espace de Strichartz de régularité s et son dual

$$S^s := \bigcap_{(q,r)\text{ admissible}} L^q_T W^{s,r} \text{ and } Y^s := \sum_{(q,r)\text{ admissible}} L^{q'}_T W^{s,r'}_x\,,$$

où (q,r) est dit admissible si $(q,r,d) \neq (2,\infty,2)$ et $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$.

Théoreme 1.23 (Dispersion et estimations de Strichartz)**.** *Soit $s \geqslant 0$ et $u_0 \in H^s(\mathbb{R}^d)$. Alors*

(i) *Dispersion : pour $p \geqslant 2$ on a $\|e^{it\Delta}u_0\|_{L^p_x} \lesssim t^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_{L^{p'}}$.*

(ii) *Strichartz homogène : on a $\|e^{it\Delta}u_0\|_{S^s} \lesssim \|u_0\|_{H^s}$*

(iii) *Strichartz inhomogène : on a $\|\int_0^t e^{i(t-t')\Delta}(F(u(t')))\,\mathrm{d}t'\|_{S^s} \lesssim \|F(u)\|_{Y^s}$.*

On voit que la condition d'admissibilité sur (q, r) limite la puissance de ce résultat alors que la version probabiliste de ces estimations est inconditionnelle en q, r finis.

À l'aide de ces estimations de Strichartz on peut démontrer et par une preuve similaire à celle de 1.4 on peut étudier le problème de Cauchy pour

$$\left\{ \begin{array}{rcl} i\partial_t u + \Delta u & = & |u|^2 u \\ u(0) & = & u_0 \in H^s(\mathbb{R}^4) \, . \end{array} \right. \tag{1.14}$$

Théoreme 1.24 (Théorie locale déterministe)**.** *Soit $s > 1$. Alors (1.14) est uniformément localement bien posé dans $H^s(\mathbb{R}^4)$.*

Grâce à la version probabiliste des estimations de Strichartz, nous pouvons énoncer le résultat suivant dont la preuve est esquissée en Annexe B.

Théoreme 1.25 (Benyi, Oh et Pocovnicu [BOP15b, BOP15a])**.** *Soit $s > \frac{3}{5}$ et $u_0 \in H^s(\mathbb{R}^3)$. On note u_0^ω sa randomisation donnée par (1.13). Fixons également $\sigma > 1$. Alors pour presque tout ω, il existe une unique solution à (1.14) associée à u_0^ω dans l'espace*

$$u_L + \mathcal{C}^0([0, T], H^\sigma(\mathbb{R}^d) \, ,$$

où l'on note $u_L(t) = e^{it\Delta} u_0^\omega$.

Pour conclure cette section, signalons que d'autres randomisations sont possibles. Par exemple, plutôt que la randomisation de Wiener en fréquences donnée par (1.13), on peut introduire une randomisation qui agit également en espace. Une telle randomisation, dite *microlocale* est définie comme suit. On se donne $u_0 \in L^2(\mathbb{R}^d)$ que l'on décompose sur des cubes de taille unité à la fois en espace et fréquences en écrivant

$$u_0(x) = \sum_{k,n \in \mathbb{Z}^d} \psi(x-k)\chi(D-n)u \, ,$$

avec χ et ψ des fonctions à supports compacts telles que $\mathbf{1}_{|\cdot| \leqslant 1} \leqslant \chi, \psi \leqslant \mathbf{1}_{|\cdot| \leqslant 2}$. La randomisation associée est alors

$$u_0^\omega(x) = \sum_{k,n \in \mathbb{Z}^d} X_{k,n}^\omega \psi(x-k)\chi(D-n)u \, ,$$

où les $(X_{k,n})_{k,n \in \mathbb{Z}^d}$ sont des variables sous-gaussiennes indépendantes et identiquement distribuées.

L'utilisation d'une telle randomisation est récente, nous renvoyons à [Bri18a] et [BK19] pour des applications.

3. Globaliser avec une méthode d'énergie

Cette section concerne le chapitre 3 de cette thèse et s'insère dans le cadre de l'étude du problème de Cauchy global à données aléatoires pour les équations d'ondes non-linéaires. Les travaux qui sont présentés font l'objet d'une publication dans *Journal of Differential Equations* [Lat21].

On considère l'équation des ondes non-linéaires posée sur un domaine U qui peut être $\mathbb{R}^3$ ou $\mathbb{T}^3$ et qui s'écrit

$$(\mathrm{NLW}_p) \qquad \begin{cases} \partial_t^2 u - \Delta u + |u|^{p-1}u &= 0 \\ (u(0), \partial_t u(0)) &= (u_0, u_1) \in H^s(U) \times H^{s-1}(U)\,. \end{cases}$$

L'espace $H^s(U) \times H^{s-1}(U)$ sera plus simplement noté $\mathcal{H}^s$ dans la suite.

Une solution u à (NLW_p) conserve, au moins formellement, la quantité suivante

$$\mathcal{E}(u(t)) := \int_U \left(\frac{|\partial_t u(t)|^2}{2} + \frac{|\nabla u(t)|^2}{2} + \frac{|u(t)|^{p+1}}{p+1} \right) \mathrm{d}x\,,$$

à laquelle on référera en tant que « énergie », de niveau de régularité H^1.

3.1. Problème de Cauchy local et global pour (NLW_p)**.** Les travaux qui concernent l'étude du problème de Cauchy local et global pour (NLW_p) sont trop nombreux pour tous les évoquer. On se limite aux travaux les plus pertinents en rapport avec les travaux qui seront présentés.

3.1.1. *La théorie déterministe.* Le problème de Cauchy local pour l'équation (NLW_p) est bien compris depuis les années 1990 [LS95, GV95, Kap90]. En particulier, dès que $s > \frac{3}{2} - \frac{2}{p-1}$, ($\mathrm{NLW}_p$) est uniformément localement bien posé dans $\mathcal{H}^s$. Une telle théorie est développée à l'aide d'estimations de Strichartz. En particulier, cela permet la construction de solutions locales dans l'espace d'énergie $\mathcal{H}^1$ dès que $p < 5$, qui sont alors des solutions globales. Pour $p = 5$ le problème est dit « énergie-critique » et l'obtention d'une théorie globale dans $\mathcal{H}^1$ est plus difficile [Kap94]. Pour $p > 5$, aucune théorie globale n'est connue au niveau $\mathcal{H}^1$, l'équation étant « sur-critique » par rapport à l'énergie.

En particulier, dans le cas $p = 3$, le problème de Cauchy est localement bien posé dans $\mathcal{H}^s$ pour $s \geqslant \frac{1}{2}$, et globalement bien posé dans $\mathcal{H}^1$. Dans [GP03, KPV00, BC06], les auteurs étendent ce résultat et construisent une théorie globale au niveau $\mathcal{H}^s$ pour $s > \frac{3}{4}$. Dans le cas radial, Dodson construit des solutions globales pour $s > \frac{1}{2}$, [Dod19]. À l'inverse, pour $s \in]0, \frac{1}{2}[$, le problème de Cauchy est mal posé [BT08a].

Notons que pour tout $p \geqslant 1$ on peut toujours construire des solutions faibles globales au niveau de régularité de l'énergie, c'est à dire dans $\mathcal{H}^1$. C'est le contenu du résultat suivant.

Théoreme 1.26 (Strauss, [Str70]). *Soit f une fonction réelle lisse et F une primitive de f. On suppose que $F(v) \gtrsim -|v|^2$ et que $\frac{|F(u)|}{|f(u)|} \to \infty$ quand $|u| \to \infty$. Soit $(u_0, u_1) \in \mathcal{H}^1(\mathbb{R}^3)$ d'énergie finie, c'est à dire*

$$E(u_0, u_1) := \int_{\mathbb{R}^3} \left(\frac{|u_1|^2}{2} + \frac{|\nabla u_0|^2}{2} + F(u_0) \right) \mathrm{d}x < \infty \,.$$

Alors l'équation

$$\left\{ \begin{array}{rcl} \partial_t^2 u - \Delta u + f(u) & = & 0 \\ (u(0), \partial_t u(0)) & = & (u_0, u_1) \in \mathcal{H}^1(\mathbb{R}^3)\,, \end{array} \right. \tag{1.15}$$

admet une solution faible, c'est à dire qu'il existe une solution u au sens des distributions, $u : \mathbb{R} \to \mathcal{H}^1(\mathbb{R}^3)$ qui est faiblement continue en temps et telle que

$$E(u(t), \partial_t u(t)) \leqslant E(u_0, u_1) \text{ pour tout } t \geqslant 0\,.$$

3.1.2. *Améliorations probabilistes.* L'étude du problème de Cauchy pour (NLW$_p$) avec des données aléatoires est entreprise par Burq et Tzvetkov, qui développent une théorie locale [BT08a] et globale [BT08b]. Leurs méthodes culminent dans [BT14] avec l'obtention d'une théorie globale probabiliste dans l'espace $\mathcal{H}^0$ lorsque $p = 3$.

Le cas $p = 5$ se révèle plus compliqué, car énergie-critique. Oh et Pocovnicu étudient ce problème dans [Poc17, OP16] et construisent une théorie globale dans ce cas. En particulier, lorsque $d = 3$ et $p = 5$ ils obtiennent une théorie globale probabiliste dans $\mathcal{H}^s$ pour $s > \frac{1}{2}$.

Dans la gamme des non-linéarités $p \in]3, 5[$, on s'attend à construire une théorie globale probabiliste dans $\mathcal{H}^s$ pour $s > \frac{p-3}{p-1}$. Des premiers travaux de Lührmann et Mendelson [LM14, LM16] traitent cette question avant que le résultat pour $s > \frac{p-3}{p-1}$ soit prouvé par Sun et Xia [SX16].

3.2. Contribution de la thèse. L'objectif principal est de compléter les résultats obtenus par Burq et Tzvetkov dans le cas $p = 3$, Oh et Pocovnicu dans le cas $p = 5$ et Sun et Xia dans le cas intermédiaire $p \in]3, 5[$. Puisque pour $p > 5$ l'énergie est une quantité sur-critique pour (NLW$_p$) on ne s'attend pas à produire facilement des solutions fortes, c'est à dire des solutions uniques de haute régularité. On construit plutôt des solutions faibles globales pour des données initiales dans $\mathcal{H}^s$ pour certains $s < 1$, donnant ainsi une extension du théorème 1.26 dans un cadre probabiliste.

De façon plus précise, rappelons que la méthode probabiliste locale de la section 2.2 est la suivante. On cherche une solution $u \in \mathcal{C}^0(\mathbb{R}, \mathcal{H}^s)$ à (NLW$_p$) sous la forme $u = u_L + v$ avec u_L la solution globale de l'équation linéaire

$$\left\{ \begin{array}{rcl} \partial_t^2 u - \Delta u & = & 0 \\ (u(0), \partial_t u(0)) & = & (u_0, u_1) \in \mathcal{H}^s\,, \end{array} \right. \tag{1.16}$$

et $v \in \mathcal{C}^0(\mathbb{R}, \mathcal{H}^1)$ qui résout formellement

$$\left\{\begin{array}{rcl} \partial_t^2 v - \Delta v & = & -|u_L + v|^{p-1}(u_L + v) \\ (v(0), \partial_t v(0)) & = & 0\,. \end{array}\right. \tag{1.17}$$

On cherche v au niveau $\mathcal{H}^1$ car on souhaite exploiter la conservation de l'énergie pour (NLW$_p$) qui contrôle la norme $\mathcal{H}^1$.

Pour une donnée initiale $(u_0, u_1) \in \mathcal{H}^s$ on introduit la randomisation associée à u_0 et u_1 respectivement construites avec des variables aléatoires indépendantes pour chacune des deux fonctions via (1.13) dans le cas $\mathbb{R}^3$ et (1.12) dans le cas $\mathbb{T}^3$. On note $\mu_{(u_0,u_1)}$ la mesure image par une telle randomisation, qui est une mesure sur l'espace $\mathcal{H}^s$. Enfin on note

$$\mathcal{M}^s(U) \coloneqq \bigcup_{(u_0,u_1)\in\mathcal{H}^s(U)} \mu_{(u_0,u_1)}\,,$$

l'ensemble formé par toutes ces mesures au niveau $\mathcal{H}^s(U)$.

Le résultat principal s'énonce alors sous la forme suivante.

Théorème A (L. [Lat20a])**.** *Soient $p > 5$ et $s > \frac{p-3}{p-1}$. Soit U qui désigne $\mathbb{R}^3$ ou $\mathbb{T}^3$. Soit $\mu \in \mathcal{M}^s(U)$. Alors pour μ-presque toute donnée initiale $(u_0, u_1) \in \mathcal{H}^s(U)$ il existe une solution globale $u \in \mathcal{C}^0(\mathbb{R}, \mathcal{H}^s(U))$ à (NLW$_p$) dans le sens faible, c'est à dire que pour toute fonction $\varphi \in \mathcal{C}^2(\mathbb{R} \times U)$ on a l'égalité,*

$$\int_{\mathbb{R}\times U} \Big(\partial_t v \partial_t \varphi - \nabla v \cdot \nabla \varphi - (u_L + v)|u_L + v|^{p-1}\varphi(t)\Big)\, \mathrm{d}x\, \mathrm{d}t = 0\,,$$

où u_L est solution de (1.16) et v est formellement une solution de (1.17) de sorte que $u = u_L + v$. De plus v est tel que pour tout $T > 0$ on a

$$v \in H^1(]-T, T[\times U) \cap L^{p+1}(]-T, T[\times U)\,.$$

Remarque 1.27. La solution faible construite par le théorème possède la même régularité que les solutions fortes. Cependant, il faut bien retenir que ce résultat ne donne aucune information quant à l'unicité de telles solutions. Ainsi on parlera toujours de solutions *faibles.*

Le deuxième résultat principal que l'on démontre concerne le cas $p \in [3, 5[$. On sait qu'une théorie globale probabiliste est connue dans $\mathcal{H}^s$ pour $s > \frac{p-3}{p-1}$, et par ailleurs, dans le cas $p = 3$, la régularité limite $s = 0$ est aussi traitée dans [BT14]. On se propose de généraliser cet argument à toute la gamme des non-linéarités $p \in]3, 5[$.

Théorème B (L. [Lat21])**.** *Soient $p \in]3, 5[$, $s \coloneqq \frac{p-3}{p-1}$ et U qui désigne $\mathbb{R}^3$ ou $\mathbb{T}^3$. Soit $\mu \in \mathcal{M}^s$, alors pour μ presque toute donnée initiale $(u_0, u_1) \in \mathcal{H}^s(U)$, il existe une unique solution à (NLW$_p$) dans l'espace $u_L + \mathcal{C}^0(\mathbb{R}, \mathcal{H}^1)$ où u_L résout (1.16).*

Pour une explication des idées principales de la preuve de ce théorème, on renvoie à l'introduction du chapitre 3.

La fin de cette section est consacrée à l'explication des idées de la preuve du théorème A. Pour commencer, on évacue le problème de Cauchy local et on suppose que celui-ci a été résolu de façon probabiliste sous la forme $u(t) = u_L^\omega(t) + v(t)$ avec u_L^ω qui résout (1.16) avec données initiales (u_0^ω, u_1^ω) et v qui résout (1.17) dans $\mathcal{C}^0([-T,T], \mathcal{H}^1)$. Afin de montrer que u est une solution globale, et puisque u_L est globale par définition, il suffit de montrer que v résout globalement (1.17). Pour cela, comme v est de régularité $\mathcal{H}^1$, qui est le niveau de conservation de l'énergie pour (NLW_p), on voudrait utiliser cette conservation. Cependant v ne satisfait pas la même équation que u donc ne conserve pas l'énergie associée à (NLW_p), il va donc falloir utiliser une méthode d'énergie *perturbée*.

Remarque 1.28. Le problème traité est typique des travaux de Burq-Tzvetkov, qui permettent de construire des solutions dans $\mathbb{R}^d$ ou $\mathbb{T}^d$ (et même d'autres variétés) à divers niveaux de régularité, et non uniquement celui prescrit par la mesure de Gibbs lorsqu'elle existe. Ceci illustre la flexibilité de cette approche, la contrepartie étant que la globalisation est rendue bien plus difficile et moins automatique, à l'inverse de ce que permet le méta-théorème de la section 2.1 en présence d'une mesure invariante.

On introduit l'énergie modifiée associée à v, définie par

$$E(t) := \frac{\|\partial_t v(t)\|_{L^2}^2}{2} + \frac{\|\nabla v(t)\|_{L^2}^2}{2} + \frac{\|v(t)\|_{L^{p+1}}^{p+1}}{p+1},$$

dont on étudie l'évolution. Son contrôle global fournira un contrôle de la norme $\mathcal{H}^1$ et sera suffisant pour montrer que v est une solution globale. On calcule l'évolution de $E(t)$,

$$E'(t) = -\int_{\mathbb{R}^3} \partial_t v(t) \left(|u_L^\omega(t) + v(t)|^{p-1}(u_L^\omega(t) + v(t)) - |v(t)|^{p-1}v(t) \right) \mathrm{d}x\,.$$

À ce stade, il convient de rappeler que $(u_0, u_1) \in \mathcal{H}^s$, et donc d'après les inégalités de Strichartz aléatoires données par le corollaire 1.15, qui sont facilement adaptables au cas de l'équation considérée, on a presque sûrement $u_L^\omega \in L_T^q W^{s,p}$ pour tous p et q finis. Disons que heuristiquement on peut même mener les calculs avec $u_L^\omega \in L_T^q W^{s,\infty}$ pour tout q. On doit donc répondre à la question suivante : pour quelles valeurs de $s < 1$, et quelles valeurs correspondantes de p peut-on prouver une inégalité du type $E'(t) \lesssim E(t)$?

On peut simplifier l'expression de $E'(t)$ en écrivant que

$$|u_L^\omega + v|^{p-1}(u_L^\omega + v) \simeq |v|^{p-1}v + u_L^\omega v^{p-1} + \{\text{termes d'ordres inférieur}\}\,,$$

Le terme, $|v|^{p-1}v$ est compensé dans l'expression de E' et le terme suivant, zv^{p-1} apparaît comme le terme principal restant car il est le plus non-linéaire en v. On obtient donc, toujours à un niveau informel :

$$E'(t) \simeq \int_{\mathbb{R}^3} \partial_t v(t) u_L^\omega(t) v(t)^{p-1} \, \mathrm{d}x \, .$$

Première tentative. En plus de la borne $u_L \in W^{s,\infty}$, on peut utiliser les bornes $\partial_t v \in L^2$ et $v^{p-1} \in L^{\frac{p+1}{p-1}}$ qui sont contrôlées par des puissances de l'énergie E. Après application de l'inégalité de Hölder et de la définition de E on a :

$$\begin{aligned} E'(t) &\lesssim \|\partial_t v\|_{L^2} \|u_L^\omega(t)\|_{L^\infty} \|v(t)\|_{L^{p+1}}^{p-1} \\ &\lesssim E(t)^{\frac{1}{2}+\frac{p-1}{p+1}} \, . \end{aligned}$$

Remarquons que l'on a $\frac{1}{2} + \frac{p-1}{p+1} \leqslant 1$ dès que $p \leqslant 3$. On a utilisé uniquement la borne $u_L \in L_x^\infty$, accessible presque sûrement dès que $s > 0$. C'est ce qui conduit au résultat dans [BT14].

Deuxième tentative. Lorsque s est grand, l'inégalité précédente n'utilise pas de façon optimale l'information $u_L \in W^{s,\infty}$. De plus, l'analyse ci-dessus est limitée à $p \leqslant 3$. Pour aller plus loin on transfère des dérivées sur u_L^ω via des intégrations par parties en temps. Les dérivées temporelles sont alors converties en dérivées spatiales puisque l'on a $\partial_t u_L^\omega \simeq |\nabla| u_L^\omega$. Ainsi on a

$$\begin{aligned} E(t) &\simeq \int_0^t \int_{\mathbb{R}^3} \partial_t (v(t')^p) u_L^\omega(t') \, \mathrm{d}t' \\ &\simeq \int_{\mathbb{R}^3} \int_0^t v(t')^p \partial_t u_L^\omega(t') \, \mathrm{d}t' \, \mathrm{d}x \\ &\simeq \int_{\mathbb{R}^3} \int_0^t v(t')^p |\nabla| u_L^\omega(t') \, \mathrm{d}t' \, \mathrm{d}x \, , \end{aligned}$$

où l'on a négligé les termes de bord dans l'intégration par parties. Puisque l'on veut faire apparaître s dérivées en espace sur u_L^ω, on écrit $\nabla = \nabla^{1-s}\nabla^s$ et on « intègre par parties » avec ∇^{1-s}, ce qui donne

$$E(t) \simeq \int_0^t \int_{\mathbb{R}^3} \nabla^s u_L^\omega \nabla^{1-s} v v^{p-1} \, \mathrm{d}x \, \mathrm{d}t' \, .$$

Cette observation est effectuée par Oh et Pocovnicu dans le cas $s = 1/2$ et $p = 5$. Il suffit alors de placer $v \in L^{p+1}$, $\nabla^s z \in L^\infty$ et $\nabla^{1-s} v \in L^{\frac{p+1}{2}}$, terme pour lequel on peut utiliser l'inégalité d'interpolation de Gagliardo-Nirenberg :

$$\|\nabla^{1-s} v\|_{L^{\frac{p+1}{2}}} \lesssim \|\nabla v\|_{L^2}^{1-\alpha} \|v\|_{L^{p+1}}^\alpha \lesssim E(t)^{\frac{1-\alpha}{2}+\frac{\alpha}{p+1}} \, ,$$

valable dès que $\frac{2}{p+1} - \frac{1-s}{3} = \frac{1-\alpha}{6} + \frac{\alpha}{p+1}$ et $s \geqslant \alpha$. On trouve $\alpha = \frac{p-3}{p-1}$ et donc

$$E(t) \lesssim \|u_L^\omega\|_{W^{s,\infty}} \int_0^t E(t') \, \mathrm{d}t' \, ,$$

ce qui permet de conclure par l'inégalité de Grönwall. On voit ainsi que cet argument ne donne pas de limitation en p. On réfère au chapitre 3 de cette thèse pour les détails dans le cas $p > 5$.

3.3. Perspectives. La construction des solutions globales, continues en temps à valeurs H^s, pour $s < 1$ qui est donnée par le théorème A n'est pas pleinement satisfaisante. En effet, une telle construction ne dit rien de l'unicité des solutions produites.

On rappelle que de façon générale, on peut montrer que le problème de Cauchy pour (NLW$_p$) est uniformément localement bien posé au niveau H^s dès que $s > \frac{3}{2} - \frac{2}{p-1}$ en utilisant des estimations de Strichartz. Prenons un exemple typique du régime sur-critique pour l'énergie, le cas $p = 7$. On est alors capable de construire,

(1) Des solutions globales, non nécessairement uniques, au niveau $\mathcal{C}_t H^s$ pour $s > \frac{2}{3}$ grâce au théorème A.
(2) Des solutions fortes locales, uniques, au niveau $\mathcal{C}_t H^s$ pour $s > \frac{7}{6}$, d'après la théorie locale déterministe.

Une question de long-terme à laquelle on voudrait répondre est la suivante.

Question 1.1. *Peut-on construire des solutions uniques et globales pour pour* (NLW$_p$) *au niveau H^1 lorsque $d = 7$, pour presque toute donnée initiale ?*

Afin d'aborder une telle question, une première approche pourrait être celle d'étudier la relation entre les solutions faibles et fortes à notre disposition.

(1) On peut chercher une théorie probabiliste locale au niveau H^1 par exemple : on cherche $u(t) = u_L(t) + v(t)$, avec $v(t)$ le terme Duhamel, qui est plus régulier que u_L. Pour $u_0 \in H^1$ on a $u_L \in H^1$ et $v \in H^2$ donc il est raisonnable d'essayer de résoudre localement le problème de Cauchy local dans $u_L + \mathcal{C}^0([0,T], H^{\frac{7}{6}^+}(\mathbb{R}^3))$.
(2) On peut alors chercher à montrer une propriété d'unicité fort-faible. C'est à dire que si l'on note u_S la solution construite en (1) sur l'intervalle $[0,T]$ et que l'on note u_W la solution du théorème A (ou même du théorème 1.26) avec même condition initiale, on veut démontrer que $u_S(t) = u_W(t)$ pour tout $t \in [0,T]$. Cette étape semble également raisonnable.

Une autre question est celle de l'optimalité du résultat pour $p \in [3,5]$.

Question 1.2. *Peut-on montrer que lorsque $p \in [3,5]$ et $s < \frac{p-3}{p-1}$, le problème de Cauchy pour* (NLW$_p$) *est mal posé pour presque toute donnée initiale ?*

Cette question est étudiée dans [ST20] mais dans un cadre légèrement différent. En effet, les auteurs démontrent que pour $p \in [3,5]$ et des valeurs de s comprises entre $\max\{0, \frac{3}{2} - \frac{2}{p-2}\}$ et $\frac{3}{2} - \frac{2}{p-1}$ le problème de Cauchy est localement mal posé pour un ensemble *dense* de données initiales.

Il n'est pas exclu que cet ensemble dense soit de mesure nulle, ainsi la question reste entièrement ouverte.

4. Scattering pour l'équation de Schrödinger au niveau L^2

Cette section concerne le chapitre 4 de cette thèse et s'insère dans le cadre de l'étude du *scattering* pour les équations de Schrödinger non-linéaires à faible régularité. Les travaux qui sont présentés font l'objet d'un article soumis [Lat20a].

Dans la suite on considère l'équation de Schrödinger non-linéaire suivante :

$$\begin{cases} i\partial_t u + \Delta u &= |u|^{p-1}u \\ u(0) &= u_0 \in H^s(\mathbb{R}^d)\,. \end{cases} \tag{1.18}$$

Avant d'expliquer les travaux récents qui portent sur le *scattering* des solutions de (1.18) à données aléatoires, nous expliquons quelques résultats de base de scattering.

4.1. Méthode générale de scattering. Dans cette sous-section, on suppose qu'une théorie de Cauchy globale est connue pour (1.18). La question à laquelle on souhaite répondre est la description du comportement en temps long de ces solutions globales. En particulier, les solutions sont-elles proches de solutions linéaires ? Pour tenter de répondre à cette question, on écrit (1.18) sous forme intégrale :

$$u(t) = e^{it\Delta}u_0 - \int_0^t e^{i(t-t')\Delta}\left(|u|^{p-1}u\right)\,\mathrm{d}t'\,,$$

de sorte que l'on identifie une partie linéaire $u_L(t) = e^{it\Delta}u_0$ de la solution, réduisant l'étude du comportement asymptotique au comportement du terme

$$-i\int_0^t e^{i(t-t')\Delta}(|u|^{p-1}u)\,\mathrm{d}t' = -ie^{it\Delta}\int_0^t e^{-it'\Delta}(|u|^{p-1}u)\,\mathrm{d}t'\,,$$

de sorte que

$$\|u(t) - e^{it\Delta}u_0\|_{H^s} \lesssim \left\|\int_0^t e^{-it'\Delta}|u(t')|^{p-1}u(t')\,\mathrm{d}t'\right\|_{H^s}\,. \tag{1.19}$$

Si le terme $\displaystyle\int_0^t e^{-it'\Delta}|u(t')|^{p-1}u(t')\,\mathrm{d}t'$ converge dans H^s lorsque $t\to\infty$, alors il existe $u_+ \in H^s$ tel que $e^{-it\Delta}u(t) - u_0 \underset{t\to\infty}{\longrightarrow} u_+$, ce que l'on peut écrire sous la forme $u(t) = e^{it\Delta}u_0 + v(t)$ avec $v(t) \simeq e^{it\Delta}u_+$ lorsque $t\to\infty$, c'est à dire que u exhibe un comportement linéaire en temps long. Il reste donc à trouver des conditions pour que (1.19) converge.

Remarque 1.29. On peut comprendre le scattering comme un comportement linéaire intermédiaire. En effet, on aurait pu imaginer que $e^{-it\Delta}u(t) - u_0 \to 0$ dans H^s, donnant ainsi une approximation totalement linéaire ne dépendant que de la donnée initiale. Dans le cas du scattering on s'autorise un comportement asymptotique linéaire où u_0 est remplacé par $u_0 + u_+$, c'est à dire $e^{-it\Delta}u(t) - (u_0 + u_+) \underset{t\to\infty}{\longrightarrow} 0$ dans H^s.

Une première condition pour que (1.19) converge dans H^s est la convergence absolue de l'intégrale

$$\int_0^\infty e^{i(t-t')\Delta}|u(t')|^{p-1}u(t')\,\mathrm{d}t'\,,$$

dans H^s, c'est à dire

$$\int_0^\infty \||u(t)|^{p-1}u(t)\|_{H^s_x}\,\mathrm{d}t < \infty\,.$$

On peut donner une condition suffisante moins contraignante grâce à l'estimation de Strichartz inhomogène du théorème 1.23 (*iii*),

$$\left\|\int_0^t e^{i(t-t')\Delta}|u(t')|^{p-1}u(t')\,\mathrm{d}t'\right\|_{L^\infty_t H^s_x} \lesssim \||u|^{p-1}u\|_{Y^s(\mathbb{R})}\,.$$

Une telle approche est utilisée dans la plupart des travaux déterministes. Un résultat intermédiaire important dans ces travaux est souvent que le contrôle d'une norme de Strichartz globalement en temps implique le scattering.

Lemme 1.30. *Soit $d = 3$, $p = 3$ et $q \in [\frac{10}{3}, 10]$. Alors le contrôle global $\|u\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^3)} < \infty$ pour les solutions de (1.18) entraîne le scattering dans H^1.*

Obtenir une borne globale en temps est en général difficile. La famille des estimations de Morawetz offre des contrôles qui permettent d'exclure les phénomènes de concentration, ainsi de telle estimations apportent parfois les estimations espace-temps nécessaires au scattering. En particulier, dans une séries de travaux qui culminent avec [CKS+10] l'existence globale et le scattering dans H^1 sont démontrés pour (1.18) lorsque $d = 3$ et $p = 5$. On consultera [Tao06] pour une discussion des méthodes employées.

Pour le scattering au niveau L^2, on peut résumer quelques résultats en dimension $d \geqslant 2$ de la façon suivante.

Théoreme 1.31 (Scattering au niveau L^2)**.** *Soit $p \geqslant 1$ et $d \geqslant 2$. Alors on a*

(*i*) *Pour $p \in \left[1, 1 + \frac{4}{d}\right]$ l'équation (1.18) est globalement bien posée dans $L^2(\mathbb{R}^d)$ et si $p > 1 + \frac{4}{d}$ le problème de Cauchy est mal posé.*

(*ii*) *Pour $p \leqslant 1 + \frac{2}{d}$ et tout $u_0 \in L^2$, les solutions construites ne font pas scattering dans L^2.*

(*iii*) *Pour $d \geqslant 2$, $p \in \left]1 + \frac{2}{d}, 1 + \frac{4}{d-2}\right[$ et des données initiales H^1, le scattering a lieu dans L^2.*

(*iv*) *Enfin, pour $d \geqslant 2$, et $p = 1 + \frac{4}{d}$ et des données initiales dans L^2, le scattering a lieu dans L^2.*

La partie (i) est la théorie de Cauchy standard lorsque $p < 1 + \frac{4}{d}$, voir par exemple [Tao06], chapitre 3 et la globalité vient de la conservation de la norme L^2. Dans le cas critique on réfère à [Dod16a] et le cas $p > 1 + \frac{4}{d}$ dans [CCT03, AC09]. (ii) est prouvé dans [Bar84], (iii) dans [TY84] et (iv) dans [TVZ07]. Pour la dimension 2 on réfère à [Dod16a, Dod16b].

4.2. Travaux de nature probabiliste. On peut distinguer deux types de travaux qui traitent du scattering pour (1.18) à données aléatoires.

Les travaux du premier type étudient le scattering à un niveau H^1 en améliorant les résultat déterministes. Par exemple, dans une série de travaux parmi lesquels on peut citer les travaux de Killip, Murphy, Visan [KMV19] et Dodson, Lührmann et Mendelson [DLM19], le scattering pour presque toute donnée initiale dans H^s pour des $s < 1$ est étudié. La méthode consiste à chercher des solutions sous la forme $u = u_L^\omega + v$ où v est dans H^1 et appliquer les méthodes déterministes à v et obtenir des bornes pour des normes $L^q_{t,x}$.

Les travaux du second type exploitent l'existence de mesures invariantes et de mesures quasi invariantes. On a déjà évoqué le fait que les mesures invariantes au niveau $L^2(\mathbb{R}^d)$ pour (1.18) sont nécessairement triviales. Afin de contourner cette difficulté, Burq, Thomann et Tzvetkov utilisent une transformation appelée *transformée de lentille*, $\mathcal{L} : \mathcal{S}'(\mathbb{R}\times\mathbb{R}^d) \to \mathcal{S}'\left]\left(-\frac{\pi}{4}, \frac{\pi}{4}\right[\times \mathbb{R}^d\right)$ et d'inverse noté $\mathcal{L}^{-1}$ définis par les formules suivantes

$$v(t,x) := \mathcal{L}u(t,x) = \frac{1}{\cos(2t)^{\frac{d}{2}}} u\left(\frac{\tan(2t)}{2}, \frac{x}{\cos(2t)}\right) \exp\left(-\frac{i|x|^2\tan(2t)}{2}\right),$$

$$u(s,y) = \mathcal{L}^{-1}v(s,y) = (1-4s^2)^{-\frac{d}{4}} v\left(\frac{\arctan(2s)}{2}, \frac{y}{\sqrt{1-4s^2}}\right) \exp\left(\frac{i|y|^2 s}{\sqrt{1-4s^2}}\right).$$

D'un point de vue purement formel u résout

$$i\partial_s u + \Delta_y u = |u|^{p-1}u \text{ sur } \mathbb{R}_s \times \mathbb{R}^d_y \text{ avec } u(0) = u_0$$

si et seulement si $v = \mathcal{L}u$ résout

$$i\partial_t v - Hv = \cos(2t)^{\frac{p-5}{2}}|v|^{p-1}v \text{ sur } \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[\times \mathbb{R}^d_x \text{ et } v(0) = u_0\,. \tag{1.20}$$

Avec cette transformation on a donc échangé (1.18) en une version où $-\Delta$ est remplacé par l'oscillateur harmonique $H = -\Delta + x^2$.

Le premier avantage de cette transformation est le fait que H possède un spectre discret, ce qui rend possible la construction de mesures invariantes ou quasi-invariantes, suivant les valeurs de p et d, ceci résulte de l'application de la méthode de la section 2.1. Fixons désormais $d = 1$ dans le reste de cette sous-section.

Les fonctions propres de H sont données par les fonctions de Hermite h_n qui satisfont :

$$Hh_n = (2n+1)h_n\,.$$

On peut alors construire la mesure

$$\mathrm{d}\mu = \exp\left(-\|H^{1/2}u\|_{L^2}^2\right) \mathrm{d}u$$

comme mesure image de la série aléatoire

$$u_0^\omega := \sum_{n\geqslant 0} \frac{g_n^\omega}{\sqrt{2n+1}} h_n(x)\,.$$

Notons $\mathcal{W}^{s,p}$ l'espace de Sobolev associé à l'opérateur H, de norme définie par $\|u\|_{\mathcal{W}^{s,p}} := \|H^{\frac{s}{2}}u\|_{L^p}$ et $\mathcal{H}^s = \mathcal{W}^{s,2}$.

La mesure μ_0 est supportée au niveau $\mathcal{H}^{0-}$, ce qui se voit grâce à un calcul similaire à celui de la section 2.1. De façon similaire au cas du tore, les normes L^p de u_0^ω sont finies pour tout p fini. En réalité on a même un peu mieux car les fonctions h_n vérifient une estimation de décroissance du type

$$\|h_n\|_{L^p} \lesssim n^{-\frac{1}{3}}\,,$$

dès que $p \geqslant 4$. En combinant cette observation avec la remarque 1.16, on établit en particulier que presque sûrement $u_0^\omega \in \mathcal{W}^{\frac{1}{6}^-,p}$ pour tout $p \geqslant 4$. Ainsi dans les échelles L^p pour $p \geqslant 4$, cette randomisation régularise les données.

La transformée de lentille échange le problème de scattering lorsque $t \to \infty$ pour u solution de (1.18) en un problème de scattering pour v solution de (1.20) lorsque $t \to \frac{\pi}{4}$. D'après la section 4.1 on cherche à étudier la convergence de

$$-\,i\int_0^t e^{-it'H}\cos(2t')^{\frac{p-5}{2}}(|v(t')|^{p-1}v(t'))\,\mathrm{d}t' \tag{1.21}$$

dans un espace $\mathcal{H}^{-\sigma}$, car on se place au niveau de régularité de μ_0, c'est à dire $\mathcal{H}^{0^-}$. En particulier, par l'injection $L^{p+1} \hookrightarrow \mathcal{H}^{-\sigma}$ il suffit de montrer que

$$\int_t^{\frac{\pi}{4}} \cos(2t')^{\frac{p-5}{2}}\|v(t')\|_{L^{p+1}}^p\,\mathrm{d}t' < \infty\,.$$

L'intérêt d'une mesure invariante devient clair : l'invariance de μ_0 (ou plutôt d'une mesure construite à partir de μ_0) ou sa quasi-invariance permet la construction de solutions globales à (1.20) dont les normes de Sobolev, et donc en particulier les normes L^{p+1} pour $p+1$ assez petit, sont de croissance limitée.

Lorsque $p = 5$, ce programme est mené par Burq, Thomann et Tzvetkov, et lorsque $p \in]3,5[$ ce résultat a été obtenu récemment par Burq et Thomann. Ce dernier résultat utilise des mesures quasi-invariantes, dont l'exploitation pour (1.18) trouve sa source dans les travaux de Tzvetkov [Tzv15] puis Oh et Tzvetkov [OT17].

Théoreme 1.32 (Cas $p = 5$ dans [BTT13] et $p \neq 5$ dans [BT20])**.** *Soit $d = 1$ et $p \in]3,5]$. Il existe une mesure notée μ_0 supportée par $X^0 := \bigcap_{s>0} H^{-s}$ et un* *nombre réel $\sigma > 0$ telle que pour μ_0-presque tout $u_0 \in X^0$, il existe une unique solution globale u à (1.18) dans l'espace*

$$e^{is\Delta}u_0 + \mathcal{C}^0(\mathbb{R}, H^\sigma)\,.$$

De plus, il existe $u_\pm \in H^\sigma$ tels que l'on ait le scattering

$$\|u(t) - e^{it\Delta}(u_0 + u_\pm)\|_{H^\sigma} \underset{t\to\pm\infty}{\longrightarrow} 0\,.$$

On réfère à [BTT13, BT20] pour des énoncés plus complets.

4.3. Contribution de la thèse. Le but du travail qui suit et qui sera présenté au chapitre 4 est l'extension du résultat de Burq-Thomann [BT20] à des dimensions supérieures, dans le cas radial.

Théorème C (L. [Lat20a])**.** *Soit $d \in \{2, \dots, 10\}$ et $p \in \left]1 + \frac{2}{d}, 1 + \frac{4}{d}\right]$. On note $X^0_{\mathrm{rad}} = \bigcap_{\sigma>0} H^{-\sigma}_{\mathrm{rad}}(\mathbb{R}^d)$. Il existe une mesure μ_0 supportée sur X^0_{rad} et $\sigma = \sigma(p,d)$ tels que pour μ_0-presque toute donnée initiale $u_0 \in X^0_{\mathrm{rad}}$, il existe une unique solution globale u à (1.18) dans l'espace*

$$e^{it\Delta}u_0 + \mathcal{C}(\mathbb{R}, H^\sigma)\,.$$

De plus, il existe $u_\pm \in H^\sigma$ tel que

$$\|u(t) - e^{it\Delta}(u_0 + u_\pm)\|_{H^\sigma} \underset{t\to\pm\infty}{\longrightarrow} 0\,.$$

L'usage des données radiales est rendu nécessaire pour utiliser la quasi-invariance de μ au niveau de régularité H^{0-}.

Par rapport à la dimension 1, les estimées des fonctions de Hermite h_n diffèrent. En particulier, lorsque $d \geqslant 2$ ces estimations entraînent que $u_0^\omega \in \mathcal{W}^{1/2^-,\infty}$ presque sûrement, ce qui est un gain de régularité plus grand que dans le cas $d = 1$.

La différence principale apparaît pour $d \geqslant 3$ pour étudier les normes L^{p+1} dont le contrôle implique le scattering.

En dimension $d = 1$ ou 2, le contrôle des normes $\mathcal{H}^s$ par l'argument de mesure invariante et l'injection de Sobolev suffisent pour obtenir ce contrôle. Lorsque $d \geqslant 3$, un argument supplémentaire est nécessaire. On part d'une observation utilisé dans [BT20], qui assure que $\mu(\{\|v\|_{L^{p+1}} > \lambda\}) \lesssim e^{-\frac{\lambda^{p+1}}{p+1}}$, de sorte qu'il soit suffisant de contrôler la variation de la norme L^{p+1} sur un intervalle $[t_1, t_2]$. Une telle estimation s'obtient en exploitant l'estimée de dispersion du théorème 1.23 (i) ainsi qu'une écriture sous forme Duhamel pour u entre les temps t_1 et t_2. On renvoie au chapitre 4 pour les détails.

4.4. Perspectives. On peut envisager plusieurs prolongements du théorème C. Le premier est une optimisation de la preuve en incorporant des méthodes plus fines.

Question 1.3. *Pour quelles valeurs de d et quelles valeurs de $\sigma > 0$ peut-on montrer un résultat similaire au théorème C, c'est à dire que pour tout $p \in]1+\frac{2}{d}, 1+\frac{4}{d}]$, presque toute donnée initiale $u_0^\omega \in X^0_{\mathrm{rad}}$ donne lieu à une solution globale pour (1.18) dans l'espace*

$$e^{it\Delta}u_0^\omega + \mathcal{C}^0(\mathbb{R}, H^\sigma(\mathbb{R}^d))\,,$$

et qui fait scattering lorsque $t \to \pm\infty$?

Parmi les raisons qui font penser qu'un tel programme est envisageable :

(1) On développe une théorie de Cauchy dans les espaces H^σ à l'aide d'estimations de Strichartz, on pourrait aussi travailler dans les espaces $X^{s,b}$ pour optimiser la construction locale et tirer profit des estimations bilinéaires (voir Lemme 1.40).

(2) Enfin il faut garder à l'esprit que le schéma de la preuve de scattering est rudimentaire. Plutôt que de faire l'inégalité triangulaire pour majorer (1.21) on pourrait appliquer l'inégalité de Strichartz inhomogène du théorème 1.23 (*ii*).

De telles améliorations ne pourront pas dépasser les deux limitations essentielles dont souffre le théorème C : la régularité des données initiales est donnée par H^{0^-} et les données sont radiales, en raison de l'utilisation des mesures quasi-invariantes. Ainsi une question de long-terme est la suivant.

Question 1.4. *Peut-on développer une preuve alternative du théorème C qui n'utilise pas de mesure quasi-invariante ?*

5. L'équation d'Euler et la croissance des normes de Sobolev

Cette section concerne le chapitre 5 de cette thèse et s'insère dans le cadre de l'étude de la croissance des normes de Sobolev pour les équations d'Euler. Les travaux qui sont présentés font l'objet d'un article soumis [Lat20b].

On considère les équations d'Euler posées sur le tore, qui s'écrivent

$$\text{(E)} \qquad \left\{\begin{array}{rcl} \partial_t u + u\cdot\nabla u & = & -\nabla p \\ \nabla\cdot u & = & 0 \\ u(0) & = & u_0 \in H^s(\mathbb{T}^d)\,, \end{array}\right.$$

où $d = 2, 3$, $u(t) : \mathbb{T}^d \longrightarrow \mathbb{R}^d$ représente le champs de vitesse du fluide et $p(t) : \mathbb{T}^d \longrightarrow \mathbb{R}$ représente le champ de pression.

On peut écrire cette équation sous une forme avantageuse pour la discussion qui va suivre. En appliquant le projecteur de Leray : $\mathbf{P} : L^2 \to L^2_{\mathrm{div}}$, qui est la projection orthogonale sur l'espace des fonctions L^2 à divergence nulle, on obtient l'équation

$$\left\{ \begin{array}{rcl} \partial_t u + B(u,u) & = & 0 \\ u(0) & = & u_0 \in H^s_{\mathrm{div}}(\mathbb{T}^d)\,, \end{array} \right. \tag{1.22}$$

où $B(u,v) := \mathbf{P}\,(u \cdot \nabla v)$ et où p se déduit de u en résolvant

$$-\Delta p = \nabla \cdot (u \cdot \nabla u)\,.$$

Dans cette section on note X_{div} l'espace des fonctions de X à divergence nulle.

5.1. Présentation du problème de la croissance en dimension 2. Afin de mettre en évidence la question de la croissance des normes de Sobolev pour les solutions de (E), on rappelle que pour $s > \frac{d}{2} + 1$, (E) est uniformément localement bien posé dans H^s. Ceci s'obtient à l'aide des injections de Sobolev. On peut même en déduire une théorie globale avec une estimation de la croissance des normes de Sobolev lorsque la dimension vaut 2.

Théoreme 1.33 (Wolibner, Yudovich [Wol33, Yud63])**.** *Soit $s > 2$. Le problème de Cauchy pour (E) en dimension 2 est globalement bien posé dans $\mathcal{C}^0(\mathbb{R}_+, H^s(\mathbb{T}^2))$ et il existe $C > 0$ tel que :*

$$\|u(t)\|_{H^s} \leqslant Ce^{e^{Ct}} \text{ pour tout } t > 0\,. \tag{1.23}$$

Démonstration. On se contente de prouver une estimation *a priori* pour $\|u(t)\|_{H^s}$ lorsque $s > 2$. On verra également ce qui rend la dimension 2 si particulière dans la démonstration. On commence par calculer l'évolution de $\|u(t)\|^2_{H^s}$ via l'équation (1.22), ce qui nous donne

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{H^s} = -(B(u,u),u)_{H^s}\,.$$

On étudie la non-linéarité en ne gardant que les termes de plus grand ordre en termes de dérivées, les autres termes seront plus simples à contrôler, on ne les considérera pas dans la suite. On écrit donc :

$$\begin{aligned}(B(u,u),u)_{H^s} &\sim (\nabla^s(u\cdot\nabla u), \nabla^s u)_{L^2} \\ &\sim (u \cdot \nabla\nabla^s u, \nabla^s u)_{L^2} + (\nabla u \cdot \nabla^s u, \nabla^s u)_{L^2} \\ &\quad + \{\text{termes d'ordres inférieurs}\}\,.\end{aligned}$$

On utilise alors une propriété essentielle de B, à savoir que pour tous u, v à divergence nulle on a l'identité $(B(u,v),v)_{L^2} = 0$, ce qui permet d'en déduire $(u \cdot \nabla\nabla^s u, \nabla^s u)_{L^2} = 0$ et donc

$$(B(u,u),u)_{H^s} \sim (\nabla u \cdot \nabla^s u, \nabla^s u)_{L^2}\,,$$

de sorte que par l'inégalité d'Hölder on obtienne

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{H^s}^2 \lesssim \|\nabla u\|_{L^\infty}\|u(t)\|_{H^s}^2\,. \tag{1.24}$$

Cette estimation *a priori* permet déjà d'en déduire une théorie locale pour $s > \frac{d}{2}+1$, valable en toute dimension. En effet, par injection de Sobolev on a $\|\nabla u\|_{L^\infty} \lesssim \|\nabla u\|_{H^{s-1}} \lesssim \|u\|_{H^s}$ car $s-1 > \frac{d}{2}$.

Pour obtenir une théorie globale on peut contrôler la croissance de $\|u(t)\|_{H^s}$. Ainsi dans l'inégalité (1.24) on voudrait contrôler $\|\nabla u(t)\| \in L^\infty$, dont on ne sait *a priori* rien. En revanche, et comme on va le voir, on possède des informations sur les normes L^p de $u(t)$ en dimension deux, que l'on exploitera grâce à l'inégalité obtenue par interpolation :

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{H^s}^2 \lesssim \|\nabla u(t)\|_{L^p}\|u(t)\|_{H^s}^{2+\frac{2}{p}}\,,$$

qui est valable pour $p \geqslant 2$ et où l'on souhaite prendre p très grand.

On se place maintenant dans le cas de la dimension $d=2$. Dans ce cas, la vorticité, définie par $\Omega(t) = \nabla \wedge u(t)$ satisfait une équation de transport

$$\partial_t\Omega + u\cdot\nabla\Omega = 0\,, \tag{1.25}$$

ainsi les normes $\|\Omega(t)\|_{L^p}$ sont conservées par le flot. Il s'agit maintenant de transférer cette information à ∇u, qui est de même ordre que Ω. Grâce à la théorie de Calderón-Zygmund, on obtient

$$\|\nabla u(t)\|_{L^p} \lesssim p\|\Omega(t)\|_{L^p} \lesssim p\|\Omega_0\|_{L^p}\,.$$

En notant $y(t) \coloneqq \|u(t)\|_{H^s}$ on a donc

$$y'(t) \leqslant Cpy(t)^{1+\frac{2}{p}}\,,$$

que l'on peut optimiser en p pour obtenir $y'(t) \leqslant y(t)\log y(t)$, équation qui donne $y(t) \lesssim e^{e^{Ct}}$ comme souhaité. □

La borne double exponentielle (1.23) est-elle optimale ? Peut-on exhiber une donnée initiale qui sature cette inégalité ?

Cette question a été étudiée dans d'autres domaines que le tore. En particulier dans le cas du disque $D = \{x \in \mathbb{R}^2, \in |x| \leqslant 1\}$, Kiselev et Šverák [KŠ14] construisent des solutions qui saturent la croissance double exponentielle. Dans le cas du tore on ne sait produire qu'une croissance exponentielle [Zla15]. Dans les deux cas, la démonstration repose sur l'exploitation d'un scénario hyperbolique introduit par Bahouri et Chemin [BC94].

Théoreme 1.34 (Kiselev-Šverák, Zlatoš [KŠ14, Zla15])**.** *Soit U un domaine bidimensionel qui désigne le disque unité D ou bien le tore $\mathbb{T}^2$. Il existe une donnée initiale u_0 de classe $\mathcal{C}^\infty$ dans le cas $U = D$ et seulement $\mathcal{C}^{1,\alpha}$ (pour* *un certain* $\alpha \in (0,1)$*) si* $U = \mathbb{T}^2$*, tel que l'unique solution globale* Ω *à* (1.25) *vérifie :*

(i) Dans le cas $U = D$*, il existe* $C > 0$ *tel que*

$$\|\nabla\Omega(t)\|_{L^\infty} \geqslant C \exp\left(Ce^{Ct}\right) \text{ pour tout } t \geqslant 0\,.$$

(ii) Dans le cas $U = \mathbb{T}^2$*, il existe* $t_0 > 0$ *tel que pour* $t \geqslant t_0$ *on a :*

$$\sup_{t' \leqslant t} \|\nabla\Omega(t')\|_{L^\infty} \geqslant e^t\,.$$

Pour compléter ce résultat dans le cas du tore il faudrait construire des données initiales qui donnent lieu à une croissance double exponentielle, mais il est également possible que de telles solutions n'existent pas. La construction de Kiselev et Šverák [KŠ14] exploite de façon crucial la frontière de D. Dans le cas $\mathbb{T}^2$, ce phénomène n'a pas lieu à cause de l'absence de frontière. La question suivante se pose dès lors : les normes de Sobolev des solutions de (E) dans $\mathbb{T}^2$ exhibent-elles *géneriquement* une croissance au plus exponentielle, ou même polynomiale ?

Ici le terme *générique* pourrait être compris comme « presque sûrement par rapport à une mesure définie sur $H^s(\mathbb{T}^2)$ ».

5.2. La méthode de Kuksin pour les mesures invariantes. Avant d'expliquer la méthode introduite par Kuksin, rappelons que l'on dispose déjà d'un moyen formel pour construire des mesures invariantes, il s'agit des mesures de Gibbs introduites dans la section 2.1. Pour construire une telle mesure, on développe Ω_0 en série de Fourier, $\Omega_0(x) = \sum_{n\in\mathbb{Z}^2} \hat{\Omega}_n e^{in\cdot x}$, la loi de conservation de $t \mapsto \|\Omega(t)\|_{L^2}$ s'écrit explicitement sous forme

$$E = \|\Omega(t)\|_{L^2}^2 = \|\Omega_0\|_{L^2}^2 = \sum_{n\in\mathbb{Z}^2} |\hat{\Omega}_n|^2\,.$$

On devine ainsi l'expression d'une mesure formellement invariante donnée par

$$\mathrm{d}\mu(u) = \bigotimes_{n\in\mathbb{Z}^2} e^{-|\hat{\Omega}_n|^2}\,\mathrm{d}\hat{\Omega}_n\,,$$

qui est la mesure image de

$$\Omega_0^\omega(x) = \sum_{n\in\mathbb{Z}^2} g_n(\omega) e^{in\cdot x}\,,$$

où les $(g_n)_{n\in\mathbb{Z}^2}$ sont des gaussiennes indépendantes, centrées réduites. Une telle mesure est supportée au niveau $H^{-1^-}(\mathbb{T}^2)$. Sachant que $\Omega_0 = \nabla \wedge u_0$, cela veut dire que la mesure en termes de variable u est supportée à un niveau $H^{0^-}(\mathbb{T}^2)$, ce qui est très peu régulier. On se référera aux travaux de Flandoli [Fla18]. La régularité $H^{0^-}(\mathbb{T}^2)$ est trop faible pour l'étude de la croissance des normes Sobolev à un niveau de régularité compatible avec une théorie locale ou globale.

Une méthode introduite par Kuksin permet la construction de telles mesures. On en donne les idées principales et on réfère à [KS15, Kuk04] pour une présentation détaillée.

Plutôt que de construire directement une mesure invariante pour (E), on introduit la régularisation parabolique et stochastique suivante, il s'agit de l'équation de Navier-Stokes stochastique, qui s'écrit pour $\nu > 0$,

$$\left\{\begin{array}{rcl} \partial_t u_\nu + B(u_\nu, u_\nu) - \nu\Delta u_\nu & = & \sqrt{\nu}\eta \\ u_\nu(0) & = & u_0 \in H^s_{\mathrm{div}}(\mathbb{T}^d)\,, \end{array}\right. \tag{1.26}$$

avec $\eta = \partial_t \xi$ et

$$\xi(t) = \sum_{n\in\mathbb{Z}^2} \phi_n \beta_n^\omega(t) e^{in\cdot x}\,, \tag{1.27}$$

un bruit blanc en temps et régulier en espace. Ici les $(\beta_n)_{n\in\mathbb{Z}^2}$ sont des mouvements Browniens identiquement distribués et indépendants et les $(\phi_n)_{n\in\mathbb{Z}^2}$ sont des nombres qui caractérisent ce bruit, en particulier on note

$$\mathcal{B}_k := \sum_{n\in\mathbb{Z}^2} |n|^{2k}|\phi_n|^2\,. \tag{1.28}$$

La méthode de Kuksin consiste à mener le programme suivant.

(1) Construire des mesures invariantes μ_ν aux approximations (1.26) en utilisant un argument de Krylov-Bogolioubov, ce qui se révèle être une étape qui ne pose aucune difficulté majeure.

(2) Prouver des estimations uniformes pour les μ_ν, il s'agit de l'étape clé.

(3) Passer à la limite $\nu \to 0$, ce qui se trouve être une étape qui ne pose aucune difficulté.

Ainsi on se contente d'expliquer comment obtenir les estimations de (2). Le point de départ est l'équation intégrale satisfaite par u_ν, qui s'écrit

$$u_\nu(t) = u_0 + \int_0^t (\nu\Delta u_\nu - B(u_\nu, u_\nu))\,\mathrm{d}t' + \sqrt{\nu}\sum_n \phi_n\beta_n(t)e_n\,.$$

Pour une fonctionnelle F que l'on choisira plus tard, on applique alors la formule d'Itô au processus stochastique $t \mapsto F(u_\nu(t))$. On obtient alors

$$\begin{aligned}\mathbb{E}[F(u_\nu(t)] - \mathbb{E}[F(u_0)] = \nu\int_0^t \mathbb{E}[F'(u_\nu;\Delta u_\nu)\,\mathrm{d}t' \\ - \int_0^t \mathbb{E}[F'(u_\nu; B(u_\nu,u_\nu)]\mathrm{d}t' \\ + \frac{\nu}{2}\int_0^t \sum_n |\phi_n|^2\mathbb{E}[F''(u;e_n,e_n)]\,\mathrm{d}t'\,.\end{aligned}$$

On voit ainsi que le terme $\sqrt{\nu}\eta$ apparaît dans la formule d'Itô au carré, d'où le choix du $\sqrt{\nu}$ dans (1.26)

Avec $F(u) := \|u\|_{H^1}^2$ on obtient

$$\mathbb{E}\left[\|u_\nu(t)\|_{H^1}^2\right] - \mathbb{E}\left[\|u_0\|_{H^1}^2\right] = \int_0^t (A(t') + B(t'))\,\mathrm{d}t'\,,$$

où $A(t')$ est la contribution déterministe donnée par

$$A(t') := -\mathbb{E}\left[(B(u_\nu, u_\nu), u_\nu)_{H^1} - \nu(\Delta u_\nu, u_\nu)_{H^1}\right]\,,$$

et $B(t') = B$ est la correction d'Itô $B := \frac{\nu}{2}\sum_n |n|^2|\phi_n|^2 = \frac{\nu}{2}\mathcal{B}_1$.

L'autre ingrédient essentiel de la démonstration est un usage de la dimension 2 : la forme bilinéaire B satisfait l'identité $(B(u_\nu, u_\nu), u_\nu)_{H^1} = 0$, ce qui se démontre efficacement en utilisant (1.25). Cette égalité est fausse en dimension 3.

En utilisant d'autre part que $\nu(-\Delta u, u)_{H^1} = \nu\|u\|_{H^2}^2$ et enfin l'invariance de la mesure μ_ν pour laquelle $u_\nu(t)$ et un processus invariant (construite à l'étape (1)), on obtient finalement

$$\int_0^t \left(\nu\mathbb{E}[\|u_\nu(t')\|_{H^2}^2] - \nu\frac{\mathcal{B}_1}{2}\right)\mathrm{d}t' = \mathbb{E}\int_0^t (B(u_\nu, u_\nu), u_\nu)_{H^1}\,\mathrm{d}t' = 0\,,$$

ce qui permet de diviser par ν pour obtenir l'estimation uniforme suivante :

$$\mathbb{E}\left[\|u_\nu(t)\|_{H^2}^2\right] = \frac{\mathcal{B}_1}{2}\,.$$

Le résultat démontré par Kuksin s'énonce alors sous la forme suivante.

Théoreme 1.35 (Kuksin, [Kuk04])**.** *Il existe une mesure μ supportée dans $H^2(\mathbb{T}^2)$ telle que :*

(*i*) *La mesure μ est non triviale et est de régularité H^2 :*

$$\mathbb{E}_\mu[\|u\|_{H^2}^2] \leqslant \frac{\mathcal{B}_1}{2} \text{ et } \mathbb{E}_\mu[\|u\|_{L^2}^2] \geqslant \frac{\mathcal{B}_0^2}{2\mathcal{B}_1}\,.$$

(*ii*) *Pour μ-presque tout $u_0 \in H^2$, il existe une unique solution globale à* (E) *associée à u_0 dans l'espace $L^2_{\mathrm{loc}}(\mathbb{R}_+, H^2_{\mathrm{div}}) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, H^1_{\mathrm{div}}) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+, L^p)$, pour tout $p \in [1, 2[$.*

5.3. Contribution de la thèse. L'objectif est d'étudier le problème de la croissance des normes de Sobolev pour (E) en dimensions 2 et 3. Le méta-théorème de la section 2.1 semble adapté à cette étude. De plus la mesure μ construite par le théorème 1.35 est telle que :

(1) μ est invariante pour le flot de (E).
(2) μ satisfait les bornes $\mu(\{u_0 \in H^2, \|u_0\|_{H^2(\mathbb{T}^2)} > \lambda\}) \lesssim \lambda^{-2}$, ce qui n'est pas une borne de type gaussienne mais serait suffisante pour appliquer le méta-théorème.

Malheureusement, l'hypothèse (3) du méta-théorème n'est pas vérifiée, car aucune *bonne* théorie locale pour (E) n'est connue dans $H^2(\mathbb{T}^2)$. En revanche une telle théorie est accessible dans $H^{2+\delta}(\mathbb{T}^2)$ dès que $\delta > 0$. Ainsi, afin d'appliquer l'argument du méta-théorème pour les mesures de Kuksin, il faut adapter la construction de telles mesures dans $H^{2+\delta}(\mathbb{T}^2)$, puis assouplir la condition (2) du méta-théorème. On démontre le résultat suivant dans le chapitre 5.

Théorème D (L. [Lat20b])**.** *Soit $d = 2$ (resp. $d = 3$) et $s > 2$ (resp. $s > 7/2$). Il existe une mesure μ telle que $\mu(H^s(\mathbb{T}^d)) = 1$ et qui satisfait :*

- (i) *Pour μ-presque toute donnée initiale il existe une unique solution globale u à (E) qui vérifie pour tout $t > 0$,*
$$\|u(t)\|_{H^s} \lesssim t^\alpha ,$$
où α dépend de s et d.
- (ii) *Si $d = 2$ alors $\mathbb{E}_\mu[\|u\|^2_{H^1}] > 0$ et μ n'a pas d'atome. De plus μ charge les grandes données initiales : pour tout $R > 0$ on a $\mu(u \in H^s,\ \|u\|_{H^s} > R) > 0$.*
- (iii) *Si $d = 3$ alors on a l'alternative suivante. Ou bien $\mu = \delta_0$, ou bien μ n'a pas d'atome.*

Une telle combinaison entre l'argument de Kuksin (mesures invariantes à haute régularité) et celui de Bourgain (globalisation avec faible croissance à partir d'une mesure invariante) dans le cadre des équations de la mécanique des fluides semble nouveau. Toutefois, en milieu dispersif une telle combinaison est déjà exploitée par Sy [Sy19a, Sy18, Sy19b]. On trouve également un exemple de construction de mesures invariantes à un haut niveau de régularité par la méthode de Kuksin dans les travaux récents de Flödes et Sy [FS20] pour une équation proche de l'équation d'Euler.

Afin de modifier la méthode de Kuksin on introduit une régularisation parabolique différente de celle utilisée dans la section 5.2,

$$\left\{ \begin{array}{rcl} \partial_t u_\nu + B(u_\nu, u_\nu) + \nu L u_\nu & = & \sqrt{\nu}\eta \\ u_\nu(0) & = & u_0 \in H^s_{\mathrm{div}}(\mathbb{T}^d)\,, \end{array} \right. \tag{1.29}$$

où $L = (-\Delta)^{1+\delta}$, et $\eta = \partial_t \xi$. Ici ξ et $\mathcal{B}_k$ sont définis comme en (1.27) et (1.28).

Il faut alors porter une attention particulière à la réadaptation de la méthode de Kuksin pour prouver des estimatinos uniformes. En particulier on utilise l'annulation fondamentale donnée par $(B(u,u),u)_{L^2} = 0$ valable en dimensions 2 et 3. Dans le cadre de la dimension 2 on utilise aussi la deuxième annulation fondamentale $(B(u,u),u)_{H^1} = 0$. Les détails sont apportés dans le chapitre 5 de cette thèse.

Dans le cas de la dimension 3 on doit aussi tenir compte du fait que aucune théorie globale n'est connue au niveau $H^{2+\delta}(\mathbb{T}^3)$. Heureusement, l'argument du méta-théorème de la section 2.1 est robuste et est capable de fournir l'existence de solutions globales à partir de l'information formelle de l'invariance de la mesure μ. Cela demande en particulier d'approcher μ par des mesures μ_N qui sont invariantes pour des approximations de (E) tronquées en fréquences et dont les solutions sont construites globalement. Enfin, un passage à la limite en N permet de conclure.

5.4. Comparaison des méthodes de construction de mesures invariantes. Les résultats précédents ne sont pas complets, dans le sens où peu d'informations sur les mesures μ sont données. Pour compléter le théorème principal, on peut prouver le résultat suivant.

Théorème E (L. [Lat20b])**.** *Les mesures μ satisfont :*

(i) *Lorsque $d = 2$, pour tout ensemble borélien $\Gamma \subset \mathbb{R}_+$ on a*

$$\mu(u \in H^s\,, \|u\|_{L^2} \in \Gamma) \lesssim |\Gamma|\,, \tag{1.30}$$

ce qui implique que μ n'a pas d'atome.

(ii) *Lorsque $d = 3$, (1.30) est vérifiée uniquement pour des ensembles tels que $\mathrm{dist}(\Gamma, 0) > \varepsilon$, la constante implicite dépendant de ε. Ainsi μ ne peut avoir d'autre atome que 0.*

Dans le cas de l'équation d'Euler en dimension 2 et des mesures de Kuksin de niveau H^2, les résultats sont meilleurs. En particulier Kuksin démontre le résultat suivant.

Théoreme 1.36 (Kuksin, [Kuk04, KS15])**.** *Considérons une mesure μ construite par le théorème 1.35, invariante pour* (E) *concentrée sur $H^2(\mathbb{T}^2)$. On a la description plus précise :*

(i) *Il existe une fonction continue p strictement croissante et qui satisfait $p(0) = 0$, telle que pour tout ensemble borélien $\Gamma \subset \mathbb{R}$ on a*

$$\mu(\|u\|_{L^2} \in \Gamma) + \mu(\|\nabla u\|_{L^2} \in \Gamma) \leqslant p(|\Gamma|)\,.$$

En particulier, μ est sans atome.

(ii) *Pour tout ensemble $K \subset H^2(\mathbb{T}^2)$ de dimension de Hausdorff finie, on a $\mu(K) = 0$.*

Ce résultat montre que la mesure construite est « infinie-dimensionelle ». Malgré ces propriétés supplémentaires, les deux résultats que nous venons de citer souffrent du même défaut : ils sont incapables d'exclure qu'une telle mesure invariante μ se concentre exclusivement sur l'ensemble des solutions indépendantes du temps pour (E), cas dans lequel une étude de la croissance des normes de Sobolev devient sans intérêt. Il s'agit d'un axe de recherche important concernant les mesures de Kuksin.

Profitons de cette discussion pour expliquer que dans le cas des mesures de Gibbs de la section 2.1 et plus généralement des mesures sous-gaussiennes construites par (1.12), on peut donner bien plus d'informations sur leur support, à l'inverse des mesures de Kuksin.

Dans le reste de cette sous-section on fixe une mesure μ concentrée sur $L^2(\mathbb{T}^d)$ définie comme mesure image par la randomisation

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}^d} g_n^\omega u_n e^{in\cdot x}\,,$$

avec

$$u_0 \coloneqq \sum_{n\in\mathbb{Z}^d} u_n e^{in\cdot x} \in L^2 \setminus \bigcup_{\varepsilon>0} H^\varepsilon\,,$$

et les $(g_n)_{n\in\mathbb{Z}^d}$ sont des variables aléatoires gaussiennes centrées réduites, indépendantes. On démontre alors le résultat suivant.

Lemme 1.37 (Appendix B de [BT08a])**.** *Avec les notations ci-dessus et en supposant de plus que pour tout $n \in \mathbb{Z}^d$ on a $u_n \neq 0$, on en déduit que* $\operatorname{supp}\mu = L^2$*.*

Démonstration. On fixe une fonction $v = \sum_{n\in\mathbb{Z}^d} v_n e^{in\cdot x}$ et $\varepsilon > 0$. On souhaite construire un ensemble Ω_0 tel que $\mathbb{P}(\Omega_0) > 0$ et tel que $\|u_0^\omega - v\|_{L^2} \leqslant \varepsilon$ pour tout $\omega \in \Omega_0$. Dans la suite on note $\mathbf{P}_{\leqslant N}$ la projections sur les fréquences $|n| \leqslant N$. Pour N à ajuster dans la suite on écrit que

$$\|v - u_0^\omega\|_{L^2} \leqslant \|(\mathrm{id} - \mathbf{P}_{\leqslant N})v\|_{L^2} + \|(\mathrm{id} - \mathbf{P}_{\leqslant N})u_0^\omega\|_{L^2} + \|\mathbf{P}_{\leqslant N}(u_0 - v)\|_{L^2}\,.$$

Le terme $\|(\mathrm{id} - \mathbf{P}_{\leqslant N})v\|_{L^2}$ tend vers 0 lorsque N tend vers l'infini car v est une fonction L^2. On fixe donc $N > 0$ tel que ce terme soit majoré par $\frac{\varepsilon}{3}$.

Pour le terme $\|(\mathrm{id} - \mathbf{P}_{\leqslant N})u_0^\omega\|_{L^2}$ on utilise l'estimation issue du théorème 1.14 qui donne

$$\mathbb{P}\left(\|(\mathrm{id} - \mathbf{P}_{\leqslant N})u_0^\omega\|_{L^2} > \lambda\right) \leqslant \exp\left(-c\lambda^2\left(\sum_{|n|>N}|u_n|^2\right)^{-\frac{1}{2}}\right)\,.$$

Appliquée à $\lambda = \varepsilon$ on en déduit un ensemble Ω_1 de probabilité non nulle pour lequel on a $\|(\mathrm{id} - \mathbf{P}_{\leqslant N})u_0^\omega\|_{L^2} \leqslant \frac{\varepsilon}{3}$ pour tout $\omega \in \Omega_1$.

Enfin, on définit l'ensemble

$$\Omega_2 \coloneqq \bigcap_{|n|\leqslant N}\left\{|u_n g_n^\omega - v_n| \leqslant \frac{\varepsilon}{3}N^{-\frac{1}{2}}\right\}\,,$$

de probabilité strictement positive, grâce au fait que $u_n \neq 0$ pour tout $n \in \mathbb{Z}^d$. Ainsi pour $\omega \in \Omega_0 \coloneqq \Omega_1 \cap \Omega_2$ on a $\|v - u_0^\omega\|_{L^2} \leqslant \varepsilon$, ce qui achève la démonstration car par indépendance de Ω_1 et Ω_2 on a $\mathbb{P}(\Omega) = \mathbb{P}(\Omega_1)\mathbb{P}(\Omega_2) > 0$. □

5.5. Perspectives. Comme le montre la discussion de la sous-section précédente, il est possible que les mesures produites par le théorème 1.35 et le théorème D soient supportées exclusivement par des solutions indépendantes du temps. Une question, proablement difficile mais naturelle est la suivante.

Question 1.5. *Peut-on décrire le support des mesures de Kuksin produites par les théorème 1.35 ou le théorème D ? Peut-on, dans certains cas, exclure le scénario de concentration sur les solutions indépendantes du temps ?*

On peut déjà pointer deux difficultés majeures qui empêchent même de développer une quelconque intuition quand à la réponse à cette question.

(1) La première raison est le manque de description des mesures, qui sont construites par compacité. Les propriétés de ces mesures doivent être étudiées uniformément à partir des mesures approchées μ_ν, comme c'est le cas dans le théorème 1.36 par exemple.
(2) La deuxième raison est le manque de description des solutions indépendantes du temps pour (E), qui forment un très vaste ensemble. [CŠ12]

Aborder frontalement ce problème pour l'équation d'Euler incompressible semble difficile, peut-être que pour d'autres équations, comme les équations dispersives qui ont moins de solutions stationnaires, le problème serait plus simple à étudier. On peut citer les travaux récents [BCZGH16, GHŠV15] qui étudient des équations modèles. Enfin, signalons que dans un contexte simplifié, de dimension finie, le support des mesures de Kuksin a été étudié [MP14].

6. Problème de Cauchy probabiliste pour NLS pour d'autres géométries

Cette section concerne le chapitre 5 de cette thèse et s'insère dans le cadre de l'étude du problème de Cauchy local à données aléatoires. Les travaux qui sont présentés sont le fruit d'une collaboration avec Louise Gassot et font l'objet d'un article pré-publié [GL21].

On s'intéresse au problème de Cauchy local pour l'équation de Schrödinger posée sur le groupe de Heisenberg. Cette équation peut être écrite sous la forme

$$\text{(NLS-}\mathbb{H}^1\text{)}\qquad \begin{cases} i\partial_t u + \Delta_{\mathbb{H}^1} u &= |u|^2 u\,, \\ u(0) &= u_0 \in H^s(\mathbb{H}^1)\,, \end{cases}$$

où $u(t) : \mathbb{R}^3 \to \mathbb{R}^3$. En notant $(x, y, s) \in \mathbb{R}^3$ on a

$$\Delta_{\mathbb{H}^1} = \frac{1}{4}\left(\partial_x^2 + \partial_y^2\right) + (x^2 + y^2)\partial_s^2\,,$$

les espaces $H^k(\mathbb{H}^1)$ sont associés à ce sous-laplacien.

Un modèle simplifié, en deux dimensions d'espace, mais qui retient une grande partie des difficultés liées au sous-laplacien de Heisenberg est l'opérateur de Grushin. Ainsi une version simplifiée de (NLS-$\mathbb{H}^1$) à laquelle on s'intéresse est la suivante :

$$\text{(NLS-G)} \qquad \begin{cases} i\partial_t u + \Delta_G u &= |u|^2 u\,, \\ u(0) &= u_0 \in H^s_G\,, \end{cases}$$

avec $\Delta_G = \partial_x^2 + x^2\partial_y^2$ et $(x,y) \in \mathbb{R}^2$.

6.1. Problème de Cauchy pour (NLS-$\mathbb{H}^1$) et (NLS-G). L'intérêt d'étudier de telles équations provient du manque de théorie déterministe satisfaisante en ce qui concerne le problème de Cauchy local. Plus précisément et avant d'énoncer les résultats existants, remarquons que (NLS-$\mathbb{H}^1$) (et de même pour (NLS-G)) conservent au moins à un niveau informel la quantité

$$E(t) = \frac{1}{2}\|(-\Delta_{\mathbb{H}^1})^{\frac{1}{2}} u\|^2_{L^2(\mathbb{H}^1)} + \frac{1}{4}\|u\|^4_{L^4(\mathbb{H}^1)}\,,$$

l'énergie pour (NLS-$\mathbb{H}^1$) dont le niveau de régularité est $H^1(\mathbb{H}^1)$. Ainsi si l'on parvenait à écrire une théorie de Cauchy locale pour (NLS-$\mathbb{H}^1$) dans $H^1(\mathbb{H}^1)$, on pourrait construire des solutions globales. Or la meilleure théorie connue à ce jour est donnée par le résultat suivant.

Proposition 1.38 (Problème de Cauchy pour (NLS-$\mathbb{H}^1$) et (NLS-G))**.** *On définit $k_C = 2$ (resp. $k_C = \frac{3}{2}$). Soit $k > k_C$, alors le problème de Cauchy pour* (NLS-$\mathbb{H}^1$) *(resp.* (NLS-G)*) est uniformément localement bien posé dans $\mathcal{C}^0([0,T^*[, H^k)$ avec $T^* \gtrsim \|u_0\|^{-2}_{H^k)}$.*

Une démonstration dans le cas Heisenberg peut être trouvée dans [BG01], et dans le cas Grushin une démonstration est proposée dans l'annexe du chapitre 6. La démonstration utilise l'injection de Sobolev $H^k \hookrightarrow L^\infty$, valable dès que $k > k_C$.

Ainsi, même dans le cas Grushin, la théorie locale ne s'applique pas dans H^1_G. Pour $k < k_C$, l'obtention d'une théorie locale est un problème ouvert. Pour aller plus loin on pourrait essayer de démontrer des estimations de dispersion comme dans le cas euclidien. Or il est connu que l'équation de Schrödinger sur le groupe de Heisenberg n'est pas dispersive,ce qui a été observé dans [BGX00] et implique que le flot de (NLS-$\mathbb{H}^1$) (resp. (NLS-G)) n'est pas régulier dans les espaces de Sobolev H^k dès que $k < k_C$. Une telle démonstration peut être trouvée dans l'introduction de [GG10] et la remarque 2.12 de [BGT04]. À cause de cette absence de dispersion, le paradigme dispersif ne s'applique plus. Des estimations de Strichartz qui utilisent des méthodes de restriction ont été obtenues récemment [BBG19] mais ne permettent pas à ce jour une étude du problème de Cauchy pour (NLS-$\mathbb{H}^1$).

6.2. Contribution de la thèse. Le résultat principal que nous démontrons est une construction probabiliste de solutions uniques pour (NLS-G). Un résultat similaire pour (NLS-$\mathbb{H}^1$) fera sûrement l'objet d'un travail futur. Notre résultat s'énonce pour des données initiales dans un sous-espace de H^k_G noté $\mathcal{X}^k_\rho$, qui correspond à un sous-espace de régularité partielle en y plus élevé, il s'agit d'un espace anisotrope. Afin d'introduire les résultats, on dispose d'une décomposition fréquentielle des fonctions de H^k_G, quelque peu analogue à la décomposition fréquentielle de Fourier sur le tore. Toute fonction $u_0 \in H^k_G$ s'écrit sous une forme

$$u_0 = \sum_{(a,m)\in\mathbb{N}^*\times\mathbb{N}} u_{a,m}\,,$$

et l'on peut ainsi introduire une mesure μ_{u_0} sur H^k_G image par la randomisation

$$u_0^\omega = \sum_{(a,m)\in\mathbb{N}^*\times\mathbb{N}} g^\omega_{a,m} u_{a,m}\,, \tag{1.31}$$

où les $\{g^\omega_{a,m}\}_{(a,m)\in\mathbb{N}^*\times\mathbb{N}}$ sont des variables aléatoires gaussiennes complexes indépendantes et identiquement distribuées. On consultera l'introduction et la section 2 du chapitre 6 pour plus de précision.

Théorème F (Gassot-L. [GL21])**.** *Soit* $k \in]1, \frac{3}{2}]$ *et* $u_0 \in \mathcal{X}^k_1 \subset H^k_G$.

(i) Pour tout $\ell \in]\frac{3}{2}, k + \frac{1}{2}[$ *et presque tout* $\omega \in \Omega$*, il existe* $T > 0$ *tel qu'il existe une unique solution locale à* (NLS-G) *ayant pour donnée initiale* u_0^ω*, dans l'espace*

$$e^{it\Delta_G}u_0^\omega + \mathcal{C}^0([0,T[, H^\ell_G) \subset \mathcal{C}^0([0,T[, H^k_G)\,.$$

Plus précisément, il existe $c > 0$ *tel que pour tout* $R > 0$*, en dehors d'un ensemble de mesure au plus* e^{-cR^2}*, on peut choisir* $T \geqslant \frac{1}{R^2}$.

(ii) (Pas de régularisation aléatoire) Si $u_0 \in H^k_G \setminus (\bigcup_{\varepsilon>0} H^{k+\varepsilon}_G)$ *alors*

$$\operatorname{supp}(\mu_{u_0}) \subset H^k_G \setminus (\bigcup_{\varepsilon>0} H^{k+\varepsilon}_G)\,.$$

(iii) (Densité du support des mesures) Soit $\varepsilon > 0$*, alors il existe* $v_0 \in \mathcal{X}^k_2 \setminus (\bigcup_{\varepsilon'>0} H^{k+\varepsilon'}_G)$ *tel que*

$$\operatorname{supp}(\mu_{v_0}) \cap B_{H^k_G}(u_0, \varepsilon) \neq \varnothing\,.$$

Les assertions (*ii*) et (*iii*) montrent que les mesures produites à partir des fonctions H^k_G sont supportées par H^k_G mais pas par des fonctions plus régulières, ce qui est tout à fait similaire à l'appendice de [BT08a], la démonstration étant similaire. D'autre part l'union des support de telles mesures épuise H^k_G. C'est un résultat de densité que l'on peut qualifier d'analogue aux résultats de l'appendice de [BT14].

Comme nous l'avons déjà signalé, une difficulté inhérente liée à l'équation de Schrödinger est son manque de régularisation de l'opérateur de Duhamel. Dans le cas d'un domaine comme $\mathbb{R}^d$ les estimations bilinéaires permettent de surmonter un tel manque de régularisation afin de développer une théorie de

Cauchy locale probabiliste similaire à celle de la section 2.2. Dans le cas du sous-laplacien de Grushin, ce sont de telles estimations bilinéaires et trilinéaires dont il faudrait disposer. Nous en obtenons des version valables lorsque au moins l'un des termes est aléatoire.

Théorème G (Estimations multi-linéaires probabilistes)**.** *Soit* $k \in]1, \frac{3}{2}]$ *et* $u_0 \in \mathcal{X}_1^k \subset H_G^k$*. Soit* u_0^ω *comme dans*(1.31)*, et notons* $z^\omega = e^{it\Delta_G} u_0^\omega \in \mathcal{C}^0(\mathbb{R}, H_G^k)$ *la solution de l'équation de Schrödinger linéaire associée à* u_0^ω*. Alors il existe* $c > 0$ *tel que que les énoncés suivants soient vérifiés. Soit* $q \in [2, \infty)$*.*

(*i*) *Pour tout* $R \geqslant 1$ *and* $T > 0$*, en dehors d'un ensemble de probabilité au plus* e^{-cR^2}*, on a*

$$\|(z^\omega)^2\|_{L_T^q H_G^{k+\frac{1}{2}}} + \||z^\omega|^2\|_{L_T^q H_G^{k+\frac{1}{2}}} \leqslant R^2 T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_1^k}^2 \,, \tag{1.32}$$

$$\||z^\omega|^2 z^\omega\|_{L_T^q H_G^{k+\frac{1}{2}}} \leqslant R^3 T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_1^k}^3 \,. \tag{1.33}$$

(*ii*) *On requiert de plus que* $u_0 \in \mathcal{X}_{1+\varepsilon_0}^k$ *pour un certain* $\varepsilon_0 > 0$*. Soit* $\ell < k + \frac{1}{2}$*. Alors pour tout* $R \geqslant 1$ *et tout* $T > 0$*, il existe un ensemble* $E_{R,T}$ *de probabilité au moins* $1 - e^{-cR^2}$ *tel que l'énoncé suivant soit vérifié. Soit* $\omega \in E_{R,T}$ *et* $v, w \in L_T^\infty H_G^\ell$*, alors*

$$\|z^\omega v w\|_{L_T^q H_G^\ell} \leqslant R T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_{1+\varepsilon_0}^k} \|v\|_{L_T^\infty H_G^\ell} \|w\|_{L_T^\infty H_G^\ell} \,. \tag{1.34}$$

Les fonctions v *et* w *peuvent dépendre de* ω*.*

Remarque 1.39*.* À la différence du cas $\mathbb{R}^d$ ces estimations sont ponctuelles en temps et ne résultent pas d'un phénomène de moyenne temporelle, mais plutôt d'un gain probabiliste qui est lié aux propriétés des fonctions de Hermite. En effet, comme nous l'expliquerons plus en détails dans le chapitre 6, la transformée de Fourier partielle en $y \to \eta$ des fonctions $u \in L_G^2$ se décompose suivant la base des fonctions de Hermite après changement d'échelle :

$$\mathcal{F}_{y\to\eta} u(x;\eta) = \sum_{m\geqslant 0} f_m(\eta) h_m(|\eta|^{\frac{1}{2}} x) \,,$$

ainsi les produits uv font apparaître des produits de convolution des fonctions de Hermite. La bonne connaissance de ces fonctions permet d'en déduire un gain de régularité partiel.

Équipe de ces estimations multi-linéaires on peut alors résoudre le problème de Cauchy pour une donnée aléatoire u_0^ω en cherchant une solution sous la forme $u(t) = z^\omega(t) + v(t)$, $v \in \mathcal{C}^0([0, T[, H_G^\ell)$, $\ell > \frac{3}{2}$. On peut alors appliquer la théorie locale déterministe à v qui satisfait formellement l'équation, intégrale

$$v(t) = -i \int_0^t e^{i(t-t')\Delta_G} \left(|z^\omega + v|^2 (z^\omega + v) \right) \mathrm{d}t' \,.$$

En développant le cube, le terme en $|v|^2 v$ ne pose alors aucun problème, c'est le terme contrôlé par la théorie déterministe à base d'injections de Sobolev. Pour les autres termes il faut appliquer des estimations multi-linéaires. L'estimation du terme purement probabiliste, c'est à dire le terme $|z^\omega|^2 z^\omega$, s'obtient par application de découplage probabiliste et d'estimations multi-linéaires déterministes. Le terme le plus difficile à traiter est un terme mixte entre partie déterministe et probabiliste, $|v|^2 z^\omega$ qui demande un traitement particulier, voir la section 7 du chapitre 6.

6.3. Perspectives. Plusieurs questions naturelles se posent et pourraient fournir de bons prolongement à l'étude de l'équation de Schrödinger pour l'opérateur de Grushin et de Heisenberg. En particulier, le théorème F laisse ouvert le problème de Cauchy dans l'espace d'énergie H^1_G.

Question 1.6. *Est-il possible d'étendre le résultat pour $k = 1$? Est-il possible de produire des solutions globales dans ce cas ?*

S'il est fort possible que les techniques développées nous permettent de construire une théorie locale au niveau H^1_G, celle ci serait alors construite dans le sous-espace $e^{it\Delta_G} u_0^\omega + H_G^{\frac{3}{2}^+}$, mais ne permettrait pas d'en déduire pour autant une théorie globale dans H^1_G. Il faudrait démontrer que la partie $v \in H_G^{\frac{3}{2}}$ est globale, ce qui semble difficile.

Enfin, on peut s'intéresser au cas du sous-laplacien de Heisenberg que nous avions écarté.

Question 1.7. *Peut-on prouver un résultat similaire au théorème F dans le cas de l'équation de Schrödinger posée sur le groupe de Heisenberg ?*

Annexe A. Preuve du théorème 1.14

Démonstration du théorème 1.14. La deuxième inégalité découle de la première par intégration en écrivant que toute variable aléatoire Y vérifie

$$\|Y\|_{L^p_\Omega}^p = p \int_0^\infty \lambda^{p-1} \mathbb{P}\left(|Y| > \lambda\right) \mathrm{d}\lambda \,.$$

Pour la première inégalité, on remarque que l'on peut se contenter de prouver l'inégalité sans valeurs absolues, grâce à la remarque que pour toute variable aléatoire Y, on a $\mathbb{P}(|Y| > \lambda) \leqslant \mathbb{P}(Y > \lambda) + \mathbb{P}(-Y > \lambda)$.

On introduit un paramètre $t > 0$ que l'on ajustera en fin de démonstration et l'on remarque que

$$\mathbb{P}\left(\sum_{n \geqslant 0} a_n X_n > \lambda\right) = \mathbb{P}\left(\exp\left(t \sum_{n \geqslant 0} a_n X^n\right) > e^{t\lambda}\right) \,.$$

L'inégalité de Markov, l'indépendance des X_n et (1.11) impliquent

$$\mathbb{P}\left(\sum_{n\geqslant 0} a_n X_n > \lambda\right) \leqslant e^{-t\lambda} \prod_{n\geqslant 0} \mathbb{E}[e^{ta_n X_n}] \leqslant \exp\left(-t\lambda + ct^2 \sum_{n\geqslant 0} a_n^2\right),$$

de sorte que le résultat escompté s'obtienne par optimisation en t. □

Annexe B. Grandes lignes de la preuve du théorème 1.25

La démonstration de ce théorème utilise un élément essentiel et spécifique à $\mathbb{R}^d$, il s'agit des *estimations bilinéaires* pour les solutions de l'équation de Schrödinger linéaire

$$\left\{ \begin{array}{rcl} i\partial_t u + \Delta u & = & 0 \\ u(0) & = & u_0 \in H^s(\mathbb{R}^4)\,. \end{array} \right. \tag{1.35}$$

Lemme 1.40 (Estimations bilinéaires)**.** *Soient u, v deux solutions de* (1.35) *ayant pour données initiales $u_0, v_0 \in L^2$. On note $u_{N_1} = \mathbf{P}_{N_1} u$ et $v_{N_2} = \mathbf{P}_{N_2} v$ les projections fréquentielles de ces fonctions sur les fréquences $[N_i, 2N_i]$. On suppose que $N_2 \geqslant N_1$. Alors on a*

$$\|u_{N_1} v_{N_2}\|_{L^2_{t,x}} \lesssim N_1 \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \|u_{N_1}(0)\|_{L^2_x} \|v_{N_2}(0)\|_{L^2_x}\,.$$

Remarque 1.41. Remarquons que lorsque $N_1 = N_2$ ce lemme découle des inégalités de Hölder et Bernstein. Le facteur (N_2/N_1) exprime alors un gain de presque une demie dérivée lorsque $N_2 \gg N_1$.

Afin que ce résultat puisse être utilise pour des solutions *proches* des solutions linéaires, on travaille dans des espaces adaptés à l'évolution linéaire, introduit dans ce contexte dans [Bou93]. On définit l'espace $X^{s,b}$ par la norme

$$\|u\|_{X^{s,b}} := \|\langle \tau - |\xi|^2 \rangle^b \langle \xi \rangle^s \tilde{u}(\tau, \xi)\|_{L^2_{\tau,\xi}}\,,$$

où $\tilde{u}$ et la transformée de Fourier temps-espace. L'estimation bilinéaire précédente peut être « transférée » aux espaces $X^{s,b}$. De façon plus précise, on fera usage des inégalités suivantes, rassemblées ci-dessous.

Corollaire 1.42 (Estimations linéaires et bilinéaires dans $X^{s,b}$)**.** *Soit $\eta_T(\cdot)$ un cuttof en temps, lisse et tel que $\mathbf{1}_{[0,T]} \leqslant \eta_T \leqslant \mathbf{1}_{[0,2T]}$. Soit $s \in \mathbb{R}$, $b \in]\frac{1}{2}, \frac{3}{2}[$ et $\theta \in [0, \frac{3}{2} - b[$.*

(*i*) *Estimations linéaires : on a*

$$\left\| \eta_T(t) \int_0^t e^{i(t-t')\Delta} F(t')\,\mathrm{d}t' \right\|_{X^{s,b}} \lesssim T^\theta \|F\|_{X^{s,b-1+\theta}}\,. \tag{1.36}$$

(*ii*) *Estimations bilinéaires : soient $N_2 \geqslant N_1$, $b > \frac{1}{2}$ et deux fonctions $u_{N_1}, u_{N_2} \in X^{0,b}$ localisées en fréquences (spatiales) autour de N_1 (resp.*

N_2). Alors on a

$$\|u_{N_1} v_{N_2}\|_{L^2_{t,x}} \lesssim N_1 \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \|u_{N_1}\|_{X^{0,b}} \|v_{N_2}\|_{X^{0,b}} \,, \tag{1.37}$$

On voit donc que l'effet de l'opérateur intégral de Duhamel en norme $X^{s,b}$ est directement lisible sur l'indice b.

Démonstration du théorème 1.25. On ne donne pas une démonstration complète, le but est uniquement de montrer en quoi l'estimation bilinéaire est utile dans la preuve. On reprend le schéma introduit pour étudier le problème de Cauchy local en cherchant v tel que $u(t) = u_L(t) + v(t)$ résolve (1.14), c'est à dire que l'on doit résoudre

$$v(t) = \Phi(v)(t) := -i \int_0^t e^{i(t-t')\Delta}(|u_L + v|^2(u_L + v)) \, \mathrm{d}t' \,.$$

On montre alors que Φ est une contraction $X^{\sigma,b} \to X^{\sigma,b}$, avec $b = \frac{1}{2}^+$ et $\sigma = 1^+$. Grâce à une inégalité de Strichartz (1.36), on obtient

$$\begin{aligned}\|\Phi v\|_{X^{1^+,\frac{1}{2}^+}} &\lesssim T^\theta \||u_L + v|^2(u_L + v)\|_{X^{1^+,\frac{1}{2}^-}} \\ &\lesssim T^\theta \sup_{\|w\|_{X^{0,\frac{1}{2}^+}} \leqslant 1} \int_{\mathbb{R}_t \times \mathbb{R}^4_x} \langle \nabla \rangle^{1^+} |u_L + v|^2 (u_L + v) w \, \mathrm{d}t\mathrm{d}x \,.\end{aligned}$$

où on a utilisé la dualité dans la dernière ligne.

Le développement de $u_L + v|^2(u_L + v)$ apporte essentiellement deux termes dont le traitement est radicalement différence : le terme en $|v|^2 v$ qui se traite par les méthodes déterministes et le terme $|u_L|^2 u_L$ sur lequel on concentre notre étude. Les autres termes se traitent en combiant les méthodes employées pour ces deux termes.

Par opposition au cas (fNLS) avec $\beta > 0$, l'opérateur de Duhamel ne fait pas gagner des dérivées, ainsi en fonction du terme sur lequel agit la dérivée $\langle \nabla \rangle^{1^+}$, il n'est pas évident que l'on ait $\||u_L|^2 u_L\|_{X^{1^+,\frac{1}{2}^-}} < \infty$.

Pour compenser, on va utiliser les estimations bilinéaires et pour cela, on commence par découper chaque fonction u_L dyadiquement en fréquences spatiales, grâce à une décomposition de Littlewood-Paley $u_L = \sum_{N \in 2^{\mathbb{N}}} u_N$. On en déduit alors

$$\int_{\mathbb{R}_t \times \mathbb{R}^4_x} \langle \nabla \rangle^{1^+} (|u_L|^2 u_L) w \, \mathrm{d}t\mathrm{d}x \simeq \sum_{\substack{N_1, N_2, N_3, N_4 \\ \max\{N_1, N_2, N_3\} \sim N_4}} (\langle \nabla \rangle^{1^+} (u_{N_1} u_{N_2} u_{N_3}), u_{N_4})_{L^2_{t,x}} \,,$$

la condition sur N_4 provenant du théorème de Parseval et du produit scalaire L^2. De plus, quitte à considérer les sommes symétriques, on peut supposer que dans cette somme on a $N_4 \sim N_3 \geqslant N_2, N_1$.

Le cas $N_1 \sim N_2 \sim N_3$ est favorable car alors $\langle \nabla \rangle$ agit comme N_1 est peut être distribué suivant chacun des termes. Dans la suite on se concentre sur le cas $N_4 \sim N_3 \gg N_1, N_2$ avec la condition supplémentaire $N_1, N_2 \ll N_3^{\frac{1}{3}}$, ce qui empêche d'échanger les dérivées. On s'attend à ce que ce cas soit l'un des plus difficiles de l'analyse.

Au niveau fréquentiel $u_{N_1}u_{N_2}u_{N_3}$ est supporté autour de $\max\{N_i\} = N_3$, on peut estimer le « coût » de la dérivation par N_3. On utilise également que $N_3 \gg N_1$ et $N_4 \gg N_2$ pour tirer partie de l'estimation bilinéaire (1.37) :

$$
\begin{aligned}
(\langle \nabla \rangle^{1^+}(u_{N_1}u_{N_2}u_{N_3}), u_{N_4})_{L^2_{t,x}} &\lesssim N_3^{1^+} \|u_{N_1}u_{N_3}\|_{L^2_{t,x}} \|w_{N_4}u_{N_2}\|_{L^2_{t,x}} \\
&\lesssim N_3^{1^+} N_1 \left(\frac{N_1}{N_3}\right)^{\frac{1}{2}} N_2 \left(\frac{N_2}{N_4}\right)^{\frac{1}{2}} \|w_{N_4}\|_{X^{0,b}} \prod_{i=1}^{3} \|u_{N_i}\|_{X^{0,b}} .
\end{aligned}
$$

En utilisant alors que $\|u_{N_i}\|_{X^{0,b}} \sim N_i^{-s}\|u_{N_i}\|_{X^{s,b}}$ et en tenant compte des tailles des N_i on en déduit

$$
(\langle \nabla \rangle^{1^+}(u_{N_1}u_{N_2}u_{N_3}), u_{N_4})_{L^2_{t,x}} \lesssim N_3^{1^+ - \frac{5}{3}s} \|w_{N_4}\|_{X^{0,b}} \prod_{i=1}^{3} \|u_{N_i}\|_{X^{s,b}} ,
$$

ce qui est une contribution acceptable dès lors que $s > \frac{3}{5}$. Une preuve complète peut être trouvée dans [BOP15b]. $\square$

Chapitre 2

General Introduction

The study of partial differential equations with random initial data is the main object of the work presented in this dissertation. The equations in concern are of various physical origins: the wave equation and the Schrödinger equation, both qualified as *dispersive*, but also the Euler equation which models a perfect incompressible fluid.

The purpose of this introduction is twofold. It is both to present the general methodology that is used to study these partial differential equations with random initial data, but also to introduce the works presented in this dissertation, replacing them in context.

To this end, Section 2 should be read as an introduction to two methods which I believe are the most general and adaptable, and are probably useful in other contexts. The following sections, numbered from 3 to 6 each introduce a chapter of the dissertation. They put the work in context and highlight some of the difficulties that arise. Finally, this introduction does not put the emphasis on rigour rather than the main ideas. In particular, I will provide ideas of the proofs rather than complete proofs, those can be read in the referred articles.

All of the works of this dissertation have resulted in a publication or a pre-publication. The original results of the dissertation, which are stated in this introduction, are numbered with letters and have been published or pre-published in the following articles:

- *Almost Sure Existence of Global Weak Solutions for Supercritical Wave Equations*, Journal of Differential Equations, Vol 273, pp. 83-121.
- *Almost Sure Scattering at Mass Regularity for Radial Schrödinger Equations*, pre-print arXiv:2011.06309.
- *Construction of High Regularity Invariant Measures for the Euler Equations and Remarks on the Growth of the Solutions*, pre-print arXiv:2002.11086
- *Probabilistic Local Well-Posedness for the Schrödinger Equation Posed for the Grushin Laplacian*, pre-print arXiv:2103.03560.

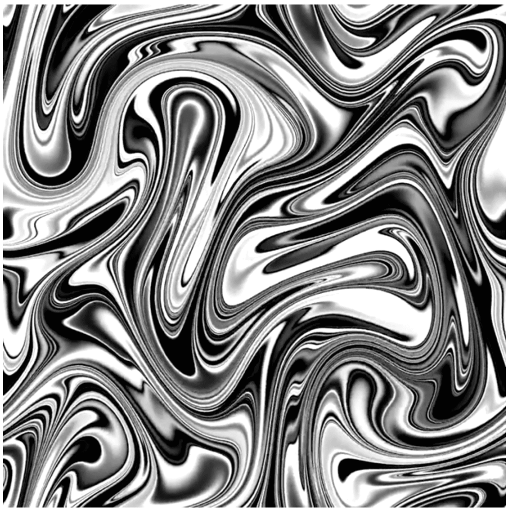
Before introducing the two fundamental methods in the study of random data equations, let us give some physical and mathematical reasons for our interest of *random* data.

1. Why introducing randomness in PDE?

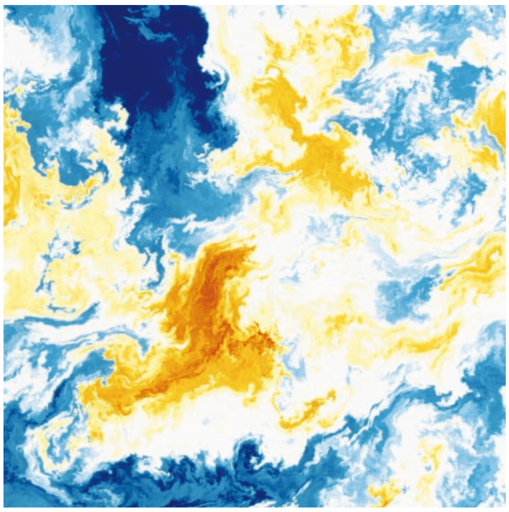
In this section, we limit ourselves to mentioning a few reasons that make the introduction of random data useful – even desirable – for the study of partial differential equations. This section does not claim any mathematical rigour.

1.1. Motivation from physics. A main reason for studying initial random data is the need for a statistical description of solutions to specific equations. A particularly important example is the phenomena of turbulence and mixing in the equations of fluid mechanics, which are still very poorly understood to this day. Being able to describe them or to establish rigorously some observed empirical "laws" is a major challenge. For example, one can consult [BBPS19] for recent progress in this direction.

Even though everyone can experience the phenomena of turbulence, for example by looking at the flow of air around the wing of an aircraft, it is even more common to experience the phenomenon of mixing: milk poured on the surface of a cup of coffee will tend to mix with the coffee in a stretch-fold process, rapidly forming highly complex structures. Figure 1 (a) illustrates this phenomenon, and Figure 1 (b) gives a numerical simulation of the equations describing the phenomenon.



(a) Mixing of milk and coffee



(b) Numerical simulation

Figure 1. The mixing phenomena[1]

One can thus understand the complexity of describing such phenomena. Faced with such a complexity of the observed solutions one can try to describe the "average" behaviour of such solutions, deduce the *typical* behaviour of the solutions and characterise the deviation from this average behaviour, find invariants, quantities which characterise the behaviour of the great majority of the solutions, and hence the idea of studying these equations with random data.

Here is another reason to consider random data. One can criticise the notion of solutions considered in mathematical statements for their lack of physical meaning. In fluid mechanics, a striking example is the non-uniqueness of the weak solutions of the Euler equation. Without going into details, Sheffer and Shnirelman [Sch93, Shn97] and more recently [DLS09, Ise18] proved that it is possible to produce such weak solutions which are equal to zero between time $t = 0$ and $t = 1$, non-zero for $t \in [1, 2]$ and vanish again for all $t \geqslant 2$. In concrete terms, this means that a fluid initially at rest suddenly becomes agitated at time $t = 1$ without any external force and then returns to rest at time $t = 2$. One may wonder about the relevance of such a notion. These *solutions* are excluded by fluid mechanics and contradictory to experimentation. Should we deduce that the Euler equations are a bad model of the physical phenomena they are supposed to model? Or is it the notion of solutions as considered by mathematicians that is not the right one, because the space of admissible solutions contains pathological examples? One may thus want to develop a generic notion of solutions, so that these pathological constructions are not the typical case of the considered solutions.

Remark 2.1. One should not fall into the opposite excess and think that the class of functions considered in physics should be limited to very regular functions. The resolution of elementary questions in quantum mechanics requires the use of rough functions which belong to the Sobolev spaces. To give just one example, the solution of the semi-infinite well problem in quantum mechanics (discussed in [BD02]) requires that the functions considered have discontinuous derivatives. In a similar way, the notion of point charge at a point P considered in electrostatics, is formalised by a Dirac distribution δ_P, which is not even a function.

1.2. Overcoming the deterministic limitations. There are also purely mathematical reasons for studying random data. One of the first questions which a study of a partial differential equation should address is to make sense of its solutions. As a reminder, let us recall what is usually meant by "solving"

[1](a) is extracted from Quanta magazine at `www.quantamagazine.org/mathematicians-prove-batchelors-law-of-turbulence-20200204` and (b) is the result of a numerical simulation [Kra96]

locally and globally the abstract evolution equation:

$$\left\{ \begin{array}{rcl} \partial_t u(t) & = & F(u(t)) \\ u(0) & = & u_0 \in X \,, \end{array} \right. \tag{2.1}$$

where $F : X \to Y$ and with X, Y some Banach spaces, of unknown variable $u : \mathbb{R} \to X$, the evolution variable t generally representing *time*. This is also called *solving the Cauchy problem*. With the previous notations we have the following definition.

Definition 2.1. The equation (2.1) is said to be locally well-posed in the space X if for any bounded set $B \subset X$ and for any initial data $u_0 \in X$ there exist $T = T(u_0) > 0$ and a unique solution $u \in \mathcal{C}^0([0,T], X)$ to (2.1), and additionally, the application $u_0 \mapsto u$ defines a continuous function $B \to \mathcal{C}^0([0,T], X)$. The problem is also called *uniformly well-posed* when this application is uniformly continuous on B.

We refer to [Tzv06] for more details. Finally, the problem is said to be globally well-posed if it is possible to take $T = \infty$ in this definition.

This definition can be seen as the basic specifications of the notion of solution. In addition to the existence and uniqueness, it requires in particular a property of continuity with respect to the initial data which makes such solutions very robust.

For some equations, such local or global Cauchy theories are unknown or even false in spaces where one would however like to construct solutions. Consider the example of the Schrödinger equation on a compact Riemannian manifold (M, g) of dimension 2:

$$i\partial_t u + \Delta_g u = |u|^2 u \,, \tag{2.2}$$

with the initial data $u(0) = u_0 \in H^s(M)$, the usual Sobolev spaces associated to the manifold M.

The local Cauchy problem for (2.2) is well-posed in the Sobolev spaces $H^s(M)$ provided $s > \frac{1}{2}$, see [BGT04]. In the case of the sphere, this threshold is even lowered to $s > \frac{1}{4}$, [BGT05]. For the torus, whose arithmetic properties help to lower this threshold, the threshold is even lower: $s > 0$, see [Bou93].

In the latter two cases, the results are essentially optimal, since it is known that for $s < \frac{1}{4}$ (resp. $s < 0$), the Cauchy problem is not uniformly well-posed [BGT02, Ban04] (resp. [BGT02, CCT03, Mol09]). More precisely, in the articles [CCT03] and [CKS+10], the authors show that for any small $\varepsilon, \tau > 0$ and any large $K > 0$, one can construct initial data which are ε close and give rise to solutions such that at time $t = \tau$ they are K-separated.

The conservation, at least at a formal level, of the quantity $t \mapsto \|u(t)\|^2_{L^2(M)}$ by the equation (2.2) would make it possible to build global solutions, provided that local solutions can first be produced. However, a local Cauchy theory is not known at this level of regularity. The introduction of random data makes it possible to overcome these regularity thresholds in certain situations that will be studied. The basis of such a strategy will be explained in Section 2.

In the case where the Cauchy problem is ill-posed at a fixed level of regularity, one may also wonder whether this ill-posedness applies to *all* initial data, or whether on the contrary one can identify a class for which solutions can be uniquely constructed. In this case, a *probabilistic* theory would at least play the role of an "example provider". Indeed, let us assume that we have an equation whose local (or global) Cauchy theory in H^{s_0} is unknown. Also assume that no explicit solution in H^{s_0} is known, except for a few trivial examples like the zero function. A probabilistic construction of the type "for almost any initial data in H^{s_0} there exists a unique associated solution" *a fortiori* produces at least one solution which was not known by deterministic methods.

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we will always denote by ω an element of Ω in the rest of this introduction.

2. The core of the probabilistic approach in two themes

We are now going to introduce two probabilistic methods which are certainly central in the study of random data equations. This dissertation will use these methods, or variants of these methods. It is the large number of applications and the robustness of the ideas which make these methods so important in study of partial differential equations with random data. We will present them in their chronological order of appearance, first the method of invariant measures which globalises solutions and goes back at least to [Bou94] then the local method used in [BT08a]. These articles can be considered as the founding works of the whole study of dispersive equations with random data.

The review of the existing works will be minimal in this section, in particular the very recent developments of these methods will not be presented. The presentation of the two main methods will be carried out through the example of a fractional Schrödinger equation with derivative gain:

$$\text{(fNLS)} \qquad \begin{cases} i\partial_t u - (-\Delta)^\alpha u &= \langle \nabla \rangle^{-\beta}(|u|^2 u) \\ u(0) &= u_0 \in H^s(\mathbb{T}^d)\,. \end{cases}$$

We will take $(d, \alpha, \beta) = (3, 2, 1)$, but the results of Section 2.1 and Section 2.2 will be written for general values of d, $\alpha, \beta > 0$.

The choice of such an equation is motivated by exposition reasons. We choose an equation of order 1 in time in order to reduce the length of our expressions and the gain of derivative in the non-linearity given by $\beta > 0$ greatly simplifies all the arguments presented. Note that if $\beta = 0$ then studying (fNLS) with random data becomes much more difficult, we refer to [Bou96] for such an example.

Before delving into the details of these methods we briefly recall the usual deterministic method for constructing local and global solutions.

In order to prove that the Cauchy problem for (fNLS) is uniformly locally well-posed in H^s for s large enough, we use a fixed point argument in which the main ingredient is the Sobolev embedding, which we will use through this chapter. More precisely, we use that:

(i) For $s > \frac{d}{2}$, the embedding $H^s(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ is continuous.

(ii) For $s \in [0, \frac{d}{2})$, the embedding $H^s(\mathbb{T}^d) \hookrightarrow L^p(\mathbb{T}^d)$ is continuous as soon as $\frac{1}{p} \geqslant \frac{1}{2} - \frac{s}{d}$.

(iii) For $\varepsilon > 0$ and $p \geqslant 2$ such that $\varepsilon p > d$ there holds $W^{\varepsilon,p}(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$.

Lemma 2.2. *Let $s > \frac{d}{2}$. The Cauchy problem for (fNLS) is uniformly locally well-posed in the space $H^s(\mathbb{T}^d)$.*

Proof. In integral form, (fNLS) writes

$$u(t) = e^{-it(-\Delta)^\alpha} u_0 - i \int_0^t \frac{e^{-i(t-t')(-\Delta)^\alpha}}{\langle \nabla \rangle^\beta} \left(|u(t')|^2 u(t') \right) \mathrm{d}t' \,, \tag{2.3}$$

which will be referred to as the "Duhamel formulation". Solving (2.3) then boils down to solving a fixed-point equation $\Phi(u) = u$ for the functional $\Phi : \mathcal{C}([0,T], H^s) \longrightarrow \mathcal{C}([0,T], H^s)$ defined as the right-hand side of (2.3). The goal is to apply the fixed-point theorem to the restriction $\Phi : B_{R,T} \to B_{R,T}$ where $B_{R,T}$ stands for the closed ball of $\mathcal{C}([0,T], H^s)$, centred at 0 and of radius R.

Let $u_0 \in H^s(\mathbb{T}^d)$. For a reason which will be clear at the end of the proof we fix $R > 2\|u_0\|_{H^s}$. We seek for a time $T > 0$ such that Φ stabilises the ball $B_{R,T}$ (which is not immediate *a priori*) and a contraction mapping on this same ball. As often in this situation, the second requirements follows from an inspection of the proof that Φ stabilises $B_{R,T}$ for T small enough, which we are going to show. In other words, we want to prove that for u satisfying $\|u\|_{L^\infty_T H^s} \leqslant R$, there holds $\|\Phi(u)\|_{L^\infty_T H^s} \leqslant R$.

For such u and $t \in [0,T]$, we start with an application of the triangle inequality:

$$\|\Phi(u)(t)\|_{H^s} \leqslant \|e^{-it(-\Delta)^\alpha} u_0\|_{H^s} + \int_0^t \left\| e^{-i(t-t')(-\Delta)^\alpha} \left(|u(t')|^2 u(t') \right) \right\|_{H^{s-\beta}} \mathrm{d}t' \,.$$

We now use that $e^{-it(-\Delta)^\alpha}$ is an isometry in $H^s(\mathbb{T}^d)$, which follows from the Parseval theorem. Therefore we have obtained

$$\begin{aligned} \|\Phi(u)(t)\|_{H^s} &\leqslant \|u_0\|_{H^s} + \int_0^T \||u(t')|^2 u(t')\|_{H^{s-\beta}} \\ &\leqslant \frac{R}{2} + T \||u|^2 u\|_{L^\infty_T H^{s-\beta}} \,. \end{aligned}$$

In order to conclude, we wish to prove

$$\||u|^2u\|_{H^{s-\beta}} \lesssim \|u\|_{H^s}^3 .$$

Such inequalities are rigorously proven using *product laws* in Sobolev spaces, we only give the intuition. When differentiating u^3, s times, one obtains many terms of which the one carrying the more derivatives on a single term is $u^2\nabla^s u$. One then expects this term to be the leading term, that is we should think as $\nabla^s(|u|^2u) \simeq |u|^2\nabla^s u$, and thus the Hölder inequality gives

$$\||u|^2u\|_{H^s} \lesssim \|u\|_{L^\infty}^2\|u\|_{H^s} .$$

Using that $\||u|^2u\|_{H^{s-\beta}} \leqslant \||u|^2u\|_{H^s}$ and the Sobolev embedding $\|u\|_{L^\infty} \lesssim \|u\|_{H^s}$ (since $s > \frac{d}{2}$) we obtain

$$\|\Phi(u)\|_{L^\infty_T H^s} \leqslant \frac{R}{2} + TR^3 ,$$

which is bounded by R as soon as $T \leqslant \frac{1}{2}R^2$. □

This local theory can be sharpened. For instance we did not use the full power of the Sobolev embedding as we only bounded $\||u|^2u\|_{H^{s-\beta}} \leqslant \||u|^2u\|_{H^s}$, not exploiting the β gain. Fixing this issue leads to the following local theory.

Lemma 2.3. *For $s > \frac{d-\beta}{2}$, the Cauchy problem for (fNLS) is uniformly locally well-posed in $H^s(\mathbb{T}^d)$.*

Proof. Going back to the above computation, we have

$$\|\Phi(u)\|_{L^\infty H^s} \leqslant \frac{R}{2} + T\||u|^2u\|_{L^\infty H^{s-\beta}} ,$$

so that it remains to prove that for any $s > \frac{d-\beta}{2}$ and any $u \in H^s$ there holds

$$\||u|^2u\|_{H^{s-\beta}} \lesssim \|u\|_{H^s}^3 . \tag{2.4}$$

To this end we make another use of the heuristics given by

$$\nabla^{s-\beta}(|u|^2u) \simeq \nabla^{s-\beta}u|u|^2 + \{\text{lower order terms}\} ,$$

which allows us to apply the Hölder inequality with finite p and q satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and the Sobolev embedding to obtain

$$\begin{aligned}\||u|^2u\|_{H^{s-\beta}} &\lesssim \|\nabla^{s-\beta}u\|_{L^p}\|u\|_{L^{2q}}^2 \\ &\lesssim \|u\|_{H^{s-\beta+d\left(\frac{1}{2}-\frac{1}{p}\right)}}\|u\|^2_{H^{d\left(\frac{1}{2}-\frac{1}{2q}\right)}} .\end{aligned}$$

Therefore (2.4) is obtained if there holds

$$s - \beta + d\left(\frac{1}{2} - \frac{1}{p}\right) \leqslant s \text{ and } d\left(\frac{1}{2} - \frac{1}{2q}\right) \leqslant s .$$

The first condition translates in $\frac{1}{q} \leqslant \frac{\beta}{d}$ and the second in $\frac{1}{q} \geqslant 1 - \frac{2s}{d}$. These two conditions are therefore simultaneously satisfied as soon as $1 - \frac{2s}{d} \leqslant \frac{\beta}{d}$, that is for $s \geqslant \frac{d-\beta}{2}$, which is the assumption. □

Remark 2.4. The proof above is not optimal either. No specific property of (fNLS) has been used, except that the linear semigroup is an isometry in L^2. A more refined local theory should take advantages of the dispersive nature of (fNLS). For exposition reasons however, we restrict ourselves to the elementary local theory of Lemma 2.3.

The obtained local theory can be compactly written in the following form: for $u_0 \in H^s(\mathbb{T}^d)$ there exists $\tau \sim \|u_0\|_{H^s}^{-2}$ such that the local Cauchy problem for (fNLS) associated to u_0 is well-posed on $[0,\tau]$. Moreover for any $t \in [0,\tau]$ there holds $\|u(t)\|_{H^s} \leqslant 2\|u_0\|_{H^s}$.

In order to construct global solutions, a natural attempt is to iterate the local argument above at time $t = \tau$. Therefore one can see that a sufficient condition for extending a local solution until any target time $T > 0$ is that $t \mapsto \|u(t)\|_{H^s}$ is bounded. To this end one could use an *a priori* estimate on $\|u\|_{H^s}$, which in general turns to be very difficult to obtain. As an example, for (fNLS), multiplying by $\langle\nabla\rangle^{\beta}\bar{u}$ and integrating gives that

$$E(t) := \frac{1}{2}\|u(t)\|^2_{H^{\alpha+\frac{\beta}{2}}} + \frac{1}{4}\|u\|^4_{L^4}\,,$$

is formally conserved and then the $H^{\alpha+\frac{\beta}{2}}$ norm of $u(t)$ is bounded. this allows for extending solutions of Lemma 2.3 of regularity $H^{\alpha+\frac{\beta}{2}}$, into global solutions. In the case $(d,\alpha,\beta) = (3,2,1)$ we have constructed a local theory in $H^s(\mathbb{T}^3)$ as soon as $s > 1$, and since $\alpha + \frac{\beta}{2} = 2$ we infer a global theory in H^2.

We are now going to explain another method for producing global solutions based on a *statistical conservation* which will eventually lead to the construction of global solutions at regularity lower than H^2.

2.1. Invariant measures: the globalisation argument. Let us first recall what is meant by an invariant measure. Given the flow $\phi_t : X \to X$ of a differential equation, a measure μ on the space X is said to be invariant under ϕ_t if for all measurable (or Borelian) set $A \subset X$ and all $t \geqslant 0$ there holds

$$\mu(\phi_{-t}A) = \mu(A)\,.$$

Intuitively, this means that the statistical properties encoded in μ are conserved by the flow, that is the statistics of the dynamics remains unchanged upon time evolution.

Obtaining invariant measures for dynamical systems such as partial differential equations is a problem of great importance. In particular, let X be a space on which a flow ϕ_t acts. Let us assume that a measure μ is invariant under ϕ_t, then two important results apply:

(1) The Poincaré recurrence theorem, which asserts that in this case, for any measurable set $A \subset X$ with $\mu(A) > 0$, almost all $u \in A$ is recurrent, in the sense that $\phi_t u \in A$ for infinitely many t.
(2) The Birkhoff theorem, which relates spatial averages of $f \circ \phi_t$ and statistical averages of f for a class of functions $f : X \to X$ called μ-ergodic functions.

The class of Hamiltonian differential equations naturally admit invariant measures. Such equations write

$$\left\{ \begin{array}{rcl} \dot{q} & = & \partial_p H(q,p) \\ \dot{p} & = & -\partial_q H(q,p)\,, \end{array} \right. \tag{2.5}$$

with initial conditions $(q(0), p(0)) = (q_0, p_0) \in \mathbb{R}^N \times \mathbb{R}^N$ where $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$.

In this case an invariant probability measure is given by

$$d\mu = Z^{-1} e^{-H(q,p)}\, \mathrm{d}q_0\, \mathrm{d}p_0\,,$$

where $\mathrm{d}q_0\, \mathrm{d}p_0$ stands for the Lebesgue measure of $\mathbb{R}^{2N}$ and Z is a normalisation constant.

In order to prove invariance, one verifies that $t \mapsto H(q(t), p(t))$ has zero derivative, hence $H(q,p)$ is conserved by the flow of (2.5). The invariance of μ then follows from the following result.

Lemma 2.5 (Liouville)**.** *The measure* $\mathrm{d}q_0\, \mathrm{d}p_0$ *is invariant by the flow of* (2.5).

Proof. Let ϕ_t be the flow associated to (2.5) and consider $f : (q,p) \mapsto f(q,p)$. Writing

$$((\phi_t)_* \mathrm{d}q_0\, \mathrm{d}p_0, f) := \int f(\phi_t(q_0,p_0))\, \mathrm{d}q_0\, \mathrm{d}p_0 = \int f(q(t), p(t))\, \mathrm{d}q_0\, \mathrm{d}p_0\,,$$

we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}((\phi_t)_* \mathrm{d}q_0\, \mathrm{d}p_0, f) = \int \left(\dot{q}\partial_q f(q,p) + \dot{p}\partial_p f(q,p)\right) \mathrm{d}q_0\, \mathrm{d}p_0\,.$$

It is then sufficient to use (2.5) and integrate by parts to obtain that $t \mapsto (\phi_t^* \mathrm{d}q_0\, \mathrm{d}p_0, f)$ is constant, thus the invariance of $\mathrm{d}q_0\, \mathrm{d}p_0$. ☐

Although it is infinite dimensional, (fNLS) enjoys a similar structure. In order to see this, we write $u \in L^2(\mathbb{T}^d)$ in Fourier series:

$$u(x) = \sum_{n \in \mathbb{Z}^d} u_n e^{in\cdot x}\,,$$

and write $u_n = q_n + ip_n$. The real variables $q = (q_n)_{n\in\mathbb{Z}^d}$ and $p = (p_n)_{n\in\mathbb{Z}^d}$ are sufficient to describe u. We check that $u = q + ip$ solves (fNLS) if and only if (p, q) is a solution of

$$\begin{cases} \partial_t q &= \partial_p H(q,p) \\ \partial_t p &= -\partial_q H(q,p)\,, \end{cases}$$

supplemented with initial data $(q(0), p(0)) = (\mathrm{Re}(\hat{u}_0), \mathrm{Im}(\hat{u}_0))$, with $H(q,p) = H(u)$ defined by

$$H(u) \coloneqq \frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}}u\|_{L^2}^2 + \Big\langle \langle\nabla\rangle^{-\beta}(|u|^2u), u\Big\rangle_{L^2}\,,$$

which is conserved under the flow of (fNLS) and is obtained by multiplication of (fNLS) by $\bar{u}$ and integration by part.

Let us write $\mathrm{d}u \coloneqq \mathrm{d}p\,\mathrm{d}q$, a measure which should be rigorously defined in the following. Therefore we expect the measure

$$\mathrm{d}\mu(u)\text{``}=\text{''}Z^{-1}\exp\left(-\frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}}u\|_{L^2}^2 - \Big\langle\langle\nabla\rangle^{-\beta}(|u|^2u), u\Big\rangle\right)\mathrm{d}u\,, \tag{2.6}$$

to be invariant for (fNLS), where Z is a renormalisation constant.

Note that in addition to the problem of giving a proper meaning to $\mathrm{d}u$ is the problem that the above setting is finite dimensional, whereas (fNLS) is infinite dimensional.

Let us briefly explain how to construct $\mathrm{d}u$. More precisely we will directly construct

$$\mathrm{d}\mu_0(u)\text{``}=\text{''}Z^{-1}\exp\left(-\frac{1}{2}\|u\|_{L^2}^2 - \frac{1}{2}\|(-\Delta)^{\frac{\alpha}{2}}u\|^2\right)\mathrm{d}u\,.$$

The measure $\mathrm{d}\mu$ can then be constructed as an absolutely continuous measure with respect to $\mathrm{d}\mu_0$. We refer to Chapter 4 for an example.

At least formally, one can picture $Z^{-1}\mathrm{d}u$ as the product measure $\bigotimes_{n\in\mathbb{Z}^d}\mathrm{d}u_n$. Writing u with Fourier coefficients u_n we construct the measure

$$\mathrm{d}\mu_0(u) = \bigotimes_{n\in\mathbb{Z}^d}(2\pi)^{-1}e^{-\frac{1}{2}(1+|n|^{2\alpha})|u_n|^2}\,\mathrm{d}u_n\,, \tag{2.7}$$

where the introduction of $(2\pi)^{-1}$ serves as a renormalisation in order to recognise independent Gaussian densities on each Fourier coefficient. More precisely, the measure μ_0 is obtained as the image measure by the map

$$\sum_{n\in\mathbb{Z}^d}\frac{g_n^\omega}{\sqrt{1+|n|^{2\alpha}}}e^{in\cdot x}\,, \tag{2.8}$$

where the $(g_n)_{n\in\mathbb{Z}^d}$ are independent standard complex Gaussian random variables.

Remark 2.6. Without the term $\|u\|_{L^2}^2$ in the definition of the measure μ_0, there is a problem of division by $n = 0$ which could be problematic in the definition of μ_0.

We now turn to the main idea of the *globalisation argument* used in [Bou94]. The great flexibility of this method explains its success. We will state this result as a meta-theorem because we believe that it is more of a general strategy rather than a theorem, which can be turned into a rigorous argument most of the time.

Meta-theorem (Globalisation argument)**.** *Let $s \in \mathbb{R}$ and ϕ_t be the flow of an evolution equation of type* (2.1)*. Assume that there exists a measure μ on H^s satisfying:*

(*i*) *μ is invariant under the flow ϕ_t.*

(*ii*) *For any $\lambda > 0$, there holds $\mu(\|u\|_{H^s} > \lambda) \lesssim e^{-c\lambda^2}$.*

(*iii*) *A good local well-posedness theory is available in H^s, that is, if $\|u_0\|_{H^s} \leqslant \lambda$ then a unique local solution exists on $[0, \tau]$, associated to u_0 and where one can take $\tau \sim \lambda^{-K}$ for a numerical constant $K > 0$, and furthermore $\|u(t)\|_{H^s} \leqslant 2\lambda$ for any $t \leqslant [0, \tau]$.*

Under these assumptions, there holds that for μ-almost every initial data $u_0 \in H^s$, there exists a unique global solution to (2.1) *associated to u_0. Moreover, for any $t > 0$ one has*

$$\|u(t)\|_{H^s} \leqslant C(u_0) \log^{\frac{1}{2}}(1 + t)\,.$$

Remark 2.7. Gaussian measures such as μ_0 defined by (2.7) satisfy (*ii*). This will indeed be a consequence of Theorem 2.12. Assumption (*iii*) should be understood as a local theory given by standard subcritical fixed-point arguments, such as the local theory obtained in Lemma 2.3.

Proof of the meta-theorem. Rather than directly constructing global solutions, we start by constructing solutions until a target time $T > 0$ which we fix, for a set of initial data of measure larger than $1 - \varepsilon$, where $\varepsilon > 0$ is also arbitrary but fixed.

In order to do so, we fix $\lambda > 0$ to be chosen in the proof. We recall that by (*iii*) there is a uniform local well-posedness time $\tau \sim \lambda^{-K}$ for initial data u_0 satisfying $\|u_0\|_{H^s} \leqslant \lambda$. Therefore, initial data in the set

$$G_{\lambda,T} \coloneqq \left\{ u_0, \text{ such that for all } n \leqslant \left\lfloor \frac{T}{\tau} \right\rfloor, \|u(n\tau)\|_{H^s} \leqslant \lambda \right\} ,$$

give rise to solutions on $[0, T]$ such that $\|u(t)\|_{H^s} \leqslant 2\lambda$ for all $t \in [0, T]$. Figure 2 represents such solutions.

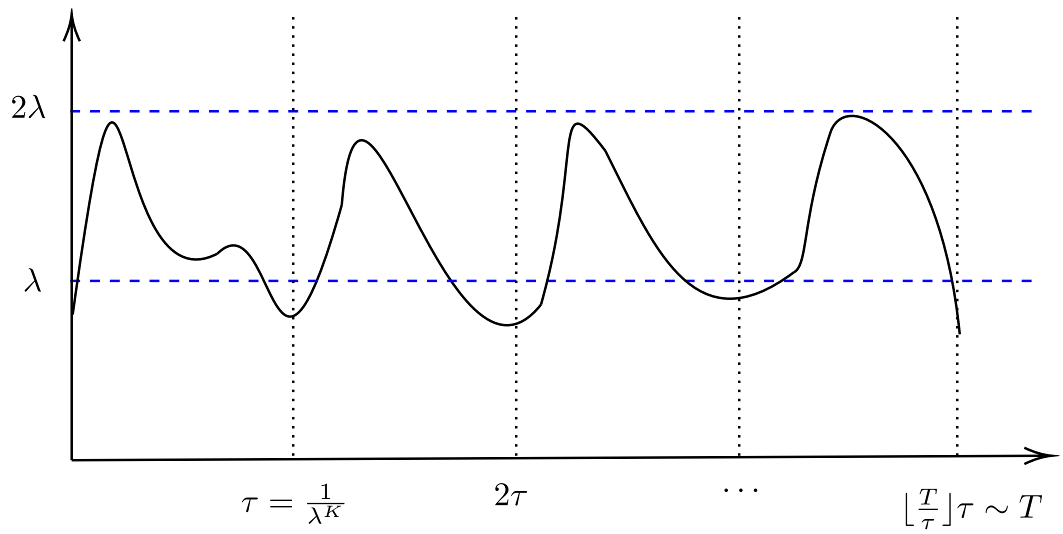


Figure 2. Solutions issued from $G_{\lambda,T}$

We evaluate the probability of $G_{\lambda,T}$ by estimating the probability of its complement. More precisely the complement is given by

$$B_{\lambda,T} := H^s \setminus G_{\lambda,T} = \bigcup_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \{u_0, \|u(n\tau)\|_{H^s} > \lambda\} \ ,$$

so that

$$\mu(B_{\lambda,T}) \leqslant \sum_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \mu\left(\{u_0, \|\phi_{n\tau} u_0\|_{H^s} > \lambda\}\right) \ .$$

Then we use that $u(n\tau) = \phi_{n\tau} u_0$ and the invariance of the measure (i) to write

$$\begin{aligned} \mu\left(\{u_0, \|\phi_{n\tau} u_0\|_{H^s} > \lambda\}\right) &= \mu\left(\phi_{-n\tau}\left(\{u_0 \in H^s, \|u_0\|_{H^s} > \lambda\}\right)\right) \\ &= \mu\left(\{u_0 \in H^s, \|u_0\|_{H^s} > \lambda\}\right) \ , \end{aligned}$$

and then by (ii) we arrive at:

$$\mu(B_{\lambda,T}) \leqslant \sum_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \mu\left(\{u_0 \in H^s, \|u_0\|_{H^s} > \lambda\}\right) \leqslant \frac{T}{\tau} e^{-c\lambda^2} \ .$$

The fact that $\tau \sim \lambda^{-K}$ leads to

$$\mu(B_{\lambda,T}) \lesssim T\lambda^K e^{-c\lambda^2} \ ,$$

so that $\mu(B_{\lambda,T}) \leqslant \varepsilon$ as soon as $\lambda \sim \log^{1/2}\left(\frac{T}{\varepsilon}\right)$. We choose such a λ. We have therefore constructed a set $G_{T,\varepsilon}$ of measure at least $1-\varepsilon$ and such that $\|u(t)\| \leqslant 2\log^{\frac{1}{2}}(\frac{T}{\varepsilon})$ for all $t \in [0, T]$.

The conclusion is now obtained by an application of a Borel-Cantelli argument. We introduce the set

$$G := \bigcup_{k\geqslant 0}\bigcap_{n\geqslant k} G_{n,\frac{1}{n^2}}\,.$$

Firstly, initial data belonging to G give rise to unique global solutions not growing faster that $C(u_0)\log^{\frac{1}{2}}(1+t)$. Finally, by monotone convergence there holds

$$\mu(H^s\setminus G)\leqslant \limsup_{k\to\infty}\mu\left(\bigcup_{n\geqslant k} H^s\setminus G_{n,\frac{1}{n^2}}\right)\leqslant \limsup_{k\to\infty}\sum_{n\geqslant k}\frac{1}{n^2}\,,$$

which is 0 since the series of the $\frac{1}{n^2}$ is convergent. □

Remark 2.8. This argument can be thought as the probabilistic version of the globalisation argument using a conserved quantity. Indeed, rather than using a deterministic information on $t\mapsto \|u(t)\|_{H^s}$ one uses a statistical information on the $\|u(t)\|_{H^s}$, which makes this quantity controlled *for all time* $t = n\tau$, because of the control at $t = 0$, using invariance. Then the local Cauchy theory is able to estimate the variation of the considered quantity in the intervals $[n\tau,(n+1)\tau]$.

The assumptions of the meta-theorem are very flexible. In fact, we will use modified versions of this result in Chapter 4 and Chapter 5 of this dissertation. The two assumptions which can we weakened are the following. The first is that the invariance of the measure μ assumed in (i) can be relaxed to a weaker assumption, only requiring an estimate of $t\mapsto\mu(\phi_t A)$ in terms of $\mu(A)$. This will lead to the use of quasi-invariant measures in Chapter 4. The second assumptions which can be relaxed is (ii) where estimates of the form

$$\mu(\{u_0, \|u_0\|_{H^s} > \lambda\})\lesssim \lambda^{-\gamma}\,,$$

are sufficient to apply the argument of the meta-theorem. We refer to Chapter 4 for an example.

In contrast, is seems difficult to weaken the assumption (iii).

Finally, let us point that the idea of the meta-theorem is so flexible that its core idea was recently used in the context of number theory [Tao19].

When an invariant measure is known, at least at a formal level, one should have access to a local well-posedness theory at the regularity level of the invariant measure in order to apply the meta-theorem. Let us assume that a given equation possesses an invariant measure at regularity H^s. Then there are three cases.

(1) *Very favourable case.* At regularity H^s a local and global well-posedness theory is known. In this case the globalisation argument is still useful. The meta-theorem indeed constructs global solutions for μ-almost any initial data, which do not grow faster than logarithmically, which is

often a much better bound than the deterministic growth bounds, usually exponential (standard local theory) or polynomial.

(2) *Favourable case.* At regularity H^s a good local Cauchy theory is known, but no global theory. Then the meta-theorem constructs global solutions for μ-almost every initial data. This is the case in [Bou94] where it is proven in dimension 1 that the quintic defocusing nonlinear Schrödinger equation is almost-surely globally well-posed on the support of its Gibbs measure, that is in $H^{\frac{1}{2}^-}$, and moreover, $\|u(t)\|_{H^s} \lesssim \log^{\frac{1}{2}}(1+t)$ for all $t > 0$. This results fits in the framework where a local theory is already known on the support of the Gibbs measure. When Bourgain wrote his article, a local theory in H^{0^+} was already known [Bou93], whereas the Gibbs measure has regularity $H^{\frac{1}{2}^-}$.

(3) *Unfavourable case.* At regularity H^s, no deterministic local Cauchy theory is known. The idea is then to try to construct a local theory in a probabilistic fashion, for initial data of the form (2.8). This is the case studied in [Bou96] which proves that (fNLS) in the case $(d, \alpha, \beta) = (2, 1, 0)$ is almost surely globally well-posed on the support of the Gibbs measure, of regularity H^{0^-}. In comparison, the best deterministic local theory is at regularity H^{0^+}.

Remark 2.9. The meta-theorem suggests that in order to apply the argument one readily needs a global flow to be constructed, that is global solutions. In order to circumvent this difficulty one can approximate (fNLS) with finite dimension equations, by truncation in frequencies for example and produce associated global solutions u_N, and invariant measures μ_N under the approximate flows ϕ_N. Then one applies the meta-theorem to the (μ_N, u_N) which gives a growth bound for the u_N, uniformly in N. In order to pass to the limit $N \to \infty$ it remains to see that $(\phi_N, \mu_N) \to (\phi, \mu)$ in a certain sense.

We go back to (fNLS) and recall that $(d, \alpha, \beta) = (3, 2, 1)$. On the one hand, the local Cauchy theory of Lemma 2.3 is accessible in $H^{1^+}(\mathbb{T}^3)$. On the other hand, the formal invariant measure μ whose support is made of functions of the form (2.8) have regularity $H^{\alpha-\frac{d}{2}^-} = H^{\frac{1}{2}^-}$. We can indeed compute for $s \in \mathbb{R}$,

$$\mathbb{E}\left[\|u_0^\omega\|_{H^s}^2\right] = \sum_{n\in\mathbb{Z}^d} \frac{\langle n\rangle^{2s}}{1+|n|^{2\alpha}} \mathbb{E}[|g_n|^2]\,,$$

and since $\mathbb{E}[|g_n|^2] = 1$, convergence holds only for $2s - 2\alpha < -d$. Therefore (fNLS) for $(d, \alpha, \beta) = (3, 2, 1)$ belongs to the case (3) and the meta-theorem cannot be directly applied, a probabilistic local Cauchy theory should first be developed. à la disposition

2.2. Probabilistic local Cauchy theory. We now establish a local theory for random initial data of the form (2.8), that is, we develop a probabilistic local Cauchy theory at regularity H^s for some sufficiently low values of s in order to apply the meta-theorem to (fNLS) and its invariant measure.

Starting in the 70's and mostly during the 90's, the Cauchy problem for (fNLS) has been studied. The deterministic methods relies on Strichartz estimates, which we will present later. We believe that in order to better understand why Strichartz estimate are useful, that is, estimating solutions in $L^p_T L^q_x$, we try to provide intuition for both deterministic and probabilistic theories, lowering the threshold of Lemma 2.3. For the moment we forget about probability and try to lower the s for which one can construct solution to (fNLS) in H^s.

Our starting point is the heuristics suggested by the Duhamel formulation (2.3) for (fNLS), which we recall:

$$u(t) = e^{-it(-\Delta)^\alpha} u_0 - i \int_0^t \frac{e^{-i(t-t')(-\Delta)^\alpha}}{\langle \nabla \rangle^\beta} \left(|u(t')|^2 u(t') \right) \mathrm{d}t' \,.$$

The linear term $u_L(t) := e^{-it(-\Delta)^\alpha} u_0$ is explicit, it is the solution to the linear equation

$$i\partial_t u - (-\Delta)^\alpha u = 0 \,, \tag{fLS}$$

with initial condition $u_L(0) = u_0$. This equation is solvable in H^s for any s. Hence this suggest the decomposition

$$u(t) = u_L(t) + v(t) \,, \tag{2.9}$$

with v defined by

$$v(t) := -i \int_0^t \frac{e^{-i(t-t')(-\Delta)^\alpha}}{\langle \nabla \rangle^\beta} (|u(t')|^2 u(t')) \mathrm{d}t' \,.$$

It is important to understand how such a decomposition is helpful: since $u \in H^s$, we also have $|u|^2 u \in H^s$ when $s > \frac{d}{2}$ (see the proof of Lemma 2.3) so that thanks to $\langle \nabla \rangle^{-\beta}$ we obtain that $v \in H^{s+\beta}$, that is, this terms gains $\beta > 0$ regularity, which we assume from now on.

Remark 2.10. Without the regularisation $\langle \nabla \rangle^{-\beta}$ in (fNLS), v is not *a priori* more regular than H^s and this strategy becomes more difficult to implement.

In (2.9) there is a global and explicit part $u_L \in H^s$ but also a more regular, unknown $v \in H^{s+\beta}$ which should be constructed. If we choose s such that $s + \beta$ is greater than the threshold of (fNLS), given by $\frac{d-\beta}{2}$, that is $s > \frac{d}{2} - \frac{3\beta}{2}$, one expects to construct v using a local Cauchy theory similar to that of Lemma 2.3.

The equation formally satisfied by $v \in H^{s+\beta}$, takes the form $v = \Psi(v)$ where

$$\Psi(v)(t) \coloneqq -i \int_0^t \frac{e^{-i(t-t')(-\Delta)^\alpha}}{\langle \nabla \rangle^\beta} \Big(|u_L + v|^2 (u_L + v) \Big) \, \mathrm{d}t' \,, \tag{2.10}$$

which is slightly different from the setting in Lemma 2.3 because of the term u_L. In our analysis we will see some norm conditions of the type $\|u_L\|_{L^p_T L^q_x} < \infty$ in order to prove that Ψ is a contraction. Note that similarly to Lemma 2.3 we are seeking for an estimation of the form

$$\|\Psi(v)\|_{L^\infty_T H^{s+\beta}} \leqslant C \|v\|^3_{L^\infty_T H^s} \,,$$

from which a local Cauchy theory follows. The estimates in Lemma 2.3 imply

$$\|\Psi(v)\|_{L^\infty_T H^{s+\beta}} \leqslant \||u_L + v|^2 (u_L + v)\|_{L^1_T H^s} \,.$$

We recall that we now that $u_L(t) \in H^s$ and $v(t) \in H^{s+\beta}$.

Our next step is to expand the cubic term and estimate it in $L^1_T H^s$. In the expansion, the terms $|u_L|^2 u_L$ and $|v|^2 v$ appear, along with other terms which can be interpolated between these two terms. The term $|v|^2 v$ is handled using the techniques of Lemma 2.3 and for $|u_L|^2 u_L$ we use product laws in Sobolev spaces so that

$$\|\Psi(v)\|_{L^\infty_T H^{s+\beta}} \lesssim \|u_L\|^2_{L^2_T L^\infty_x} \|u_L\|_{L^\infty_T H^s} + T \|v\|^3_{L^\infty_T H^{s+\beta}} \,.$$

First, $\|u_L\|_{L^\infty_T H^s_x} \leqslant \|u_0\|_{H^s_x}$ is finite because $u_0 \in H^s$. Then the most difficult estimation to obtain seems to be $\|u_L\|_{L^2_T L^\infty_x} < \infty$, which should be estimated in terms of $\|u_0\|_{H^s_x}$.

Using the Sobolev embedding on u_L requires $s > \frac{d}{2}$ to conclude, which will not lead to any improvement on the deterministic theory. We ask the following question: is there a class of initial data $u_0 \in H^s$, $s < \frac{d-\beta}{2}$ for which $u_L \in L^2_T L^\infty$?

Answering such a question is precisely the role of random data in this context. Let us consider a sequence $(u_n)_{n \in \mathbb{Z}^d}$ such that

$$u_0(x) \coloneqq \sum_{n \in \mathbb{Z}^d} u_n e^{in\cdot x} \in H^s(\mathbb{T}^d) \,.$$

We consider identically distributed independent random variables $(X_n)_{n \in \mathbb{Z}^d}$, such that there exists $c > 0$ such that for all $\gamma > 0$ there holds

$$\mathbb{E}[e^{\gamma X_n}] \leqslant e^{c\gamma^2} \,. \tag{2.11}$$

Such X_n are called sub-Gaussian random variables. Then we introduce the *randomisation* attached to u_0, defined by

$$u_0^\omega(x) \coloneqq \sum_{n \in \mathbb{Z}^d} X_n^\omega u_n e^{in\cdot x} \,. \tag{2.12}$$

We readily see that

$$\mathbb{E}\left[\|u_0^\omega\|_{H^s}^2\right] = C\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2s}|u_n|^2 = C\|u_0\|_{H^s}^2 < \infty\,,$$

so that this randomisation has regularity H^s. A more stronger statement is the following.

Lemma 2.11 (Randomisation in H^s)**.** *Consider the randomisation (2.12) where the X_n are moreover identically distributed and satisfy $(X_n)_{n\in\mathbb{Z}^d}$ satisfied $\mathbb{P}(|X_n|>0)>0$. Assume that for any $\varepsilon>0$, there holds $u_0\in H^s\setminus H^{s+\varepsilon}$. Then for almost all ω there holds:*

$$u_0^\omega \in H^s\setminus\bigcup_{\varepsilon>0}H^{s+\varepsilon}\,.$$

A proof of this result can be found in [BT08a], Lemma B.1. This result tells us that the data of the form (2.12) have exactly the same regularity as u_0. It is indeed important that a randomisation procedure does not regularise the data.

The advantage of random data appear when estimating the L^p_x norms of u_0^ω. In particular, for finite p this norm is almost-surely controlled by the L^2 norm of u_0. The key ingredient in this proof is known at least since the 30's. The proof is postponed to Appendix A.

Theorem 2.12 (Kolmogorov, Paley, Zygmund)**.** *Let $(a_n)_{n\geqslant 0}$ be a real sequence and $(X_n)_{n\geqslant 0}$ be identically distributed, independent random variables satisfying (2.11). Then*

$$\mathbb{P}\left(\left|\sum_{n\geqslant 0}a_nX_n\right|>\lambda\right)\leqslant \exp\left(-c\frac{\lambda^2}{\|a_n\|_{\ell^2}^2}\right)\,,$$

and for $r\in[2,\infty)$ there holds

$$\left\|\sum_{n\geqslant 0}a_nX_n\right\|_{L^r_\omega}\lesssim\sqrt{r}\left(\sum_{n\geqslant 0}|a_n|^2\right)^{\frac12}\,.$$

This results allows us to estimate the $L^q_TL^p_x$ norms of

$$e^{-it(-\Delta)^\alpha}u_0^\omega = \sum_{n\in\mathbb{Z}^d}e^{it|n|^{2\alpha}}a_ne^{in\cdot x}X_n\,.$$

Indeed, we use the Minkowski inequality for $r\geqslant p,q$, Theorem 2.12 and $|e^{it|n|^{2\alpha}}a_ne^{in\cdot x}| = |a_n|$ to infer

$$\left\|\sum_{n\in\mathbb{Z}^d}e^{it|n|^{2\alpha}}a_ne^{in\cdot x}X_n\right\|_{L^r_\omega L^q_TL^p_x}\lesssim\sqrt{r}\left\|\left(\sum_n|a_n|^2\right)^{\frac12}\right\|_{L^q_TL^p_x}\,.$$

Applying Parseval's theorem an using that the torus has finite volume leads to

$$\left\| \left(\sum_n |a_n|^2 \right)^{\frac{1}{2}} \right\|_{L^q_T L^p_x} \lesssim T^{\frac{1}{q}} \|u_0\|_{L^2_x} .$$

The Markov inequality then implies the following.

Corollary 2.13 (Random "Strichartz" estimates)**.** *Let $u_0 \in H^s(\mathbb{T}^d)$ and u_0^ω the randomisation defined by (2.12). Let $q, p \in [2, \infty)$. Then outside a set of measure at most e^{-cR^2} there holds*

$$\|u_L^\omega\|_{L^q_T W^{s,p}_x} \leqslant R T^{\frac{1}{q}} \|u_0\|_{H^s} ,$$

where $u_L^\omega(t) = e^{-it(-\Delta)^\alpha} u_0^\omega$.

Remark 2.14*.* One should keep in mind that these considerations are specific to the torus. In the case of eigenfunctions $e_n(x)$ of the Laplace operator on a manifold M, the Minkowski inequality only implies that outside a set of probability at most e^{-cR^2} there holds

$$\left\| e^{it\Delta} u_0 \right\|_{L^q_T L^p_x} \leqslant R T^{\frac{1}{q}} \left(\sum_{n \in \mathbb{Z}^d} |a_n|^2 \|e_n\|^2_{L^p(M)} \right)^{\frac{1}{2}} .$$

Remark 2.15 (Probabilistic decoupling)*.* In order to understand what the point of such random estimates is, consider N independent random variables following a uniform law on $\{-1, 1\}$, denoted by $(\varepsilon_n)_{1 \leqslant n \leqslant N}$. As a corollary of Theorem 2.12 one obtains that outside a set of probability at most e^{-cR^2}, there holds

$$\left| \sum_{n=1}^N \varepsilon_n^\omega a_n \right| \leqslant R \left(\sum_{n=0}^N |a_n|^2 \right)^{\frac{1}{2}} .$$

The comparison with a deterministic estimate is enlightening. In the latter case, the Cauchy-Schwarz inequality writes

$$\left| \sum_{n=1}^N \varepsilon_n^\omega a_n \right| \leqslant N^{\frac{1}{2}} \left(\sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}} ,$$

hence the probabilistic decoupling effects is the absence of the $N^{\frac{1}{2}}$ factor.

We are now ready to state a probabilistic Cauchy theory for (fNLS).

Proposition 2.2. *Let $s > \max\{\frac{d}{2} - \frac{3\beta}{2}, 0\}$ and $u_0 \in H^s(\mathbb{T}^d)$. We consider initial data u_0^ω of the form (2.12). For almost any u_0^ω, there exists $T = T(\omega) > 0$ such that there exists a unique solution u to (fNLS) in the space*

$$u_L^\omega + \mathcal{C}^0([0, T), H_x^{s+\beta}) ,$$

where $u_L^\omega(t) = e^{-it(-\Delta)^\alpha} u_0^\omega$.

Proof. We seek for a solution u to (fNLS) writing $u(t) = u_L + v$, where v solves $\Psi(v) = v$ and Ψ is defined by (2.10). We get

$$\|\Psi v\|_{L^\infty_T H^{s+\beta}_x} \leqslant \|u_L\|^2_{L^4_T L^\infty_x} \|u_L\|_{L^2_T H^s_x} + T\|v\|^3_{L^\infty_T H^{s+\beta}_x} .$$

Corollary 2.13 implies that outside a set of probability less than e^{-cR^2} there holds

$$\|u_L\|_{L^2_T H^s_x} \leqslant RT^{\frac{1}{2}} \|u_0\|_{H^s_x} .$$

Note that the corollary cannot be applied to the $L^4_T L^\infty_x$ norm. We circumvent this difficulty by an application of the Sobolev embedding for $\varepsilon \in]0, s]$ such that $u_0 \in H^\varepsilon$, leading to

$$\|u_L\|_{L^4_T L^\infty_x} \lesssim \|u_L\|_{L^4_T W^{\varepsilon,p(\varepsilon)}_x} ,$$

for a $p(\varepsilon) \in [2, \infty[$ such that $\varepsilon p(\varepsilon) > d$. Then Corollary 2.13 applies. More precisely we let $R > 0$, and introduce

$$\Omega_R := \{\|u_L\|_{L^4_T L^\infty_x} \lesssim RT^{\frac{1}{4}} \|u_0\|_{H^s_x}\} \cap \{\|u_L\|_{L^2_T H^s} \leqslant RT^{\frac{1}{2}} \|u_0\|_{H^s_x}\} ,$$

which satisfies $\mathbb{P}(\Omega \setminus \Omega_R) \leqslant 2e^{-cR^2}$. Moreover for all $\omega \in \Omega_R$ we have

$$\|\Psi(v)\|_{L^\infty_T H^{s+\beta}_x} \lesssim T \left(R^3 + \|v\|^3_{H^{s+\beta}_x} \right) .$$

Hence $\|v\|_{L^\infty_T H^{s+\beta}} \leqslant R$ and for $T = \frac{1}{CR^2}$ there holds

$$\|\Psi(v)\|_{L^\infty_T H^{s+\beta}_x} \leqslant \frac{1}{2} \|v\|_{L^\infty_T H^{s+\beta}_x} ,$$

sot that for any $\omega \in \Omega_R$ we are able to solve (fNLS) up to time $T \sim \frac{1}{R^2}$. Upgrading this local theory for $\omega \in \Omega_R$ to an almost-sure local theory is obtained by considering $\omega \in \bigcup_{R \in \mathbb{N}^*} \Omega_R$. ☐

In conclusion we have constructed solutions in H^s for s lower than $\frac{d}{2} - \frac{\beta}{2}$. The probabilistic method did allow us to gain β derivatives. The key ingredient in the above proof was that for $u_0 \in H^\varepsilon$, we can deduce that almost-surely $u^\omega_L \in L^q_T L^\infty$, where a deterministic argument would have cost $\frac{d}{2}$ derivatives and not ε. Deterministic estimates of the form $u_L \in L^q_T L^p_x$ can be derived, such estimates are referred to as Strichartz estimates and are verified only for certain values of (p, q), whereas in the probabilistic framework any finite p, q are available.

Finally, combining the analysis of this subsection and the previous one we have the following result.

Theorem 2.16 (Global solutions to (fNLS))**.** *Consider* (fNLS) *with* $(d, \alpha, \beta) = (3, 2, 1)$. *Then for almost every initial data* u^ω_0 *of the form* (2.8) *there exists a unique global solution to* (fNLS) *in the space*

$$e^{-it(-\Delta)^\alpha} u^\omega_0 + \mathcal{C}^0(\mathbb{R}, H^{1^+})$$

such that for all $s < \frac{1}{2}$ and $t > 0$ there holds

$$\|u(t)\|_{H^s} \lesssim \log^{\frac{1}{2}}(1+t)\,.$$

Sketch of the proof. This is essentially a verification of the assumption of the meta-theorem for the measure μ, formally constructed by (2.6) and invariant for (fNLS). Hence (i) and (ii) are met. finally, μ is supported in $H^{\alpha-\frac{d}{2}^-} = H^{\frac{1}{2}^-}$ and the local Cauchy theory of Proposition 2.2 applies and (iii) is satisfied. □

2.3. Defects of Gibbs measures. In Section 2.1 we managed to construct invariant measures following the Hamiltonian method. From such invariant measures the meta-theorem is then able to construct global solutions with very slow growth. This method suffers from a limitation, which is that the Gibbs measure nor its regularity could be chosen. The Gibbs measures are indeed rare and their regularity is directly computable using the conservation law from which the Gibbs measure is derived. For example, in the case of (fNLS), the Gibbs measure has regularity $H^{\alpha-\frac{d}{2}^-}(\mathbb{T}^d)$. One also sees that this regularity deteriorates when the dimension increases, which makes it difficult to construct a probabilistic Cauchy theory when the dimension is large.

There is also an obstruction to the existence of invariant measures in other settings. In Section 2.1 we heavily relied on the structure of the torus, allowing to expand in Fourier series. What happens for (fNLS) posed on $\mathbb{R}^d$? In this case it is the dispersion which prevents invariant measures to exist in L^2.

Proposition 2.3 (Proposition 3.2 in [BT20])**.** *Let μ be an invariant probability measure in $L^2(\mathbb{R})$, under the flow ϕ_t of*

$$i\partial_t u + \partial_x^2 u = |u|^4 u\,.$$

Then $\mu = \delta_0$.

Proof. This equation is globally well-posed in L^2 and solutions *scatter*, that is there exists $u^+ \in L^2$ such that $\|u(t) - e^{it\Delta}u^+\|_{L^2} \underset{t\to\infty}{\longrightarrow} 0$. This is proved in [Dod16a]. Let χ be a smooth and compactly supported function. For all t there holds

$$\int_{L^2} \frac{\|\chi u\|_{L^1}}{1+\|u\|_{L^2}}\,\mathrm{d}\mu = \int_{L^2} \frac{\|\chi\phi_t u\|_{L^1}}{1+\|u\|_{L^2}}\,\mathrm{d}\mu(u)\,,$$

where invariance of μ was used. By Cauchy-Schwarz and conservation of the L^2 norm we have

$$\frac{\|\chi\phi_t u\|_{L^1}}{1+\|u\|_{L^2}} \leqslant \frac{\|\chi\|_{L^2}\|\phi_t u\|_{L^2}}{1+\|u\|_{L^2}} \lesssim 1\,.$$

With the information that $\|\chi\phi_t u\|_{L^1} \leqslant \|\chi e^{it\Delta}u^+\|_{L^1} + \|\chi\|_{L^2}\|\phi_t u - e^{it\Delta}u^+\|_{L^2} \to 0$ obtained by scattering and dispersion, the dominated convergence theorem implies

$$\int_{L^2} \frac{\|\chi u\|_{L^1}}{1+\|u\|_{L^2}}\,\mathrm{d}\mu = 0\,,$$

hence $u = 0$, μ-almost surely. □

Constructing invariant measures in $\mathbb{R}^d$ is then not always possible, which makes the tasks of constructing such measures more difficult. If one only wants to construct local solutions, it is still possible to use the method of Section 2.2.

In order to do so, one needs to introduce a randomisation in the space $\mathbb{R}^d$. The random series introduced in the case of the torus (2.12) is a randomisation at a unit scale of frequencies. Therefore one should introduce a similar randomisation in the Euclidean space. Let $u_0 \in H^s(\mathbb{R}^d)$ and let $\sum_{n\in\mathbb{Z}^d} \chi(\xi - n) = 1$ be a partition of unity where χ is a smooth compactly supported function. Let also $(X_n)_{n\in\mathbb{Z}^d}$ be identically distributed independent subgaussian random variables satisfying (2.11). The randomisation u_0^ω is defined in the Fourier space by

$$\mathcal{F}_{x\to\xi}(u_0^\omega)(\xi) = \sum_{n\in\mathbb{Z}^d} X_n^\omega \chi(\xi - n)\hat{u}_0(\xi)\,,$$

which we write as

$$u_0^\omega = \sum_{n\in\mathbb{Z}^d} X_n^\omega \chi(D-n) u_0\,. \tag{2.13}$$

Such a randomisation is called *Wiener randomisation*, and used for example in [BOP15b]. We derive probabilistic Strichartz estimates just as in the case of the torus. The Bernstein inequality applied to a function that is frequency localised in the cube cnetered at n of length 1 ensures that for $p \geqslant 2$,

$$\|\chi(D-n)u\|_{L^p(\mathbb{R}^d)} \lesssim \|u\|_{L^2_x(\mathbb{R}^d)}\,.$$

Combining this remark with the proof of Theorem 2.12 one obtains the following.

Theorem 2.17. *Let $u_0 \in H^s(\mathbb{R}^d)$ and $p, q \in [2,\infty)$. Let u_0^ω be its randomisation given by (2.13). Then, outside a set or probability at most e^{-cR^2} there holds*

(i) $\|u_0^\omega\|_{L^p_x} \leqslant R\|u_0\|_{L^2}$.

(ii) $\|e^{it\Delta}u_0^\omega\|_{L^q_T L^p_x} \leqslant RT^{\frac{1}{q}}\|u_0\|^2_{L^2_x}$.

Let us mention that in the case of $\mathbb{R}^d$ the dispersion phenomenon allows to prove deterministic Strichartz estimates. In order to write these estimates, introduce for any s, the space

$$S^s := \bigcap_{(q,r)\text{ admissible}} L^q_T W^{s,r} \text{ and } Y^s := \sum_{(q,r)\text{ admissible}} L^{q'}_T W^{s,r'}_x\,,$$

where (q,r) is said to be admissible if $(q,r,d) \neq (2,\infty,2)$ and $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}$.

Theorem 2.18 (Dispersion and Strichartz estimates)**.** *Let $s \geqslant 0$ and $u_0 \in H^s(\mathbb{R}^d)$. Then*

(*i*) *Dispersion: for $p \geqslant 2$ there holds $\|e^{it\Delta}u_0\|_{L^p_x} \lesssim t^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_{L^{p'}}$.*

(*ii*) *Homogeneous Strichartz: there holds $\|e^{it\Delta}u_0\|_{S^s} \lesssim \|u_0\|_{H^s}$*

(*iii*) *Non-homogeneous Strichartz: there holds $\| \int_0^t e^{i(t-t')\Delta}(F(u(t')))\,\mathrm{d}t'\|_{S^s} \lesssim \|F(u)\|_{Y^s}$.*

We observe that the admissibility condition on (q,r) limits the scope of the deterministic Strichartz estimates, compared to the probabilistic version.

Thanks to Strichartz estimates it is possible to study the Cauchy problem for

$$\left\{\begin{array}{rcl} i\partial_t u + \Delta u &=& |u|^2 u \\ u(0) &=& u_0 \in H^s(\mathbb{R}^4)\,. \end{array}\right. \tag{2.14}$$

Theorem 2.19 (Deterministic local theory)**.** *Let $s>1$. Then (2.14) is uniformly locally well-posed in $H^s(\mathbb{R}^4)$.*

Using the probabilistic Strichartz estimates and bilinear estimations, the following result, which proof is sketched in Appendix B can be obtained.

Theorem 2.20 (Benyi, Oh and Pocovnicu [BOP15b, BOP15a])**.** *Let $s > \frac{3}{5}$ and $u_0 \in H^s(\mathbb{R}^3)$. Let u_0^ω be its randomisation given by (2.13). Let $\sigma > 1$. Then for almost every ω, there exists a unique solution to (2.14) in the space*

$$u_L + \mathcal{C}^0([0,T], H^\sigma(\mathbb{R}^d))\,,$$

where $u_L(t) = e^{it\Delta}u_0^\omega$.

Other randomisations are possible. For example, rather than the Wiener randomisation (2.13), one can also randomise in space. Such a randomisation, said to be *microlocal* is defined as follows. For $u_0 \in L^2(\mathbb{R}^d)$ we write

$$u_0(x) = \sum_{k,n\in\mathbb{Z}^d} \psi(x-k)\chi(D-n)u\,,$$

with χ and ψ some smooth compactly supported functions satisfying $\mathbf{1}_{|\cdot|\leqslant 1} \leqslant \chi,\psi \leqslant \mathbf{1}_{|\cdot|\leqslant 2}$. The associated randomisation is then

$$u_0^\omega(x) = \sum_{k,n\in\mathbb{Z}^d} X_{k,n}^\omega \psi(x-k)\chi(D-n)u\,,$$

where the $(X_{k,n})_{k,n\in\mathbb{Z}^d}$ are identically distributed independent subgaussian random variables.

We refer to [Bri18a] and [BK19] for the use of such a randomisation.

3. Globalisation using an energy method

This section introduces Chapter 3 of this dissertation which is concerned by the random data global Cauchy problem for nonlinear wave equations. The work presented in this section is the content of an article published in *Journal of Differential Equations* [Lat21].

We consider the nonlinear wave equation on a domain U which is $\mathbb{R}^3$ or $\mathbb{T}^3$ and which writes

$$(\mathrm{NLW}_p) \qquad \begin{cases} \partial_t^2 u - \Delta u + |u|^{p-1}u &= 0 \\ (u(0), \partial_t u(0)) &= (u_0, u_1) \in H^s(U) \times H^{s-1}(U)\,. \end{cases}$$

The space $H^s(U) \times H^{s-1}(U)$ will be denoted $\mathcal{H}^s$ in the following.

A solution u to (NLW_p) formally conserves the quantity

$$\mathcal{E}(u(t)) := \int_U \left(\frac{|\partial_t u(t)|^2}{2} + \frac{|\nabla u(t)|^2}{2} + \frac{|u(t)|^{p+1}}{p+1} \right) \mathrm{d}x\,,$$

which we refer to as an "energy", and has regularity H^1.

3.1. Local and global Cauchy problem for (NLW_p). The works pertaining to the well-posedness theory of (NLW_p) are too numerous to be all mentioned. We limit ourselves to some key works which are directly related to our main theorem.

3.1.1. *Deterministic theory.* The local Cauchy theory for (NLW_p) is well understood since the 90's [LS95, GV95, Kap90]. In particular, as soon as $s > \frac{3}{2} - \frac{2}{p-1}$, ($\mathrm{NLW}_p$) is uniformly locally well-posed in $\mathcal{H}^s$. Such a local theory is built on Strichartz estimates. This local theory allows us to construct local solutions in the energy space $\mathcal{H}^1$ for $p < 5$, which are then global solutions. When $p = 5$ the Cauchy problem is said to be "energy-critical" and the a global theory in $\mathcal{H}^1$ is more difficult to obtain [Kap94]. For $p > 5$, no global theory is known in $\mathcal{H}^1$, the equation is said to be energy-supercritical.

In the case $p = 3$, the Cauchy is locally well-posed in $\mathcal{H}^s$ for $s \geqslant \frac{1}{2}$, and globally well-posed in $\mathcal{H}^1$. In [GP03, KPV00, BC06], the authors extend this result and construct a global Cauchy theory in $\mathcal{H}^s$ for $s > \frac{3}{4}$ (and $s > \frac{1}{2}$ for radial initial data [Dod19]). On the opposite direction, for $s \in (0, \frac{1}{2})$, the Cauchy problem is ill-posed [BT08a].

Note that for all $p \geqslant 1$ one can construct global weak solutions in the energy space, $\mathcal{H}^1$. This is the content of the following result.

Theorem 2.21 (Strauss, [Str70])**.** *Let f be a smooth real function and F and anti-derivative of f. Assume that $F(v) \gtrsim -|v|^2$ and $\frac{|F(u)|}{|f(u)|} \to \infty$ as $|u| \to \infty$.*

Let $(u_0, u_1) \in \mathcal{H}^1(\mathbb{R}^3)$ *of finite energy, that is*

$$E(u_0, u_1) := \int_{\mathbb{R}^3} \left(\frac{|u_1|^2}{2} + \frac{|\nabla u_0|^2}{2} + F(u_0) \right) \mathrm{d}x < \infty\,.$$

Then the equation

$$\left\{ \begin{array}{rcl} \partial_t^2 u - \Delta u + f(u) & = & 0 \\ (u(0), \partial_t u(0)) & = & (u_0, u_1) \in \mathcal{H}^1(\mathbb{R}^3)\,, \end{array} \right. \tag{2.15}$$

admits a global weak solution, that is there exists a solution u *in the sense of distributions,* $u : \mathbb{R} \to \mathcal{H}^1(\mathbb{R}^3)$ *weakly continuous in time and such that*

$$E(u(t), \partial_t u(t)) \leqslant E(u_0, u_1) \textit{ for all } t \geqslant 0\,.$$

3.1.2. *Probabilistic improvements.* The Cauchy problem for (NLW_p) with random initial data has been studied by Burq and Tzvetkov, leading to a local [BT08a] and global [BT08b] theory. In [BT14] they construct unique global solutions in $\mathcal{H}^0$ when $p = 3$.

The case $p = 5$ is more difficult because it is energy-critical. Oh and Pocovnicu studied this problem in [Poc17, OP16] and constructed a global theory in this case. In particular, when $d = 3$ and $p = 5$ they constructed unique global solutions in $\mathcal{H}^s$ for $s > \frac{1}{2}$.

When $p \in (3, 5)$, one expects a probabilistic Cauchy theory in $\mathcal{H}^s$ for $s > \frac{p-3}{p-1}$. After the results of Lührmann and Mendelson [LM14, LM16] the results when $s > \frac{p-3}{p-1}$ is obtained by Sun and Xia [SX16].

3.2. Contribution. Our main goal is to complete the above results of Burq and Tzvetkov in the case $p = 3$, Oh and Pocovnicu in the case $p = 5$ and Sun and Xia in the intermediate case $p \in (3, 5)$. Since for $p > 5$ the energy is a supercritical quantity for (NLW_p) one does not expect to easily construct strong solutions. We instead construct global weak solutions for random initial data in the space $\mathcal{H}^s$ for some $s < 1$, providing a probabilistic extension of Theorem 2.21.

More precisely, we recall that the probabilistic method of Section 2.2 consists of the following steps. We seek for a solution $u \in \mathcal{C}^0(\mathbb{R}, \mathcal{H}^s)$ to (NLW_p) of the form $u = u_L + v$ with u_L the global solution to

$$\left\{ \begin{array}{rcl} \partial_t^2 u - \Delta u & = & 0 \\ (u(0), \partial_t u(0)) & = & (u_0, u_1) \in \mathcal{H}^s\,, \end{array} \right. \tag{2.16}$$

and $v \in \mathcal{C}^0(\mathbb{R}, \mathcal{H}^1)$ which formally solves

$$\left\{ \begin{array}{rcl} \partial_t^2 v - \Delta v & = & -|u_L + v|^{p-1}(u_L + v) \\ (v(0), \partial_t v(0)) & = & 0\,. \end{array} \right. \tag{2.17}$$

We seek for v at regularity level $\mathcal{H}^1$ since we want to use some energy conservation for (NLW_p), known to control the $\mathcal{H}^1$ norm.

For an initial data $(u_0, u_1) \in \mathcal{H}^s$ we introduce the associated randomisation respectively constructed by (2.13) in the case $\mathbb{R}^3$ and (2.12) in the case $\mathbb{T}^3$. Note that the induced measure $\mu_{(u_0,u_1)}$ by such a randomisation is a measure on $\mathcal{H}^s$. Let

$$\mathcal{M}^s(U) \coloneqq \bigcup_{(u_0,u_1)\in\mathcal{H}^s(U)} \mu_{(u_0,u_1)}\,,$$

the set of these measures.

Our main result is the following.

Theorem A (L. [Lat20a])**.** *Let $p > 5$ and $s > \frac{p-3}{p-1}$. Let U, which stands for $\mathbb{R}^3$ or $\mathbb{T}^3$. Let $\mu \in \mathcal{M}^s(U)$. Then for μ-almost every $(u_0, u_1) \in \mathcal{H}^s(U)$ there exists a global solution $u \in \mathcal{C}^0(\mathbb{R}, \mathcal{H}^s(U))$ to (NLW$_p$) in a weak sense, that is for all $\varphi \in \mathcal{C}^2(\mathbb{R} \times U)$ there holds,*

$$\int_{\mathbb{R}\times U} \Big(\partial_t v \partial_t \varphi - \nabla v \cdot \nabla \varphi - (u_L + v)|u_L + v|^{p-1}\varphi(t)\Big)\, \mathrm{d}x\, \mathrm{d}t = 0\,,$$

where u_L solves (2.16) and v is a solution to (2.17) such that $u = u_L + v$. Moreover v is such that for all $T > 0$ we have

$$v \in H^1(]-T, T[\times U) \cap L^{p+1}(]-T, T[\times U)\,.$$

Remark 2.22. The weak solution constructed by this theorem has the same regularity as the strong solutions. Since our theorem does not provide uniqueness we still write weak solutions.

Our second result concerns the case $p \in [3, 5)$. We know that a global theory is known in $\mathcal{H}^s$ for $s > \frac{p-3}{p-1}$, and that in the special case $p = 3$, the regularity threshold $s = 0$ is also addressed in [BT14]. We generalise the argument to the case $p \in (3, 5)$.

Theorem B (L. [Lat21])**.** *Let $p \in (3, 5)$, $s \coloneqq \frac{p-3}{p-1}$ and U standing for $\mathbb{R}^3$ or $\mathbb{T}^3$. Let $\mu \in \mathcal{M}^s$. Then for μ-almost every initial data $(u_0, u_1) \in \mathcal{H}^s(U)$, there exists a unique solution to (NLW$_p$) in the space $u_L + \mathcal{C}^0(\mathbb{R}, \mathcal{H}^1)$ where u_L solves (2.16).*

The ideas of the proof of Theorem B are given in the introduction of Chapter 3. The end of this section focuses on explaining the main idea of the proof of Theorem A. To start with, we assume that the local Cauchy problem has been solved in the form $u(t) = u_L^\omega(t) + v(t)$ where u_L^ω solves (2.16) with initial data (u_0^ω, u_1^ω) and v solves (2.17) in $\mathcal{C}^0([-T, T], \mathcal{H}^1)$. In order to prove that u is a global solution, and since u_L is global by definition, it boils down to proving that v is a global solution to (2.17). To this end, since v has regularity $\mathcal{H}^1$, which is the regularity of the energy conservation for (NLW$_p$), one might want to use the energy. However, v does not satisfy the same equation as u and does not conserves the energy. We use a modified energy method.

Remark 2.23. Such a problem is typical of the Burq-Tzvetkov method, allowing us to construct solutions in $\mathbb{R}^d$ or $\mathbb{T}^d$ (and even other manifolds) at various regularity levels, not only the regularity prescribed by a Gibbs measure. This shows the flexibility of this method, the counterpart being that extending local solutions to global solutions becomes much more difficult. This is in sharp contrast to the globalisation argument of the meta-theorem of Section 2.1.

We introduce the modified energy associated to v and defined by

$$E(t) := \frac{\|\partial_t v(t)\|_{L^2}^2}{2} + \frac{\|\nabla v(t)\|_{L^2}^2}{2} + \frac{\|v(t)\|_{L^{p+1}}^{p+1}}{p+1},$$

whose evolution remains to be studied. A global control of E would imply a control of the $\mathcal{H}^1$ norm and would be sufficient to prove that v is global. We start by computing

$$E'(t) = -\int_{\mathbb{R}^3} \partial_t v(t) \left(|u_L^\omega(t) + v(t)|^{p-1}(u_L^\omega(t) + v(t)) - |v(t)|^{p-1} v(t) \right) \mathrm{d}x\,.$$

At this stage we recall that $(u_0, u_1) \in \mathcal{H}^s$, so that thanks to probabilistic Strichartz estimates given in Corollary 2.13, (easily adapted to our setting) there holds that almost-surely, $u_L^\omega \in L_T^q W^{s,p}$ for any finite p and q. Heuristically, one can even run the computations with $u_L^\omega \in L_T^q W^{s,\infty}$ for all q. The question which we want to answer is the following: for which values of $s < 1$, and corresponding values of p does one have $E'(t) \lesssim E(t)$?

The expression of $E'(t)$ can be simplified as

$$|u_L^\omega + v|^{p-1}(u_L^\omega + v) \simeq |v|^{p-1} v + u_L^\omega v^{p-1} + \{\text{lower order terms}\}\,,$$

The term $|v|^{p-1}v$ in E' is cancelled, so that the next term zv^{p-1} appears as the principal term, as the more nonlinear in v. Therefore we expect

$$E'(t) \simeq \int_{\mathbb{R}^3} \partial_t v(t) u_L^\omega(t) v(t)^{p-1}\,\mathrm{d}x\,.$$

First attempt. In addition to the bound $u_L \in W^{s,\infty}$, one can use the bounds of $\partial_t v \in L^2$ and $v^{p-1} \in L^{\frac{p+1}{p-1}}$ which are controlled by some powers of the energy E. After an application of the Hölder inequality and the definition of E we have,

$$\begin{aligned} E'(t) &\lesssim \|\partial_t v\|_{L^2} \|u_L^\omega(t)\|_{L^\infty} \|v(t)\|_{L^{p+1}}^{p-1} \\ &\lesssim E(t')^{\frac{1}{2}+\frac{p-1}{p+1}}\,. \end{aligned}$$

Remark that $\frac{1}{2} + \frac{p-1}{p+1} \leqslant 1$ as soon as $p \leqslant 3$. Since we have only used that $u_L \in L_x^\infty$, which is accessible (almost-surely) as soon as $s > 0$. This is what led Burq and Tzvetkov to their result [BT14].

Second attempt. When s is larger then the previous estimate does not use the information $u_L \in W^{s,\infty}$ optimally. Moreover the analysis above is limited to $p \leqslant 3$. In order to go beyond we transfer derivatives on u_L^ω using integration by parts in time. Time derivatives are then converted into spatial derivatives since $\partial_t u_L^\omega \simeq |\nabla| u_L^\omega$. Therefore we have

$$\begin{aligned} E(t) &\simeq \int_0^t \int_{\mathbb{R}^3} \partial_t (v(t')^p) u_L^\omega(t') \,\mathrm{d}t' \\ &\simeq \int_{\mathbb{R}^3} \int_0^t v(t')^p \partial_t u_L^\omega(t') \,\mathrm{d}t' \,\mathrm{d}x \\ &\simeq \int_{\mathbb{R}^3} \int_0^t v(t')^p |\nabla| u_L^\omega(t') \,\mathrm{d}t' \,\mathrm{d}x \,, \end{aligned}$$

where boundary terms in the integration by parts have been neglected. Since we want to bring s space derivatives on u_L^ω, we would write $|\nabla| = |\nabla|^{1-s}|\nabla|^s$ and integrate by parts with $|\nabla|^{1-s}$, which leads to

$$E(t) \simeq \int_0^t \int_{\mathbb{R}^3} |\nabla|^s u_L^\omega |\nabla|^{1-s} v v^{p-1} \,\mathrm{d}x \,\mathrm{d}t' \,.$$

This was observed by Oh and Pocovnicu in the case $s = 1/2$ and $p = 5$. Then we put $v \in L^{p+1}$, $\nabla^s z \in L^\infty$ and $\nabla^{1-s} v \in L^{\frac{p+1}{2}}$, term which requires a Gagliardo-Nirenberg inequality to be estimated by

$$\|\nabla^{1-s} v\|_{L^{\frac{p+1}{2}}} \lesssim \|\nabla v\|_{L^2}^{1-\alpha} \|v\|_{L^{p+1}}^{\alpha} \lesssim E(t)^{\frac{1-\alpha}{2} + \frac{\alpha}{p+1}} \,,$$

and valid as soon as $\frac{2}{p+1} - \frac{1-s}{3} = \frac{1-\alpha}{6} + \frac{\alpha}{p+1}$ and $s \geqslant \alpha$. One finds $\alpha = \frac{p-3}{p-1}$ so that

$$E(t) \lesssim \|u_L^\omega\|_{W^{s,\infty}} \int_0^t E(t') \,\mathrm{d}t' \,,$$

which by Grönwall implies a global exponential bound on E. This argument is not limited in p. We refer to Chapter 3 of this dissertation for the details in the case $p > 5$.

3.3. Perspectives. The construction of global solutions, continuous in time with values in H^s, $s < 1$ and given by Theorem A is not fully satisfactory, since such a construction does not give any information on the possible uniqueness of the constructed solutions.

We recall that in a similar fashion, it is possible to prove that the Cauchy problem for (NLW$_p$) is uniformly locally well-posed in H^s for $s > \frac{3}{2} - \frac{2}{p-1}$, using Strichartz estimates. A typical energy-supercritical case is given by $p = 7$, for which:

(1) Global solution, not necessarily unique are constructed in $\mathcal{C}_t H^s$ for $s > \frac{2}{3}$ thanks to Theorem A.
(2) Unique local solutions exist in $\mathcal{C}_t H^s$ for $s > \frac{7}{6}$.

A long-term question is the following.

Question 2.1. *Is it possible to construct global unique solutions to* (NLW$_p$) *in* H^1 *when* $p = 7$*?*

In order to tackle such a question, a first step could be to study the relation between the weak and strong notions of solutions.

(1) First, at regularity H^1 one can try to construct a probabilistic local Cauchy theory: we seek for a solution $u(t) = u_L(t) + v(t)$, where $v(t)$ is more regular than u_L. For $u_0 \in H^1$ there holds $u_L \in H^1$ so that $v \in H^2$ and thus the local Cauchy problem could be solved in $u_L + \mathcal{C}^0([0,T], H^{\frac{7}{6}^+}(\mathbb{R}^3))$ for random initial data.

(2) One could then study weak-strong uniqueness. More precisely, if u_S is the solution constructed in (1) on $[0,T]$ and u_W is given by Theorem A (or even Theorem 2.21) with the same initial condition, one could try to prove that $u_S(t) = u_W(t)$ for all $t \in [0,T]$.

Another question is that of the optimality of Theorem B for $p \in [3,5]$.

Question 2.2. *When $p \in [3,5]$ and $s < \frac{p-3}{p-1}$, is it true that the Cauchy problem for* (NLW_p) *is ill-posed on a set of full probability?*

This question has been studied in [ST20] but in a slightly different framework. The authors indeed prove that for $p \in [3,5]$ and values of s in $[\max\{0, \frac{3}{2} - \frac{2}{p-2}\}, \frac{3}{2} - \frac{2}{p-1}]$ the Cauchy problem is locally ill-posed for a dense set of initial data.

It is possible that this set has zero measure. Thus the question remains entirely open.

4. Scattering for the nonlinear Schrödinger equation in L^2

This sections introduces Chapter 4 of this dissertation and deals with deals with *scattering* for nonlinear Schrödinger equations at weak regularity. The work presented in this section is the content of a preprint [Lat20a].

In the following we consider the nonlinear Schrödinger equation defined by

$$\begin{cases} i\partial_t u + \Delta u &= |u|^{p-1}u \\ u(0) &= u_0 \in H^s(\mathbb{R}^d)\,. \end{cases} \tag{2.18}$$

Before explaining some recent works related to *scattering* of solutions to (2.18) with random initial data, we explain some basic results in the scattering theory.

4.1. General method of scattering. In this subsection we suppose that a global Cauchy theory for (2.18) is already known. The question that we ask is the description of the long-time behaviour. In particular, is it true that the solutions are closed to linear solutions? A first start is to write (2.18) as

$$u(t) = e^{it\Delta}u_0 - \int_0^t e^{i(t-t')\Delta}\left(|u|^{p-1}u\right)\,\mathrm{d}t'\,,$$

so that we readily identify a linear part of the solution, given by $u_L(t) = e^{it\Delta}u_0$ reducing the asymptotic behaviour to the study of

$$-i\int_0^t e^{i(t-t')\Delta}(|u|^{p-1}u)\,\mathrm{d}t' = -ie^{it\Delta}\int_0^t e^{-it'\Delta}(|u|^{p-1}u)\,\mathrm{d}t'\,,$$

so that

$$\|u(t) - e^{it\Delta}u_0\|_{H^s} \lesssim \left\|\int_0^t e^{-it'\Delta}|u(t')|^{p-1}u(t')\,\mathrm{d}t'\right\|_{H^s}. \tag{2.19}$$

If $\int_0^t e^{-it'\Delta}|u(t')|^{p-1}u(t')\,\mathrm{d}t'$ converges in H^s as $t\to\infty$, then there exists $u_+\in H^s$ such that $e^{-it\Delta}u(t)-u_0 \underset{t\to\infty}{\longrightarrow} u_+$, which can be written as $u(t) = e^{it\Delta}u_0+v(t)$ where $v(t)\simeq e^{it\Delta}u_+$ as $t\to\infty$, that is, u exhibits a linear behaviour in long time. The matter boils down to finding conditions on which (2.19) converges.

Remark 2.24. The scattering behaviour should be understood as the first non "fully linear" behaviour. Indeed, one can have $e^{-it\Delta}u(t)-u_0\to 0$ in H^s, which forms a fully linear approximation. of the equation. In the case of scattering, the limiting behaviour is linear but we authorise u_0 to be replaced by some u_0+u_+, that is $e^{-it\Delta}u(t)-(u_0+u_+) \underset{t\to\infty}{\longrightarrow} 0$ in H^s.

A first condition for (2.19) to converge in H^s is the absolute convergence of

$$\int_0^\infty e^{i(t-t')\Delta}|u(t')|^{p-1}u(t')\,\mathrm{d}t'\,,$$

in H^s, that is

$$\int_0^\infty \||u(t)|^{p-1}u(t)\|_{H^s_x}\,\mathrm{d}t < \infty\,.$$

A less brutal condition can be derived using the non-homogeneous Strichartz estimates from Theorem 2.18, (*iii*),

$$\left\|\int_0^t e^{i(t-t')\Delta}|u(t')|^{p-1}u(t')\,\mathrm{d}t'\right\|_{L^\infty_t H^s_x} \lesssim \||u|^{p-1}u\|_{Y^s(\mathbb{R})}\,.$$

Such an approach is used in most of the deterministic works. Some key step in such works is the reduction of scattering to a control of a global in time Strichartz norm.

Lemma 2.25. *Let $d=3$, $p=3$ and $q\in[\frac{10}{3},10]$. The global control $\|u\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^3)}<\infty$ for solutions to (2.18) implies scattering in H^1.*

It is the obtaining of a global bound for Strichartz norm which is difficult. The Morawetz estimates sometimes offer control of Strichartz norms needed to prove scattering. In particular, in a series of work culminating with [CKS+10] global existence and scattering in H^1 are proven for (2.18) when $d=3$ and $p=5$. We refer to [Tao06] for an introduction to the methods.

For the scattering at low regularity, in L^2, we gather some results in the case $d \geqslant 2$.

Theorem 2.26 (Scattering in L^2)**.** *Let $p \geqslant 1$ and $d \geqslant 2$. Then:*

(*i*) *For $p \in \left[1, 1+\frac{4}{d}\right]$ the equation (2.18) is globally well-posed in $L^2(\mathbb{R}^d)$ and if $p > 1+\frac{4}{d}$ the Cauchy problem is ill-posed.*

(*ii*) *For $p \leqslant 1+\frac{2}{d}$ and all $u_0 \in L^2$, the solutions do not scatter in L^2.*

(*iii*) *For $d \geqslant 2$, $p \in \left(1+\frac{2}{d}, 1+\frac{4}{d-2}\right)$ and initial data in H^1, scattering holds in L^2.*

(*iv*) *For $d \geqslant 2$, $p = 1+\frac{4}{d}$ and initial data in L^2, scattering holds in L^2.*

Part (i) is the standard local Cauchy theory for $p < 1+\frac{4}{d}$, see for example [Tao06], Chapter 3. Global well-posedness follows from the conservation of the L^2 norm. In the critical case we refer to [Dod16a] and in the case $p > 1+\frac{4}{d}$ to [CCT03, AC09]. (ii) is proven in [Bar84], (iii) in [TY84] and (iv) in [TVZ07]. For dimension 2 we refer to [Dod16a, Dod16b].

4.2. Probabilistic works. One can distinguish two types of works pertaining to the theory of scattering for (2.18) with random initial data.

The works of the first type study the scattering at a regularity H^1 and extend the deterministic results. For example, in a series of works among which one can find the work of Killip, Murphy and Visan [KMV19], Dodson, Lührmann and Mendelson [DLM19], scattering for energy-critical nonlinear Schrödinger equations is studied in H^s for some $s < 1$. The method consists of seeking solutions of the form $u = u_L^\omega + v$ where v is in H^1 and apply deterministic methods to v leading to global estimates in the norms $L^q_{t,x}$.

The works of the second type use invariant measure theoretic arguments and quasi-invariant measures. Even if no invariant measure for (2.18) exists in $L^2(\mathbb{R}^d)$, this problem is circumvent by Burq, Thomann and Tzvetkov by the use of the *lens transform*, $\mathcal{L} : \mathcal{S}'(\mathbb{R}\times\mathbb{R}^d) \to \mathcal{S}'\left(\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)\times\mathbb{R}^d\right)$ and inverse $\mathcal{L}^{-1}$ defined by

$$v(t,x) := \mathcal{L}u(t,x) = \frac{1}{\cos(2t)^{\frac{d}{2}}} u\left(\frac{\tan(2t)}{2}, \frac{x}{\cos(2t)}\right) \exp\left(-\frac{i|x|^2\tan(2t)}{2}\right),$$

$$u(s,y) = \mathcal{L}^{-1}v(s,y) = (1-4s^2)^{-\frac{d}{4}} v\left(\frac{\arctan(2s)}{2}, \frac{y}{\sqrt{1-4s^2}}\right) \exp\left(\frac{i|y|^2 s}{\sqrt{1-4s^2}}\right).$$

At a formal level, u solves

$$i\partial_s u + \Delta_y u = |u|^{p-1}u \text{ on } \mathbb{R}_s \times \mathbb{R}^d_y \text{ with } u(0) = u_0$$

if and only if $v = \mathcal{L}u$ solves

$$i\partial_t v - Hv = \cos(2t)^{\frac{p-5}{2}}|v|^{p-1}v \text{ on } \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \times \mathbb{R}^d_x \text{ with } v(0) = u_0 \,. \tag{2.20}$$

With this transformation, we trade (2.18) for a version in which $-\Delta$ is replaced by the harmonic oscillator $H = -\Delta + x^2$.

The first advantage of such a transformation is that H possesses only discrete spectrum which allows for constructing invariant measures, or quasi-invariant measures, depending on the values of p and d, resulting from an application of the method of Section 2.1. In the remaining of this subsection let us fix $d = 1$.

The eigenfunctions of H are given by the Hermite functions h_n which satisfy

$$Hh_n = (2n+1)h_n \,.$$

Therfore one can construct the measure

$$\mathrm{d}\mu = \exp\left(-\|H^{1/2}u\|^2_{L^2}\right) \,\mathrm{d}u$$

as the image measure under the randomisation map

$$u_0^\omega := \sum_{n\geqslant 0} \frac{g_n^\omega}{\sqrt{2n+1}} h_n(x) \,.$$

Let $\mathcal{W}^{s,p}$ stands for the Sobolev space associated to H, defined by the norm $\|u\|_{\mathcal{W}^{s,p}} := \|H^{\frac{s}{2}}u\|_{L^p}$ and let us denote $\mathcal{H}^s = \mathcal{W}^{s,2}$.

The measure μ_0 is supported at regularity $\mathcal{H}^{0-}$, which is seen using a similar computation to that of Section 2.1. Similarly to the case of the torus, the L^p norms of u_0^ω are finite for all p. Even more is true since the h_n satisfy estimates of the form

$$\|h_n\|_{L^p} \lesssim n^{-\frac{1}{3}} \,,$$

as soon as $p \geqslant 4$. Combining this observation with Remark 2.14 proves that almost-surely $u_0^\omega \in \mathcal{W}^{\frac{1}{6}^-,p}$ for all $p \geqslant 4$. Hence in the L^p scales and for $p \geqslant 4$, the randomisation regularises the data.

The lens transform exchange a scattering problem as $t \to \infty$ for a solution u to (2.18) into a scattering problem for a solution v to (2.20) as $t \to \frac{\pi}{4}$. According to Section 4.1 we need to study the convergence of

$$-i \int_0^t e^{-it'H} \cos(2t')^{\frac{p-5}{2}} (|v(t')|^{p-1}v(t')) \,\mathrm{d}t' \tag{2.21}$$

in some space $\mathcal{H}^{-\sigma}$. In particular, thanks to the embedding $L^{p+1} \hookrightarrow \mathcal{H}^{-\sigma}$ it is enough to show that

$$\int_t^{\frac{\pi}{4}} \cos(2t')^{\frac{p-5}{2}} \|v(t')\|^p_{L^{p+1}} \,, \mathrm{d}t' < \infty \,.$$

The interest of an invariant measure becomes clear: the invariance or quasi-invariance allows us to construct global solutions to (2.20) for which Sobolev norms grow slowly. Since for $p+1$ small enough, L^{p+1} is controlled by the Sobolev norms, we infer the scattering.

In the case $p=5$, this program is achieved by Burq, Thomann and Tzvetkov, and when $p\in]3,5[$ this result has been recently obtained by Burq and Thomann. The latter results relies on quasi-invariant measures, whose use in the context of (2.18) dates back to the works of Tevetkov [Tzv15] and then Oh and Tzvetkov [OT17].

Theorem 2.27 (Case $p=5$ in [BTT13], case $p\neq 5$ in [BT20])**.** *Let $d=1$ and $p\in(3,5]$. There exists a measure μ supported by $X^0 := \bigcap_{s>0} H^{-s}$ and a real number $\sigma>0$ such that for μ-almost every $u_0\in X^0$, there exists a unique global solution u to (2.18) in the space*

$$e^{is\Delta}u_0+\mathcal{C}^0(\mathbb{R},H^\sigma)\,.$$

Moreover, there exists $u_\pm\in H^\sigma$ such that the following scattering property holds

$$\|u(t)-e^{it\Delta}(u_0+u_\pm)\|_{H^\sigma}\underset{t\to\pm\infty}{\longrightarrow}0\,.$$

We refer to [BTT13, BT20] for more precise statements.

4.3. Contribution. The goal of the following work, which is the content of Chapter 4 is the extension of the result of Burq-Thomann [BT20] to higher dimensions in the radial case.

Theorem C (L. [Lat20a])**.** *Let $2\leqslant d\leqslant 10\}$ and $p\in\left(1+\frac{2}{d},1+\frac{4}{d}\right]$. We let $X^0_{\mathrm{rad}}=\bigcap_{\sigma>0}H^{-\sigma}_{\mathrm{rad}}(\mathbb{R}^d)$. There exists a measure μ_0 supported by X^0_{rad} and $\sigma=\sigma(p,d)$ such that for μ_0-almost every initial data $u_0\in X^0_{\mathrm{rad}}$, there exists a unique global solution u to (2.18) in the space*

$$e^{it\Delta}u_0+\mathcal{C}(\mathbb{R},H^\sigma)\,.$$

Moreover, there exists $u_\pm\in H^\sigma$ such that the following scattering property holds:

$$\|u(t)-e^{it\Delta}(u_0+u_\pm)\|_{H^\sigma}\underset{t\to\pm\infty}{\longrightarrow}0\,.$$

The use of radial random data is imposed by the quasi-invariant measure μ_0, in order to be of regularity H^{0-}.

Compared to dimension 1, the estimates of the Hermite functions h_n differ. When $d\geqslant 2$ these estimates imply that $u_0^\omega\in\mathcal{W}^{1/2^-,\infty}$ almost-surely, which is higher regularity gain than in the case $d=1$. The main difference appear for $d\geqslant 3$ when studying the L^{p+1} norms.

In dimension $d=1$ or 2, the control of the $\mathcal{H}^s$ norms by the invariant measure argument and the Sobolev embedding are sufficient to obtain such a control.

This is not the case when $d \geqslant 3$, a new argument is then required. We start from an observation of [BT20], which ensures that $\mu(\{\|v\|_{L^{p+1}} > \lambda\}) \lesssim e^{-\frac{\lambda^{p+1}}{p+1}}$, so that controlling the growth of the L^{p+1} norms amounts to controlling the variation of this norm on an interval $[t_1, t_2]$. Such an estimation is obtained using the dispersion estimate of Theorem 2.18 (i) and the Duhamel formula for u between times t_1 and t_2. We refer to Chapter 4 for details.

4.4. Perspectives. Several extensions of Theorem C can be considered. The first is a careful optimisation of its proof, incorporating refined methods in the analysis.

Question 2.3. *For which values of d and which values of $\sigma > 0$ is it possible to extend Theorem C, that is, for all $p \in (1 + \frac{2}{d}, 1 + \frac{4}{d}]$, almost every initial data in $u_0^\omega \in X^0_{\mathrm{rad}}$ give rise to unique global solutions to (2.18) in the space*

$$e^{it\Delta}u_0^\omega + \mathcal{C}^0(\mathbb{R}, H^\sigma(\mathbb{R}^d))\,,$$

and which scatters as $t \to \pm\infty$?

Among the reasons which makes this program reasonable are:

(1) The local Cauchy theory is developed in H^σ spaces only using Strichartz estimates. It is possible that working in $X^{s,b}$ spaces would optimise the local Cauchy theory and allow for using bilinear estimates (see Lemma 2.34).

(2) Finally, one should bear in mind that the scheme of the proof of scattering is very basic. Rather than applying the triangle inequality to bound (2.21) one could use non-homogeneous Strichartz estimates of Theorem 2.18 (ii).

However, note that Theorem C and the evoked extentions are limited by the regularity of the initial data in H^{0^-} and that the initial data are radial. Thus, a along-term question is the following

Question 2.4. *Is it possible to develop an alternative prof of Theorem C which does not rely on invariant measure theoretic arguments?*

5. Growth for Sobolev norms and Euler equation

This section introduces Chapter 5 of this dissertation and is concerned with the growth of the Sobolev norms for the Euler equations. The work presented in this section is the object of a submitted preprint [Lat20b].

We consider the Euler equations posed on the torus,

$$\text{(E)} \qquad \begin{cases} \partial_t u + u \cdot \nabla u &= -\nabla p \\ \nabla \cdot u &= 0 \\ u(0) &= u_0 \in H^s(\mathbb{T}^d)\,, \end{cases}$$

where $d = 2, 3$, $u(t) : \mathbb{T}^d \longrightarrow \mathbb{R}^d$ represents the velocity field and $p(t) : \mathbb{T}^d \longrightarrow \mathbb{R}$ represents the pressure field.

One can rewrite this equation in a more convenient fashion. Applying Leray's projector: $\mathbf{P} : (L^2)^d \to (L^2_{\mathrm{div}})^d$, which is the orthogonal projection on divergence-free L^2 vector fields, we arrive at

$$\begin{cases} \partial_t u + B(u,u) &= \quad 0 \\ \quad u(0) &= u_0 \in H^s_{\mathrm{div}}(\mathbb{T}^d)\,, \end{cases} \tag{2.22}$$

where $B(u, v) \coloneqq \mathbf{P}\,(u \cdot \nabla v)$ and where p is recovered from u by solving

$$-\Delta p = \nabla \cdot (u \cdot \nabla u)\,.$$

In this section we write X_{div} for the space of divergence-free functions of X.

5.1. The problem of the growth in dimension 2. In order to highlight the question of the growth of the Sobolev norms for solutions of (E), we recall that for $s > \frac{d}{2} + 1$, (E) is locally well-posed in H^s. This mostly follows from the Sobolev embedding. In dimension 2, one can even construct a global theory with an upper bound for the growth of the Sobolev norms.

Theorem 2.28 (Wolibner, Yudovich [Wol33, Yud63])**.** *Let $s > 2$. The Cauchy problem for (E) in dimension 2 is globally well-posed in $\mathcal{C}^0(\mathbb{R}_+, H^s(\mathbb{T}^2))$. Furthermore, there exists $C > 0$ such that :*

$$\|u(t)\|_{H^s} \leqslant Ce^{e^{Ct}} \textit{ for all } t > 0\,. \tag{2.23}$$

Proof. We only prove *a priori* estimates for $\|u(t)\|_{H^s}$ when s is large enough, and focus on pointing out where dimension 2 is used in the argument. We start by estimating $\|u(t)\|^2_{H^s}$ via the equation (2.22), which gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{H^s} = -(B(u,u),u)_{H^s}\,.$$

We study the non-linearity by only keeping track of the highest order terms, the other being easier to handle. Therefore we write

$$\begin{aligned}(B(u,u),u)_{H^s} &\sim (\nabla^s(u\cdot\nabla u), \nabla^s u)_{L^2} \\ &\sim (u \cdot \nabla\nabla^s u, \nabla^s u)_{L^2} + (\nabla u \cdot \nabla^s u, \nabla^s u)_{L^2} \\ &\quad + \{\text{lower order terms}\}\,.\end{aligned}$$

We then use an essential property of B, namely that for any divergence free u, v the identity $(B(u,v),v)_{L^2} = 0$ holds, from which one can infer that $(u \cdot \nabla\nabla^s u, \nabla^s u)_{L^2} = 0$, thus

$$(B(u,u),u)_{H^s} \sim (\nabla u \cdot \nabla^s u, \nabla^s u)_{L^2}\,,$$

so that after an application of Hölder inequality we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{H^s}^2 \lesssim \|\nabla u\|_{L^\infty}\|u(t)\|_{H^s}^2 \,. \tag{2.24}$$

This *a priori* estimate readily allows us to construct a local Cauchy theory for $s > \frac{d}{2}+1$, in any dimension. By the Sobolev embedding there indeed follows that $\|\nabla u\|_{L^\infty} \lesssim \|\nabla u\|_{H^{s-1}} \lesssim \|u\|_{H^s}$ since $s-1 > \frac{d}{2}$, leading to the local theory.

In order to obtain a global theory, we estimate the evolution of $\|u(t)\|_{H^s}$. In view of (2.24) we would like to control $\|\nabla u(t)\| \in L^\infty$, on which we do not have *a priori* information. However, we will see that the L^p norms of $u(t)$ in dimension 2 can be estimated, which will be used using the interpolation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{H^s}^2 \lesssim \|\nabla u(t)\|_{L^p}\|u(t)\|_{H^s}^{2+\frac{2}{p}} \,,$$

valid for $p \geqslant 2$ and where we would like to take p very large.

In the rest of the proof we set $d = 2$. In this case, the vorticity, defined by $\Omega(t) = \nabla \wedge u(t)$ satisfies the transport equation

$$\partial_t \Omega + u \cdot \nabla \Omega = 0 \,. \tag{2.25}$$

Hence the norms $\|\Omega(t)\|_{L^p}$ are conserved by the flow. Our goal is now to transfer this information to ∇u, which has same order as Ω. By the Calderón-Zygmund theory, we obtain

$$\|\nabla u(t)\|_{L^p} \lesssim p\|\Omega(t)\|_{L^p} \lesssim p\|\Omega_0\|_{L^p} \,.$$

Letting $y(t) := \|u(t)\|_{H^s}$ we have

$$y'(t) \leqslant Cp y(t)^{1+\frac{2}{p}} \,,$$

which after optimisation in p gives $y'(t) \leqslant y(t)\log y(t)$, which in turn gives $y(t) \lesssim e^{e^{Ct}}$ as expected. □

Is the double exponential bound (2.23) optimal? Is it possible to construct an initial data exhibiting a double exponential growth?

This question has been tackled in the case of the torus and the disc $D = \{x \in \mathbb{R}^2, \in |x| \leqslant 1\}$, in which case Kiselev and Šverák [KŠ14] are able to prove that the bound (2.23) is sharp. For the torus, only exponential growth is known [Zla15]. Note that in both cases, the proof relies on a hyperbolic scenario introduced by Bahouri and Chemin [BC94].

Theorem 2.29 (Kiselev-Šverák, Zlatoš [KŠ14, Zla15]). *Let U, standing for the unit disk D in $\mathbb{R}^2$ or $\mathbb{T}^2$. Then there exists an initial data u_0 belonging to $\mathcal{C}^\infty$ in the case of $U = D$, and only $\mathcal{C}^{1,\alpha}$ (for some $\alpha \in (0,1)$) if $U = \mathbb{T}^2$, such that the unique global associated solution Ω to (2.25) satisfies:*

(i) If $U = D$ then there exists $C > 0$ such that

$$\|\nabla\Omega(t)\|_{L^\infty} \geqslant C\exp\left(Ce^{Ct}\right) \text{ for all } t \geqslant 0\,.$$

(ii) If $U = \mathbb{T}^2$ then there exists $t_0 > 0$ such that for all $t \geqslant t_0$,

$$\sup_{t' \leqslant t} \|\nabla\Omega(t')\|_{L^\infty} \geqslant e^t\,.$$

In order to complete this result in the case of the torus, one should construct initial data which give rise to a double exponential growth, or prove that such solutions do not exist. We point out that the construction of Kiselev and Šverák [KŠ14] crucially exploits that the domain D has a boundary, which is not the case for $\mathbb{T}^2$. The following question arises: do the Sobolev norms of solutions to (E) in $\mathbb{T}^2$ *generically* grow not faster than exponentially, or even polynomially?

Here *generic* should be understood as "almost-surely with respect to a measure in $H^s(\mathbb{T}^2)$".

5.2. Kuksin's method. Before getting to the method introduced by Kuksin, let us recall that we are already in possession of a formal method for producing invariant measures in Section 2.1.

In order to construct an invariant measure, we write Ω_0 in Fourier series as $\Omega_0(x) = \sum_{n\in\mathbb{Z}^2}\hat{\Omega}_n e^{in\cdot x}$. The conservation law for $t \mapsto \|\Omega(t)\|_{L^2}$ then formally writes

$$E = \|\Omega(t)\|^2_{L^2} = \|\Omega_0\|^2_{L^2} = \sum_{n\in\mathbb{Z}^2}|\hat{\Omega}_n|^2\,.$$

From this point, we can guess that the following measure is invariant:

$$\mathrm{d}\mu(u) = \bigotimes_{n\in\mathbb{Z}^2} e^{-|\hat{\Omega}_n|^2}\,\mathrm{d}\hat{\Omega}_n\,,$$

which is the image measure of

$$\Omega_0^\omega(x) = \sum_{n\in\mathbb{Z}^2} g_n(\omega)e^{in\cdot x}\,,$$

where the $(g_n)_{n\in\mathbb{Z}^2}$ are independent standard Gaussian random variables. Such a measure is supported at regularity $H^{-1^-}(\mathbb{T}^2)$. Recalling that $\Omega_0 = \nabla\wedge u_0$, this means that in terms of u such a measure is supported in $H^{0^-}(\mathbb{T}^2)$, which a very low regularity. We refer to the work of Flandoli [Fla18].

The regularity $H^{0^-}(\mathbb{T}^2)$ is too weak for studying the growth of Sobolev norms since this regularity is incompatible with the local well-posedness theory.

The construction introduced by Kuksin is able to produce high regularity invariant measures. We only give the main idea and refer to [KS15, Kuk04] for a detailed presentation.

Rather than directly constructing an invariant measure for (E), one introduces parabolic-stochastic regularisations, which are given by stochastic Navier-Stokes equations, for any $\nu > 0$:

$$\begin{cases} \partial_t u_\nu + B(u_\nu, u_\nu) - \nu \Delta u_\nu &= \sqrt{\nu}\eta \\ u_\nu(0) &= u_0 \in H^s_{\mathrm{div}}(\mathbb{T}^d)\,, \end{cases} \tag{2.26}$$

with $\eta = \partial_t \xi$ and

$$\xi(t) = \sum_{n\in\mathbb{Z}^2} \phi_n \beta_n^\omega(t) e^{in\cdot x}\,, \tag{2.27}$$

white in time, smooth in space. The $(\beta_n)_{n\in\mathbb{Z}^2}$ are independent Brownian motions, the $(\phi_n)_{n\in\mathbb{Z}^2}$ are numbers which characterise the noise, in particular we define

$$\mathcal{B}_k \coloneqq \sum_{n\in\mathbb{Z}^2} |n|^{2k} |\phi_n|^2\,. \tag{2.28}$$

Kuksin's method consists of the following steps.

(1) Construct invariant measures μ_ν to approximate equations (2.26) using a Krylov-Bogolioubov argument, which turns out to be a standard argument.

(2) Prove uniform estimates of the μ_ν, which is the key step.

(3) Pass to the limit $\nu \to 0$, which do not bring any difficulty.

We only explain (2). The starting point is the integral equation satisfied by u_ν, which writes

$$u_\nu(t) = u_0 + \int_0^t (\nu\Delta u_\nu - B(u_\nu, u_\nu))\,\mathrm{d}t' + \sqrt{\nu}\sum_n \phi_n \beta_n(t) e_n\,.$$

For a functional F to be chosen, one applies the Itô formula to the stochastic process $t \mapsto F(u_\nu(t))$. Therefore

$$\begin{aligned} \mathbb{E}[F(u_\nu(t)] - \mathbb{E}[F(u_0)] = \nu \int_0^t \mathbb{E}[F'(u_\nu; \Delta u_\nu)\,\mathrm{d}t' \\ - \int_0^t \mathbb{E}[F'(u_\nu; B(u_\nu, u_\nu)]\mathrm{d}t' \\ + \frac{\nu}{2}\int_0^t \sum_n |\phi_n|^2 \mathbb{E}[F''(u; e_n, e_n)]\,\mathrm{d}t'\,. \end{aligned}$$

One can see that the term $\sqrt{\nu}\eta$ appears as squared in the Itô formula hence the choice for $\sqrt{\nu}$ in (2.26)

With $F(u) \coloneqq \|u\|^2_{H^1}$ one obtains

$$\mathbb{E}\left[\|u_\nu(t)\|^2_{H^1}\right] - \mathbb{E}\left[\|u_0\|^2_{H^1}\right] = \int_0^t (A(t') + B(t'))\,\mathrm{d}t'\,,$$

where $A(t')$ is the deterministic contribution

$$A(t') \coloneqq -\mathbb{E}\left[(B(u_\nu, u_\nu), u_\nu)_{H^1} - \nu(\Delta u_\nu, u_\nu)_{H^1}\right]\,,$$

and $B(t') = B$ is the Itô correction $B \coloneqq \frac{\nu}{2}\sum_n |n|^2|\phi_n|^2 = \frac{\nu}{2}\mathcal{B}_1$.

The other important ingredient in the proof is the use of dimension 2: the bilinear form B satisfies $(B(u_\nu, u_\nu), u_\nu)_{H^1} = 0$, which is efficiently seen using (2.25). Note that this identity is false in dimension 3.

Using that $\nu(-\Delta u, u)_{H^1} = \nu\|u\|_{H^2}^2$ and invariance of μ_ν for which $u_\nu(t)$ is an invariant process (constructed in (1)), we finally obtain

$$\int_0^t \left(\nu\mathbb{E}[\|u_\nu(t')\|_{H^2}^2] - \nu\frac{\mathcal{B}_1}{2}\right) \mathrm{d}t' = \mathbb{E}\int_0^t (B(u_\nu, u_\nu), u_\nu)_{H^1} \, \mathrm{d}t' = 0\,,$$

which enables us to divide by ν and obtain:

$$\mathbb{E}\left[\|u_\nu(t)\|_{H^2}^2\right] = \frac{\mathcal{B}_1}{2}\,.$$

Kuksin's result then reads:

Theorem 2.30 (Kuksin, [Kuk04])**.** *There exists a measure μ supported in $H^2(\mathbb{T}^2)$ and such that:*

(*i*) *The measure μ is non-trivial ans has regularity H^2,*

$$\mathbb{E}_\mu[\|u\|_{H^2}^2] \leqslant \frac{\mathcal{B}_1}{2} \text{ and } \mathbb{E}_\mu[\|u\|_{L^2}^2] \geqslant \frac{\mathcal{B}_0^2}{2\mathcal{B}_1}\,.$$

(*ii*) *For μ-almost every $u_0 \in H^2$, there exists a unique global solution to (E) associated to u_0, in the space $L^2_{\mathrm{loc}}(\mathbb{R}_+, H^2_{\mathrm{div}}) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, H^1_{\mathrm{div}}) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+, L^p)$, for all $p \in [1, 2[$.*

5.3. Contribution. Our goal is to study the possible growth of the Sobolev norms of solutions to (E) in dimensions 2 and 3. The meta-theorem of Section 2.1 seems perfectly fit for this purpose. Moreover, the measure μ constructed in Theorem 2.30 is such that

(*i*) μ is invariant under the flow of (E).

(*ii*) μ satisfies the estimates $\mu(\{u_0 \in H^2, \|u_0\|_{H^2(\mathbb{T}^2)} > \lambda\}) \lesssim \lambda^{-2}$, which is not a Gaussian bound but would be sufficient to apply the argument of the meta-theorem.

Unfortunately, assumption (3) of the meta-theorem is not satisfied since no good local Cauchy theory is known for (E) in $H^2(\mathbb{T}^2)$. However, such a theory is known in $H^{2+\delta}(\mathbb{T}^2)$ as soon as $\delta > 0$, therefore in order to apply the meta-theorem to Kuksin's measures, one should generalise the construction of such measures in $H^{2+\delta}(\mathbb{T}^2)$, and then soften hypothesis (2) of the meta-theorem. The following result is proven in Chapter 5.

Theorem D (L. [Lat20b])**.** *Let $d = 2$ (resp. $d = 3$) and $s > 2$ (resp. $s > 7/2$). There exists a measure μ such that $\mu(H^s(\mathbb{T}^d)) = 1$ and satisfying :*

(*i*) *For μ-almost any initial data, there exists a unique global solution u to* (E) *such that for all $t > 0$,*

$$\|u(t)\|_{H^s} \lesssim t^\alpha \,,$$

where α depends on s and d.

(*ii*) *If $d = 2$ then $\mathbb{E}_\mu[\|u\|^2_{H^1}] > 0$ and μ does not possess any atom. Furthermore μ charges large norm data: for any $R > 0$ there holds $\mu(u \in H^s,\ \|u\|_{H^s} > R) > 0$.*

(*iii*) *If $d = 3$ then we have the following alternative. Either $\mu = \delta_0$, or μ does not possess any atom.*

Such a combination of arguments, that is the use of Kuksin's construction of high regularity invariant measures and Bourgain's argument of constructing slowly growing global solution construction seems new in the setting of the fluid mechanics equations. In the dispersive setting however, such a combination has already been used by Sy [Sy19a, Sy18, Sy19b]. One can also find an example of construction of high regularity invariant measures in the recent work of Földes and Sy [FS20] for an equation similar to the Euler equation.

In order to modify Kuksin's method, one introduces a different parabolic regularisation than (2.26), namely

$$\left\{ \begin{array}{rcl} \partial_t u_\nu + B(u_\nu, u_\nu) + \nu L u_\nu &=& \sqrt{\nu}\eta \\ u_\nu(0) &=& u_0 \in H^s_{\mathrm{div}}(\mathbb{T}^d)\,, \end{array} \right. \tag{2.29}$$

where $L = (-\Delta)^{1+\delta}$ and $\eta = \partial_t \xi$. Here ξ and $\mathcal{B}_k$ are defined as in (2.27) and (2.28).

In order to prove uniform estimates by adapting Kuksin's method, one should also use the fundamental cancellation given by $(B(u,u),u)_{L^2} = 0$ valid in dimensions 2 and 3. In dimension 2 one also use the other fundamental cancellation given by $(B(u,u),u)_{H^1} = 0$. Details are given in Chapter 5 of this dissertation.

In dimension 3 one should keep in mind that no global well-posedness theory in $H^{2+\delta}(\mathbb{T}^3)$ is known. However, the meta-theorem of Section 2.1 is robust enough to construct global solutions only from the formal invariance of μ. A rigorous proof is obtained by approximating μ by measures μ_N which are invariant under the flow of approximations of (E) obtained by truncation in frequencies. Then one applies the meta-theorem to the solutions of these approximate equations uniformly with respect to the approximation parameter and pass to the limit in N.

5.4. A comparison of invariant measure constructions. The above theorems are not complete in the sense that only few information on the measures μ is given by the theorem. In order to complete the main theorem one can prove the following.

Theorem E (L. [Lat20b]). *The measures μ satisfy:*

(*i*) *If $d=2$, then for any Borelian set $\Gamma\subset\mathbb{R}_+$ there holds*

$$\mu(u\in H^s\,,\|u\|_{L^2}\in\Gamma)\lesssim|\Gamma|\,,\tag{2.30}$$

which implies that μ does not possess any atom.

(*ii*) *If $d=3$, then (2.30) is satisfied only for sets satisfying* $\operatorname{dist}(\Gamma,0)>\varepsilon$, *where the implicit constants depends on ε. Therefore μ can not possess any atom other than 0.*

In the case of the Euler equation in dimension 2 and for Kuksin measures of regularity H^2, a stronger result holds.

Theorem 2.31 (Kuksin, [Kuk04, KS15]). *Let μ be a measure constructed by Theorem 2.30, invariant for (E) and concentrated on $H^2(\mathbb{T}^2)$. We have:*

(*i*) *There exists a continuous increasing function p satisfying $p(0)=0$ and such that for all Borelian set $\Gamma\subset\mathbb{R}$,*

$$\mu(\|u\|_{L^2}\in\Gamma)+\mu(\|\nabla u\|_{L^2}\in\Gamma)\leqslant p(|\Gamma|)\,.$$

In particular, μ does not have any atom.

(*ii*) *For any set $K\subset H^2(\mathbb{T}^2)$ of finite Hausdorff dimension, there holds $\mu(K)=0$.*

This results shows that the constructed measures are “infinite-dimensional”. Despite these additional properties, the two presented results have the same lack as they are not able to preclude that such invariant measures μ do not concentrate exclusively on time-independent solutions of (E), in which case the study of the growth of the Sobolev norms becomes useless. It should be viewed as an open ended interest in the study of Kuksin's measures.

Building on this discussion, let us explain that in the case of Gibbs measures of Section 2.1 and more generally for subgaussian measures constructed by (2.12), one has many information on the support of the measures, contrary to the case of Kuksin's measures.

In the remaining of this sub-section we fix a measure μ concentrated on $L^2(\mathbb{T}^d)$ and defines it as the image measure under the randomisation map

$$u_0^\omega(x)=\sum_{n\in\mathbb{Z}^d}g_n^\omega u_n e^{in\cdot x}\,,$$

with

$$u_0\coloneqq\sum_{n\in\mathbb{Z}^d}u_n e^{in\cdot x}\in L^2\setminus\bigcup_{\varepsilon>0}H^\varepsilon\,,$$

and the $(g_n)_{n\in\mathbb{Z}^d}$ are identically distributed independent normalised Gaussian random variables. We prove the following.

Lemma 2.32 ([BT08a], Appendix B)**.** *With the above notation and further assuming that for all $n \in \mathbb{Z}^d$, $u_n \neq 0$, we have* $\operatorname{supp}\mu = L^2$.

Proof. Fix a function $v = \sum_{n\in\mathbb{Z}^d} v_n e^{in\cdot x}$ and $\varepsilon > 0$. We want to construct a set Ω_0 such that $\mathbb{P}(\Omega_0) > 0$ and such that $\|u_0^\omega - v\|_{L^2} \leqslant \varepsilon$ for all $\omega \in \Omega_0$. In the following we write $\mathbf{P}_{\leqslant N}$ the projection on frequencies $|n| \leqslant N$. For N to adjust, we write

$$\|v - u_0^\omega\|_{L^2} \leqslant \|(\mathrm{id} - \mathbf{P}_{\leqslant N})v\|_{L^2} + \|(\mathrm{id} - \mathbf{P}_{\leqslant N})u_0^\omega\|_{L^2} + \|\mathbf{P}_{\leqslant N}(u_0 - v)\|_{L^2}\,.$$

The term $\|(\mathrm{id} - \mathbf{P}_{\leqslant N})v\|_{L^2}$ goes to 0 as N goes to infinity since v is an L^2 function, thus fix $N > 0$ such that this term is bounded by $\frac{\varepsilon}{3}$.

For $\|(\mathrm{id} - \mathbf{P}_{\leqslant N})u_0^\omega\|_{L^2}$ we use the estimations obtained in Theorem 2.12

$$\mathbb{P}\left(\|(\mathrm{id} - \mathbf{P}_{\leqslant N})u_0^\omega\|_{L^2} > \lambda\right) \leqslant \exp\left(-c\lambda^2\left(\sum_{|n|>N}|u_n|^2\right)^{-\frac{1}{2}}\right).$$

Using this inequality to $\lambda = \varepsilon$ it follows that there exists a set Ω_1 of positive probability such that $\|(\mathrm{id} - \mathbf{P}_{\leqslant N})u_0^\omega\|_{L^2} \leqslant \frac{\varepsilon}{3}$ for all $\omega \in \Omega_1$.

Finally, let us define

$$\Omega_2 \coloneqq \bigcap_{|n|\leqslant N}\left\{|u_n g_n^\omega - v_n| \leqslant \frac{\varepsilon}{3}N^{-\frac{1}{2}}\right\},$$

of positive probability since $u_n \neq 0$ for all $n \in \mathbb{Z}^d$. Hence for $\omega \in \Omega_0 \coloneqq \Omega_1 \cap \Omega_2$ one has $\|v - u_0^\omega\|_{L^2} \leqslant \varepsilon$, which end the proof since by independence of Ω_1 and Ω_2 there holds $\mathbb{P}(\Omega) = \mathbb{P}(\Omega_1)\mathbb{P}(\Omega_2) > 0$. ☐

5.5. Perspectives. As the discussion from the previous sub-section shows it, it is possible that the measures produced by Theorem 2.30 and Theorem D are exclusively supported by time-independent solutions. A natural question, which may be difficult to answer is the following.

Question 2.5. *Is it possible to describe the support of Kuksin's measures produced by Theorem 2.30 and Theorem D? In some cases, is it possible to exclude the scenario in which the Kuksin measures are supported only by time-independent solutions?*

There are at least two difficulties which prevent from developing intuition on this question.

(1) The first reason is the lack of description of the measures, which are constructed using compactness methods, thus the properties of these limiting measures should be studied using uniform estimates on the measures μ_ν, just as in Theorem 2.31 for example.

(2) The second reason is the lack a description for the time-independent solutions to (E), which form a very large set [CŠ12].

Directly trying to study these problems for the incompressible Euler equations seems too difficult, maybe for other equations, such as dispersive equations which have less stationary solutions, the problem might be easier to study. In [BCZGH16, GHŠV15] this question is tackled for model equations. Finally, let us mention that the support of Kuksin measures has been studied in the simpler context of finite dimension in [MP14].

6. The Cauchy problem for the Grushin-NLS equation

This section introduces Chapter 6 of this dissertation and is concerned with the local Cauchy problem with random initial data for some nonlinear Schrödinger equations. The work presented in this section has been done in collaboration with Louise Gassot and form the content of a preprint [GL21].

We consider the local Cauchy problem for the nonlinear Schrödinger equations posed on the Heisenberg group. This equations reads

$$\text{(NLS-}\mathbb{H}^1\text{)} \qquad \begin{cases} i\partial_t u + \Delta_{\mathbb{H}^1} u &= |u|^2 u \,, \\ u(0) &= u_0 \in H^s(\mathbb{H}^1)\,. \end{cases}$$

Denoting $(x, y, s) \in \mathbb{R}^3$ we can write

$$\Delta_{\mathbb{H}^1} = \frac{1}{4}\left(\partial_x^2 + \partial_y^2\right) + (x^2 + y^2)\partial_s^2\,.$$

We denote by $H^k(\mathbb{H}^1)$ the Sobolev spaces in this setting.

A simplified two-dimensional model of this equation, still grasping the majority of the difficulty linked to the Heisenberg sub-laplacian is the Grushin-Schrödinger equation, hence a simplification of (NLS-$\mathbb{H}^1$) which we will study is the following,

$$\text{(NLS-G)} \qquad \begin{cases} i\partial_t u + \Delta_G u &= |u|^2 u \,, \\ u(0) &= u_0 \in H^s_G\,, \end{cases}$$

where $\Delta_G = \partial_x^2 + x^2\partial_y^2$ and $(x, y) \in \mathbb{R}^2$.

6.1. The Cauchy problem for (NLS-$\mathbb{H}^1$) and (NLS-G). The interest of studying such equations comes from the lack of a satisfactory local well-posedness theory for (NLS-$\mathbb{H}^1$) and (NLS-G). Note that the following quantity (and similarly for the Grushin case),

$$E(t) = \frac{1}{2}\|(-\Delta_{\mathbb{H}^1})^{\frac{1}{2}} u\|^2_{L^2(\mathbb{H}^1)} + \frac{1}{4}\|u\|^4_{L^4(\mathbb{H}^1)}\,,$$

is conserved, and referred to as the energy. This quantity has regularity at level $H^1(\mathbb{H}^1)$, hence if one constructs a local Cauchy theory in $H^1(\mathbb{H}^1)$, a global theory would immediately follow. However, to this day the best local Cauchy theory is the following.

Proposition 2.4 (Local Caychy theory for (NLS-$\mathbb{H}^1$) and (NLS-G))**.** *Define $k_C = 2$ (resp. $k_C = \frac{3}{2}$). Let $k > k_C$, then the Cauchy problem for* (NLS-$\mathbb{H}^1$) *(resp.* (NLS-G)*) is uniformly locally well-posed in $\mathcal{C}^0([0,T^*), H^k)$ with $T^* \gtrsim \|u_0\|_{H^k}^{-2}$.*

A proof in the Heisenberg case can be found in [BG01], as for the Grushin case we give a proof in the Appendix of Chapter 6. The proof essentially relies on the Sobolev embedding $H^k \hookrightarrow L^\infty$, which holds as soon as $k > k_C$.

Thus, even in the Grushin case, the local theory does not apply in H^1_G. For $k < k_C$, obtaining a local Cauchy theory is an open problem. In order to go further, one could try to obtain dispersive estimates as in the Euclidean case. It is however known that the Schrödinger equation on the Heisenberg group is not dispersive, as observed in [BGX00]. In this work it is even proven that the flow of (NLS-$\mathbb{H}^1$) (resp. (NLS-G)) is irregular in H^k as soon as $k < k_C$, precluding contraction-mapping arguments to yield a local Cauchy theory better than Proposition 2.4. We also refer to the introduction of [GG10] and Remark 2.12 of [BGT04]. In view of this lack of dispersion, the dispersive paradigm does not apply. Note that, however, Strichartz estimates proven by Fourier restriction methods have been obtained recently [BBG19], but do not seem to yield any improvement in the Cauchy theory yet.

6.2. Contribution. Our main result is a probabilistic construction of unique solutions to (NLS-G). Our result is stated in a subspace of H^k_G denoted by $\mathcal{X}^k_\rho$, which corresponds to H^k_G functions with additional partial regularity in the y variable. The randomisation procedure is similar to the randomisation in the torus case. More precisely, the functions of H^k_G, admit a frequency decomposition. For $u_0 \in H^k_G$ one can write

$$u_0 = \sum_{(a,m)\in\mathbb{N}^*\times\mathbb{N}} u_{a,m}\,,$$

which enables us to introduce the measure μ_{u_0} as the image measure on H^k_G by the randomisation map

$$u_0^\omega = \sum_{(a,m)\in\mathbb{N}^*\times\mathbb{N}} g^\omega_{a,m} u_{a,m}\,, \tag{2.31}$$

where the $\{g^\omega_{a,m}\}_{(a,m)\in\mathbb{N}^*\times\mathbb{N}}$ are identically distributed independent standard complex Gaussian random variables. We refer to the introduction and Section 2 of Chapter 6 for more details on this randomisation procedure.

The main result reads as follows.

Theorem F (Gassot-L. [GL21])**.** *Let $k \in (1, \frac{3}{2}]$ and $u_0 \in \mathcal{X}^k_1 \subset H^k_G$.*

(i) For all $\ell \in (\frac{3}{2}, k + \frac{1}{2})$ and almost every $\omega \in \Omega$, there exists $T > 0$ such that there exists a unique local solution to (NLS-G) *with initial data*

u_0^ω in the space

$$e^{it\Delta_G}u_0^\omega + \mathcal{C}^0([0,T), H_G^\ell) \subset \mathcal{C}^0([0,T), H_G^k)\,.$$

More precisely, there exists $c>0$ such that for all $R>0$, outside a set of probability at most e^{-cR^2}, we have $T \geqslant \frac{1}{R^2}$.

(ii) (No smoothing) If $u_0 \in H_G^k \setminus (\bigcup_{\varepsilon>0} H_G^{k+\varepsilon})$ then

$$\operatorname{supp}(\mu_{u_0}) \subset H_G^k \setminus (\bigcup_{\varepsilon>0} H_G^{k+\varepsilon})\,.$$

(iii) (Density of the support of the measures) Let $\varepsilon>0$ then there exists $v_0 \in \mathcal{X}_1^k \setminus (\bigcup_{\varepsilon'>0} H_G^{k+\varepsilon'})$ such that

$$\operatorname{supp}(\mu_{v_0}) \cap B_{H_G^k}(u_0,\varepsilon) \neq \varnothing\,.$$

Assertions (ii) and (iii) prove that the measures produced with functions in H_G^k are supported by H_G^k exactly, which is similar to the results obtained in Appendix B of [BT08a], the proof being similar. Moreover, the union of the supports of all these measure is entirely filling H_G^k. This is a density result akin to that of Appendix B in [BT14].

As we already pointed, an inherent difficulty to the Schrödinger equation is its lack of direct smoothing in the Duhamel term. In the case of $\mathbb{R}^d$, bilinear estimates can circumvent this difficulty in order to develop a probabilistic local theory similar that of Section 2.2. In the case of the Grushin sub-laplacian, it is such bilinear and trilinear estimates which we would like to possess. We obtain some random versions of such estimates.

Theorem G (Bilinear and trilinear estimates for random solutions)**.** *Let $k \in (1, \frac{3}{2}]$ and $u_0 \in \mathcal{X}_1^k \subset H_G^k$. Let u_0^ω as in (2.31), and let us denote by $z^\omega = e^{it\Delta_G}u_0^\omega \in \mathcal{C}^0(\mathbb{R}, H_G^k)$ the solution to the linear Schrödinger equation associated to u_0^ω. Then there exists $c>0$ such that the following statements hold. Fix $q \in [2,\infty)$.*

(i) For all $R \geqslant 1$ and $T>0$, outside a set of probability at most e^{-cR^2}, one has

$$\|(z^\omega)^2\|_{L_T^q H_G^{k+\frac{1}{2}}} + \||z^\omega|^2\|_{L_T^q H_G^{k+\frac{1}{2}}} \leqslant R^2 T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_1^k}^2\,, \tag{2.32}$$

$$\||z^\omega|^2 z^\omega\|_{L_T^q H_G^{k+\frac{1}{2}}} \leqslant R^3 T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_1^k}^3\,. \tag{2.33}$$

(ii) We further require that $u_0 \in \mathcal{X}_{1+\varepsilon_0}^k$ for some $\varepsilon_0>0$. Let $\ell < k+\frac{1}{2}$. For all $R \geqslant 1$ and $T>0$, there exists a set $E_{R,T}$ of probability at least $1-e^{-cR^2}$ such that the following holds. Fix $\omega \in E_{R,T}$ and $v,w \in L_T^\infty H_G^\ell$, then

$$\|z^\omega v w\|_{L_T^q H_G^\ell} \leqslant R T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_{1+\varepsilon_0}^k} \|v\|_{L_T^\infty H_G^\ell} \|w\|_{L_T^\infty H_G^\ell}\,. \tag{2.34}$$

Note that v and w may depend on ω.

Remark 2.33. Unlike the $\mathbb{R}^d$ case, the multilinear estimates are point-wise in time and do not result from averaging in time, but rather on a probabilistic gain due to bilinear estimates for Hermite functions. Indeed, as it will be shown in details in Chapter 6, the partial Fourier transform $y \to \eta$ of functions $u \in L^2_G$ has a natural decomposition along the basis of the rescaled Hermite functions

$$\mathcal{F}_{y\to\eta} u(x;\eta) = \sum_{m\geqslant 0} f_m(\eta) h_m(|\eta|^{\frac{1}{2}} x)\,,$$

hence we see that in the products uv, appear some convolution of Hermite functions. A good understanding of these functions is instrumental in proving the partial regularity gain for uv.

Equipped with such multilinear estimates it is then possible to solve the Cauchy problem with random data u_0^ω by seeking solutions of the form $u(t) = z^\omega(t) + v(t)$, $v \in \mathcal{C}^0([0,T[, H^\ell_G)$, $\ell > \frac{3}{2}$. The goal is then to construct a deterministic local Cauchy to v which formally satisfies

$$v(t) = -i\int_0^t e^{i(t-t')\Delta_G}\left(|z^\omega + v|^2(z^\omega + v)\right)\,\mathrm{d}t'\,.$$

Expanding the cubic term, we see that $|v|^2 v$ is easily controlled by the deterministic theory based on Sobolev embeddings. For the other terms, one should apply the multi-linear estimates. The estimate for the purely probabilistic term $|z^\omega|^2 z^\omega$ follows from probabilistic decoupling and deterministic multi-linear estimates. Finally, the most difficult term to handle is a mixed term between the deterministic and probabilistic part $|v|^2 z^\omega$ which requires a delicate treatment, see Section 7 of Chapter 6.

6.3. Perspectives. Several questions arise and could provide good extensions to the study of the Schrödinger equation for the Grushin operator. In particular, Theorem F leaves open the Cauchy problem in the energy space H^1_G.

Question 2.6. *Is it possible to generalise Theorem F to the end point $k = 1$? Can one construct global solutions in this case?*

It is highly possible that the techniques developed in the proof of our theorem lead to construction of unique solutions at regularity H^1_G, which would then be constructed in the space $e^{it\Delta_G} u_0^\omega + H_G^{\frac{3}{2}^+}$. Such a construction would not yield global existence in H^1_G, as one should rather prove that the part $v \in H_G^{\frac{3}{2}}$ is global, which seems very difficult, due to the absence of *a priori* control of this norm.

Finally, one can try to prove similar results in the case of the Heisenberg, for which we expect that most of the arguments apply with small modification.

Question 2.7. *Is it possible to prove an analogue to Theorem F for* (NLS-$\mathbb{H}^1$)*?*

Appendix A. Proof of Theorem 2.12

Proof of Theorem 2.12. The second inequality follows from the first by integration using that any random variable Y verifies

$$\|Y\|_{L^p_\Omega}^p = p\int_0^\infty \lambda^{p-1}\mathbb{P}\left(|Y|>\lambda\right)\,\mathrm{d}\lambda\,.$$

For the first inequality, it reduces to proving the inequality without absolute values because for any random variable Y there holds $\mathbb{P}(|Y|>\lambda)\leqslant \mathbb{P}(Y>\lambda)+\mathbb{P}(-Y>\lambda)$.

We introduce $t>0$ which has to be chosen.

$$\mathbb{P}\left(\sum_{n\geqslant 0} a_n X_n > \lambda\right) = \mathbb{P}\left(\exp\left(t\sum_{n\geqslant 0} a_n X^n\right) > e^{t\lambda}\right)\,.$$

The Markov inequality, the independence of the X_n and (2.11) imply

$$\mathbb{P}\left(\sum_{n\geqslant 0} a_n X_n > \lambda\right) \leqslant e^{-t\lambda}\prod_{n\geqslant 0}\mathbb{E}[e^{t a_n X_n}] \leqslant \exp\left(-t\lambda + ct^2\sum_{n\geqslant 0} a_n^2\right)\,,$$

and the result is obtained after optimisation in t. □

Appendix B. Idea of the proof of Theorem 2.20

The proof of this theorem relies on a specific feature of the $\mathbb{R}^d$ setting: the *bilinear estimates* for solutions to the linear Schrödinger equation

$$\left\{\begin{array}{rcl} i\partial_t u + \Delta u &=& 0 \\ u(0) &=& u_0\in H^s(\mathbb{R}^4)\,. \end{array}\right. \tag{2.35}$$

Lemma 2.34 (Bilinear estimates)**.** *Let u, v be two solutions of* (2.35) *with initial data $u_0, v_0\in L^2$ and write $u_{N_1} = \mathbf{P}_{N_1}u$ and $v_{N_2} = \mathbf{P}_{N_2}v$ the frequency projections of these functions on $[N_i, 2N_i]$. Assume that $N_2\geqslant N_1$. Then one has*

$$\|u_{N_1}v_{N_2}\|_{L^2_{t,x}} \lesssim N_1\left(\frac{N_1}{N_2}\right)^{\frac12}\|u_{N_1}(0)\|_{L^2_x}\|v_{N_2}(0)\|_{L^2_x}\,.$$

Remark 2.35. If $N_1 = N_2$ the this follows from Hölder and Bernstein inequalities, thus the factor (N_2/N_1) represents a half derivative soothing when $N_2\gg N_1$.

In order for this result to be used for almost linear solutions one works in spaces adapted to the linear evolution. In this context, these spaces introduced in [Bou93] are the $X^{s,b}$ spaces defined by the norm

$$\|u\|_{X^{s,b}} \coloneqq \|\langle\tau - |\xi|^2\rangle^b\langle\xi\rangle^s\tilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}}\,,$$

where $\tilde{u}$ stands for the space-time Fourier transform of u. The bilinear estimates can be "transfered" to $X^{s,b}$ spaces. More precisely we will use the following estimates, gathered in the following result.

Corollary 2.36 (Linear and bilinear estimations in $X^{s,b}$)**.** *Let $\eta_T(\cdot)$ be a smooth cutoff in time such that $\mathbf{1}_{[0,T]} \leqslant \eta_T \leqslant \mathbf{1}_{[0,2T]}$. Let $s \in \mathbb{R}$, $b \in (\frac{1}{2}, \frac{3}{2})$ and $\theta \in [0, \frac{3}{2} - b)$.*

(*i*) *Linear estimate: one has*

$$\left\| \eta_T(t) \int_0^t e^{i(t-t')\Delta} F(t') \,\mathrm{d}t' \right\|_{X^{s,b}} \lesssim T^\theta \|F\|_{X^{s,b-1+\theta}} \,. \tag{2.36}$$

(*ii*) *Bilinear estimate: let $N_2 \geqslant N_1$, $b > \frac{1}{2}$ and two functions $u_{N_1}, u_{N_2} \in X^{0,b}$ frequency localised (in spatial frequencies) around N_1 (resp. N_2). Then there holds*

$$\|u_{N_1} v_{N_2}\|_{L^2_{t,x}} \lesssim N_1 \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \|u_{N_1}\|_{X^{0,b}} \|v_{N_2}\|_{X^{0,b}} \,, \tag{2.37}$$

Hence the effect of the Duhamel operator in $X^{s,b}$ norms can be read on b.

Proof of Theorem 2.20. We do not give a complete proof and should only emphasis the use of the bilinear estimates.

We go back to the scheme introduced to study the local Cauchy problem, seeking v such that $u(t) = u_L(t) + v(t)$ solves (2.14), that is

$$v(t) = \Phi(v)(t) := -i \int_0^t e^{i(t-t')\Delta} (|u_L + v|^2 (u_L + v)) \,\mathrm{d}t' \,.$$

Then we prove that Φ is a contraction mapping $X^{\sigma,b} \to X^{\sigma,b}$, with $b = \frac{1}{2}^+$ and $\sigma = 1^+$. Thanks to Strichartz's inequality (2.36), we have

$$\begin{aligned}
\|\Phi v\|_{X^{1^+,\frac{1}{2}^+}} &\lesssim T^\theta \||u_L + v|^2 (u_L + v)\|_{X^{1^+,\frac{1}{2}^-}} \\
&\lesssim T^\theta \sup_{\|w\|_{X^{0,\frac{1}{2}^+}} \leqslant 1} \int_{\mathbb{R}_t \times \mathbb{R}^4_x} \langle \nabla \rangle^{1^+} |u_L + v|^2 (u_L + v) w \,\mathrm{d}t\mathrm{d}x \,.
\end{aligned}$$

where on the last line we used duality.

Developing $u_L + v|^2(u_L + v)$ essentially leads to two terms which treatment is different: the term $|v|^2 v$, handled using deterministic methods and the term $|u_L|^2 u_L$ which we will study. The other terms are treated using combination of arguments used in the treatment of these two terms.

In contrast with the study of (fNLS) where $\beta > 0$, the Duhamel operator does not gains derivatives, hence depending on which terms the derivation $\langle \nabla \rangle^{1^+}$ acts, it is not clear that there holds $\||u_L|^2 u_L\|_{X^{1^+,\frac{1}{2}^-}} < \infty$.

In order to circumvent this difficulty, one can use bilinear estimates. In order to do so, one splits each u_L dyadically in spatial frequencies using a Littlewood-Paley decomposition $u_L = \sum_{N\in 2^{\mathbb{N}}} u_N$. Thus one has

$$\int_{\mathbb{R}_t\times\mathbb{R}^4_x} \langle\nabla\rangle^{1^+}(|u_L|^2u_L)w\,\mathrm{d}t\mathrm{d}x \simeq \sum_{\substack{N_1,N_2,N_3,N_4\\ \max\{N_1,N_2,N_3\}\sim N_4}} (\langle\nabla\rangle^{1^+}(u_{N_1}u_{N_2}u_{N_3}),u_{N_4})_{L^2_{t,x}}\,,$$

where the condition on N_4 comes from Parseval's theorem and the scalar product in L^2. Moreover by symmetry one can further assumes that $N_4\sim N_3 \geqslant N_2, N_1$ in the above sum.

The $N_1\sim N_2\sim N_3$ case is favourable since in this case $\langle\nabla\rangle$ acts as multiplication by N_1 which can be split. In the following we only address the case $N_4\sim N_3 \gg N_1, N_2$ with the supplementary condition $N_1, N_2 \ll N_3^{\frac{1}{3}}$, which prevents derivative trade-off. This case is expected to be one of the most difficult in the analysis.

In frequency space, $u_{N_1}u_{N_2}u_{N_3}$ is supported around $\max\{N_i\} = N_3$, hence the "cost" of derivation is estimated by N_3. One also uses that $N_3\gg N_1$ and $N_4\gg N_2$ in order to use (2.37) :

$$\begin{aligned}(\langle\nabla\rangle^{1^+}(u_{N_1}u_{N_2}u_{N_3}),u_{N_4})_{L^2_{t,x}} &\lesssim N_3^{1^+}\|u_{N_1}u_{N_3}\|_{L^2_{t,x}}\|w_{N_4}u_{N_2}\|_{L^2_{t,x}}\\ &\lesssim N_3^{1^+}N_1\left(\frac{N_1}{N_3}\right)^{\frac12}N_2\left(\frac{N_2}{N_4}\right)^{\frac12}\|w_{N_4}\|_{X^{0,b}}\prod_{i=1}^3\|u_{N_i}\|_{X^{0,b}}\,.\end{aligned}$$

Using that $\|u_{N_i}\|_{X^{0,b}}\sim N_i^{-s}\|u_{N_i}\|_{X^{s,b}}$ one infers that

$$(\langle\nabla\rangle^{1^+}(u_{N_1}u_{N_2}u_{N_3}),u_{N_4})_{L^2_{t,x}} \lesssim N_3^{1^+-\frac53 s}\|w_{N_4}\|_{X^{0,b}}\prod_{i=1}^3\|u_{N_i}\|_{X^{s,b}}\,,$$

which is an acceptable contribution as soon as $s>\frac35$. A complete proof is given in [BOP15b]. □

Chapitre 3

Almost Sure Existence of Global Solutions for Supercritical Semilinear Wave Equations

Abstract. We prove that for almost every initial data $(u_0, u_1) \in H^s \times H^{s-1}$ with $s > \frac{p-3}{p-1}$ there exists a global weak solution to the supercritical defocusing semilinear wave equation $\partial_t^2 u - \Delta u + |u|^{p-1}u = 0$ where $p > 5$, in both $\mathbb{R}^3$ and $\mathbb{T}^3$. This improves in a probabilistic framework the classical result of Strauss [Str70] who proved global existence of weak solutions associated to $H^1 \times L^2$ initial data. The proof relies on techniques introduced by Oh and Pocovnicu in [OP16] based on the pioneer work of Burq and Tzvetkov in [BT08a]. We also improve the global well-posedness result in [SX16] for the subcritical regime $p < 5$ to the endpoint $s = \frac{p-3}{p-1}$.

1. Introduction

1.1. Supercritical semilinear wave equations. We consider the Cauchy problem for the energy-supercritical defocusing semilinear wave equation in dimension 3, that is for $p > 5$ and $s \geqslant 0$:

$$\text{(SLW}_p\text{)} \qquad \begin{cases} \partial_t^2 u - \Delta u + |u|^{p-1}u = 0 \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H^s(U) \times H^{s-1}(U) \,, \end{cases}$$

where $u(t)$ is a real-valued function defined on $U = \mathbb{R}^3$ or $\mathbb{T}^3$. We also consider the associate linear wave equation:

$$\text{(LW)} \qquad \begin{cases} \partial_t^2 z - \Delta z = 0 \\ (z(0), \partial_t z(0)) = (z_0, z_1) \in H^s(U) \times H^{s-1}(U) \,. \end{cases}$$

The formal conserved energy for a solution u to (SLW$_p$) is

$$E(u(t)) \coloneqq \int_U \left(\frac{|\partial_t u(t,x)|^2}{2} + \frac{|\nabla u(t,x)|^2}{2} + \frac{|u(t,x)|^{p+1}}{p+1} \right) \mathrm{d}x \,.$$

Moreover (SLW$_p$) is known to be invariant under the dilation symmetry

$$u(t,x) \mapsto u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x) \,.$$

A necessary condition, for a function u to belong to the *energy space*, $E(u(t)) < \infty$, is that $(u(0), \partial_t u(0)) \in \dot{H}^1(U) \times L^2(U)$. We observe that

$$\begin{aligned} \|(u_\lambda(0), \partial_t u_\lambda(0))\|_{\dot{H}^1(U) \times L^2(U)} &= \lambda^{\frac{2}{p-1} - \frac{1}{2}} \|(u(0), \partial_t u(0))\|_{\dot{H}^1(U) \times L^2(U)} \\ &\underset{\lambda \to 0}{\gg} \|(u(0), \partial_t u(0))\|_{\dot{H}^1(U) \times L^2(U)} \,, \end{aligned}$$

since $p > 5$, which explains why for such p, (SLW_p) is called *energy-supercritical.*

We first recall the classical result of existence of *weak solutions* to (SLW_p).

Theorem 3.1 (Strauss, 1970, [Str70])**.** *Let f be a real smooth function and F be an antiderivative of f. Assume that $F(v) \gtrsim -|v|^2$ and*

$$\frac{|F(u)|}{|f(u)|} \to \infty \text{ as } |u| \to \infty\,.$$

Let $(u_0, u_1) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ satisfying

$$E(u_0, u_1) := \int_{\mathbb{R}^3} \left(\frac{|u_1|^2}{2} + \frac{|\nabla u_0|^2}{2} + F(u_0) \right) \mathrm{d}x < \infty\,.$$

Then the equation

$$\left\{ \begin{array}{rcl} \partial_t^2 u - \Delta u + f(u) & = & 0 \\ (u(0), \partial_t u(0)) & = & (u_0, u_1) \in H^1 \times L^2\,, \end{array} \right. \tag{3.1}$$

admits a weak solution, that is a distributional solution $(u, \partial_t u) : \mathbb{R} \to \mathcal{H}^1(\mathbb{R}^3)$ which is weakly continuous in time and such that

$$E(u(t), \partial_t u(t)) \leqslant E(u_0, u_1) \text{ for every } t \in \mathbb{R}\,.$$

In the following we will seek for global solutions to (SLW_p) in the following sense.

Definition 3.1 (Weak solutions for (SLW_p))**.** A function $u : \mathbb{R} \times U \to \mathbb{R}$ is said to be a *weak solution* to (SLW_p) with initial data $(u_0, u_1) \in H^s(U) \times H^{s-1}(U)$ if and only if one can write $u = z + v$ where z is a strong solution to (LW) with initial data (u_0, u_1) and v is such that $(v(0), \partial_t v(0)) = (0, 0)$ and satisfies for every $T > 0$, $v \in H^1((-T, T) \times U) \cap L^{p+1}((-T, T) \times U)$, and for every compactly supported $\varphi \in \mathcal{C}^2(\mathbb{R} \times U)$:

$$\int_{\mathbb{R}} \int_U \left(\partial_t v \partial_t \varphi - \nabla v \cdot \nabla \varphi - (z + v)|z + v|^{p-1} \varphi \right) \mathrm{d}x\, \mathrm{d}t = 0\,.$$

Remark 3.2. Note that this definition differs from the definition of weak solutions in Theorem 3.1. In our setting we ask for the solution u to be written in the form $u = z + v$, where z solves the associate linear problem (LW): the reason why we ask for such a decomposition will appear clearly in the proof of the main result, Theorem 3.4. Note that these weak solutions are *a fortiori* weak solution as in Theorem 3.1.

In contrast, we recall the notion of strong solution that we will use, slightly different from usual definitions, adapted to a decomposition $u = z + v$. More precisely, we have the following.

Definition 3.2 (Strong solutions for (SLW$_p$))**.** Let $T > 0$. A function $u : (-T,T) \times U \to \mathbb{R}$ is said to be a *strong solution* to (SLW$_p$) on the time interval $(-T,T)$, with initial data $(u_0,u_1) \in H^s(U) \times H^{s-1}(U)$ if there is a decomposition $u = z + v$ where $(z, \partial_t z) \in \mathcal{C}^0((-T,T), H^s(U) \times H^{s-1}(U))$ is a solution to (LW) associated to (u_0,u_1) and $(v, \partial_t v)$ belongs to $\mathcal{C}^0(\mathbb{R}, H^s(U) \cap L^{p+1}(U)) \times \mathcal{C}^0(\mathbb{R}, H^{s-1}(U))$ and is a solution to

$$\partial_t^2 v - \Delta v + |z+v|^{p-1}(z+v) = 0\,,$$

with initial condition $(v_0,v_1) = (0,0)$. Such a solution is *global* if one can take $T = \infty$.

Remark 3.3. We remark that these *strong* solutions are such that

$$(u, \partial_t u) \in \mathcal{C}^0((-T,T), H^s) \times \mathcal{C}^0((-T,T), H^{s-1})\,,$$

which can be considered to be the usual space for *strong* solutions. We will see that, however, differences appear when one considers the uniqueness of such solutions; see Theorem 3.8.

1.2. Previous works on probabilistic well-posedness for wave equations. Our purpose is to construct solutions to (SLW$_p$), in the sense of Definition 3.1 using a probabilistic method when initial data are below the energy space. Indeed, up to the knowledge of the author, no global existence result is known for (SLW$_p$), $p > 5$ and initial data $(u_0,u_1) \in H^s \times H^{s-1}$ with $s < 1$.

Probabilistic methods have been implemented in order to construct solutions to (SLW$_p$) associated to initial data below the energy space. As a result, the local and global well-posedness theory have been widely improved. We briefly recall the existing results in the context of semilinear wave equations, our list being not exhaustive.

The probabilistic well-posedness theory goes back to J. Bourgain who proved global existence for the two-dimensional nonlinear Schrödinger equation in [Bou94]. Building on Bourgain's ideas, N. Burq and N. Tzvetkov published a series of two articles [BT08a, BT08b], introducing a randomisation procedure that allows to choose random initial data in Lebesgue and Sobolev spaces. They developed the local and global probabilistic well-posedness theory. They later considered the global well-posedness of (SLW$_p$) for $p = 3$ in [BT14] proving that, although for $s < \frac{1}{2}$, (SLW$_p$) is ill-posed, there exist unique global solutions for almost every initial data in $H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ as soon as $s \geqslant 0$. The proof relies on a probabilistic improvement of the Strichartz estimates. In [BTT15], Burq-Thomann-Tzvetkov considered (SLW$_p$) in higher dimensions and proved the almost sure existence of global infinite energy solutions of (SLW$_p$), for $p = 3$ in $\mathbb{T}^d$, $d \geqslant 3$. Their argument uses compactness techniques just like the ones presented in this article. We mention that in the context of the Navier-Stokes equation on $\mathbb{T}^3$, A.R. Nahmod, N. Pavlović and G. Staffilani proved existence of global weak solutions almost surely in [NPS13].

The work of Lührman-Mendelson in [LM14, LM16] deals with the global well-posedness theory for (SLW$_p$) in the case $3 < p < 5$. They prove an almost-sure global well-posedness result associated to initial data

$$(u_0, u_1) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3) \text{ as long as } s > \frac{p^3 + 5p^2 - 11p - 3}{9p^2 - 6p - 3}$$

which improves the deterministic theory when $\frac{1}{4}(7 + \sqrt{73}) \simeq 3.88 < p < 5$. In [LM16] they improved their result to $\frac{p-1}{p+1} < s < 1$ using Oh-Pocovnicu's ideas from [OP16].

In [Poc17], Pocovnicu proved almost-sure global well-posedness for the energy critical wave equation (SLW$_p$) in the Euclidean space $\mathbb{R}^d$ of dimension $d = 4, 5$. The proof relies on the deterministic perturbation theory for critical dispersive equations as well as the probabilistic improvements of the Strichartz estimates coming from the work of Burq-Tzvetkov. With some more efforts in the domains $\mathbb{R}^3$ and $\mathbb{T}^3$ the global well-posedness theory for (SLW$_p$), $p = 5$ has been treated in the joint work of Oh-Pocovnicu in [OP16, OP17]. In their proof they used a new energy estimate and a new probabilistic Strichartz estimate. Their result shows that almost-sure global well-posedness in known to hold for initial data in $H^s \times H^{s-1}$, $s > \frac{1}{2}$.

The global well-posedness theory for (SLW$_p$) and when $3 < p < 5$ was then studied in the work of Sun-Xia, in [SX16]. They proved global existence and uniqueness for $s > \frac{p-3}{p-1}$ interpolating between the results of Oh-Pocovnicu [OP16] and Burq-Tzvetkov [BT14].

Finally it should be mentioned that in the context of the semilinear Shrödinger equation, similar probabilistic well-posedness have been investigated, we refer to [BOP15b, OOP17] for the Euclidean setting and [BB14a, BB14b] for the setting of the unit ball.

1.3. Main results and notations.

1.3.1. *Statement of the main results.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and a randomisation map $(u_0, u_1) \longmapsto (u_0^\omega, u_1^\omega)$ that will be described in Section 2.1, see (3.5) and (3.7). Let U be an open set. The measure $\mu_{(u_0,u_1)}$ is defined as the pushforward probability measure of $\mathbb{P}$ by the above randomisation map. We define

$$\mathcal{M}^s(U) := \bigcup_{(u_0,u_1)\in H^s(U)\times H^{s-1}(U)} \left\{\mu_{(u_0,u_1)}\right\}.$$

We now state our results. The first is an existence result in the supercritical case.

Theorem 3.4. *Let $p > 5$, $s > \frac{p-3}{p-1}$ and U which stands for $\mathbb{R}^3$ or $\mathbb{T}^3$. Let $\mu \in \mathcal{M}^s(U)$. Then for μ almost every $(u_0, u_1) \in H^s(U) \times H^{s-1}(U)$ there exists a global weak solution u to (SLW$_p$), in the sense of Definition 3.1.*

Remark 3.5. Note that no information is given concerning the uniqueness of the solutions constructed.

The solutions constructed in Theorem 3.4 enjoy additional properties:

Corollary 3.6. *Under the hypothesis of Theorem 3.4, and given $\mu \in \mathcal{M}^s(U)$, there exists a set $\Sigma \subset H^s(U) \times H^{s-1}(U)$ of full μ-measure which is invariant under the flow of (SLW$_p$). If $s < 1$ then for every initial data $(u_0, u_1) \in \Sigma$, the solutions u constructed by Theorem3.4 satisfy $(u, \partial_t u) \in \mathcal{C}^0(\mathbb{R}, H^s(U) \times H^{s-1}(U))$*

Corollary 3.7. *For $U = \mathbb{R}^3$, the solutions constructed by Theorem 3.4 enjoy the finite speed of propagation property with speed at most* 1*.*

The next result is an extension of Theorem 1.2 from [SX16] to the endpoint $s = \frac{p-3}{p-1}$.

Theorem 3.8. *Let $3 < p < 5$, $s := \frac{p-3}{p-1}$ and U stands for $\mathbb{R}^3$ or $\mathbb{T}^3$. Let $\mu \in \mathcal{M}^s$. Then for almost every $(u_0, u_1) \in H^s(U) \times H^{s-1}(U)$ there exists a unique global* strong *solution to (SLW$_p$) in the sense of Definition 3.2, where uniqueness for $(u, \partial_t u)$ holds in the space*

$$(S(\cdot)(u_0, u_1), \partial_t S(\cdot)(u_0, u_1)) + \mathcal{C}^0(\mathbb{R}, H^s(U) \cap L^{p+1}(U)) \times \mathcal{C}^0(\mathbb{R}, H^{s-1}(U)) \,,$$

with $S(\cdot)(u_0, u_1)$ standing for the solution to (LW) associated to (u_0, u_1).

Again a consequence of the proof of Theorem 3.8 is the following:

Corollary 3.9. *Under the hypothesis of Theorem 3.8, and given $\mu \in \mathcal{M}^s$, there exists a set $\Sigma \subset H^s(U) \times H^{s-1}(U)$ of full μ-measure which is invariant under the flow of (SLW$_p$).*

Remark 3.10. One can also prove probabilistic continuous dependence of the flow in the sense of [BT14], using the proof given in [Poc17], Theorem 1.4 and Remark 1.6 (iii) but this Chapter does not focus on this matter.

Remark 3.11. Combining these results with the existing results in the case $p \leqslant 5$, see [BT14, OP16, OP17, SX16] yields the following classification:

(*i*) In the *energy sub-critical* setting *i.e* $p \in [3, 5)$, there exists a unique strong solution to (SLW$_p$) for almost every initial data in $H^s(U) \times H^{s-1}(U)$ as soon as $s \geqslant \frac{p-3}{p-1}$. The case $p = 3$ is treated entirely in [BT14] in the case of the torus, but similar arguments work in the

Euclidean setting. The case $p \in (3,5)$ and $s > \frac{p-3}{p-1}$ is treated in [SX16], and the case $p \in (3,5)$, $s = \frac{p-3}{p-1}$ is Theorem 3.8. In [BT14], additional results of continuous dependence and flow-invariant set are proven in the case $p = 3, s \geqslant 0$ which remain valid in the case $p \in (3,5)$, $s \geqslant \frac{p-3}{p-1}$.

(*ii*) In the *energy critical* setting $p = 5$, there exists a unique strong solution to (SLW$_p$) with initial data in $H^s(U) \times H^{s-1}(U)$ as soon as $s > \frac{1}{2}$. This result is proven in [OP16, OP17] both in $\mathbb{R}^3$ and $\mathbb{T}^3$ and comes with a continuous dependence result and flow-invariant set construction.

(*iii*) In the *energy super-critical* setting $p > 5$ there exists a global weak solution as long as $s \in (\frac{p-3}{p-1}, 1)$: this is Theorem 3.4. For $s > 1$ there still exists a weak solution, in the sense of Definition 3.1. In both cases such solutions can be constructed in a flow-invariant way, this is Corollary 3.6.

Remark 3.12. One can wonder if it is possible, for $p > 5$ (resp. $p = 5$), to extend the result of Theorem 3.4 (resp. Theorem 3.8) to the endpoint $s = \frac{p-3}{p-1}$. As it will be explained in Remark 3.26, this endpoint remains out of reach of the techniques presented on this article. Thus the extension to the endpoint proven in Theorem 3.8 may be viewed as subcritical.

1.3.2. *General notation.* $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space if Ω is a set and $\mathcal{F}$ is a σ-algebra, endowed with a probability measure $\mathbb{P}$. The expectation of a random variable X will be denoted by $\mathbb{E}[X]$.

The notation $A \lesssim B$ means that there exists a constant C such that $A \leqslant CB$. The notation $A \lesssim_x B$ is used to specify that the constant C depends on x.

For a real number x, $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) denotes the lower integer part (resp. upper integral part).

We adopt widely used notations for functional spaces: $\mathcal{C}^k$ denotes the set of k differentiable functions with continuous derivatives up to order k, L^p stands for the Lebesgue spaces and $W^{s,p} := \{u, (\mathrm{Id} - \Delta)^{s/2} u \in L^p\}$ (resp. $\dot{W}^{s,p}$) denotes the usual nonhomogeneous Sobolev spaces (resp. homogeneous spaces). The hilbertian Sobolev spaces are denoted $H^s := W^{s,2}$. Besov spaces are denoted $B^s_{p,r}$, see Appendix A for more details. The Fourier transform of a function f is denoted by $\hat{f}$ or equivalently $\mathcal{F}(f)$. $\mathcal{D}$ denotes the space of test functions, that is $\mathcal{C}^\infty$ functions with compact support. $\mathcal{S}'$ stands for the space of tempered distributions. If X is a Banach space we often write L^pX or L^p_TX for $L^p((0,T), X)$. For $x \in \mathbb{R}^d$, $\langle x \rangle = (1 + |x|^2)^{1/2}$, and $\langle \nabla \rangle$ denotes the Fourier multiplier of symbol $\langle \xi \rangle$.

$S^m := S^m_{1,0}$ denotes the class of classical symbols of order $m \in \mathbb{R}$, that is smooth functions $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \lesssim_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|}, \text{ for every } \alpha \in \mathbb{N}^d \text{ and } x, \xi \in \mathbb{R}^d .$$

In the rest of this chapter $\mathcal{H}^s(U)$ will be used as a shorthand notation for $H^s(U) \times H^{s-1}(U)$.

1.4. Outline of the chapter, heuristic arguments. The purpose of this section is to provide the reader with heuristic arguments that will help to understand the main ideas behind the proofs of the two main results. Note that this section is not mathematically needed in order to follow the rigorous proofs of the result.

1.4.1. *Energy estimates.* In this paragraph we set $U = \mathbb{R}^3$ and drop the reference to it. As in [BT14, OP16], the method to construct global solutions to (SLW$_p$) is to seek for solutions u of the form $u(t) = z(t) + v(t)$ with initial data $v(0) = \partial_t v(0) = 0$, and z being the solution to (LW) associated to initial data $(u_0, u_1) \in \mathcal{H}^s$. Note that z is globally defined. It is thus sufficient to prove that v globally solves an equation in the energy space $\mathcal{H}^1$. A direct computation shows that v formally satisfies

$$\partial_t^2 v - \Delta v + |z+v|^{p-1}(z+v) = 0 .$$

We now assume that v locally solves this equation in $\mathcal{H}^1$ and that the local well-posedness result comes with a blow-up criteria that only depends on the size of $\|(v(t), \partial_t v(t))\|_{\mathcal{H}^1}$. In this case, proving global existence v reduces to proving that the (not conserved) nonlinear energy

$$E(t) := \frac{\|\partial_t v(t)\|_{L^2}^2}{2} + \frac{\|\nabla v(t)\|_{L^2}^2}{2} + \frac{\|v(t)\|_{L^{p+1}}^{p+1}}{p+1}$$

is bounded on every time interval.

In order to do so, the standard way is to estimate $E(t)$ using a Grönwall-type estimate and hope for a sublinear estimate that will give non-blowup for $E(t)$. We first begin by writing that

$$E'(t) = -\int_{\mathbb{R}^3} \partial_t v(t) \left(|z(t)+v(t)|^{p-1}(z(t)+v(t)) - |v(t)|^{p-1}v(t) \right) \mathrm{d}x .$$

In the following we will provide a rough argument, ignoring lower order terms and using fractional integration by parts. The terms in powers of z will be estimated using the probabilistic Strichartz estimates from Proposition 3.4 and Proposition 3.5 so they constitute the “good part” when expanding the quantity $(z+v)^p - v^p$, if we assume p to be an odd integer for instance. These considerations lead to the “worse order approximation” $(z+v)^p - v^p \simeq$

$v^{p-1}z+$(better terms). The other terms are expected to be handled in an easier way so that we write :

$$E'(t) \simeq \int_{\mathbb{R}^3} \partial_t v\, v^{p-1} z \,\mathrm{d}x + \underbrace{(\text{better terms})}_{\lesssim E(t)} .$$

From now on, we will dismiss the better terms and study the worse term.

Remark 3.13. A crude estimate using the Hölder inequality, putting $z \in L^\infty$, $\partial_t v \in L^2$ and $v^{p-1} \in L^{\frac{p+1}{p-1}}$ leads to

$$E'(t) \lesssim E(t')^{\frac{1}{2}+\frac{p-1}{p+1}}$$

which is sublinear if $\frac{1}{2}+\frac{p-1}{p+1} \leqslant 1$ *i.e.* $p \leqslant 3$. Thus such an argument would provide an energy estimate that does not blow-up for $p \leqslant 3$ and $s > 0$. This was the idea behind the energy estimates in [BT14, Poc17].

In order to obtain energy estimates for $p = 5$, Oh-Pocovnicu introduced in [OP16] an appropriate method that we describe: if one accepts a loss of regularity for the initial data then one can transfer time regularity into space regularity using an integration by parts in time, and properties of the wave equation, which state that roughly $\partial_t z \simeq \nabla z$. Since $E(0) = 0$ we have after integration in time:

$$E(t) = \int_0^t E'(t')\,\mathrm{d}t' \simeq \int_0^t \int_{\mathbb{R}^3} \partial_t v v^{p-1} z \,\mathrm{d}x\,\mathrm{d}t' .$$

Then using that $\partial_t v v^{p-1} \sim \partial_t(v^p)$ an integration by parts yields

$$\begin{aligned} E(t) &\simeq \int_{\mathbb{R}^3} \int_0^t v(t')^p \partial_t z(t')\,\mathrm{d}t'\,\mathrm{d}x \\ &\simeq \int_{\mathbb{R}^3} \int_0^t v(t')^p \nabla z(t')\,\mathrm{d}t'\,\mathrm{d}x . \end{aligned}$$

Pick $s \in [0,1]$ which will be chosen later. We write that $\nabla z = \nabla^{1-s}\nabla^s z$ and integrate by parts in space with the operator ∇^{1-s}, neglecting the boundary terms:

$$E(t) \simeq \int_0^t \int_{\mathbb{R}^3} \nabla^{1-s}(v(t')^p)\nabla^s z(t')\,\mathrm{d}x\,\mathrm{d}t' .$$

We expect that $\nabla^{1-s}(v^p) \simeq v^{p-1}\nabla^{1-s}v$ so that Hölder's inequality yields

$$\begin{aligned} E(t) &\simeq \int_0^t \int_{\mathbb{R}^3} \nabla^{1-s}v(t')v(t')^{p-1}\nabla^s z(t')\,\mathrm{d}x\,\mathrm{d}t' \\ &\lesssim \|\nabla^s z\|_{L^\infty_{T,x}} \int_0^t E(t')^{\frac{p-1}{p+1}} \|v(t')\|_{\dot{W}^{1-s,\frac{p+1}{2}}}\,\mathrm{d}t' \end{aligned} \tag{3.2}$$

The term $\|v(t')\|_{\dot{W}^{1-s,\frac{p+1}{2}}}$ will be estimated by interpolating the estimates for $v(t')$ in L^{p+1} and $\dot{H}^1$. The standard tool to do so is the Gagliardo-Nirenberg inequality, see Theorem 3.32. We obtain

$$\|\nabla^{1-s}v\|_{L^{\frac{p+1}{2}}} \lesssim \|\nabla v\|_{L^2}^{1-\alpha}\|v\|_{L^{p+1}}^{\alpha} \lesssim E(t)^{\frac{1-\alpha}{2}+\frac{\alpha}{p+1}}\,,$$

where s satisfies the homogeneity conditions

$$\begin{cases} s \geqslant \alpha \\ \frac{2}{p+1} - \frac{1-s}{3} = \frac{1-\alpha}{6} + \frac{\alpha}{p+1} \end{cases}$$

which gives $\alpha = s_p \leqslant \frac{p-3}{p-1}$, so that

$$E(t) \lesssim \|\nabla^{s_p} z\|_{L^\infty_{T,x}} \int_0^t E(t')\,\mathrm{d}t'$$

and the Grönwall lemma proves the non blow-up of $E(t)$, provided $\|\nabla^{s_p} z\|_{L^\infty_{T,x}} < +\infty$ which is the case for initial data in $\mathcal{H}^{s_p+\varepsilon}$, thanks to probabilistic improvement of the Strichartz estimates that we will prove later.

In our context it will not be possible to construct such a global strong solution v. However the strategy used in [BTT15] applies: this is the strategy one uses to construct Leray solutions in the context of the Navier-Stokes equations, which consists of first finding approximate solutions $u_n = z_n + v_n$ that are global in time. Then the previous energy estimate provides uniform bounds for v_n that allow strong compactness arguments in order to pass to the limit.

1.4.2. *Yudovich-Wolibner argument.* The Yudovich-Wolibner argument was first presented in the work of Wolibner in the context of the 2-dimensional Euler equations, see [Wol33]. A similar argument was provided by Yudovich in the same context, see [Yud63]. We will recall this argument, in its simplest version. Note that this kind of argument has been widely used since, in particular in the study of (SLW$_p$), $p = 3, s = 0$, see [BT14].

Let us explain it in the context of the Schrödinger equation on $\mathbb{R}^2$:

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u + \Delta u + u|u|^2 &= 0 \\ u(0) &= u_0 \in H^1(\mathbb{R}^2)\,. \end{cases}$$

Let $u_1, u_2 \in \mathcal{C}^0([0,T), H^1)$ be two solutions to (NLS). The following aims at proving uniqueness of solutions, that is $u_1 = u_2$. In order to do so consider $E(t) := \|u_1(t) - u_2(t)\|_{L^2}^2$. Then a computation, using that u_1, u_2 solve (NLS) yields

$$E'(t) \lesssim \int_{\mathbb{R}^2} |u_1(t) - u_2(t)|^2 (|u_1(t)|^2 + |u_2(t)|^2)\,\mathrm{d}x\,.$$

If the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ were true, then we would have $E'(t) \lesssim E(t)$ and would deduce that $E(t) = 0$ for $t \in [0,T]$. Unfortunately there is no such continuous embedding, and we only have the *Trudinger type* estimate [BCD11]

$$\|u\|_{L^p} \lesssim \sqrt{p}\|u\|_{H^1} \text{ for all } p > 2\,.$$

Using the previous inequality and the Hölder inequality gives

$$E'(t) \lesssim pE(t)^{1-\frac{1}{p}} \tag{3.3}$$

for all $p > 1$. After integration by separation of variables this implies $E(t) \leqslant (Ct)^p$ for a constant $C > 0$. Thus for a fixed $t < \frac{1}{C}$ and letting $p \to \infty$, we get $E(t) = 0$. This argument can be iterated on time intervals $\left[\frac{n}{C}, \frac{n+1}{C}\right]$ for $n \geqslant 0$ so that $E(t) = 0$ for all $t \geqslant 0$.

Remark 3.14. Another way to conclude is to optimize in p in (3.3) so that $E'(t) \lesssim -E(t)\log(E(t))$ and gives the same result.

As mentioned before our setting will be a little more complicated: we will need some bootstrap argument to conclude rather than this simple integration techniques. The method will be used to prove existence of global solutions in a limiting case rather than proving uniqueness, the framework being very similar.

1.4.3. *Organization of the paper.* Section 2 explains the the randomisation procedure and recalls the probabilistic improvement for the Strichartz estimates. We then provide a generalization of these estimates in the context of Besov spaces, see Proposition 3.5.

Section 3 is devoted to the proof of Theorem 3.4 in the case of the Euclidean space $\mathbb{R}^3$. More precisely, Sub-section 3.1 proves existence of global solutions u_n to approximate equations, Sub-section 3.2 provides uniform bounds in n for the nonlinear energies that will allow to use a compactness argument in Sub-section 3.3.

Section 4 provides the proof of Theorem 3.8 and its corollary.

For reader's convenience some useful facts concerning Sobolev and Besov spaces are gathered in Appendix A.

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2. Probabilistic estimates

2.1. The probabilistic setting. We first recall some standard notation in Littlewood-Paley analysis.

Let $\mathcal{C}$ be the annulus $\{\xi \in \mathbb{R}^3,\ 3/4 \leqslant |\xi| \leqslant 8/3\}$, then there exists radial functions χ, φ taking values in $[0,1]$ belonging to $\mathcal{D}(B(0,4/3))$ and $\mathcal{D}(\mathcal{C})$

satisfying

$$\chi(\xi)+\sum_{j\geqslant 0}\varphi(2^{-j}\xi)=1 \text{ for all } \xi\in\mathbb{R}^3 \text{ and } \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1 \text{ for all } \xi\in\mathbb{R}^3\setminus\{0\}$$

and such that

$$\begin{array}{rcl} |j-j'|\geqslant 2 & \Longrightarrow & \operatorname{supp}\varphi(2^{-j}\cdot)\cap\operatorname{supp}\varphi(2^{-j'}\cdot)=\varnothing \\ j\geqslant 1 & \Longrightarrow & \operatorname{supp}\chi\cap\operatorname{supp}\varphi(2^{-j}\cdot)=\varnothing\,. \end{array}$$

We now define the nonhomogeneous Littlewood-Paley projectors:

$$\Delta_j u := \begin{cases} 0 & \text{for } \ j\leqslant -2 \\ \chi(D)u & \text{for } \ j=-1 \\ \varphi(2^{-j}D)u & \text{for } \ j\geqslant 0, \end{cases}$$

where $\chi(D)$ (resp. $\varphi(2^{-j}D)$) denotes the Fourier multiplier of symbol χ (resp. $\varphi(2^{-j}\cdot)$). As a homogeneous Littlewood decomposition will be needed, we set $\dot{\Delta}_j:=\varphi(2^{-j}D)$ for all $j\in\mathbb{Z}$. We set :

$$\mathbf{P}_j=\sum_{j'\leqslant j-1}\Delta_{j'} \text{ and } \dot{\mathbf{P}}_j=\chi(2^{-j}D).$$

In the case of the torus $\mathbb{T}^3$ we construct a similar decomposition, with a bump function $\varphi\in\mathcal{D}(B(0,2))$ such that $\varphi=1$ on $B(0,1)$. Let $(e_n)_{n\in\mathbb{Z}^3}$ be the hilbertian sequence of $L^2(\mathbb{T}^3)$ defined by $x\mapsto e_n(x)=e^{2i\pi n\cdot x}$. For a function $u=\sum_{n\in\mathbb{Z}^3}c_ne_n$, and for $j\geqslant 0$, we set $\mathbf{P}_ju=\sum_{n\in\mathbb{Z}^3}\varphi(2^{-j}|n|)c_ne_n$ and $\Delta_ju:=\mathbf{P}_ju-\mathbf{P}_{j-1}u$, with the convention that $\mathbf{P}_{-1}=0$

An account of useful facts in Littlewood-Paley theory is given in Appendix A.

The randomisation that is widely used in the context of the torus $\mathbb{T}^3$ or more generally a compact manifold is presented in [BT08a]. Consider $(X_n)_{n\in\mathbb{Z}^3}$ a sequence of random variables defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ which satisfy the following definition.

Definition 3.3. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\left(X_n^{(i)}\right)_{n\in\mathbb{Z}^3;i=0,1}$ be a sequence of complex random variables defined on Ω, satisfying the symmetry property $X_{-n}^{(i)}=\overline{X_n^{(i)}}$ and such that the random variables

$$\left(X_0^{(i)},\operatorname{Re}\left(X_n^{(i)}\right),\operatorname{Im}\left(X_n^{(i)}\right)\right)_{n\in I;i=0,1}$$

where

$$I=\left(\mathbb{Z}_+\times\{0\}^2\right)\cup\left(\mathbb{Z}\times\mathbb{Z}_+\times\{0\}\right)\cup\left(\mathbb{Z}^2\times\mathbb{Z}_+\right),$$

are independent; and that there exists a constant $c>0$ such that for any $Y\in\{X_0^{(i)},\operatorname{Re}\left(X_n^{(i)}\right),\operatorname{Im}\left(X_n^{(i)}\right)\}_{n\in I;i=0,1}$ there holds

$$\left|\mathbb{E}\left[e^{\gamma Y}\right]\right|\leqslant e^{c\gamma^2} \tag{3.4}$$

for all $\gamma\in\mathbb{R}$.

Every $u \in L^2(\mathbb{T}^3)$ can be written in the hilbertian basis $(e_n)_{n\in\mathbb{Z}}$ as $u = \sum_{n\in\mathbb{Z}^d} u_n e_n$ with u_n the Fourier coefficients of u. We introduce the *Fourier randomisation* associated to a couple (u_0, u_1) with the randomisation map:

$$\Theta_{(u_0,u_1)} : \begin{cases} \Omega & \longrightarrow \qquad \{ \text{ map from } \mathbb{T}^3 \text{ to } \mathbb{C}\} \\ \omega & \longmapsto \left(\sum_{n\in\mathbb{Z}^3} X_n^{(0)}(\omega) u_{0,n} e_n, \sum_{n\in\mathbb{Z}^3} X_n^{(1)}(\omega) u_{1,n} e_n \right) . \end{cases} \tag{3.5}$$

There is a similar procedure in the Euclidean setting. In this case the standard randomisation setup is called the *Wiener randomisation* and randomizes frequencies cubes in a way that we explain. Note that a rough version of that randomisation was introduced by Wiener in [Wie32], and that the smooth version used here has been developed by Benyi-Oh-Pocovnicu in [BOP15b]. Let $\psi \in \mathcal{D}(]-1,1[^3)$ with the symmetry property $\psi(-\xi) = \overline{\psi(\xi)}$ for all $\xi \in \mathbb{R}^3$ and satisfying the unit partition condition $\sum_{n\in\mathbb{Z}^3} \psi(\cdot - n) = 1$, so that for every $u \in \mathcal{S}'$ there holds

$$u = \sum_{n\in\mathbb{Z}^3} \psi(D-n) u \, .$$

One readily sees that

$$\xi \longmapsto \sum_{n\in\mathbb{Z}^3} |\psi(\xi - n)|^2 \text{ is bounded.} \tag{3.6}$$

The randomisation of a couple (u_0, u_1) is then defined with the randomisation map

(3.7)

$$\Theta_{(u_0,u_1)} : \begin{cases} \Omega & \longrightarrow \qquad \{ \text{ map from } \mathbb{R}^3 \text{ to } \mathbb{C}\} \\ \omega & \longmapsto \left(\sum_{n\in\mathbb{Z}^3} X_n^{(0)}(\omega) \psi(D-n) u_0, \sum_{n\in\mathbb{Z}^3} X_n^{(1)}(\omega) \psi(D-n) u_1 \right) . \end{cases}$$

In both cases (randomisation in $\mathbb{T}^3$ or $\mathbb{R}^3$) we set $\mu_{(u_0,u_1)}(A) := \mathbb{P}\left(\Theta^{-1}_{(u_0,u_1)}(A)\right)$ for every measurable set A, and define

$$\mathcal{M}^s(U) := \bigcap_{(u_0,u_1)\in H^s(U)\times H^{s-1}(U)} \mu_{(u_0,u_1)} \, ,$$

the set of all measures that we will work with.

Remark 3.15. In the following, when there is no possible confusion we will denote by (u_0, u_1) the couple of random variables defined by the randomisation maps (3.5) and (3.7), rather than (u_0^ω, u_1^ω) or any other notation.

These randomisations have been studied in [BT08a], in which Burq-Tzvetkov proved the following theorem. For a proof see Lemma B.1 in [BT08a] and also Lemma 2.2 in [BOP15b].

Theorem 3.16 (Non-smoothing effect of the randomisation setup)**.** *Let $\mu = \mu_{(u_0,u_1)} \in \mathcal{M}^s(U)$, $U = \mathbb{R}^3$ or $\mathbb{T}^3$ and for $i = 0, 1$ there exists a $c > 0$ such that* $\limsup_n \mathbb{P}(|X_n^{(i)}| \leqslant c) < 1$. *We have the following:*

(i) The measure μ is supported by $\mathcal{H}^s(U)$.

(ii) For $s' > s$ if $(u_0, u_1) \notin \mathcal{H}^{s'}(U)$ then $\mu(\mathcal{H}^{s'}(U)) = 0$.

(iii) In the case of the torus, if all the Fourier coefficients of u_0 and u_1 are different from zero and the support of the distributions of the $\left(X_0^{(i)}, \mathrm{Re}(X_n^{(i)}), \mathrm{Im}(X_n^{(i)})\right)_{n\in I\,;\,i=0,1}$ are $\mathbb{R}$ then the support of μ is exactly $\mathcal{H}^s(U)$.

This theorem proves that there is no gain in regularity by randomisation. However a gain in integrability is known, see Theorem 3.17, and is responsible for the improvement in Strichartz inequalities and thus the local wellposedness and global well-posedness theory in dispersive equations.

2.2. Probabilistic semigroup estimates. The starting point of every probabilistic improvement of the Strichartz estimates is the following well-known theorem in the theory of random Fourier series. For a proof see Lemma 3.1 in [BT08a] and Lemma 2.1 in [OP16].

Theorem 3.17 (Kolmogorov-Paley-Zygmund)**.** *Let $(X_n)_{n\in\mathbb{Z}}$ be a sequence of independent and identically distributed real-valued random variables such that there exists a constant $c > 0$ such that for every $\gamma \in \mathbb{R}$, the inequality (3.4) holds. Let $(a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ be a complex valued sequence with the symmetry property $a_{-n} = \overline{a_n}$ for every integer n (resp. a real-valued sequence $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$). For $q \in [1, \infty)$ one has:*

$$\left\| \sum_{n\in\mathbb{Z}} a_n X_n \right\|_{L^q} \lesssim \sqrt{q} \|(a_n)_{n\in\mathbb{Z}}\|_{\ell^2}. \tag{3.8}$$

We turn to the probabilistic improvement of Strichartz estimates and introduce semi-groups associated to the linear wave equation. Let $s \geqslant 0$ and $(u_0, u_1) \in \mathcal{H}^s(U)$. We set:

$$z(t) := S(t)(u_0, u_1) := \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|}u_1,$$

which is a solution to (LW). One of the key features of the linear wave equation is that "two time derivatives equal two space derivatives" so that one expects that "one time derivative equals one space derivative" which can be turned more rigorously writing $\partial_t z(t) = \langle\nabla\rangle \tilde{z}(t)$ where

$$\tilde{z}(t) := \tilde{S}(t)(u_0, u_1) := -\frac{|\nabla| \sin(t|\nabla|)}{\langle\nabla\rangle}u_0 + \frac{\cos(t|\nabla|)}{\langle\nabla\rangle}u_1.$$

For our purposes we will need a smooth version of both $z(t)$ and $\tilde{z}(t)$, namely

$$\left\{ \begin{array}{l} z_n(t) := S_n(t)(u_0,u_1) := \mathbf{P}_n S(t)(u_0,u_1) \\ \tilde{z}_n(t) := \tilde{S}_n(t)(u_0,u_1) := \mathbf{P}_n \tilde{S}(t)(u_0,u_1)\,, \end{array} \right. \tag{3.9}$$

for every integer $n \geqslant 1$. We recall the probabilistic Strichartz estimates, proven in [Poc17, OP17] for (3.10) and [OP16, OP17] for (3.11).

Proposition 3.4 (Probabilistic Strichartz estimates for the wave operator)**.** *Let U standing for either $\mathbb{T}^3$ or $\mathbb{R}^3$. Let $(u_0,u_1) \in \mathcal{H}^s(U)$ and still write (u_0,u_1) its randomisation (Fourier randomisation procedure or Wiener randomisation procedure). Let z^* stand for either $z, \tilde{z}, z_n$ or $\tilde{z}_n$. For any $q_1 \in [1,\infty)$ and $q_2 \in [2,\infty]$:*

$$\mathbb{P}(\|z^*\|_{L^{q_1}((0,T),W^{s,q_2}(U))} > \lambda) \lesssim_\varepsilon \exp\left(-\frac{c\lambda^2}{\max\left\{T^{\frac{2}{q_1}}, T^{2+\frac{2}{q_1}}\right\} \|(u_0,u_1)\|^2_{\mathcal{H}^{s+\varepsilon}}} \right) \tag{3.10}$$

with $\varepsilon = 0$ if $q_2 < \infty$ and $\varepsilon > 0$ arbitrarily small otherwise.
For any $q_2 \in [2,\infty]$ and arbitrarily small $\varepsilon > 0$:

$$\mathbb{P}(\|z^*\|_{L^\infty((0,T),W^{s,q_2}(U))} > \lambda) \lesssim_\varepsilon (1+T) \exp\left(-\frac{c\lambda^2}{\max\{1,T^2\}\|(u_0,u_1)\|^2_{\mathcal{H}^{s+\varepsilon}}} \right)\,. \tag{3.11}$$

For our purposes we will need a counterpart of Proposition 3.4 in the context of Besov spaces:

Proposition 3.5 (Besov norm probabilistic Strichartz estimates)**.** *Let $(u_0,u_1) \in \mathcal{H}^s(U)$ where U stands for either $\mathbb{T}^3$ or $\mathbb{R}^3$ and still write (u_0,u_1) its randomisation (Fourier randomisation procedure or Wiener randomisation procedure). Let z^* stand for either $z, \tilde{z}, z_n$ or $\tilde{z}_n$. For any $q_1 \in [1,\infty)$, $q_2 \in [2,\infty]$ and $r \in [1,\infty]$:*

$$\mathbb{P}\left(\|z^*\|_{L^{q_1}((0,T),B^s_{q_2,r})} > \lambda\right) \lesssim_\varepsilon \exp\left(-c\frac{\lambda^2}{\max\{T^{\frac{2}{q_1}}, T^{2+\frac{2}{q_1}}\}\|(u_0,u_1)\|^2_{\mathcal{H}^{s+\varepsilon}}} \right) \tag{3.12}$$

with $\varepsilon = 0$ if $q_2 < \infty$ and $r \geqslant 2$; $\varepsilon > 0$ otherwise.
For any $q_2 \in [2,\infty]$ and $r \in [1,\infty]$:

$$\mathbb{P}\left(\|z^*\|_{L^\infty((0,T),B^s_{q_2,r})} > \lambda\right) \lesssim_\varepsilon \exp\left(-c(\varepsilon)\frac{\lambda^2}{\langle T\rangle^{2\varepsilon}\|(u_0,u_1)\|^2_{\mathcal{H}^{s+\varepsilon}}} \right) \tag{3.13}$$

with $\varepsilon > 0$.

Remark 3.18. Note that the estimate (3.13) differs slightly from (3.11). The proof presented here will indeed differ from the one in [OP16] which appears in the context of Lebesgue spaces and relies on a series representation for $z(t)$, a method that we decided not to use and present an alernative method. However, by applying the method of Oh-Pocovnicu one can prove a similar estimate.

Proof. The proof follows closely the one in [OP16, OP17, Poc17] as only the parameter r has been added in the analysis. Nonetheless the proof is given in quite extensive details for reader's convenience. We will only give the proof in the case $U=\mathbb{R}^3$ since the computations are almost the same in $\mathbb{T}^3$. It is indeed only the randomisation setup which differs but in both cases they satisfy the same smoothing properties, see Appendix A. We will also assume that $z^*(t)=z(t)$, other cases could be treated in the exact same way.

Step 1. Assume $s=0$, $q_1<\infty$ and $q_2<\infty$. Without loss of generality we can assume that $r<\infty$ thanks to the inequality $\|z\|_{L^{q_1}((0,T),B^0_{q_2,\infty})}\lesssim \|z\|_{L^{q_1}((0,T),L^{q_2})}$ and (3.10). Assume first that $r\geqslant 2$. We will prove that for $p\geqslant \max\{q_1,q_2,r\}$ one has

$$\|z\|_{L^p(\Omega,L^{q_1}((0,T),B^0_{q_2,r}))}\lesssim \sqrt{p}T^{1+\frac{1}{q_1}}\|(u_0,u_1)\|_{\mathcal{H}^0}. \tag{3.14}$$

Assume that (3.14) is proved, then the Markov inequality (with the function $\lambda\mapsto\lambda^p$) gives

$$\mathbb{P}\left(\|z\|_{L^{q_1}((0,T),B^0_{q_2,r})}>\lambda\right)\leqslant \lambda^{-p}\,\|z\|^p_{L^p(\Omega,L^{q_1}((0,T),B^0_{q_2,r}))}$$

and minimizing in p yields (3.12). It is indeed the case when the optimizing p is such that $p\geqslant\max\{q_1,q_2,r\}$, otherwise just take C large enough to ensure $Ce^{-\max\{q_1,q_2,r\}}\geqslant 1$ and write

$$\mathbb{P}\left(\|z\|_{L^{q_1}((0,T),B^0_{q_2,r})}>\lambda\right)\leqslant 1\leqslant Ce^{-\max\{q_1,q_2,r\}}\leqslant Ce^{-p}\,,$$

which ends the proof of (3.12).

The proof now reduces to the one of (3.14). Since $z(t)=\cos(t|\nabla|)u_0+\frac{\sin(t|\nabla|)}{|\nabla|}u_1$ we assume that $z(t)=\frac{\sin(t|\nabla|)}{|\nabla|}u_1$ and only estimate this term (the other is even simpler to handle and we omit the details). Set $p\geqslant\max\{q_1,q_2,r\}$, use the integral Minkowski inequality and Theorem 3.17:

$$\begin{aligned}A &:= \left\|\frac{\sin(t|\nabla|)}{|\nabla|}u_1\right\|_{L^p(\Omega,L^{q_1}((0,T),B^0_{q_2,r}))}\leqslant \left\|\Delta_j\frac{\sin(t|\nabla|)}{|\nabla|}u_1\right\|_{L^{q_1}_T\ell^r_j(\mathbb{N},L^{q_2})L^p(\Omega)}\\ &\lesssim \sqrt{p}\left\|\left\|\psi(D-n)\frac{\sin(t|\nabla|)}{|\nabla|}\Delta_j u_1\right\|_{\ell^2_n(\mathbb{Z})}\right\|_{L^{q_1}((0,T),\ell^r_j(\mathbb{N},L^{q_2}))}\end{aligned}$$

As $q_2 \geqslant 2$, the Minkowski inequality followed by the Bernstein inequality on $\psi(D-n)$ (see 3.28) imply

$$
\begin{aligned}
A &\lesssim \sqrt{p}\left\|\left\|\psi(D-n)\frac{\sin(t|\nabla|)}{|\nabla|}\Delta_j u_1\right\|_{L^{q_2}}\right\|_{L^{q_1}((0,T),\ell^r_j(\mathbb{N},\ell^2_n(\mathbb{Z})))}\\
&\lesssim \sqrt{p}\left\|\left\|\psi(D-n)\frac{\sin(t|\nabla|)}{|\nabla|}\Delta_j u_1\right\|_{L^{2}}\right\|_{L^{q_1}((0,T),\ell^r_j(\mathbb{N},\ell^2_n(\mathbb{Z})))}\\
&\lesssim \sqrt{p}\left\|\left(t^2\left\|\frac{\psi(D)\sin(t|\nabla|)}{t|\nabla|}\Delta_j u_1\right\|_{L^2}^2+\sum_{|n|\geqslant 1}\left\|\frac{\psi(D-n)\sin(t|\nabla|)}{|\nabla|}\Delta_j u_1\right\|_{L^2}^2\right)^{1/2}\right\|_{L^{q_1}((0,T),\ell^r_j(\mathbb{N}))}.
\end{aligned}
$$

The use of the elementary inequalities $\left|\frac{\sin x}{x}\right| \leqslant 1$ and $|\sin(x)| \leqslant 1$ for all $x \in \mathbb{R}$ along with the Bernstein inequality give:

$$
\begin{aligned}
A &\lesssim \sqrt{p}\left\|\left(t^2\left\|\psi(D)\Delta_j u_1\right\|_{L^2}^2+\sum_{|n|\geqslant 1}\left\|\frac{\psi(D-n)}{|\nabla|}\Delta_j u_1\right\|_{L^2}^2\right)^{1/2}\right\|_{L^{q_1}((0,T),\ell^r_j(\mathbb{N}))}\\
&\lesssim \sqrt{p}\max\left\{T^{\frac{1}{q_1}},T^{1+\frac{1}{q_1}}\right\}\left\|\left(\sum_{n\in\mathbb{Z}}\left\|\psi(D-n)\Delta_j(\langle\nabla\rangle^{-1}u_1)\right\|_{L^2}^2\right)^{1/2}\right\|_{\ell^r_j(\mathbb{N})},
\end{aligned}
$$

and using (3.6) as well as the definition of Besov spaces this implies

$$
\left\|\frac{\sin(t|\nabla|)}{|\nabla|}u_1\right\|_{L^p(\Omega,L^{q_1}((0,T),B^0_{q_2,r}))} \lesssim \sqrt{p}\max\left\{T^{\frac{1}{q_1}},T^{2+\frac{1}{q_1}}\right\}\|u_1\|_{B^{-1}_{q_2,r}}.
$$

When $r \geqslant 2$ the conclusion follows from the fact that $H^{-1} \simeq B^{-1}_{2,2} \hookrightarrow B^{-1}_{2,r}$, see Theorem 3.31. For $r < 2$ it follows from the fact that for all $\varepsilon > 0$ arbitrarly small, $H^{-1+\varepsilon} \hookrightarrow B^{-1+\varepsilon}_{2,2} \hookrightarrow B^{-1}_{2,r}$ which explains the loss of derivatives. See also Theorem 3.31.

Step 2. The case where $s > 0$, $q_1 < \infty$ and $q_2 < \infty$ is inferred by the case $s = 0$ using that $\langle\nabla\rangle^s$ commutes with semi-groups $S(t), \tilde{S}(t)$.

Step 3. The case where $s > 0$, $q_1 < \infty$ and $q_2 = \infty$ follows from Sobolev-Besov continuous embeddings given in Theorem 3.31 in the usual manner: for $q_2 > \frac{3}{\varepsilon}$, $B^{\varepsilon}_{q_2,r} \hookrightarrow B^{\varepsilon}_{\infty,r}$ so that the conclusion follows with an ε loss of derivatives.

Step 4. Assume that $(s, q_1) = (0, \infty)$ and $r \in [2, \infty)$ which is the last case we need to address. Other cases will follow from the use of Step 2 and Step 3.

Let q large enough such that $\varepsilon q > 1$, for example $q = \frac{2}{\varepsilon}$, which ensures that the embedding $W^{\varepsilon,q}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is continuous. Then

$$
\|z\|_{L^\infty((0,T),B^0_{q_2,r})} \lesssim_\varepsilon \langle T\rangle^\varepsilon \left\|\langle t\rangle^{-\varepsilon}z\right\|_{W^{\varepsilon,q}(\mathbb{R},B^0_{q_2,r})}.
$$

Note that similarly as in Step 1-3 (see [BT08a], Proposition A.5 for details) that with $\delta := \frac{2}{q_1}$ one has

$$\mathbb{P}\left(\left\|\langle t\rangle^{-\delta} z\right\|_{L^{q_1}(\mathbb{R},B^s_{q_2,r})} > \lambda\right) \lesssim_\varepsilon \exp\left(-c\frac{\lambda^2}{\|(u_0,u_1)\|^2_{\mathcal{H}^{s+\varepsilon}}}\right) \tag{3.15}$$

with $\varepsilon = 0$ if $q_2 < \infty$ and $r \geqslant 2$; $\varepsilon > 0$ otherwise.

Now, using the representation of z in terms of exponentials rather than trigonometric we can assume without loss of generality that $z(t) = e^{it|\nabla|}\phi$ where $\phi \in B^\varepsilon_{q_2,r}$. For clarity reasons set $\chi(t) := \langle t\rangle^{-\varepsilon}$ and observe that in order to conclude we only need to prove

$$\|\chi z\|_{W^{\varepsilon,q}(\mathbb{R},B^0_{q_2,r})} \lesssim_\varepsilon \|\chi z\|_{L^q(\mathbb{R},B^\varepsilon_{q_2,r})}\,, \tag{3.16}$$

since conditionally to (3.16), the estimate (3.15) applies to the latter norm.

In order to do so, remark that $\|\chi z\|_{W^{\varepsilon,q}(\mathbb{R},B^0_{q_2,r})} \lesssim \|\langle D_t\rangle^\varepsilon(\chi z)\|_{L^q(\mathbb{R},B^0_{q_2,r})}$. Next we write

$$\begin{aligned}
\langle D_t\rangle^\varepsilon \chi &= \chi\langle D_t\rangle^\varepsilon + [\langle D_t\rangle^\varepsilon,\chi] \\
&= \left(1 + [\langle D_t\rangle^\varepsilon,\chi]\,\langle D_t\rangle^{-\varepsilon}\chi^{-1}\right)\chi\langle D_t\rangle^\varepsilon \\
&:= A\chi\langle D_t\rangle^\varepsilon.
\end{aligned}$$

Now remark that $\langle D_t\rangle^\varepsilon\chi$ is a pseudo-differential operator with symbol in S^ε and χ, χ^{-1} are pseudo-differential operators of order zero. The standard pseudo-differential calculus now shows that $[\langle D_t\rangle^\varepsilon,\chi]$ is of order $\varepsilon - 1 < 0$ and thus A is of order zero. Such operators are known to be continuous on L^q, see [Hör65] for instance. This yields

$$\|\chi z\|_{W^{\varepsilon,q}(\mathbb{R},B^0_{q_2,r})} \lesssim_\varepsilon \|\chi\langle D_t\rangle^\varepsilon z\|_{L^q(\mathbb{R},B^0_{q_2,r})}\,. \tag{3.17}$$

To finish the proof of (3.16) remark that

$$\langle D_t\rangle^\varepsilon z = \langle\nabla\rangle^\varepsilon z \text{ in } \mathcal{S}'(\mathbb{R}\times\mathbb{R}^3)$$

thus when plugged in (3.17) this implies (3.16) and concludes the proof. □

3. Proof of Theorem 3.4

We provide the proof of Theorem 3.4 in the case of $U = \mathbb{R}^3$. The case of the torus $\mathbb{T}^3$ is very similar as the Littlewood-Paley analysis works the same. For other adaptations to the case of $\mathbb{T}^3$ see proof of the probabilistic well-posedness in the subcritical regime $3 < p < 5$ in [SX16] and the proof in the critical regime $p = 5$ in [OP17].

3.1. Global strong solutions for the regularized system. In order to derive *a priori* energy estimates for (SLW$_p$) we first construct global strong solutions for approximate equations. In order to do so we use a smooth truncation in frequencies, which will prove helpful in the following.

Set $f(x) = |x|^{p-1}x$ and consider the regularized equation for $n \geqslant 1$:

$$\text{(rSLW}_p^n\text{)} \qquad \begin{cases} \partial_t^2 u_n - \mathbf{P}_n \Delta u_n + \mathbf{P}_n f(u_n) = 0, \\ (u_n(0), \partial_t u_n(0)) = (\mathbf{P}_n u_0, \mathbf{P}_n u_1) \in \mathcal{H}^1(\mathbb{R}^3)\,. \end{cases}$$

We prove existence of a unique global solution $(u_n, \partial_t u_n)$ in the space

$$X_n := L^2_n(\mathbb{R}^3) \times L^2_n(\mathbb{R}^3) \text{ where } L^2_n(\mathbb{R}^3) := \{f \in L^2(\mathbb{R}^3),\ \mathbf{P}_n g = g\}.$$

Endowed with the norm $\|(u,v)\|_{X_n} := \|u\|_{L^2} + \|v\|_{L^2}$, X_n is a Banach space.

Proposition 3.6 (Study of (rSLW$_p^n$))**.** *There exists unique global strong solutions $(u_n)_{n\geqslant 1}$to the equations (rSLW$_p^n$) that belong to the spaces X_n. Moreover $u_n \in H^1 \cap L^{p+1}$ and for every $n \geqslant 1$, every $t \in \mathbb{R}$:*

$$\begin{aligned} (3.18) \qquad E_{\text{reg}}(u_n(t), \partial_t u_n(t)) &:= \int_\Omega \left(\frac{|\partial_t u_n(t)|^2}{2} + \frac{|\nabla u_n(t)|^2}{2} + \frac{|u_n(t)|^{p+1}}{p+1} \right) \mathrm{d}x \\ &= E_{\text{reg}}(\mathbf{P}_n u_0, \mathbf{P}_n u_1). \end{aligned}$$

Proof. The proof is standard as local existence and uniqueness is achieved via the Picard-Lindelöf theorem, and the global existence will result from energy conservation. The remaining of this proof provides details of this classical scheme. Before starting the proof, remark that by the time reversibility of (rSLW$_p^n$) it is sufficient to show existence and uniqueness of global solutions on the time interval $\mathbb{R}_+$.

We start by proving that the equation (rSLW$_p^n$) is locally well-posed in $\mathcal{C}^1(\mathbb{R}_+, X_n)$. It is a consequence of the Picard-Lindelöf theorem once we have written (rSLW$_p^n$) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} U_n(t) = F_n(U_n(t))$$

with $U_n(t) := (u_n(t), \partial_t u_n(t))$ and $F_n(u,v) := (v, \mathbf{P}_n \Delta u - \mathbf{P}_n f(u))$.

In order to be applied, the Picard-Lindelöf theorem requires the map F_n to be locally Lipschitz on X_n. As $u \mapsto \mathbf{P}_n \Delta u$ is linear and continuous from L^2_n into itself, it is locally Lipschitz. Observe that for $u \in L^2_n$, the Bernstein inequality proves that $u \in L^\infty$ and more precisely observe that $\|u\|_{L^\infty} \lesssim 2^{\frac{3n}{2}} \|u\|_{L^2}$ so that F_n is well-defined from L^2_n into itself. Finally let $(u,v), (u',v') \in X_n$ satisfying

$\|(u,v)\|_{X_n}, \|(u',v')\|_{X_n} \leqslant R$ and compute:

$$\begin{aligned}\|F_n(u,v)-F_n(u',v')\|_{X_n} &\leqslant \|v-v'\|_{L^2}+\|\mathbf{P}_n\Delta(u-u')\|_{L^2}\\ &\quad+\|\mathbf{P}_n(f(u)-f(u'))\|_{L^2}\\ &\lesssim_n \|v-v'\|_{L^2}+\|u-u'\|_{L^2}+\|f(u)-f(u')\|_{L^2}\\ &\lesssim_n \|(u,v)-(u',v')\|_{X_n}.\end{aligned}$$

Notice that in the last inequality we used that

$$\||u|^{p-1}u-|v|^{p-1}v\|_{L^2} \lesssim_n (\|u\|_{L^\infty}^{p-1}+\|v\|_{L^\infty}^{p-1})\|u-v\|_{L^2} \lesssim_n R^{p-1}\|u-v\|_{L^2}.$$

The Picard-Lindelöf theorem applies and gives rise to unique solutions defined on maximal time intervals that we denote $[0,T_n)$. These solutions belong to $\mathcal{C}^1([0,T_n),X_n)$.

In order to derive the energy estimates, if we prove that u_n has regularity $\mathcal{C}^2$ in both space and time, then it is sufficient to multiply (rSLW$_p^n$) by $\partial_t u_n(t)$, integrate by parts in space and use the fact that the operator $\mathbf{P}_n$ is symmetric in L^2 to obtain (3.18). Time regularity is granted from the regularity given by the Picard-Lindelöf theorem. For space regularity observe that since $u_n \in L^2_n$, the derivation is a continuous mapping of L^2_n, thus $u_n \in H^k$ for all $k \geqslant 0$. The Sobolev embedding theorem proves the required smoothness in space for u_n.

Let $n \geqslant 1$. We prove that $T_n=+\infty$. Remark that the energy equality (3.18) proves that

$$\sup_{t\in[0,T_n)} (\|\nabla u_n(t)\|_{L^2}+\|\partial_t u_n(t)\|_{L^2}) < \infty.$$

Assume that $T_n<\infty$. Then for $t\in(0,T_n)$

$$\|u_n(t)\|_{L^2} \leqslant \|u_0\|_{L^2}+\int_0^t \|\partial_t u_n(s)\|_{L^2}\,\mathrm{d}s \leqslant \|u_0\|_{L^2}+T_n \sup_{t\in(0,T_n)} \|\partial_t u(t)\|_{L^2}$$

which yields $\sup_{t\in[0,T_n)}\|u_n(t)\|_{L^2}<\infty$.

Note that the Bernstein inequality implies $\|\mathbf{P}_n\Delta(u_n)\|_{L^2} \lesssim_n \|u_n\|_{L^2}$ and $\|\mathbf{P}_n f(u_n(t))\|_{L^2} \lesssim_n \|u_n(t)\|_{L^{2p}}^p \lesssim_n \|u_n(t)\|_{L^2}^p$ so that:

$$\sup_{t\in[0,T_n)}\left\|\frac{\mathrm{d}}{\mathrm{d}t}U_n(t)\right\|_{X_n} = \sup_{t\in[0,T_n)}\{\|\partial_t u_n(t)\|_{L^2}+\|\mathbf{P}_n\Delta u_n(t)-\mathbf{P}_n f(u_n(t))\|_{L^2}\}<\infty$$

which allow to construct a continuation for U_n at $t=T_n$ and contradicts the maximality of T_n. Finally $T_n=\infty$. □

We now write $u_n(t)=z_n(t)+v_n(t)$ where z_n has been introduced in (3.9), with initial data $(z_n(0),\partial_t z_n(0))=\mathbf{P}_n(u_0,u_1)$, and v_n satisfying

$$(3.19)\qquad \begin{cases} \partial_t^2 v_n(t)-\mathbf{P}_n\Delta v_n(t)+\mathbf{P}_n\left((z_n(t)+v_n(t))|z_n(t)+v_n(t)|^{p-1}\right)=0,\\ (v_n(0),\partial_t v_n(0))=(0,0).\end{cases}$$

3.2. A priori estimates for the regularized system. In order to pass to the limit $n \to \infty$, one needs uniform estimates for (rSLW$_p^n$). As we expect the linear part to be handled in a simple way we may focus on the nonlinear part v_n, satisfying (3.19), and introduce its nonlinear energy

$$E_n(t) := \int_{\mathbb{R}^3} \left(\frac{|\partial_t v_n(t)|^2}{2} + \frac{|\nabla v_n(t)|^2}{2} + \frac{|v_n(t)|^{p+1}}{p+1} \right) \mathrm{d}x \,.$$

Let $s_p := \frac{p-3}{p-1}$. In this subsection we prove the following uniform bound.

Proposition 3.7 (Probabilistic a priori estimates). *Let $s > s_p$, and $p > 5$. Let $T > 0$ and $\eta \in (0,1)$. There exists a measurable set $\Omega_{T,\eta} \subset \Omega$ and a constant $C(T, \eta, \|(u_0, u_1)\|_{\mathcal{H}^s})$ which depends only on $T, \eta, \|(u_0, u_1)\|_{\mathcal{H}^s}$, such that:*

(i) $\mathbb{P}(\Omega_{T,\eta}) \geqslant 1 - \eta$.

(ii) For every $\omega \in \Omega_{T,\eta}$, if the initial data for u_n is attached to ω via the randomisation map of (u_0, u_1) then:

$$\sup_{n \geqslant 0} \sup_{t \in (-T,T)} E_n(t) \leqslant C(T, \eta, \|(u_0, u_1)\|_{\mathcal{H}^s}).$$

The cornerstone of the proof of Proposition 3.7, which will allow to close the energy estimates in the Grönwall argument is the following. We introduce $\alpha_p := \lceil \frac{p-3}{2} \rceil$ and recall that $f(x) = |x|^{p-1}x$.

Lemma 3.19. *For every $1 \leqslant k \leqslant \alpha_p$ one has:*

$$\left| \int_{\mathbb{R}^3} f^{(k-1)}(v_n(t)) z_n(t)^{k-1} \langle \nabla \rangle \tilde{z}_n(t) \, \mathrm{d}x \right| \lesssim g\left(\|z_n\|_{L^\infty((-T,T),X)}, \|\tilde{z}_n\|_{L^\infty((-T,T),Y)} \right) \times (1 + E_n(t)) \,. \tag{3.20}$$

where g is a polynomial with positive coefficients,

$$X := L^\infty \cap L^{\frac{p+1}{2}} \cap B^{1-s_p}_{q_2,1} \cap B^{1-s_p}_{q_{\alpha_p},1} \text{ and } Y := L^\infty \cap B^{s_p}_{\infty,1}$$

where for $2 \leqslant k \leqslant \alpha_p$, q_k being defined by $\frac{1}{q_k} + \frac{p-k+1}{p+1} = 1$.

Remark 3.20. Observe that for $p > 5$ we have $1 - s_p < s_p$.

For exposition reasons we postpone its proof, and prove Proposition 3.7 assuming Lemma 3.19.

Proof of Proposition 3.7. Once again, by time reversibility, we will prove an *a priori* estimate on $[0, T)$ rather than $(-T, T)$. We will first find a large measure set allowing to prove the desired estimates. Note that the forthcoming constraints in the definition of $\Omega_{T,\eta}$ are designed to control all the terms requiring bounds for the linear parts z_n or $\tilde{z}_n$ that will be proven in the following. Let $\lambda > 0$, which will be chosen later. Set

$$\Omega_{T,\eta} := \Big\{ \|z_n\|_{L^\infty_T L^{p+1}} + \|z_n\|_{L^{2p}_T L^{2p}} + \|z_n\|^{\alpha_p+1}_{L^\infty_T L^{r_p(\alpha_p+1)}} + g\left(\|z_n\|_{L^\infty((0,T),X)}, \|\tilde{z}_n\|_{L^\infty((0,T),Y)}\right) \leqslant \lambda, \text{for all } n \geqslant 1 \Big\}, \tag{3.21}$$

where r_p is defined by $\frac{1}{2} + \frac{p-\alpha_p-1}{p+1} + \frac{1}{r_p} = 1$.

For λ large enough, depending only on the initial data, T and η, we can assume $\mathbb{P}(\Omega_{T,\eta}) \geqslant 1 - \eta$ thanks to Proposition 3.4 and Proposition 3.5, as soon as $s > s_p$. The fact that λ does not depend on n comes from the inequality $\|\mathbf{P}_n(u_0, u_1)\|_{\mathcal{H}^s} \leqslant \|(u_0, u_1)\|_{\mathcal{H}^s}$.

Let $\omega \in \Omega_{T,\eta}$. From now on the following estimates will be deterministic as we have fixed the initial data (attached to ω *via* the randomisation map). We will carry out the computations for $s = s_p$.

Recall that $\mathbf{P}_n$ is a symmetric operator in $L^2(\mathbb{R}^3)$ and that v_n is smooth. This allows one to compute $\frac{\mathrm{d}}{\mathrm{d}t}E_n(t)$ and obtain:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E_n(t) &= \int_{\mathbb{R}^3} \partial_t v_n(t) \left((\partial_t^2 v_n(t) - \mathbf{P}_n \Delta v_n(t) + \mathbf{P}_n(v_n(t)|v_n(t)|^{p-1}) \right) \mathrm{d}x \\ &= -\int_{\mathbb{R}^3} \partial_t v_n(t) \left(f(z_n(t) + v_n(t)) - f(v_n(t)) \right) \mathrm{d}x. \end{aligned} \tag{3.22}$$

Next we expand the nonlinearity f at the point $v(t,x)$ using the Taylor formula with integral remainder up to the order $\alpha_p = \lceil \frac{p-3}{2} \rceil$. For convenience we drop the t, x references and recall that for $k \geqslant 0$,

$$f^{(k)}(x) = \left\{ \begin{array}{lll} C_{p,k} x |x|^{p-k-1} & \text{for} & k \text{ even}, \\ C_{p,k} |x|^{p-k} & \text{for} & k \text{ odd}, \end{array} \right.$$

thus

$$f(v_n + z_n) - f(v_n) = \sum_{k=1}^{\alpha_p} f^{(k)}(v_n) z_n^k + \underbrace{\int_{v_n}^{v_n+z_n} \frac{f^{(\alpha_p+1)}(s)}{\alpha_p!} (v_n + z_n - s)^{\alpha_p} \, \mathrm{d}s}_{R(z_n, v_n)} .$$

One can integrate (3.22) and use $E(0) = 0$ and $(v_n(0), \partial_t v_n(0)) = (0,0)$ so that we can write:

$$E_n(t) = \sum_{k=1}^{\alpha_p} C_{k,p} I_n^{(k)} + C_p R_n \,, \tag{3.23}$$

where

$$\begin{aligned} I_n^{(k)} &:= -\int_0^t \int_{\mathbb{R}^3} \partial_t v_n f^{(k)}(v_n) z_n^k \, \mathrm{d}x \, \mathrm{d}t' \text{ for } 1 \leqslant k \leqslant \alpha_p, \\ R_n &:= -\int_0^t \int_{\mathbb{R}^3} \partial_t v_n R(z_n, v_n) \, \mathrm{d}x \, \mathrm{d}t' \,. \end{aligned}$$

We first estimate R_n as we expect it to be simpler to handle. Remark that for $\theta_n \in [v_n, v_n + z_n]$ one has

$$|f^{(\alpha_p+1)}(\theta_n)| \lesssim |v_n + z_n|^{p-\alpha_p-1} + |v_n|^{p-\alpha_p-1} \lesssim |v_n|^{p-\alpha_p-1} + |z_n|^{p-\alpha_p-1}\,,$$

so that

$$R(v_n, z_n) \lesssim |z_n|^p + |v_n|^{p-\alpha_p-1}|z_n|^{\alpha_p+1}\,.$$

The Hölder inequality and the Young inequality give:

$$\begin{aligned}
R_n &\lesssim \int_0^t \|\partial_t v_n(t')\|_{L^2}\|v_n(t')\|_{L^{p+1}}^{p-\alpha_p-1}\|z_n(t')\|_{L^{r_p(\alpha_p+1)}}^{\alpha_p+1}\,\mathrm{d}t' + \int_0^t \|\partial_t v_n(t')\|_{L^2}\|z_n(t')\|_{L^{2p}}^p\,\mathrm{d}t' \\
&\lesssim \int_0^t \|\partial_t v_n(t')\|_{L^2}\|v_n(t')\|_{L^{p+1}}^{p-\alpha_p-1}\|z_n(t')\|_{L^{r_p(\alpha_p+1)}}^{\alpha_p+1}\,\mathrm{d}t' + \int_0^t \|\partial_t v_n(t')\|_{L^2}^2\,\mathrm{d}t' + \|z_n\|_{L_T^{2p}L^{2p}}^{2p} \\
&\lesssim \|z_n\|_{L_T^{2p}L^{2p}}^{2p} + \left(1 + \|z_n\|_{L_T^\infty L^{r_p(\alpha_p+1)}}^{\alpha_p+1}\right)\int_0^t \max\left\{E_n(t'), E_n(t')^{\frac{1}{2}+\frac{p-\alpha_p-1}{p+1}}\right\}\mathrm{d}t'\,.
\end{aligned}$$

Observe that $\frac{p-\alpha_p-1}{p+1} \leqslant \frac{1}{2}$ since $\alpha_p = \lceil \frac{p-3}{2} \rceil \geqslant \frac{p-3}{2}$ so that

$$R_n \lesssim \|z_n\|_{L_T^{2p}L^{2p}}^{2p} + \left(1 + \|z_n\|_{L_T^\infty L^{r_p(\alpha_p+1)}}^{\alpha_p+1}\right)\int_0^t (1 + E_n(t'))\,\mathrm{d}t'\,.$$

Bounds for the terms $I_n^{(k)}$ require a more intricate analysis and will follow from Lemma 3.19. More precisely, let $1 \leqslant k \leqslant \alpha_p$. First apply Fubini's theorem and write:

$$I_n^{(k)} = -\int_{\mathbb{R}^3}\int_0^t \partial_t(f^{(k-1)}(v_n))z_n^k\,\mathrm{d}t'\,\mathrm{d}x.$$

Integrate by parts in time so that

$$I_n^{(k)} = -\int_{\mathbb{R}^3} f^{(k-1)}(v_n(t))z_n^k(t)\,\mathrm{d}x + k\int_{\mathbb{R}^3}\int_0^t f^{(k-1)}(v_n(t'))\partial_t z_n(t')z_n(t')^{k-1}\,\mathrm{d}t'\,\mathrm{d}x.$$

Observe that $\partial_t z_n = \langle\nabla\rangle\tilde{z}_n$ and bound

$$\begin{aligned}
|I_n^{(k)}| &\lesssim \int_{\mathbb{R}^3}|z_n(t)|^k|v_n(t)|^{p-k+1}\,\mathrm{d}x + \left|\int_0^t\int_{\mathbb{R}^3} f^{(k-1)}(v_n(t'))\langle\nabla\rangle\tilde{z}_n(t')z_n(t')^{k-1}\,\mathrm{d}t'\,\mathrm{d}x\right| \\
&= J_n^{(k)} + K_n^{(k)}.
\end{aligned}$$

In order to handle $J_n^{(k)}$, we use Hölder and Young's inequality:

$$J_n^{(k)}(t) \leqslant E_n(t)^{\frac{p-k+1}{p+1}}\|z_n\|_{L^{p+1}}^k \leqslant \frac{1}{2}E_n(t) + C\|z_n\|_{L_T^\infty L^{p+1}}^{p+1}.$$

$K_n^{(k)}$ is more difficult to study and is estimated via Lemma 3.19, so that

$$K_n^{(k)} \lesssim g\left(\|z_n\|_{L^\infty((-T,T),X)}, \|\tilde{z}_n\|_{L^\infty((-T,T),Y)}\right)\left(1 + \int_0^t E_n(t')\,\mathrm{d}t'\right).$$

Finally using the bounds from (3.21) we have

$$\begin{aligned}E_n(t) &\lesssim \left(1 + \|z_n\|^{\alpha_p+1}_{L^\infty_T L^{r_p(\alpha_p+1)}} + g\left(\|z_n\|_{L^\infty((0,T),X)}, \|\tilde z_n\|_{L^\infty((0,T),Y)}\right)\right)\\ &\quad\times\left(1+\int_0^t E_n(t')\,\mathrm{d}t'\right) + \|z_n\|^{2p}_{L^{2p}_T L^{2p}} + \|z_n\|^{p+1}_{L^\infty_T L^{p+1}}\\ &\lesssim 1 + \int_0^t E_n(t')\,\mathrm{d}t'\,.\end{aligned}$$

Knowing that the implicit constant does not depend on n, but only on η, T, p, the Grönwall lemma ends the proof. □

It remains to prove Lemma 3.19. Its proof will require a chain rule estimate in Besov spaces whose proof is similar to the one of Theorem 2.61 in [BCD11].

Lemma 3.21 (Chain rule estimates in Besov spaces). *Let $u \in \mathcal{S}'$, $s \in (0,1)$ and $p > 3$. Let $q, r \in [1,\infty]$ and $q_1, q_2 \in [1,\infty]$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Let f denote the function defined by $f(x) = x|x|^{p-1}$ or $f(x) = \operatorname{sgn}(x)x|x|^{p-1} = |x|^p$. Then the following identities hold:*

(i) $\|f(u)\|_{B^s_{q,r}} \lesssim \|u\|_{B^s_{q_1,r}} \||u|^{p-1}\|_{L^{q_2}}$, and $\|f(u)\|_{\dot B^s_{q,r}} \lesssim \|u\|_{\dot B^s_{q_1,r}} \||u|^{p-1}\|_{L^{q_2}}$.

(ii) $\|f(u)\|_{B^s_{q,r}} \lesssim_\varepsilon \|u\|_{W^{s+\varepsilon,q_1}} \||u|^{p-1}\|_{L^{q_2}}$ for all $\varepsilon > 0$ such that $s + \varepsilon < 1$.

Proof of Lemma 3.21. The proof uses Lemma 3.30. Since $\mathbf{P}_j u \underset{j\to\infty}{\longrightarrow} u$ in L^p and $f(0) = 0$ we have

$$f(u) = \sum_{j\geqslant 0} f_j \text{ with } f_j := f(\mathbf{P}_{j+1}u) - f(\mathbf{P}_j u).$$

The Taylor formula at order 1 writes:

$$f_j = \Delta_j u\, m_j \text{ with } m_j := \int_0^1 f'(\mathbf{P}_j u + t\Delta_j u)\,\mathrm{d}t.$$

In view of Lemma 3.30 we will focus on estimating $\|\partial^\alpha f_j\|_{L^p}$ with $|\alpha| \leqslant \lfloor s\rfloor + 1 = 1$. For $|\alpha| = 0$ write that $\|\partial^\alpha f_j\|_{L^p} \leqslant \|m_j\|_{L^{q_2}}\|\Delta_j u\|_{L^{q_1}}$. Then we estimate m_j:

$$|m_j| \leqslant p\int_0^1 |\mathbf{P}_j u + t\Delta_j u|^{p-1}\,\mathrm{d}t \lesssim |\mathbf{P}_j u|^{p-1} + |\Delta_j u|^{p-1}$$

where we used that $|f'(x)| \leqslant p|x|^{p-1}$ for $x \in \mathbb{R}$ and $(a+b)^{p-1} \lesssim a^{p-1} + b^{p-1}$ for $a, b > 0$.

Then $\|m_j\|_{L^{q_2}} \lesssim \|\mathbf{P}_j u\|^{p-1}_{L^{q_2(p-1)}}$. Similarly, when $|\alpha| = 1$ and for multi-indicies $\beta \leqslant \alpha$, the Bernstein inequality (the function to which it is applied is indeed with frequencies supported in a ball of radius $\simeq 2^j$) and the same arguments as before yield

$$\|\partial^\beta m_j\|_{L^{q_2}} \lesssim 2^{j|\beta|}\left\|\int_0^1 f'(\mathbf{P}_j u + t\Delta_j u)\,\mathrm{d}t\right\|_{L^{q_2}} \lesssim 2^{j|\beta|}\|\mathbf{P}_j u\|^{p-1}_{L^{q_2(p-1)}}\,.$$

Using the Leibniz formula $\partial^\alpha(fg) = \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\partial^{\alpha-\beta}f\partial^\beta g$ and putting all the previous estimates together we recover the estimate

$$\sup_{\alpha\leqslant\lfloor s\rfloor+1} 2^{j(s-|\alpha|)}\|\partial^\alpha u_j\|_{L^q} \lesssim 2^{js}\|\Delta_j u\|_{L^{q_1}}\|\mathbf{P}_j u\|^{p-1}_{L^{q_2(p-1)}}\,.$$

The proof of (i) now follows from the direct inequality $\|\mathbf{P}_j u\|_{L^{q_2(p-1)}} \lesssim \|u\|_{L^{q_2(p-1)}}$.

For the homogeneous counterpart of (i), replace all the appearences of $\mathbf{P}_j$ or Δ_j with $\dot{\mathbf{P}}_j$ or $\dot{\Delta}_j$ and observe that all the inequalities written still hold true. The only difficulty is proving the convergence of the series $\sum_{j\leqslant 0} f_j$ with $f_j := f(\dot{\mathbf{P}}_{j+1}u) - f(\dot{\mathbf{P}}_j u)$. This is explained in [BCD11], Lemma 2.62.

For (ii) write $2^{js} = 2^{j(s+\varepsilon)}2^{-js\varepsilon}$, and use the Hölder inequality to obtain:

$$\|f(u)\|_{B^s_{q,r}} \lesssim \|u\|_{B^{s+\varepsilon}_{q_1,\infty}} \sum_{j\geqslant 0} 2^{-j\varepsilon}\|\mathbf{P}_j u\|^{p-1}_{L^{q_2(p-1)}}. \tag{3.24}$$

Then Theorem 3.31 gives

$$\|u\|_{B^{s+\varepsilon}_{q_1,\infty}} \lesssim \|u\|_{W^{s+\varepsilon,q_1}} \text{ and } \|\mathbf{P}_j u\|^{p-1}_{L^{q_2(p-1)}} \lesssim_\varepsilon \|u\|^{p-1}_{L^{q_2(p-1)}}\,.$$

These inequalities and (3.24) end the proof. Note that there is no homogeneous counterpart of (ii). □

Proof of Lemma 3.19. We now turn the idea explained in the introduction into a mathematical proof. An efficient way of doing so is the systematic use of the Littlewood-Paley theory. Recall that $s_p = \frac{p-3}{p-1}$. In the following, estimates for $k=1$ and $k\geqslant 2$ could be different, thus we assume $k\geqslant 2$ and explain the modifications for $k=1$ at the end. The Fourier-Plancherel theorem and the fact that the contribution for $j'=-1,0,1$ are up to a universal constant identical to the case $j'=0$ yield:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} f^{(k-1)}(v_n(t))z_n(t)^{k-1}\langle\nabla\rangle\tilde{z}_n(t)\,\mathrm{d}x\right| \\
&\qquad = \left|\sum_{j'=-1}^{1}\sum_{j\geqslant 0}\int_{\mathbb{R}^3}\Delta_j(f^{(k-1)}(v_n(t))z_n(t)^{k-1})\Delta_{j+j'}(\langle\nabla\rangle\tilde{z}_n(t))\,\mathrm{d}x\right| \\
&\qquad \lesssim \sum_{j>2}\int_{\mathbb{R}^3}|\Delta_j(f^{(k-1)}(v_n(t))z_n(t)^{k-1})||\Delta_j(\langle\nabla\rangle\tilde{z}_n(t))|\,\mathrm{d}x \\
&\qquad + \sum_{j=0}^{2}\int_{\mathbb{R}^3}|\Delta_j(f^{(k-1)}(v_n(t))z_n(t)^{k-1}(t))||\Delta_j(\langle\nabla\rangle\tilde{z}_n(t))|\,\mathrm{d}x \\
&\qquad =: I_1 + I_2\,,
\end{aligned}$$

We first estimate I_2 using Hölder, Bernstein and Young's inequalities, where $r_k := \frac{(k-1)(p+1)}{k}$:

$$\begin{aligned} I_2 &\lesssim \|z_n(t)\|_{L^{r_k}}^{k-1} \|v_n(t)\|_{L^{p+1}}^{p-k+1} \sum_{j=0}^{2} \|\langle\nabla\rangle \Delta_j \tilde{z}_n(t)\|_{L^\infty_x} \\ &\lesssim \|z_n(t)\|_{L^{r_k}}^{k-1} \|\tilde{z}_n(t)\|_{L^\infty_x} E_n(t)^{\frac{p-k+1}{p+1}} \\ &\lesssim E_n(t) + \|z_n\|_{L^\infty_T L^{r_k}_x}^{r_k} \|\tilde{z}_n\|_{L^\infty_T L^\infty_x}^{\frac{p+1}{k}} . \end{aligned}$$

For I_1 observe that with Hölder and Bernstein inequalities,

$$I_1 \lesssim \sum_{j>2} 2^{j(1-s_p)} \|\Delta_j(z_n(t)^{k-1} f^{(k-1)}(v_n(t)))\|_{L^1} 2^{js_p} \|\Delta_j(\tilde{z}_n(t))\|_{L^\infty}$$

so that the Hölder inequality for series gives, as only the high frequencies appeared in the sum,

$$I_1 \lesssim \|z_n(t)^{k-1} f^{(k-1)}(v_n(t))\|_{\dot{B}^{1-s_p}_{1,\infty}} \|\tilde{z}_n(t)\|_{B^{s_p}_{\infty,1}} .$$

Then Corollary 3.35 provides us with:

$$\begin{aligned} \|z_n(t)^{k-1} f^{(k-1)}(v_n(t))\|_{\dot{B}^{1-s_p}_{1,\infty}} &\lesssim \|f^{(k-1)}(v_n(t))\|_{\dot{B}^{1-s_p}_{\frac{p+1}{p+2-k},\infty}} \|z_n(t)^{k-1}\|_{L^{\frac{p+1}{k-1}}} \\ &+ \||v_n(t)|^{p-k+1}\|_{L^{\frac{p+1}{p-k+1}}} \|z_n(t)^{k-1}\|_{\dot{B}^{1-s_p}_{q_k,\infty}} \\ &\lesssim \underbrace{\|f^{(k-1)}(v_n(t))\|_{\dot{B}^{1-s_p}_{\frac{p+1}{p+2-k},\infty}}}_{J_1} \|z_n(t)\|_{L^{p+1}}^{k-1} + E_n(t)^{\frac{p-k+1}{p+1}} \underbrace{\|z_n(t)^{k-1}\|_{\dot{B}^{1-s_p}_{q_k,\infty}}}_{J_2} , \end{aligned}$$

where we recall that $\frac{1}{q_k} + \frac{p-k+1}{p+1} = 1$. The chain rule from Lemma 3.21 will estimate the terms J_1 and J_2. For J_2, a direct application shows that:

$$J_2 \lesssim \|z_n(t)\|_{\dot{B}^{1-s_p}_{q_k,\infty}} \|z_n\|_{L^\infty}^{k-2}.$$

For J_1, also using Lemma 3.21 followed by Theorem 3.31 (*ii*) and then by the Gagliardo-Nirenberg inequality, Theorem 3.32 yield:

$$\begin{aligned} J_1 &\lesssim \|v_n(t)\|_{\dot{B}^{1-s_p}_{\frac{p+1}{2},\infty}} \||v_n(t)|^{p-k}\|_{L^{\frac{p+1}{p-k}}} \\ &\lesssim \|v_n(t)\|_{\dot{W}^{1-s_p,\frac{p+1}{2}}} E_n(t)^{\frac{p-k}{p+1}} \\ &\lesssim \|\nabla v_n(t)\|_{L^2}^{1-\alpha} \|v_n(t)\|_{L^{p+1}}^{\alpha} E_n(t)^{\frac{p-k}{p+1}}, \end{aligned}$$

where $\alpha \in [0, s_p]$ is such that $\frac{2}{p+1} = \frac{1-s_p}{3} + \frac{1-\alpha}{6} + \frac{\alpha}{p+1}$, *i.e* $\alpha = s_p = \frac{p-3}{p-1}$. Finally we get:

$$J_1 \lesssim E_n(t)^{\frac{p-k}{p+1} + \frac{\alpha}{p+1} + \frac{1-\alpha}{2}} = E_n(t)^{\frac{p-k}{p+1} + \frac{2}{p+1}}. \tag{3.25}$$

This yields $J_1 \lesssim 1 + E_n(t)$.

For $k = 1$ one can proceed in the same manner: split the left-hand side of (3.20) into I_1 and I_2 and write $I_2 \lesssim E_n(t) + \|\tilde{z}_n\|^{p+1}_{L^\infty_T L^\infty_x}$. Remark that the estimate of I_1 is handled using the same arguments as for J_1 above leading to

$$I_1 \lesssim \|f(v_n(t))\|_{\dot{B}^{1-s_p}_{1,\infty}} \lesssim E_n(t),$$

which ends the proof. □

3.3. Passing to the limit and end of the proof. The linear part is handled via the following elementary lemma:

Lemma 3.22 (Linear compactness)**.** *For every $q \in [1, \infty]$ and $r \in (1, \infty)$ one has*

$$\|z_n - z\|_{L^q((-T,T),L^r(\mathbb{R}^3))} \underset{n\to\infty}{\longrightarrow} 0\,.$$

Proof. By definition we have $z_n - z = (\mathrm{id} - \mathbf{P}_n)\, z$. As $\mathbf{P}_n$ is a mollifier it follows that for every $t \in (-T, T)$, $\|z_n(t) - z(t)\|_{L^r} \to 0$. Now observe that $\|z_n(t) - z(t)\|_{L^r} \leqslant 2\|z(t)\|_{L^r}$ and $z \in L^q((-T, T), L^r(\mathbb{R}^3))$ so that the Lebesgue convergence theorem gives the desired result. □

The nonlinear part v_n will be handled using the following compactness result.

Lemma 3.23 (Nonlinear compactness)**.** *There exists a function v that belongs to the space:*

$$H^1((-T, T) \times \mathbb{R}^3) \cap L^{p+1}((-T, T) \times \mathbb{R}^3) \cap \mathcal{C}^0([-T, T], L^2(\mathbb{R}^3))$$

such that up to a subsequence:

(i) $v_n \underset{n\to\infty}{\rightharpoonup} v$ *in* $H^1((-T, T) \times \mathbb{R}^3)$,

(ii) $v_n \underset{n\to\infty}{\longrightarrow} v$ *in* $L^2_{\mathrm{loc}}((-T, T) \times \mathbb{R}^3)$,

(iii) $v_n \underset{n\to\infty}{\longrightarrow} v$ *in* $L^p_{\mathrm{loc}}((-T, T) \times \mathbb{R}^3)$.

Proof. (i) follows from the boundedness of $(v_n)_{n\geqslant 0}$ in the space $H^1((0, T) \times \mathbb{R}^3)$ and the Banach-Alaoglu theorem in Hilbert spaces. This bound is indeed obtained via the energy control of v which immediately implies

$$\sup_{n\geqslant 0} \left\{ \|\nabla v_n\|_{L^\infty((-T,T),L^2)} + \|\partial_t v_n\|_{L^\infty((-T,T),L^2)} \right\} < \infty.$$

Since $(-T, T)$ is a bounded interval, we obtain $\sup_{n\geqslant 0} \|v_n\|_{\dot{H}^1((-T,T)\times\mathbb{R}^3)} < \infty$. The Taylor formula in time gives

$$\|v_n(t)\|_{L^2} \leqslant \|v_n(0)\|_{L^2} + \int_{-T}^{T} \|\partial_t v_n(t')\|_{L^2}\, \mathrm{d}t'.$$

Using the $L^\infty((-T, T), L^2)$ bound for $\partial_t v_n$ yields a uniform bound for v_n in $L^\infty((-T, T), L^2)$ and thus in $L^2((-T, T), L^2)$.

Finaly the sequence $(v_n)_{n\geqslant 0}$ is bounded in the space $H^1((-T,T)\times\mathbb{R}^3)$. The Banach-Alaoglu theorem proves that up to extraction we can assume that v_n is weakly convergent to a function v that belongs to $H^1((-T,T)\times\mathbb{R}^3)$. Note that the uniform bound for $\partial_t u_n$ in $L^\infty((0,T),L^2)$ allow to use the Ascoli theorem which ensures that $v\in\mathcal{C}^0([0,T],L^2)$

(ii) Let $K\subset(T,T)\times\mathbb{R}^3$ be a compact set. Then the fact that the embedding $H^1\hookrightarrow L^2_{\text{loc}}$ is compact (this is the Rellich-Kondrakov theorem), and the bound from (i) proves that up to another extraction, (ii) holds. Up to a diagonal extraction we can assume that this sequence converges for any compact set K.

(iii) We have proved local compactness for $(v_n)_{n\geqslant 0}$ in L^2 in both space and time, and a uniform bound for $(v_n)_{n\geqslant 0}$ in L^{p+1} given by Proposition 3.7. We can interpolate those two, and for every compact K:

$$\|v_n-v\|_{L^p(K)}\lesssim\|v_n-v\|^{\alpha}_{L^2(K)}\|v_n-v\|^{1-\alpha}_{L^{p+1}(K)}\lesssim\|v_n-v\|^{\alpha}_{L^2(K)},$$

with $\alpha\in(0,1)$ such that $\frac{3}{p}=\frac{\alpha}{2}+\frac{1-\alpha}{p+1}$. This proves the convergence. □

We are now ready for the proof of Theorem 3.4.

End of the proof of Theorem 3.4. Without loss of generality, assume that $s\in(\frac{p-3}{p-1},1)$, as for $s\geqslant 1$, $\mathcal{H}^s$ initial data are also in $\mathcal{H}^{1-\varepsilon}$.

Set $\Omega_{T,\frac{\eta}{2}}$ as in Proposition 3.7 and consider $\Omega'_{T,\frac{\eta}{2}}:=\{\|\langle\nabla\rangle^s z\|_{L^{p+1}((-T,T)\times\mathbb{R}^3)}\leqslant\lambda\}$ with $\lambda>0$ large enough to ensure that $\mathbb{P}(\Omega'_{T,\frac{\eta}{2}})\geqslant 1-\frac{\eta}{2}$. Now set $\tilde{\Omega}_{T,\eta}:=\Omega_{T,\eta/2}\cap\Omega'_{T,\eta/2}$ so that $\mathbb{P}(\tilde{\Omega}_{T,\eta})\geqslant 1-\eta$.

Let $m\geqslant 1$ and let $A_m=\Omega_{m,\frac{1}{m^2}}$ (i.e. take $T=m$ and $\eta=\frac{1}{m^2}$ above). Let φ be an admissible test function as in Defintion 3.1. We now work with solutions arising from $\omega\in\tilde{\Omega}:=\bigcup_{m\geqslant 0}\bigcap_{k\geqslant m}\tilde{\Omega}_{m,\frac{1}{m^2}}$.

By the compactness lemmata and up to using a diagonal extraction, we assume that along a subsequence and for all $m\geqslant 1$ there holds $u_n\rightharpoonup u$ in $H^1((-m,m)\times\mathbb{R}^3)$, and also the strong convergences $z_n\to z$ and $v_n\to v$ both in $L^p_{\text{loc}}((-m,m)\times\mathbb{R}^3)$. With the fact that φ is compactly supported in space, and supported in time in some interval $(-T,T)$ we obtain that

$$\begin{aligned}0&=\int_{-T}^{T}\int_{\mathbb{R}^3}\Big(\partial_t v_n\partial_t\varphi-\nabla v_n\cdot\nabla\varphi-(z_n+v_n)|z_n+v_n|^{p-1}\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\underset{n\to\infty}{\longrightarrow}\int_{-T}^{T}\int_{\mathbb{R}^3}\Big(\partial_t v\partial_t\varphi-\nabla v\cdot\nabla\varphi-(z+v)|z+v|^{p-1}\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t=0\,,\end{aligned}\tag{3.26}$$

that is, v is a global weak solution to (SLW$_p$).

Finally, since we have $\sum_{m\geqslant 0}\mathbb{P}(\tilde{\Omega}^c_{m,\frac{1}{m^2}})<+\infty$, the Borel-Cantelli lemma it follows that $\mathbb{P}(\tilde{\Omega})=1$. □

The proof of the continuity in time part of Corollary 3.6 is a consequence of the Aubin-Lions compactness theorem that we recall. For a proof see [BF13].

Theorem 3.24 (Aubin-Lions)**.** *Let $X_0 \hookrightarrow X \hookrightarrow X_1$ be three Banach spaces, the first embedding being compact and the second being continuous. Let $p, q \in [1, \infty]$ and*

$$W = \{u \in L^p((-T,T), X_0), \partial_t u \in L^q((-T,T), X_1)\}\,.$$

Then:

(i) If $p < \infty$ then $W \hookrightarrow L^p((-T,T), X)$ is compact.

(ii) If $p = \infty$ and $q > 1$ then $W \hookrightarrow \mathcal{C}^0([-T,T], X)$ is compact.

Proof of Corollary 3.6. Let us first prove the continuity in time. As the linear solution $(z, \partial_t z)$ has regularity $\mathcal{C}^0(\mathbb{R}, \mathcal{H}^s)$, thanks to Lemma 3.22, it is sufficient to prove the needed continuity on $(v, \partial_t v)$ on every interval $[-T, T]$. Fix such an interval and let z_n, v_n be the regularized solutions introduced in (rSLW$_p^n$). Recall that they satisfy

$$\partial_t^2 v_n = \mathbf{P}_n \Delta v_n - \mathbf{P}_n((z_n + v_n)|z_n + v_n|^{p-1}) \tag{3.27}$$

We will prove the two continuity results:

(i) $v \in \mathcal{C}^0([-T,T], H^s)$,

(ii) $\partial_t v \in \mathcal{C}^0([-T,T], H^{s-1})$.

Case of $U = \mathbb{T}^3$.

(i) results from the uniform bound for $(v_n)_{n\geqslant 1}$ in $L^\infty((-T,T), H^1)$, the uniform bound for $(\partial_t v_n)_{n\geqslant 1}$ in $L^\infty((-T,T), L^2)$ given by Proposition 3.7, and the Aubin-Lions Theorem 3.24 with $X = H^s$, $X_0 = H^1$, $X_1 = L^2$, $p = q = \infty$.

(ii) We use that

$$\sup_{n\geqslant 1} \|\partial_t v_n\|_{L^\infty((-T,T),L^2)} < \infty\,.$$

We will also need the estimate

$$\sup_{n\geqslant 1} \|\partial_t^2 v_n\|_{L^\infty((-T,T),H^{-3/2})} < \infty \tag{3.28}$$

so that another application of the Aubin-Lions theorem proves the needed continuity. In order to prove (3.28) remark that as $H^{-1} \hookrightarrow H^{-3/2}$ we have

$$\|\mathbf{P}_n \Delta v_n(t)\|_{H^{-3/2}} \lesssim \|\Delta v_n(t)\|_{H^{-1}} \lesssim \|v_n(t)\|_{H^1}$$

so that

$$\sup_{n\geqslant 1} \|\mathbf{P}_n \Delta v_n\|_{L^\infty((-T,T),H^{-3/2})} \lesssim \sup_{n\geqslant 1} \|v_n\|_{L^\infty((-T,T),H^1)} < \infty\,. \tag{3.29}$$

Remark that thanks to Proposition 3.7, $((z_n + v_n)|z_n + v_n|^{p-1})_{n\geqslant 1}$ is uniformly bounded in $L^{\frac{p+1}{p}} \hookrightarrow H^{-\frac{3(p-1)}{2(p+1)}} \hookrightarrow H^{-3/2}$. Combined with (3.29) and (3.27) gives the uniform bound (3.28) and ends the proof.

Case of $U = \mathbb{R}^3$. We only prove that $v \in \mathcal{C}^0([-T,T], H^s)$. In this case we use that the sequence $(v_n)_{\geqslant 0}$ is bounded in $L^\infty_T H^1 \cap W^{1,\infty}_T L^2$ thus v also belongs to $L^\infty_T H^1 \cap W^{1,\infty}_T L^2$, and hence continuous in time with values in L^2 since we can write:

$$\begin{aligned}\|v(t_1) - v(t_2)\|_{L^2} &\leqslant \int_{t_1}^{t_2} \|\partial_t v\|_{L^2}\,\mathrm{d}t \\ &\lesssim |t_2 - t_1|\,.\end{aligned}$$

v being $\mathcal{C}^0([-T,T], L^2)$ and bounded in H^1 is therefore continuous with values in H^s by interpolation.

For the first part of Corollary 3.6 we need to find an invariant set of full μ-measure. Consider the set

$$\Theta := \{(u_0, u_1) \in \mathcal{H}^s, \|S(t)(u_0,u_1)\|_{X\cap Y\cap L^2\cap L^\infty} \in L^\infty_{\mathrm{loc}}(\mathbb{R})\}\,.$$

Set $\Sigma := \Theta + \mathcal{H}^1(U)$. This is indeed a set invariant by the flow as $S(t)(\Theta) = \Theta$. Moreover this set is of full μ-measure since Θ is of full measure by the proof of Theorem 3.4. This set also gives rise to weak solutions. □

Finally we prove the finite speed of propagation when $U = \mathbb{R}^3$.

Proof of Corollary 3.7. For a given $s > \frac{p-3}{p-1}$, let $(u_0, u_1) \in \mathcal{H}^s$ with compact support included in $B(0,R)$, an initial data that gives rise to a global solution constructed in the proof of Theorem 3.4. We want to prove that the solution $u(t)$ to (SLW$_p$) is also compactly supported, in $B(0, R+t)$.

The finite speed of propagation is known to hold for solutions to (SLW$_p$) as well as solutions to (rSLW$_p^n$) as soon as the initial data belongs to the energy space $\mathcal{H}^1$, and propagation holds with maximum speed 1. This proves that the approximate solutions $u_n(t)$ are supported in $B\left(0, R+t\right)$. As we know that $v_n(t) \to v$ almost everywhere, and that up to extraction $z_n(t) \to z(t)$ almost everywhere, $u(t) = z(t) + v(t)$ is an almost everywhere pointwise limit of the $u_n(t)$, consequently $\operatorname{supp} u(t) \subset B(0, R+t)$. □

4. Proof of Theorem 3.8

The proof of Theorem 3.8 uses Proposition 3.8 whose proof, as explained in the introduction, differs from the one of Proposition 3.7 in avoiding any L^∞_T and L^∞_x estimates on the linear part $z_n, \tilde{z}_n$.

Let $(u_0, u_1) \in \mathcal{H}^{s_p}$ and z_n be the solution to the linear equation (LW) with initial data $(z_n(0), \partial_t z_n(0)) = (\mathbf{P}_n u_0, \mathbf{P}_n u_1)$ and v_n the unique smooth global solution to the perturbed nonlinear wave equation, which is energy subcritical ($p < 5$):

$$\left\{ \begin{array}{c} \partial_t^2 v_n - \Delta v_n + |z_n + v_n|^{p-1}(z_n + v_n) = 0, \\ (v_n(0), \partial_t v_n(0)) = (0, 0). \end{array} \right. \tag{3.30}$$

We recall, as above, the energy of the nonlinear part v_n:

$$E_n(t) := \int_{\mathbb{R}^3} \left(\frac{|\partial_t v_n(t)|^2}{2} + \frac{|\nabla v_n(t)|^2}{2} + \frac{|v_n(t)|^{p+1}}{p+1} \right) \mathrm{d}x \,.$$

We state the main result of this section.

Proposition 3.8 (Probabilistic *a priori* estimates for $s = s_p$)**.** *Let $p \in (3, 5)$, $T > 0$ and $\eta \in (0, 1)$. There exists a measurable set $\Omega_{T,\eta} \subset \Omega$ and a constant $C(T, \eta, \|(u_0, u_1)\|_{\mathcal{H}^{s_p}})$ which depends only on $T, \eta, \|(u_0, u_1)\|_{\mathcal{H}^{s_p}}$ such that:*

(i) $\mathbb{P}(\Omega_{T,\eta}) \geqslant 1 - \eta$.

(ii) For every $\omega \in \Omega_{T,\eta}$, if the initial data for u_n is attached to ω via the randomisation map of (u_0, u_1) then:

$$\sup_{n \geqslant 0} \sup_{t \in (0,T)} E_n(t) \leqslant C(T, \eta, \|(u_0, u_1)\|_{\mathcal{H}^{s_p}}).$$

Once Proposition 3.8 is obtained, the proof of Theorem 3.8 follows from the deterministic theory developed in [SX16] which asserts that local solutions exists in $\mathcal{H}^s$ and can be globalized under energy control conditions. We refer to [SX16] for details.

Proposition 3.9 (Reduction to global bounds)**.** *Let v_n be the solution to 3.30. Then if ther exists $C(T, \|u_0, u_1)\|_{\mathcal{H}^s}) < \infty$ such that*

$$\sup_{n \geqslant 0} \sup_{t \in (0,T)} E(t) \leqslant C(T, \|(u_0, u_1)\|_{\mathcal{H}^s}) \,,$$

then ther exists a unique global solution $u := z + v$ to SLW_p, where we recall that z solve LW.

Remark 3.25. This reduction to global bounds is typical of the subcritical equations. Here $p < 5$ is energy subcritical, thus (3.30) enjoys a subcritical well-posedness theory and maximal time depend only on the size of the norm. In the critical case, $p = 5$ a similar results holds, which is much harder to prove, see [Poc17], Section 4.

It remains to prove Proposition 3.8.

Proof of Proposition 3.8. We first establish energy estimates and will fix the probabilistic setting later.

The proof begins with the same energy estimates treated in Proposition 3.7, using the same integration in time technique and a Taylor expansion of $|x|^{p-1}x$ at order 1. We omit the details and obtain:

$$\begin{aligned} E_n(t) &= \int_0^t \int_{\mathbb{R}^3} \langle \nabla \rangle \tilde{z}_n(t') v_n^p(t') \,\mathrm{d}x \,\mathrm{d}t' - \int_{\mathbb{R}^3} z_n(t') v_n^p(t') \,\mathrm{d}x \\ &- \int_0^t \int_{\mathbb{R}^3} \partial_t v_n(t') N(z_n, v_n)(t') \,\mathrm{d}x \,\mathrm{d}t' \\ &=: J_1 + J_2 + J_3 \end{aligned}$$

where $|N(z_n, v_n)(t')| \lesssim |z_n(t')|^2 |v_n(t')|^{p-2} + |z_n(t')|^p$. The Hölder inequality and the Young inequality estimate the term J_2 by:

$$J_2 \leqslant \frac{1}{2} E_n(t) + C \|z_n\|_{L^{p+1}_{T,x}}^{p+1}, \tag{3.31}$$

In a similar fashion, one observes that

$$\int_0^t \int_{\mathbb{R}^3} |\partial_t v_n(t')| |z_n(t')|^p \,\mathrm{d}x \,\mathrm{d}t' \lesssim \int_0^t E_n(t') \,\mathrm{d}t' + \|z_n\|_{L^{2p}_{T,x}}^{2p}$$

and

$$\int_0^t \int_{\mathbb{R}^3} |\partial_t v_n(t')| |v_n|^{p-2} |z_n(t')|^2 \,\mathrm{d}x \,\mathrm{d}t' \lesssim \int_0^t \|z_n(t')\|^2_{L^{\frac{4(p+1)}{5-p}}} E_n(t')^{\frac{1}{2} + \frac{p-2}{p+1}} \,\mathrm{d}t' \,.$$

Along with the fact that $p < 5$ these inequalities yield

$$J_3 \lesssim \|z_n\|_{L^{2p}_{T,x}}^{2p} + \|z_n\|^2_{L^\infty_T L_x^{\frac{4(p+1)}{5-p}}} \left(1 + \int_0^t E_n(t') \,\mathrm{d}t'\right). \tag{3.32}$$

For the term J_1 we write $J_1 = \displaystyle\int_0^t K(t') \,\mathrm{d}t'$ where

$$K(t) := \int_{\mathbb{R}^3} \langle \nabla \rangle \tilde{z}_n(t) |v_n|^{p-1}(t) v_n(t) \,\mathrm{d}x \,.$$

We use the duality between $W^{s_p - 1, q}$ and $W^{1-s_p, q'}$ where $\frac{1}{q} + \frac{1}{q'} = 1$. This gives

$$K(t) \lesssim \|\langle \nabla \rangle \tilde{z}_n(t)\|_{W^{s_p-1,q}} \||v_n|^{p-1}(t) v_n(t)\|_{W^{1-s_p,q'}} \,,$$

where the constant does not depend on q. Using the chain rule in Sobolev spaces, Theorem 3.33 yields:

$$K(t) \lesssim \|\tilde{z}_n(t)\|_{W^{s_p,q}} E_n(t)^{\frac{p-1}{p+1}} \|v_n(t)\|_{W^{1-s_p,\tilde{q}}}$$

where $\frac{1}{q} + \frac{1}{\tilde{q}} + \frac{p-1}{p+1} = 1$. Since $p < 5$, the term $\|v_n(t)\|_{W^{1-s_p,\tilde{q}}}$ can be interpolated using the Gagliardo-Nirenberg theorem, see Theorem 3.32 applied between H^1 and L^{p+1} (a computation ensures that $\alpha < s_p$ in the theorem), and gives

$$K(t) \lesssim \|\tilde{z}_n(t)\|_{W^{s_p,q}} E_n(t)^{1+\alpha(q)} \,,$$

where
$$\alpha(q)=\frac{3(p-1)(2q'-p-1)}{\tilde{q}(p+1)(5-p)}=\frac{3(p-1)}{q(5-p)}=\frac{\beta_p}{q}\,,$$
with $\beta_p:=\frac{3(p-1)}{5-p}$ so that

$$J_1\lesssim\int_0^t\|z_n(t')\|_{W^{s_p,q}}E_n(t')^{1+\frac{\beta_p}{q}}\,\mathrm{d}t' \tag{3.33}$$

Finally assembling (3.33), (3.31) and (3.32) together yields the existence of a universal constant $C=C(T)$ such that

(3.34)
$$\begin{aligned}E_n(t)&\leqslant C\left(\|z_n\|^{2p}_{L^{2p}_{T,x}}+\|z_n\|^{p+1}_{L^{p+1}_{T,x}}+\|z_n\|^2_{L^\infty_T L_x^{\frac{4(p+1)}{5-p}}}\right)\\&\quad+C\|z_n\|^2_{L^\infty_T L_x^{\frac{4(p+1)}{5-p}}}\int_0^t E_n(t')\,\mathrm{d}t'+C\int_0^t\|\tilde{z}_n(t')\|_{W^{s_p,q}}E_n(t')^{1+\frac{\beta_p}{q}}\,\mathrm{d}t'\\&=:a(z_n)+b(z_n)\int_0^t E_n(t')\,\mathrm{d}t'+\int_0^t c_q(\tilde{z}_n)(t')E_n(t')^{1+\frac{\beta_p}{q}}\,\mathrm{d}t'\end{aligned}$$

We now provide the Yudovich argument, following closely [BT14], which we recalled in the introduction. Set λ_0 large enough such that the set
$$\Omega_0:=\{a(z_n)+b(z_n)\leqslant\lambda_0,\ \text{for all } n\geqslant 1\}$$
is such that $\mathbb{P}(\Omega_0)\geqslant 1-\frac{\eta}{2}$. Note that such a λ_0 exists thanks to the conclusion of Proposition 3.4 and the fact that $\|\mathbf{P}_n(u_0,u_1)\|_{L^2}\leqslant\|(u_0,u_1)\|_{L^2}$; and λ_0 depends on η,T.

Let $q_0\geqslant 1$ an integer that will be chosen later. As we have seen in the proof of Proposition 3.4, for all $\lambda>0$ and $p\geqslant q\geqslant q_0$:
$$\mathbb{P}\left(\|\tilde{z}_n\|_{L^2_TW^{s_p,q}}>\lambda\right)\leqslant\left(\frac{C\sqrt{p}}{\lambda}\right)^p$$
where $C=C(\|(u_0,u_1)\|_{\mathcal{H}^{s_p}},T)$ can be chosen independently of n as explained above. For $q\geqslant q_0$ set
$$\Omega_q:=\{\|\tilde{z}_n\|_{L^2_TW^{s_p,q}}\leqslant 2C\sqrt{q},\ \text{for all } n\geqslant 1\}\cap\Omega_0\,.$$
When applied with $p=q$ and $\lambda=2C\sqrt{q}$ we observe that $\mathbb{P}(\Omega_q^c)\leqslant 2^{-q}$.

We then use a bootstrap argument. Define
$$A_q:=\left\{t\geqslant 0,\ \sup_{n\geqslant 0}E_n(t)\leqslant\lambda_0^{\frac{q}{\beta_p}}\right\}$$
and $t_q^*:=\sup A_q$. We claim that there is a positive constant $\alpha>0$ which does not depend on q such that

$$t_q^*\geqslant\alpha q\,. \tag{3.35}$$

Indeed, for $t \in A_q$ and $n \geqslant 0$ we can write that

$$E_n(t) \leqslant \lambda_0 + \int_0^t (\lambda_0 + \|\tilde{z}_n(t')\|_{W^{s_p,q}})\, E_n(t')\, \mathrm{d}t' .$$

Then the Grönwall lemma provides us with

$$E_n(t) \leqslant \lambda_0 \exp\left(\int_0^t (\lambda_0 + \|\tilde{z}_n(t')\|_{W^{s_p,q}})\, \mathrm{d}t'\right) \leqslant \lambda_0 \exp\left(\lambda_0 t + 4C\lambda_0\sqrt{qt}\right).$$

In order to prove (3.35) it suffices to exhibit an $\alpha > 0$ independent of q such that $t := \alpha q \in A_q$. A sufficient condition for such an α to exist is to satisfy for every $q \geqslant q_0$:

$$\lambda_0 \exp(\lambda_0 q(\alpha + 4C\sqrt{\alpha})) \leqslant \lambda_0^{\frac{q}{\beta_p}} .$$

Assume that $\lambda_0 > 1$ and q_0 large enough to satisfy $\frac{q_0}{\beta_p} - 1 > 0$. Then it is sufficient for α to satisfy

$$\alpha + 4C\sqrt{\alpha} \leqslant \left(\frac{q_0}{\beta_p} - 1\right) \frac{\log \lambda_0}{\lambda_0}$$

which is satisfied for small α only depending on C, λ_0, q_0, p.

Now set q_0 larger if needed to ensure $\sum_{q \geqslant q_0} 2^{-q} \leqslant \frac{\eta}{2}$ and thus set $\tilde{\Omega} := \bigcap_{q \geqslant q_0} \Omega_q$, constructed in order to satisfy $\mathbb{P}(\tilde{\Omega}) \geqslant 1 - \eta$. Pick $\omega \in \tilde{\Omega}$ and $t \in (0,T)$. If $t \leqslant \alpha q_0$ one has $\sup_{n \geqslant 1} \sup_{t \in (0,T)} E_n(t) \leqslant \lambda_0^{\frac{q_0}{\beta_p}}$, otherwise select a dyadic integer 2^N such that $2^N \alpha \leqslant t \leqslant 2^{N+1}\alpha$. Then as $\omega \in \Omega_{2^{N+1}}$ we observe that

$$\sup_{n \geqslant 1} \sup_{t \in (0,T)} E_n(t) \leqslant \lambda_0^{\frac{2^{N+1}\alpha}{\beta_p}} .$$

Since $t \leqslant T$ and $N \simeq \log t$ we can write it as

$$\sup_{n} \sup_{t \in (0,T)} E_n(t) \leqslant C(T, \eta, \|(u_0, u_1)\|_{\mathcal{H}^s}) ,$$

just as needed. $\square$

Sketch of the proof of Corollary 3.9. Let us explain how one can define an invariant set of full μ-measure, where $\mu \in \mathcal{M}^s$ for $s = \frac{p-3}{p-1}$. We introduce

$$\begin{aligned} \Theta_k^q := &\{(u_0, u_1) \in \mathcal{H}^s, \|\tilde{S}(t)(u_0, u_1)\|_{L^2_T W^{s_p,q}} \leqslant 2Ck\sqrt{q}\} \\ &\cap \{a(S(t)(u_0, u_1)), b(S(t)(u_0, u_1)) < \infty\} \end{aligned}$$

and

$$\Sigma := \bigcap_{k \geqslant 1} \bigcap_{q \geqslant q_0} \Theta_k^q .$$

Then $\Sigma + \mathcal{H}^1$ is of full measure and invariant by the flow.

For details see the end Section 5 in [BT14]. $\square$

Remark 3.26. The threshold $s_p = \frac{p-3}{p-1}$ appears to be out of reach when $p \geqslant 5$. Let us explain, at least informally, why a uniform bound on the energy E such as in Proposition 3.7 is out of reach when using the techniques presented here. We start with (3.2) obtained in the introduction:

$$E(t) \simeq \int_0^t \int_{\mathbb{R}^3} \nabla^{1-s_p} v(t') v(t')^{p-1} \nabla^{s_p} z(t') \, \mathrm{d}x \, \mathrm{d}t' .$$

In order to conclude we would like a bound of the form

$$E(t) \lesssim c(q) \int_0^t E(t')^{1+\frac{\beta}{q}}(t') \, \mathrm{d}t' ,$$

for all large enough $q > 1$. Since $(u_0, u_1) \in \mathcal{H}^{s_p}$ and because we did not assume more regularity, and recalling that any L^∞_T or L^∞_x estimates costs derivatives (see Section 2), we should use the Hölder inequality putting ∇^{s_p} in some L^q space instead of L^∞. Therefore we have to estimate $\|v^{p-1}\nabla^{1-s_p} v\|_{L^{q'}}$ instead of its L^1 norm. We use the Hölder inequality to write:

$$\|v^{p-1}\nabla^{1-s_p} v\|_{L^{q'}} \leqslant \|v^{p-1}\|_{L^{\gamma_1}} \|\nabla^{1-s_p} v\|_{L^{\gamma_2}} ,$$

where $\frac{1}{q'} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$.

At this stage, observe that when we estimated the L^1 norm (thus $q' = 1$) we applied Hölder's inequality with $\gamma_1 = \frac{p+1}{p-1}$ and $\gamma_2 = \frac{p+1}{2}$. Then we observed the following (schematically):

(i) That $v^{p-1} \in L^{\frac{p+1}{p-1}}$ gives $\|v\|^{p-1}_{L^{p+1}}$, controllable by a power of the energy.

(ii) That $\nabla^{1-s_p} v \in L^{\frac{p+1}{2}}$ may be controlled by the Gagliardo-Nirenberg theorem (which leads to control by a power of the energy), which writes

$$\|\nabla^{1-s_p} v\|_{L^{\frac{p+1}{2}}} \lesssim \|\nabla v\|^{1-\alpha}_{L^2} \|v\|^{\alpha}_{L^{p+1}} ,$$

provided $\alpha \in [0, s_p]$ satisfies

$$\frac{2}{p+1} - \frac{1-s_p}{3} = \frac{1-\alpha}{6} + \frac{\alpha}{p+1} .$$

It turned out that $\alpha = s_p$.

Now observe that when 1 is increased to q', either γ_1 or γ_2 has to increase, in order to match the Hölder indices requirement $\frac{1}{q'} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$. Unfortunately, none of these modifications lead to a control by the energy. More precisely:

(i) Increasing γ_1 amounts to estimating $\|v\|_{L^{p+1+\varepsilon}}$ for some $\varepsilon > 0$. Since $p \geqslant 5$, we have $p+1+\varepsilon > 6$ and then the $L^{p+1+\varepsilon}$ norm is supercritical with respect to the energy, therefore not controllable by a power of the energy.

(ii) Increasing γ_2 amounts to lowering $\frac{1-\alpha}{6} + \frac{\alpha}{p+1}$, but since $p+1 \geqslant 6$ this boils down to increasing α, which is not possible since α was already maximal.

These remarks explain why the Yudovich-Wolibner argument alone fails to obtain the endpoint $s = \frac{p-3}{p-1}$ when $p \geqslant 5$.

Appendix A. Littlewood-Paley theory and Besov spaces

This appendix gathers some results dealing with harmonic analysis, analysis in Besov spaces and product rules. A comprehensive treatment of that matter, is given in [BCD11] and [Tay07].

We start with a Bernstein-type lemma. For a proof see Lemma 2.1 in [BCD11].

Theorem 3.27 (Bernstein-type lemma)**.** *Let $(p, q) \in [1, \infty]^2$ with $p \leqslant q$ and $u \in L^p(\mathbb{R}^d)$. Let B be a ball centred on 0 and C be an annulus, $k \geqslant 0$ an integer.*

(i) If $\hat{u}$ is supported in λB then $\|\nabla^k u\|_{L^q} \lesssim_k \lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)} \|u\|_{L^p}$.
(ii) If $\hat{u}$ is supported in λC then $\|\nabla^k u\|_{L^p} \simeq_k \lambda^k \|u\|_{L^p}$
(iii) The statements (i) and (ii) are true for non-integer orders of derivation.

The proof of Proposition 3.5 requires the following consequence of Theorem 3.27.

Corollary 3.28. *Let $q \geqslant p \geqslant 1$. Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a smooth function, supported in $[-1, 1]^d$. Let $n \in \mathbb{Z}^d$ and $\psi(D - n)$ denote the Fourier multiplier of symbol $\psi(\xi - n)$. Then there holds*

$$\|\psi(D - n)u\|_{L^q} \lesssim \|\psi(D - n)u\|_{L^p} ,$$

for any $u \in L^p(\mathbb{R}^d)$. The implicit constant does not depend on n nor u.

Proof. Let $u \in L^p(\mathbb{R}^d)$ and $q \geqslant p \geqslant 1$. We remark that the support of $\widehat{\psi(D - n)u}$ is contained in $n + [-1, 1]^d$. We remark that therefore the support of the Fourier transform of $e^{-in\cdot x}\psi(D - n)u$ is contained in $[-1, 1]$ and we can use the Bernstein estimate, Theorem 3.27 (i) with $k = 0$ and $\lambda = 1$, which writes

$$\|e^{in\cdot x}\psi(D - n)u\|_{L^q} \lesssim \|e^{in\cdot x}\psi(D - n)u\|_{L^p} ,$$

with implicit constant not depending on n nor u. Since the multiplication by $e^{in\cdot x}$ does not affect the L^p norms, the lemma is proved. □

We now recall the definition of Besov spaces:

Definition 3.10. A function f belongs to the nonhomogeneous Besov space $B^s_{p,r}$ if, and only if

$$\|f\|_{B^s_{p,r}} := \left(\sum_{j \geqslant 0} 2^{jsr} \|\Delta_j f\|^r_{L^p} \right)^{1/r} < \infty.$$

Similarly a function f belongs to the homogeneous Besov space $\dot{B}^s_{p,r}$ if, and only if

$$\|f\|_{\dot{B}^s_{p,r}} := \left(\sum_{j\in\mathbb{Z}} 2^{jsr}\|\dot{\Delta}_j f\|^r_{L^p}\right)^{1/r} < \infty.$$

In this text we use the two following reconstruction lemmata, that illustrate the fact that in order to study the H^s norm of a function f it is sufficient to study the frequency localizations $\Delta_j f$.

Lemma 3.29 ([BCD11])**.** *For $s\in\mathbb{R}$ and $f\in H^s(\mathbb{R}^d)$ one has $\|f\|_{B^s_{2,2}} \sim \|f\|_{H^s}$.*

Lemma 3.30 (Besov reconstruction, [BCD11])**.** *Let $s>0$ and $p,r\in[1,\infty]$. Let $(u_j)_{j\geqslant 0}$ be a sequence of smooth functions which satisfy*

$$\left(\sup_{|\alpha|\leqslant \lfloor s\rfloor+1} 2^{j(s-|\alpha|)}\|\partial^\alpha u_j\|_{L^p}\right)_{j\geqslant 0} \in \ell^r,$$

then one has $u=\sum_{j\geqslant 0} u_j \in B^s_{p,r}$ and the estimate:

$$\|u\|_{B^s_{p,r}} \lesssim \left\|\left(\sup_{|\alpha|\leqslant \lfloor s\rfloor+1} 2^{j(s-|\alpha|)}\|\partial^\alpha u_j\|_{L^p}\right)_{j\geqslant 0}\right\|_{\ell^r},$$

whith implicit constant only depending on s. The same result holds for homogeneous Besov spaces $\dot{B}^s_{p,r}$ and the ℓ^r norm taken with indices running in $\mathbb{Z}$.

Next we recall some embedding theorems:

Theorem 3.31 (Sobolev-Besov embeddings, [BCD11])**.** *Let $1\leqslant p_1\leqslant p_2\leqslant\infty$ and $1\leqslant r_1\leqslant r_2\leqslant\infty$, $p,q\in[1,\infty]$ and $s,s_1,s_2\in\mathbb{R}$. Then:*

(i) The embedding $B^s_{p_1,r_1}\hookrightarrow B^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}_{p_2,r_2}$ is continuous.

(ii) For $1\leqslant p\leqslant\infty$ the embeddings $B^s_{p,1}\hookrightarrow W^{s,p}\hookrightarrow B^s_{p,\infty}$ are continuous. The homogeneous counterpart is also true.

(iii) The embedding $W^{s_1,p}\hookrightarrow W^{s_2,q}$ is continuous as soon as $\frac{1}{p}-\frac{s_1}{d}\leqslant\frac{1}{q}-\frac{s_2}{d}$.

(iv) The embedding (iii) is locally compact whenever the inequality is strict.

(v) Let $r_1>r_2$, $s<0$, $p\in[1,\infty]$. Then for arbitrarly small $\varepsilon>0$, the embedding $B^{s+\varepsilon}_{p,r_1}\hookrightarrow B^s_{p,r_2}$ is continuous.

The next theorem is a well-known interpolation inequality.

Theorem 3.32 (Gagliardo-Nirenberg)**.** *Let $p_0, p_1 \in (1, \infty)$ and $s, t > 0$. Let $p > 1$ and $\alpha \in (0, 1)$ satisfying*

$$-\frac{s}{d} + \frac{1}{p} = (1-\alpha)\left(\frac{1}{p_0} - \frac{t}{d}\right) + \frac{\alpha}{p_1} \text{ and } s \leqslant (1-\alpha)t\,.$$

Then for $u \in W^{t,p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d)$ one has

$$\|u\|_{\dot{W}^{s,p}} \lesssim \|u\|_{\dot{W}^{t,p_0}}^{1-\alpha} \|u\|_{L^{p_1}}^{\alpha}\,,$$

and the same result holds in nonhomogeneous Sobolev spaces.

Proof. The limit embedding is given by Theorem 2.44 in [BCD11], that is:

$$\|u\|_{\dot{W}^{s,\bar{p}}} \lesssim \|u\|_{\dot{W}^{t,p_0}}^{s/t} \|u\|_{L^{p_1}}^{1-s/t}$$

where $\frac{1}{\bar{p}} = \frac{s/t}{p_0} + \frac{1-s/t}{p_1}$. On the other hand the Sobolev embedding reads $\|u\|_{\dot{W}^{s,\tilde{p}}} \lesssim \|u\|_{\dot{W}^{t,p_0}}$ where $\frac{1}{\tilde{p}} - \frac{s}{d} = \frac{1}{p_0} - \frac{t}{d}$. Let p a real number such that there exists $\tilde{\theta} \in [0, 1]$ satisfying $\frac{1}{p} = \frac{\tilde{\theta}}{\bar{p}} + \frac{1-\tilde{\theta}}{\tilde{p}}$. Then by interpolation and using the previous inequalities on have

$$\|u\|_{\dot{W}^{s,p}} \leqslant \|u\|_{\dot{W}^{s,\bar{p}}}^{\tilde{\theta}} \|u\|_{\dot{W}^{s,\tilde{p}}}^{1-\tilde{\theta}} \lesssim \|u\|_{\dot{W}^{t,p_0}}^{1-\theta} \|u\|_{L^{p_1}}^{\theta}$$

with $\theta = \left(1 - \frac{s}{t}\right)\tilde{\theta}$. A straightforward computation yields

$$\frac{1}{p} - \frac{s}{d} = (1-\theta)\left(\frac{1}{p_0} - \frac{t}{d}\right) + \frac{\theta}{p_1}. \tag{3.36}$$

Note that the imposed condition $\theta \in \left[0, 1 - \frac{s}{t}\right]$ comes from the fact that $\tilde{\theta} \in [0, 1]$. All the range of p is covered since the equality (3.36) implies $\frac{1}{p} \in \left[\frac{1}{\bar{p}}, \frac{1}{\tilde{p}}\right]$.

The proof in the nonhomogeneous case follows using $\|u\|_{W^{s,q}} \leqslant \|u\|_{L^q} + \|u\|_{\dot{W}^{s,q}}$. ☐

Theorem 3.33 (Sobolev chain rule, [Tay07])**.** *For $s \in (0, 1)$, $p \in (1, \infty)$ and a function $F \in \mathcal{C}^1(\mathbb{R})$ such that $F(0) = 0$ and such that there is a $\mu \in L^1([0, 1])$ such that for every $\theta \in [0, 1]$:*

$$|F'(\theta v + (1-\theta)w)| \leqslant \mu(\theta)\,(G(v) + G(w))$$

where $G > 0$. We have $\|F \circ u\|_{W^{s,p}} \lesssim \|u\|_{W^{s,p_0}} \|G \circ u\|_{L^{p_1}}$ as soon as $\frac{1}{p_0} + \frac{1}{p_1} = \frac{1}{p}$, provided $p_0 \in (1, \infty]$ and $p_1 \in (1, \infty)$.

In order to derive product laws in Sobolev or Besov spaces, recall the paraproduct algorithm introduced by J.M. Bony, which consists in writing for two given functions a, b:

$$ab = T_a b + T_b a + R(a, b) = \dot{T}_a b + \dot{T}_b a + \dot{R}(a, b)$$

where

$$T_a b := \sum_j \mathbf{P}_{j-1} a \Delta_j b \text{ and } R(a,b) := \sum_{|k-j|\leqslant 1} \Delta_k a \Delta_j b\,,$$

and

$$\dot{T}_a b := \sum_j \dot{\mathbf{P}}_{j-1} a \dot{\Delta}_j b \text{ and } \dot{R}(a,b) := \sum_{|k-j|\leqslant 1} \dot{\Delta}_k a \dot{\Delta}_j b\,,$$

respectively denote nonhomogeneous paraproduct and remainder (resp. homogeneous).

Theorem 3.34 (Tame estimates, [BCD11])**.** *Let $s > 0$, $p, r \in [1,\infty]$, and $p_1, p_2 \in [1,\infty]$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. Then:*

(i) $\|\dot{T}_u v\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{L^{p_1}} \|v\|_{\dot{B}^s_{p_2,r}}$.

(ii) $\|\dot{R}(u,v)\|_{\dot{B}^s_{p,r}} \lesssim \|u\|_{L^{p_1}} \|v\|_{\dot{B}^s_{p_2,r}}$.

From this one infers the next inequality which is important in our analysis:

Corollary 3.35. *Let $s > 0$, $p, r \in [1,\infty]$, and $p_1, p_2, p_1', p_2' \in [1,\infty]$ satisfying*

$$\frac{1}{p_1} + \frac{1}{p_1'} = \frac{1}{p_2} + \frac{1}{p_2'} = \frac{1}{p}\,.$$

Then:

$$\|fg\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{\dot{B}^s_{p_1,r}} \|g\|_{L^{p_1'}} + \|f\|_{L^{p_2}} \|g\|_{\dot{B}^s_{p_2',r}}\,.$$

Chapitre 4

Almost sure Scattering at Mass regularity for radial Schrödinger Equations

Abstract. We consider the radial nonlinear Schrödinger equation $i\partial_t u + \Delta u = |u|^{p-1}u$ in dimension $d \geqslant 2$ for $p \in \left(1, 1+\frac{4}{d}\right]$ and construct a natural Gaussian measure μ whose support is almost L^2_{rad} and such that μ - almost every initial data gives rise to a unique global solution. Furthermore, for $p > 1 + \frac{2}{d}$ and $d \leqslant 10$, the solutions constructed scatter in a space which is almost L^2. This paper can be viewed as a higher dimensional counterpart of the work of Burq and Thomann [BT20], in the radial case.

1. Introduction

We consider the Cauchy problem and the long time dynamics for the semilinear Schrödinger equation:

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u \\ u(0) = u_0 \in H^s(\mathbb{R}^d)\,, \end{cases}$$

where $p \geqslant 1$ need not be an integer, $s \in \mathbb{R}$ and $d \geqslant 2$. (NLS) is known to be invariant under the scaling symmetry:

$$u(t,x) \mapsto u_\lambda(s,y) := \lambda^{\frac{2}{p-1}} u(\lambda^2 s, \lambda y),$$

is such that $\|u_\lambda\|_{\dot{H}^s} = \lambda^{s(d,p)}\|u\|_{\dot{H}^s}$ with the *critical* regularity $s(d,p) := \frac{d}{2} - \frac{2}{p-1}$, where $\dot{H}^s$ stands for the homogeneous Sobolev space. (NLS) also enjoys several formal conservation laws, the most important being the conservation of the *mass* and the *energy*, that is the quantities

$$M(t) := \|u(t)\|_{L^2}^2 \text{ and } E(t) := \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{p+1}\|u(t)\|_{L^{p+1}}^{p+1}$$

are conserved under the flow of (NLS).

When $s > s(d,p)$, local well-posedness and even global well-posedness are expected and when $s < s(d,p)$ an ill-posedness behaviour is to be expected. This article deals with the exponent range $p \leqslant 1 + \frac{4}{d}$ and regularity $s = 0$, *i.e.*, the mass regularity. Remark that $s(d, 1 + \frac{4}{d}) = 0$. We gather below some known results concerning this problem. We recall that a solution u to (NLS) is said to scatter in H^s (which stands for the non-homogeneous Sobolev spaces) forward in time (resp. backward in time) if there exists $u_+ \in H^s$ (resp. u_-) such that:

$$\|u(t) - e^{it\Delta}u_\pm\|_{H^s} \underset{t\to\pm\infty}{\longrightarrow} 0\,.$$

This expresses that even if u is the solution of a nonlinear equation, its long time behaviour will eventually be close the linear evolution of some state $u_\pm$, which may not always be the initial data u_0.

1.1. Scattering for the nonlinear Schrödinger equations. The literature pertaining to the long time behaviour of (NLS) is broad and we do not claim to be exhaustive. For an introduction to the deterministic theory of the mass sub-critical and critical nonlinear Schrödinger equations we refer to [Tao06, Caz03]. The following theorem gathers some of the most important results known in the scattering theory of (NLS). We also refer to [Nak99, Dod16a] for more scattering results.

Theorem 4.1 (Deterministic theory)**.** *Let $p \geqslant 1$ and $d \geqslant 2$. Then:*

(i) For $p \in \left[1, 1+\frac{4}{d}\right]$ the Cauchy problem for (NLS) *is globally well-posed in $L^2(\mathbb{R}^d)$ and if $p > 1 + \frac{4}{d}$ the Cauchy problem is ill-posed in L^2.*

(ii) For $p \leqslant 1 + \frac{2}{d}$ and for every $u_0 \in L^2$, the solutions do not scatter in L^2, neither forward nor backward in time.

(iii) For $d \geqslant 2$, $p \in \left(1+\frac{2}{d}, 1+\frac{4}{d-2}\right)$ and initial data in H^1 scattering in L^2 holds.

(iv) For $d \geqslant 2$, $p = 1 + \frac{4}{d}$ and initial data in L^2, scattering holds in L^2.

Proof. (i) is the standard local well-posedness result when $p < 1+\frac{4}{d}$, see [Tao06], Chapter 3 and global well-posedness comes from the conservation of mass. In the critical case the global theory is more involved, see [Dod16a]. The ill-posedness part is proven in [CCT03, AC09]. (ii) is the content of [Bar84]. For (iii) see [TY84] and for (iv) see [TVZ07]. For dimension 2 see [Dod16a, Dod16b]. ☐

In order to address this gap between the previous results and the scaling heuristic, one way is to study the Cauchy problem for (NLS) with random initial data. The theory of dispersive equations with random initial data can be tracked back at least to the pioneer work of Bourgain [Bou94, Bou96] and the use of formal invariant measures. See also [BT08a] for an approach without invariant measure construction or [Tzv15, OT17, OT20, OST17] for an approach with quasi-invariant measure constructions. A key feature of the random data dispersive equation theory is that it allows for proving well-posedness results below the scaling for critical problems.

In dimension 1 the gap in the theory of mass regularity scattering for (NLS) was partially closed in [BTT13]. More precisely the authors proved almost sure global existence and scattering in L^2 with respect to a measure which typical regularity is L^2, as long as $p \geqslant 5 = 1 + \frac{4}{d}$ and thus missed the range $(3,5)$. Their method is based on the use of the *lens transform* (see Appendix Appendix B for details) which transforms the scattering problem for (NLS) into a scattering problem for a harmonic oscillator version of the nonlinear Schrödinger equation, which turns out to be more amenable. The gap was fully closed in [BT20] by studying quasi-invariant measures in a quantified manner. In both cases such results are interesting in the sense that they give large data scattering, without assuming decay at infinity.

In dimension $d = 2$, the counterpart of [BTT13] in the radial case is established in [Den12]. See also [PRT14, Tho09] where almost sure scattering in higher dimensions was studied (although a smallness assumption is required).

Note that in another setting, scattering for the nonlinear wave equations at energy regularity has been studied, and we refer to the works [DLM20] and also also [KMV19, Bri18b, Bri18a].

1.2. Notation. We adopt widely used notations such as $\lfloor x \rfloor$ for the lower integer part of a real number x and $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$. We denote $\otimes$ the tensor product (of functions, spaces or measures) and we write $[A, B] = AB - BA$ the commutator of A and B. The letters Ω and $\mathbb{P}$ will always denote a probability space and its associate probability measure with an expectation denoted by $\mathbb{E}$.

For inequalities we often write $A \lesssim B$ when there exists a universal constant $C > 0$ such that $A \leqslant CB$. In some cases we need to track explicitly the dependence of the constant C upon other constants and we denote by $A \lesssim_a B$ to indicate that the implicit constant depends on a. In other cases we use the notation C for a constant which can change from one line to another, and write $C(a, b)$ to explicitly recall the dependence of C on other parameters. We write $A \sim B$ if $A \lesssim B$ and $A \gtrsim B$.

D_t is defined as $-i\partial_t$. $\mathcal{S}$ and $\mathcal{S}'$ respectively stand for the space of Schwartz functions and its dual the space of tempered distributions. The subscript rad means radial, so for example $\mathcal{S}_{\mathrm{rad}}$ stands for the radially symmetric functions of $\mathcal{S}$. The Lebesgue spaces are denoted by L^p and for $p \in [1, \infty]$ we set p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. If X is a Banach space then $L^p_T X$ serves as a shortcut for $L^p((0,T), X)$ and L^p_w denotes the Lebesgue space with weight w. The usual Sobolev spaces are denoted by $H^s = W^{s,2}$ where $W^{s,p} = \{u \in \mathcal{S}',\ (\mathrm{id} - \Delta)^{\frac{s}{2}} \in L^p\}$. Let $H := -\Delta + |x|^2$ be the Harmonic oscillator, we define

$$\mathcal{W}^{s,p} = \{u, H^{\frac{s}{2}} u \in L^p\}, \tag{4.1}$$

and

$$\mathcal{H}^s := \mathcal{W}^{s,2}, \tag{4.2}$$

and these spaces are called the harmonic Sobolev spaces. The Hölder space $C^{0,\alpha}(I,X)$, for $\alpha\in(0,1)$, is defined as the set of functions continuous functions on X that satisfy

$$\|f\|_{C^{0,\alpha}} := \|f\|_{L^\infty(I,X)} + \sup_{t\neq s\in I}\frac{\|f(t)-f(s)\|_X}{|t-s|^\alpha} < \infty\,.$$

Smooth Littlewood-Paley projectors at frequency $N=2^n$ are denoted by $\mathbf{P}_N$ and we set $\mathbf{S}_N := \sum_{m=0}^n \mathbf{P}_{2^m}$ and $\mathbf{P}_{>N} = \mathrm{id} - \mathbf{S}_N$. Truncation in frequency in $[-N,N]$ will be denoted by Π_N.

We denote by $B_Z(\lambda)$ the closed ball of the space Z, centred at 0 and of radius λ.

If a_n, b_n are real independent standard Gaussian random variables then $g_n := a_n + ib_n$ is called a complex Gaussian random variable.

1.3. Main results. This section presents the main results we shall prove. Precise definitions of the measure μ_0 will be given in Section 2. At this stage one can picture μ_0 as a measure supported by radially symmetric functions which are almost L^2.

Keeping in mind Theorem 4.1 (ii), when $p\leqslant 1+\frac{2}{d}$ there is no scattering in L^2 and thus even in the probabilistic approach there is no possible scattering result. However we have global well-posedness. Let us introduce the parameter:

$$\alpha(p,d) := 2 - \frac{d}{2}(p-1)\,. \tag{4.3}$$

Theorem 4.2 (Almost-sure global existence and weak scattering)**.** *Let $d\geqslant 2$ and $p\in\left(1, 1+\frac{4}{d}\right]$. There exists $\sigma\in(0,\frac{1}{2})$ such that for μ_0-almost every initial data u_0, there exists a unique global solution $u = e^{it\Delta}u_0 + w$ to (NLS) in the space*

$$e^{it\Delta}u_0 + \mathcal{C}(\mathbb{R}, H^\sigma(\mathbb{R}^d))\,.$$

Ths solution satisfies

$$\|w(s)\|_{H^\sigma} \lesssim \langle s\rangle^{\frac{\alpha(p,d)}{2}} \log^{\frac{1}{2}}\langle s\rangle\,,$$

for any $s\in\mathbb{R}$.

Furthermore if $p\leqslant p_{\max}(d)$ (defined by (4.32) and (4.33)) then

$$\|u(s)\|_{L^{p+1}} \leqslant C(u_0)\log^{\frac{1}{2}}(1+|s|)\,,$$

for any $s\in\mathbb{R}$.

Finally, the constants $C(u_0)$ satisfy that there exist constants $C,c>0$ such that

$$\mu_0(u_0 : C(u_0)>\lambda) \leqslant Ce^{-c\lambda^2}\,.$$

For $p \in \left(1+\frac{2}{d}, 1+\frac{4}{d}\right]$ and $d \leqslant 10$ scattering holds in L^2 for almost every radial initial data at mass regularity. More precisely we define the following regularity parameter:

$$\sigma(p,d) := \begin{cases} \frac{1}{2} & \text{if } p \leqslant 1+\frac{3}{d-2} \\ 2-\frac{d-2}{2}(p-1) & \text{if } p \geqslant \frac{3}{d-2}\,. \end{cases} \tag{4.4}$$

Theorem 4.3 (Almost sure scattering at mass regularity)**.** *Let $d \in \{2,\dots,10\}$ and $p \in \left(1+\frac{2}{d}, 1+\frac{4}{d}\right]$.*

(i) For every $\sigma \in (0,\sigma(p,d))$ and μ_0-almost every initial data there exists a unique global solution to (NLS) satisfying

$$u(s) - e^{is\Delta}u_0 \in \mathcal{C}(\mathbb{R}, H^\sigma)\,.$$

(ii) The solutions constructed scatter at infinity, in L^2. More precisely, there exist $\sigma \in (0,\sigma(p,d))$ and $\kappa > 0$ only depending on p and d such that for μ_0-almost every u_0 there exist $u_\pm \in \mathcal{H}^\sigma$ (defined in (4.2)) such that the corresponding solution constructed in (i) satisfies

$$\|u(s) - e^{is\Delta}(u_0+u_\pm)\|_{\mathcal{H}^\sigma} \lesssim C(u_0)\langle s\rangle^{-\kappa} \underset{s\to\pm\infty}{\longrightarrow} 0\,, \tag{4.5}$$

and also

$$\|e^{-is\Delta}u(s) - (u_0+u_\pm)\|_{\mathcal{H}^\sigma} \lesssim C(u_0)\langle s\rangle^{-\kappa} \underset{s\to\pm\infty}{\longrightarrow} 0\,. \tag{4.6}$$

In both cases there exist numerical constants $C, c > 0$ such that

$$\mu_0(u_0 : C(u_0) > \lambda) \leqslant Ce^{-c\lambda}\,.$$

Remark 4.4. Since the corresponding measure μ will be such supported by the radial functions of $\bigcap_{\sigma>0} H^{-\sigma}_{\mathrm{rad}}$ we see that this result is radial in nature. We also emphasis that the convergence rate κ can be made explicit by following the computations line by line.

Remark 4.5. First note that in dimension $d=2$, the case $p=3$ has been treated in [Den12]. Similarly in dimension $d>2$ the endpoint case $p=1+\frac{4}{d}$ is an adaptation of the proof in [Den12] and we do not treat this case in view of the deterministic result [TVZ07]. Then we assume that $p \in \left(1+\frac{2}{d}, 1+\frac{4}{d}\right)$.

Remark 4.6. As the proof will make it clear, the obstruction to generalise to dimensions higher than 10 is contained in the establishment of a powerful enough almost-sure local Cauchy theory. We did not put much effort into developing such a local theory in higher space dimensions and we believe that this is possible, using refinements of the local theory in Bourgain's type spaces, building on bilinear estimates.

Remark 4.7. This result is not of small data type. The measure μ_0 indeed satisfies that for $p > 2$ sufficiently close to 2 and every $R > 0$, $\mu_0(u \in X^0, \|u\|_{L^p} > R) > 0$, see Section 2 for a proof.

1.4. Strategy of the proof and organisation of the paper. The following paragraphs outline the proof of the main results. There are three main features that we need to address: assuming a global theory, how can one prove scattering? How can one construct a good local theory? How to extend this local theory into a powerful enough global theory?

1.4.1. *Proving scattering.* Assume that one has access to an almost-sure local existence theory for (NLS), even a global theory in L^2.

In order to deal with the long-time behaviour of the solutions u we write, thanks to Duhamel's formula:

$$u(s) = e^{is\Delta}u_0 - i\int_0^s e^{i(d-s'))\Delta}\left(|u(s')|^{p-1}u(s')\right)\,\mathrm{d}s'\,,$$

so that in order to prove the existence of u_+ we only need to prove that the integral

$$\int_0^s e^{i(s-s')\Delta}\left(|u(s')|^{p-1}u(s')\right)\,\mathrm{d}s'$$

converges, which is achieved by proving that this integral actually converges absolutely. From there we can see that *a priori* bounds on u in L^r_x spaces may be needed to carry such a program. Such bounds can be obtained using Sobolev embeddings if u is more regular. This heuristic suggests that we can try to construct a better local theory in a probabilistic setting, building on some stochastic smoothing. We then will need to extend the theory to a global one.

1.4.2. *Almost-sure local well-posedness.* The standard method to prove local well-posedness is to implement a fixed-point argument in some Banach spaces. In the context of dispersive equations, Strichartz estimates allow for achieving such a goal. Heuristically, Strichartz estimates tell us that if $u_0 \in H^s$ then the free evolution $u_L(s) = e^{is\Delta}u_0$ is not be smoother than u_0 but however exhibits some gain in space-time integrability, from which we access to a wider range of (p,q) such that bounds of the form $\|u_L\|_{L^p_TL^q_x} \lesssim \|u_0\|_{L^2}$ hold. The advantage of working with random data is that improving the L^p integrability of a random L^2 function (on the torus, for example) is essentially granted by a result which appeared in the work of Paley-Zygmund. See [BT08b] Appendix A for a proof.

Theorem 4.8 (Kolmogorov-Paley-Zygmund)**.** *Let $(c_n)_{n\in\mathbb{Z}}$ an ℓ^2 sequence. Then if $(g_n)_n$ is a sequence of identically distributed centred and normalised* *complex Gaussian variables we have*

$$\Big\|\sum_{n\in\mathbb{Z}} c_n g_n\Big\|_{L^p_\Omega} \lesssim \sqrt{p}\|(c_n)\|^2_\ell\,.$$

Using this result we can prove a probabilistic Strichartz estimate which improves the classical one. Take u_0 a random initial data, then with the previous remarks we can easily control the space-time norms of $u_L(s)$, so we seek solutions of the form $u(s) = u_L(s)+w(s)$ where w is deterministic and can be taken in a smoother space, for example $w \in H^s$ for some $s > 0$. Formally w is the solution of the fixed point problem $\Phi(w) = w$ where

$$\Phi(w) = i\int_0^s e^{i(s-s')\Delta}\left((u_L(s) + w(s))^p\right)$$

and thanks to the gain in controlling the space time norms of u_L we can expect to solve this problem in H^s. For an illustration of the method see [BT08b]. For probabilistic well-posedness of (NLS) below the scaling regularity see [BOP15b].

In our context we will have to take care of the fact that since we will work with random initial data slightly below L^2 we will not really access to all the range of Strichartz estimates. These statements are made precise by Lemma 4.13. Establishing such a good local theory is the content of Proposition 4.3.

1.4.3. *The globalisation argument.* Once one has a good local well-posedness theory, there exists a general globalisation argument which can be traced back at least to Bourgain [Bou94]. In order to clarify the arguments of Proposition 4.5 and Proposition 4.7 we explain the argument in a simpler setting, the invariant measure setting. One wants to achieve an almost sure global theory. To this end we first remark that thanks to the Borel-Cantelli lemma it is sufficient to prove that for every $\delta > 0$ and every $T > 0$ there exists a set $G_{\delta,T}$ such that $\mu(X \setminus G_\delta) \leqslant \delta$ and that for every initial data in $G_{\delta,T}$ we have existence on $[0, T]$. The requirements for the argument to work are roughly the following:

- (i) A local well-posedness theory of the following flavour: for initial data at time t_0 of size (in a space X) less than λ there exists a solution on $[t_0, t_0 + \tau]$ with $\tau \sim \lambda^{-\kappa}$, in a space X.
- (ii) An invariant measure μ, that is if ϕ_t denotes the flow of the equation, $\mu(A) = \mu(\phi_{-t}A)$ for all $t > 0$ and every measurable set A, at least formally.

We further assume that $\mu(u_0, \|u_0\|_X > \lambda) \lesssim e^{-c\lambda^2}$, as this will always be the case in the following.

Let $\lambda > 0$ to be chosen later. Let G_λ be the set of *good* initial data u_0 which give rise to solutions defined on $[0, T]$ and such that $\|u(t)\|_X \leqslant 2\lambda$ for every $t \leqslant T$. Let also $A_\lambda := \{u_0,\ \|u_0\| \leqslant \lambda\}$. Choose τ small enough (which amounts to enlarging λ) so that the local theory constructed implies that a solution on

$[n\tau, (n+1)\tau]$ does not grow more than doubling its size. Then we immediately see that

$$\bigcap_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \phi_{n\tau}^{-1}(A_\lambda) \subset G_\lambda \,.$$

Taking the complementary we see that

$$\{u_0 \notin G_\lambda\} \subset \bigcup_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \phi_{n\tau}^{-1}(X \setminus A_\lambda) \,,$$

and taking into account the expression of τ and the fact that μ is invariant under the flow this gives

$$\mu(X \setminus G_\lambda) \leqslant CT\lambda^\kappa e^{-c\lambda^2} \,,$$

which is smaller than δ for $\lambda \sim \log^{\frac{1}{2}}\left(\frac{T}{\delta}\right)$.

From there we obtain the global existence and a logarithmic bound for $\|u(t)\|_X$. However let us recall that there is no non-trivial invariant measure for the linear Schrödinger equation, or the nonlinear equation in presence of scattering in L^2.

Proposition 4.1. *Let $d \geqslant 2$. Then,*

(i) Let $\sigma \in \mathbb{R}$. The only measure supported in H^σ which is invariant by the flow of the linear Schrödinger equation $i\partial_s u + \Delta_y u = 0$ with initial condition $u(0) = u_0 \in H^\sigma(\mathbb{R}^d)$ is δ_0.

(ii) Let p such that scattering holds in L^2 for solutions of (NLS). Then for any $\sigma \in \mathbb{R}$ the only measure supported in L^2 which is invariant by the flow of (NLS) is δ_0.

Proof. We refer to [BT20] where a proof is given in dimension 1. The proof in dimension d is a straightforward adaptation. □

In order to overcome this difficulty, the authors in [BTT13] use the *lens transform* (see Appendix Appendix B) to transform solutions of (NLS) with time interval $\mathbb{R}$ to solutions of

$$\text{(HNLS)} \qquad \begin{cases} i\partial_t v - Hv = \cos(2t)^{-\alpha(p,d)}|v|^{p-1}v \\ v(0) = v_0 \in \mathcal{H}^s \,, \end{cases}$$

with time interval $(-\frac{\pi}{4}, \frac{\pi}{4})$ where we recall that $H = -\Delta + |x|^2$ is the harmonic oscillator, and $\mathcal{H}^s$ stands for the associate Sobolev spaces and $\alpha(p,d)$ has been defined in (4.3).

(HNLS) admits quasi-invariant measures with quantified evolution bounds, see Proposition 4.4, which makes it possible to use a similar globalisation argument to the one presented. For this reason most of this paper studies properties of (HNLS), and we will eventually get back to (NLS) using the *lens transform* in Section 5.2.

For the global theory, the main difference when compared to [BT20] lies in the proof of Proposition 4.7. Let us recall that the proof in [BT20] uses:

(i) A pointwise bound in L^{p+1} with large probability for the solutions.

(ii) A Sobolev embedding to control the L^{p+1} norm variation by the $\mathcal{H}^s$ norm variation, controlled by the local theory.

In dimension $d \geqslant 3$ such a program would not yield the full proof of Theorem 4.3, due to the use of the Sobolev embedding which would only yield a result for $p \leqslant 1 + \frac{2}{d-1}$. Instead we control the variation of the L^{p+1} norm, writing:

$$v(t') = e^{-i(t'-t_n)H} v(t_n) - i \int_{t_n}^{t'} e^{-i(t'-s)H} \left(|v(s)|^{p-1} v(s) \right) \mathrm{d}s \,. \tag{4.7}$$

for t' in some interval $[t_n, t_{n+1}]$. The first term is controlled using linear methods. For the second term we need to control its $L^\infty L^{p+1}$ norm, which is not always Schrödinger admissible (terminology defined in Section 3). Instead we use the dispersive estimate $L^{\frac{p+1}{p}} \to L^{p+1}$. A similar idea was previously used in [PW18].

Once we have these L^{p+1} bounds, scattering in $\mathcal{H}^\varepsilon$ for (HNLS) is obtained through interpolation between scattering in $\mathcal{H}^{-\sigma}$ (obtained by Sobolev embeddings and the L^{p+1} bounds) and a bound for the solution in $\mathcal{H}^\varepsilon$. See Proposition 4.8 for the details.

1.4.4. *Organisation of the paper.* The remaining of this paper is organised as follows: Section 2 introduces the space X^0, the Gaussian measure μ and gathers their properties. Then the proof of Theorem 4.2 and Theorem 4.3 starts in Section 3 with a large probability local well-posedness theory. The evolution of the quasi-invariant measures are studied in Section 4 which allows to extend the local theory into an almost-sure global theory in Section 5 and prove Theorem 4.2 and Theorem 4.3. Finally technical estimates are gathered in Appendix Appendix A and Appendix Appendix B for results concerining the *lens transform*.

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2. The harmonic oscillator and quasi-invariant measures

2.1. The harmonic oscillator in the radial setting. We recall some facts concerning the radial harmonic oscillator, and refer to [Sze39] for proofs.

The radial harmonic oscillator is defined as $H = -\Delta + |x|^2$ acting on the space $\mathcal{S}_{\text{rad}}(\mathbb{R}^d)$ or radial Schwartz functions. It is a symmetric operator and admits a self-adjoint extension on $\mathcal{H}^1_{\text{rad}}(\mathbb{R}^d)$.

It is known that the spectrum of H, acting on $\mathcal{H}^1_{\text{rad}}(\mathbb{R}^d)$ is made of eigenvalues

$$\lambda_n^2 = 4n + d \text{ for } n \geqslant 0\,,$$

and associated (simple) eigenfunctions are denoted by e_n, which can be written explicitly, but in the sequel we do not use their expression. All that will be used in the rest of this article is that the sequence $(e_n)_{n\geqslant 0}$ satisfies the following L^p estimates. We also refer to [IRT16b] for details on the harmonic oscillator acting on radial functions.

Lemma 4.9 ([IRT16b], Proposition 2.4)**.** *Let $d \geqslant 2$. Then*

(i) $\|e_n\|_{L^p} \lesssim \lambda_n^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}$ *for* $p \in \left[2, \frac{2d}{d-1}\right)$;

(ii) $\|e_n\|_{L^p} \lesssim \lambda_n^{-\frac{1}{2}} \log^{\frac{1}{p}} \lambda_n$ *for* $p = \frac{2d}{d-1}$;

(iii) $\|e_n\|_{L^p} \lesssim \lambda_n^{d\left(\frac{1}{2}-\frac{1}{p}\right)-1}$ *for* $p \in \left(\frac{2d}{d-1}, \infty\right]$.

In the above inequalities the implicit constant may depend on d but not p nor n.

Remark 4.10. From this lemma we see that in the scale of L^p regularity, as long as $p < \frac{2d}{d-2}$, then eigenfunctions e_n exhibit some decay, which is used to prove probabilistic smoothing estimates, see for example [BTT13], Appendix A for such estimates.

2.2. Measures. We fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $(g_n)_{n\geqslant 0}$ be an identically distributed sequence of centred, normalised complex Gaussian variables. Then we define the random variable $f : \Omega \to \mathcal{S}'(\mathbb{R}^d)$ by

$$f^\omega(x) \coloneqq \sum_{n\geqslant 0} \frac{g_n(\omega)}{\lambda_n} e_n(x)\,, \tag{4.8}$$

where λ_n, e_n are the eigenvalues and eigenfunctions of the radial harmonic oscillator previously defined. Moreover we consider the approximations

$$f_N \coloneqq \sum_{n=0}^{N} \frac{g_n}{\lambda_n} e_n\,,$$

which almost surely converge to f in $\mathcal{S}'$. Moreover $(f_N)_{N\geqslant 1}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{H}^{-\sigma})$ for every $\sigma > 0$. For a proof of this fact see [BTT13], Lemma 3.3. More precisely, if we denote by $\mu \coloneqq f_*\mathbb{P}$, the law of the random variable f, then we have the following.

Lemma 4.11. *The measure μ is supported by $X \coloneqq \bigcap_{\sigma>0} \mathcal{H}^{-\sigma}_{\text{rad}}$ and moreover:*

(i) *For μ almost every u one has $u \notin L^2$.*

(ii) *For any $p > 2$ sufficiently close to 2 and any $R > 0$, $\mu(u \in X,\ \|u\|_{L^p} > R) > 0$.*

Remark 4.12. If μ_0 denotes measure given by $\mathcal{L}_*^{-1}\mu$, where $\mathcal{L}$ is the *lens transform* then we infer that μ_0 is supported by $\bigcap_{\sigma>0} H^{-\sigma}_{\mathrm{rad}}$ and satisfies (ii).

Proof. (i) The first part is a straightforward adaptation of Lemma 3.3 in [BTT13]. For the proof of the fact that for μ-almost every u, there holds $u \notin L^2$, see [Poi12], Section 4, Lemma 53 and Proposition 54. The argument is easily adapted to our case and mostly relies on a careful application of an inequality of Paley-Kolmogorov which states that for $\lambda \geqslant 0$ and $X \in L^2(\Omega)$:

$$\mathbb{P}(X \geqslant \lambda \mathbb{E}[X]) \geqslant (1-\lambda)^2 \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}\,.$$

(ii) This result is claimed in [BTT13] and we provide a proof in full details. We construct large data in the following way. Let $E \subset \Omega$ be the set defined by:

$$E \coloneqq \left\{ g_0 > 2R/\|e_0\|_{L^p}\,, (g_n)_{n\geqslant 1}\ \left\|\sum_{n\geqslant 1} \frac{g_n}{\lambda_n} e_n \right\|_{L^p} \leqslant R \right\}\,,$$

which by independence has probability

$$\mathbb{P}(E) = \mathbb{P}(g_0 > 2R)\mathbb{P}\left(\left\|\sum_{n\geqslant 1} \frac{g_n}{\lambda_n} e_n \right\|_{L^p} \leqslant R \right)\,.$$

Since g_0 is a Gaussian, $\mathbb{P}(g_0 > 2R) > 0$. Moreover, since $p > 2$ is sufficiently close to 2, and thanks to Lemma 4.13 we have

$$\mathbb{P}\left(\left\|\sum_{n\geqslant 1} \frac{g_n}{\lambda_n} e_n \right\|_{L^p} \leqslant R \right) \geqslant 1 - e^{-cR^2}\,.$$

Then on the set E we have $\|u\|_{L^p} \geqslant R$. Hence we have proven the claim. ☐

We remark that, if $\mathcal{N}_{\mathbb{C}}(0,\lambda_n^{-2})$ denotes the law of a centred complex Gaussian of variance λ_n^{-2}, then we have

$$\mu = \bigotimes_{n\geqslant 0} \mathcal{N}_{\mathbb{C}}(0,\lambda_n^{-2}) = \mu_N \otimes \mu_N^{\perp}\,,$$

where $\mu_N = \bigotimes_{0\leqslant n\leqslant N} \mathcal{N}_{\mathbb{C}}(0,\lambda_n^{-2})$, and $\mu_N^{\perp} = \bigotimes_{n>N} \mathcal{N}_{\mathbb{C}}(0,\lambda_n^{-2})$.

Writing any $u \in X$ as $u = \sum_{n\geqslant 0} u_n e_n$, we see remark that the distribution of $\frac{g_n}{\lambda_n}$ is given by $e^{-\frac{1}{2}\lambda_n^2|u_n|^2}\pi^{-1}\lambda_n^2\,\mathrm{d}u_n$, where $\mathrm{d}u_n$ is the Lebesgue measure on $\mathbb{C}$.

Therefore we have

$$\mathrm{d}\mu_N(u_N) = \left(\prod_{0\leqslant n\leqslant N} \pi^{-1}\lambda_n^2\right) e^{-\frac{1}{2}\|u_N\|^2_{\mathcal{H}^1}}\,\mathrm{d}u_N\,,$$

where $\mathrm{d}u_N = \prod_{0\leqslant n\leqslant N}\mathrm{d}u_n$, and $u_N = \sum_{0\leqslant n\leqslant N} u_n e_n \in X$. For this reason, we will write informally write that for any measurable set $A\subset X$,

$$\mu(A) = \int_A e^{-\frac{1}{2}\|u\|^2_{\mathcal{H}^1}}\,\mathrm{d}u$$

2.3. Probabilistic estimates. The main probabilistic estimate which will be used is a standard large deviation estimate in the context of the radial harmonic oscillator.

Lemma 4.13. *Let $d\geqslant 2$ and $\varepsilon>0$ arbitrarily small. Define*

$$s_r = \begin{cases} d\left(\frac{1}{2}-\frac{1}{r}\right) & \text{if} \quad r\in\left(2,\frac{2d}{d-1}\right] \\ 1-d\left(\frac{1}{2}-\frac{1}{r}\right) & \text{if} \quad r\in\left(\frac{2d}{d-1},\frac{2d}{d-2}\right] \end{cases}$$

and $s_r^- := s_r-\varepsilon$. Then:

(i) For μ-almost every $u\in X$ one has $u\in\mathcal{W}^{s_r^-,p}$.

(ii) More precisely there exists $C,c>0$ such that for every $p\geqslant 2$ and $\lambda>0$ we have

$$\mu\left(u\in X,\ \|u\|_{\mathcal{W}^{s_r^-,r}}>\lambda\right)\leqslant Ce^{-c\lambda^2}\,.$$

Proof. The proof is very similar to the proof of Lemma 3.3 in [BTT13] with $N_0=0$ and $N=\infty$, and using the bounds 4.9. See also [AT08]. □

As we can see, there is always a good choice of L^r scale to gain almost $\frac{1}{2}$ derivatives.

Lemma 4.13 immediately implies the following consequence.

Corollary 4.14. *There exists $C,c>0$ such that for every $q\geqslant 1$, $r\in\left[2,\frac{2d}{d-2}\right]$ and $\sigma\in[0,s_r^-)$ one has*

$$\mu\left(u\in X, \|e^{-itH}u\|_{L^q_{[-\pi,\pi]}\mathcal{W}^{\sigma,r}}>\lambda\right)\leqslant Ce^{-c\lambda^2}\ \textit{for all}\ \lambda\geqslant 1\,.$$

For details see [Tho09]. Note that we take norms in time on $[-\pi,\pi]$ although we work only on $[-\frac{\pi}{4},\frac{\pi}{4}]$. This is due to the need of room for the estimates in the analysis, in particular in Lemma 4.33.

2.4. Deterministic estimates. We will need dispersion and Strichartz estimates in order to construct local solutions. A pair $(q,r) \in [2,\infty]^2$ is Schrödinger admissible in dimension $d \geqslant 1$ if (q,r,d) satisfies

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \quad \text{and} \quad (q,r,d) \neq (2,\infty,2)\,.$$

Remark 4.15. We should check on many ocasions that a given pair (q,r) is admissible. One should keep in mind that the condition $q,r \geqslant 2$ is required. Taking into account the relation between q and r, the requirement $q,r \geqslant 2$ is equivalent to $q \geqslant 2$. In the same spirit, (q',r') is such that (q,r) is admissible if and only if $\frac{1}{2} \leqslant \frac{1}{r'} \leqslant \frac{1}{2} + \frac{1}{d}$.

We define the Strichartz space

$$Y^{\sigma}_{[t_0,t]} := \bigcap_{(q,r)\ \text{admissible}} L^q([t_0,t],\mathcal{W}^{\sigma,r})$$

and its dual $\tilde{Y}^{\sigma}_{[t_0,t]}$ that we respectively endow with the norms:

$$\|u\|_{Y^{\sigma}_{[t_0,t]}} := \sup_{(q,r)\ \text{admissible}} \|u\|_{L^q([t_0,t],\mathcal{W}^{\sigma,r})}$$

and

$$\|u\|_{\tilde{Y}^{\sigma}_{[t_0,t]}} := \inf_{(q,r)\ \text{admissible}} \|u\|_{L^{q'}([t_0,t],\mathcal{W}^{\sigma,r'})}\,.$$

We also write $Y^{\sigma}_{t_0,\tau}$ for $Y^{\sigma}_{[t_0,t_0+\tau]}$.

Let $\sigma > 0$. Using the Sobolev embedding, if $(q,\tilde{r})$ is Schrödinger admissible and $r \geqslant 1$ is such that $\frac{d}{\tilde{r}} = \frac{d}{r} + \sigma$ then by the Sobolev embedding,

$$\|u\|_{L^q_T L^r} \lesssim \|u\|_{L^q_T \mathcal{W}^{\sigma,\tilde{r}}} \lesssim \|u\|_{Y^{\sigma}_T}\,.$$

Thus we often refer to (q,r) being σ-Schrödinger admissible if $\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - \sigma$, and $q \geqslant 2$.

Proposition 4.2 (Dispersion and Strichartz estimates)**.** *Given $\sigma \geqslant 0$ and $t_0, T \in (0,\frac{\pi}{4})$, the following estimates hold.*

(i) (Dispersion) For $r \geqslant 2$ and $t > 0$, $\|e^{itH}\|_{L^{r'} \to L^r} \leqslant C(d,r)|t|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)}$;

(ii) (Homogeneous) $\|e^{itH}\|_{\mathcal{H}^{\sigma} \to Y^{\sigma}_{[t_0,T]}} \leqslant C(d)$;

(iii) (Non-homogeneous) $\left\|\int_0^t e^{i(t-s)H}\,\mathrm{d}s\right\|_{\tilde{Y}^{\sigma}_T \to Y^{\sigma}_T} \leqslant C(d)$.

Proof. The proof is carried out in detail in dimension two in [Den12], Proposition 2.7. For dimension $d \geqslant 3$ we refer to [Poi12]. In both cases the proof is an application of the TT^* argument and the Christ-Kiselev lemma along with a dispersive estimate for $e^{-itH} : L^1 \to L^{\infty}$ which is obtained through stationary phase estimates. □

2.5. Spaces of functions. In section 3 we will not construct local solutions to (HNLS) for any initial data in X, but rather for initial data in a subspace X^σ, but such that $\mu(X^\sigma) = 1$.

Let $d \geqslant 2$, $p \in [1, 1 + \frac{4}{d})$, then depending on whether $p \leqslant 1 + \frac{3}{d-2}$ or not we introduce de following spaces.

- *Case 1.* If $p \leqslant 1 + \frac{3}{d-2}$ then we recall that $\sigma(p,d) = \frac{1}{2}$ and define, for any $\sigma \in (0, \sigma(p,d))$ a complete space X^σ by the norm
$$\|u\|_{X^\sigma} := \|e^{-itH}\|_{L^{q_2}_{[-\pi,\pi]}\mathcal{W}^{\sigma,r_2}} + \|e^{-itH}u\|_{L^{a(p-1)}_{[-\pi,\pi]}L^{b(p-1)}} ,$$
where $(q_2, r_2) = (4, \frac{2d}{d-1})$ is Schrödinger admissible, and a, b only depend on p, d, σ and will be defined in the course of the proof of Lemma 4.21 in a way such that $(a(p-1), b(p-1))$ is σ-Schrödinger admissible and $b(p-1) \leqslant \frac{2d}{d-2}$.
- *Case 2.* If $p > 1 + \frac{3}{d-2}$ then we recall that $\sigma(p,d) = 2 - \frac{d-2}{2}(p-1)$ and for any $\sigma \in (0, \sigma(p,d))$ we define a complete space X^σ by the norm
$$\|u\|_{X^\sigma} := \|e^{-itH}\|_{L^{q_2}_{[}-\pi,\pi]\mathcal{W}^{\sigma,r_2}} + \|e^{-itH}u\|_{L^{a(p-1)}_{[-\pi,\pi]}L^{b(p-1)}} ,$$
where $(q_2, r_2) = (\frac{4}{(d-2)(p-1)-2}, \frac{2d}{d+2-(d-2)(p-1)})$ is again a Schrödinger admissible pair and a, b only depend on p and d and will be chosen in Lemma 4.21 in order for $(a(p-1), b(p-1))$ to be σ-Schrödinger admissible and also satisfy $b(p-1) \leqslant \frac{2d}{d-2}$.

As a consequence of Corollary 4.14 we have the following.

Corollary 4.16. *Let $d \geqslant 2$, $p \in [1, 1 + \frac{4}{d})$ and $\sigma \in (0, \sigma(p,d))$, then*
$$\mu(\{u \in X, \|u\|_{X^\sigma} > \lambda\}) \leqslant e^{-c\lambda^2} ,$$
and in particular $\mu(X^\sigma) = 1$.

Next, we establish some useful properties of the spaces X^σ.

Lemma 4.17. *Let $\sigma \geqslant$, then the following properties hold:*

(i) *For any $u \in X^\sigma$, $\|e^{-itH}u\|_{X^\sigma} = \|u\|_{X^\sigma}$*

(ii) *There is a continuous embedding $\mathcal{H}^\sigma \hookrightarrow X^\sigma$. As a consequence, the embedding $Y^\sigma_{[t_0,t_1]} \hookrightarrow L^\infty_{[t_0,t_1]}X^\sigma$ is continuous.*

(iii) *$\mathcal{S}_N$ acts continuously on X^σ and $Y^\sigma_{t_0,\tau}$ with continuity constant independent of N. As a consequence, if $u \in X^\sigma$ then $\mathcal{S}_N u \to u$ in X^σ.*

(iv) *If $\sigma' > \sigma$ then $X^{\sigma'} \hookrightarrow X^\sigma$ is compact and more precisely there holds*
$$\|(\mathrm{id} - \mathbf{S}_N)u\|_{X^\sigma} \lesssim N^{-(\sigma'-\sigma)}\|u\|_{X^\sigma} .$$

Proof. (i) This comes from the 2π-periodicity of e^{itH}, which is the case because the eigenvalues of H, the λ_n^2 are integers.

(ii) Since in the definition of X^σ, (q_2, r_2) is admissible and $(a(p-1), b(p-1))$ is σ-admissible, this follows from Strichartz estimates.

(iii) These uniform continuity properties boil down to the bound

$$\|\mathbf{S}_N\|_{L^r \to L^r} \lesssim_r 1\,,$$

available for any $r \in [1, \infty)$, see [BTT13], Proposition 4.1 for example.

(iv) The inequality is a consequence of Bernstein estimates and the compactness follows from the inequality and the fact that H has a spectrum made of countable many eigenfunctions. □

3. The probabilistic local Cauchy theory

In this section we construct a local Cauchy theory for (HNLS) and prove quantitative approximation results of solutions to (HNLS) by solutions of

$$\text{(HNLS}_N\text{)} \qquad \begin{cases} i\partial_t v - Hv = \cos(2t)^{p-3}\mathbf{S}_N(|\mathbf{S}_N v|^{p-1}\mathbf{S}_N v) \\ v(0) = u_0\,. \end{cases}$$

3.1. Local construction of solutions. The main result of this section is a flexible local well-posedness result, where the initial data is given at a time t_0. We state only a forward in time result in $[0, \frac{\pi}{4})$, a similar statement holds for negative times.

Proposition 4.3 (Local Cauchy theory)**.** *Let $d \geqslant 2$, $t_0 \in (0, \frac{\pi}{4})$, $p \in \left[1, 1 + \frac{4}{d}\right)$, $\sigma \in [0, \sigma(p,d))$ (defined by (4.4)) and $\lambda > 0$. There exists $\tau = \tau(t_0, \lambda) > 0$ of the form:*

$$\tau \sim \lambda^{-L}\left(\frac{\pi}{4} - t_0\right)^K, \tag{4.9}$$

for constants $L > 0$, $K \geqslant 1$ such that the following statements holds.

(Existence, Uniqueness) *For any $v_0 \in B_{X^\sigma}(\lambda)$ there exists a unique solution*

$$v \in e^{-i(t-t_0)H}v_0 + Y^\sigma_{t_0,\tau}$$

to (HNLS) on $[t_0, t_0 + \tau]$ such that $v(t_0) = v_0$, where uniqueness hold for $w = v - e^{-i(t-t_0)H}v_0$ in $Y^\sigma_{[t_0,t_0+\tau]}$. Moreover, $w \in \mathcal{C}^0([t_0, t_0+\tau], \mathcal{H}^\sigma)$.

(Continuity of the flow) *$\phi_t(v_0) := v(t)$ defines a flow for $t \in [t_0, t_0 + \tau]$ acting on $B_{X^\sigma}(\lambda)$ which is such that for any v_0, $\tilde{v}_0 \in B_{X^\sigma}(\lambda)$ there holds*

$$\|\phi_t(v_0) - \phi_t(\tilde{v}_0)\|_{X^{\sigma'}} \lesssim \|v_0 - \tilde{v}_0\|_{X^{\sigma'}}\,, \tag{4.10}$$

for any $\sigma' \in [0, \sigma)$.

(Persistence of regularity) *There exists a constant $C > 0$ such that the following holds. If we assume furthermore that $v_0 \in B_{X^{\sigma'}}(R)$ for some $\sigma' \in (\sigma, \sigma(p,d))$ and $R > 0$, then the associated constructed solutions $v = v_L + w$ satisfies for any $t \in [t_0, t_0 + \tau]$,*

$$\|v(t)\|_{X^{\sigma'}} \leqslant CR\,, \tag{4.11}$$

and

$$\|w\|_{Y^{\sigma'}_{t_0,\tau}} \leqslant CR\,, \tag{4.12}$$

Remark 4.18. Since $\mathbf{S}_N$ is continuous $L^q \to L^q$ for any $q \in [2, \infty)$ the above local Cauchy theory applies verbatim to (HNLS$_N$), uniformly in N with the same τ given by (4.3).

Remark 4.19. It is important in (4.11) that the local well-posedness time τ does not depend on R but only on λ, that is, the size of u_0 in X^σ.

Remark 4.20. Observe that for any $t_0 \in [0, t_1]$, and up to reducing the loc well-posedness time τ, we can choose $\tau(t_1, \lambda)$ as a local well-posedness time uniformly in $t_0 \in [0, t_1]$. We say that τ is chosen *uniformly* in $[0, t_1]$.

The strategy for proving local well-posedness is a fixed-point argument, which turns out to be easier than the one in [BT20]. In the latter case, the lack of sufficient regularisation in the the stochastic linear term requires a more refined analysis, whereas in our case, Lemma 4.13 is readily powerful enough for our purpose. We write $\Phi : Y^\sigma_{t_0,\tau} \longrightarrow \mathcal{S}'$ for the map defined by

$$\Phi(w)(t) := -i \int_{t_0}^t e^{-i(t-s)H} \Big(\cos(2s)^{-\alpha(p,d)} |v_L(s) + w(s)|^{p-1} (v_L(s) + w(s))\Big)\, \mathrm{d}s\,, \tag{4.13}$$

where $v_L(t) := e^{-i(t-t_0)H} v_0$ is the linear evolution of the initial data v_0. The crucial estimates needed to run the fixed point argument are gathered in the following lemma.

Lemma 4.21. *Let Φ defined by (4.13). Let $d \geqslant 2$. Let also $p \in \left[1, 1 + \frac{4}{d}\right)$, $\sigma \in [0, \sigma(p,d))$, $t_0 \in [0, \frac{\pi}{4})$ and $\tau \leqslant \frac{1}{2}\left(\frac{\pi}{4} - t_0\right)$. There exists $\kappa > 0$ which only depends on p, d and σ, such that for $v_0 \in B_{X^\sigma}(\lambda)$ and for w_1, w_2, w in the ball $B_{Y^\sigma_{t_0,\tau}}(\lambda)$, one has:*

$$\|\Phi(w)\|_{Y^\sigma_{t_0,\tau}} \lesssim_{d,p} \left(\frac{\pi}{4} - t_0\right)^{-\alpha(p,d)} \tau^\kappa \lambda^p\,, \tag{4.14}$$

and

$$\|\Phi(w_1) - \Phi(w_2)\|_{Y^0_{t_0,\tau}} \lesssim_{d,p} \left(\frac{\pi}{4} - t_0\right)^{-\alpha(p,d)} \tau^\kappa \lambda^{p-1} \|w_1 - w_2\|_{Y^0_{t_0,\tau}}\,. \tag{4.15}$$

Moreover, for $v_0, \tilde{v}_0 \in B_{X^\sigma}(\lambda)$, le us denote by $\Phi, \tilde{\Phi}$ the associated maps defined as in (4.13). Then, for any $w, \tilde{w} \in B_{Y^\sigma_{t_0,\tau}}(\lambda)$ there holds:
(4.16)
$$\|\Phi(w) - \tilde{\Phi}(\tilde{w})\|_{Y^0_{t_0,\tau}} \lesssim_{d,p} \left(\frac{\pi}{4} - t_0\right)^{-\alpha(p,d)} \tau^\kappa \lambda^{p-1} \left(\|v_0 - \tilde{v}_0\|_{X^0} + \|w - \tilde{w}\|_{Y^0_{t_0,\tau}}\right) .$$

Proof of Lemma 4.21. Proof of (4.14). Let $u_0 \in B_{X^\sigma}(\lambda)$. Recall that we need to fix the parameters a, b of the space X^σ, whih we will fo in this proof. From (4.13) and by the non-homogeneous Strichartz estimates with an admissible pair (q, r), one has

$$\begin{aligned}
\|\Phi(v)\|_{Y^\sigma_{[t_0,t_0+\tau]}} &\leqslant \| \cos(2s)^{-\alpha(p,d)} |v_L(s) + w(s)|^{p-1}(v_L(s) + w(s))\|_{L^{q'}_{[t_0,t_0+\tau]}\mathcal{W}^{\sigma,r'}} \\
&\leqslant \left(\int_{t_0}^{t_0+\tau} \left(\frac{\pi}{4} - s\right)^{-\alpha(p,d)q_1}\right)^{\frac{1}{q_1}} \||v_L + w|^{p-1}(v_L + w)\|_{L^{\tilde{q}}_{[t_0,t_0+\tau]}\mathcal{W}^{\sigma,r'}} \\
&\lesssim \left(\frac{\pi}{4} - t_0\right)^{-\alpha(p,d)} \tau^{\frac{1}{q_1}} \||v_L + w|^{p-1}(v_L + w)\|_{L^{\tilde{q}}_{[t_0,t_0+\tau]}\mathcal{W}^{\sigma,r'}} ,
\end{aligned}$$

where we used Hölder's inequality with $q_1, \tilde{q}$ to be adjusted and such that $\frac{1}{q_1} + \frac{1}{\tilde{q}} = \frac{1}{q'}$. We also used that $|\cos(2s)| \gtrsim \left(\frac{\pi}{4} - s\right)$ and $\tau \leqslant \frac{1}{2}\left(\frac{\pi}{4} - t_0\right)$.

Then by the Sobolev product estimates from Lemma 4.32 and the triangle inequality one can write

$$\begin{aligned}
\|\Phi(w)\|_{Y^\sigma_{[t_0,t_0+\tau]}} &\lesssim \left(\frac{\pi}{4} - t_0\right)^{-\alpha(p,d)} \tau^{\frac{1}{q_1}} \|v_L + w\|_{L^{q_2}\mathcal{W}^{\sigma,r_2}} \||v_L + w|^{p-1}\|_{L^{q_3}L^{r_3}} \\
&\lesssim \left(\frac{\pi}{4} - t_0\right)^{-\alpha(p,d)} \tau^{\frac{1}{q_1}} \left(\|v_L\|_{L^{q_2}\mathcal{W}^{\sigma,r_2}} + \|w\|_{L^{q_2}\mathcal{W}^{\sigma,\frac{2d}{d-1}}}\right) \\
&\times \left(\|v_L\|^{p-1}_{L^{q_3(p-1)}L^{r_3(p-1)}} + \|w\|^{p-1}_{L^{q_3(p-1)}L^{r_3(p-1)}}\right) ,
\end{aligned}$$

where the parameters q_1, q_2, q_3 and r_3 need to satisfy

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = \frac{1}{q'} ,$$

and

$$\frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r'} .$$

We recall that q', r' need to satisfiy $q', r' \in [1, 2]$ and

$$\frac{2}{q'} + \frac{d}{r'} = \frac{d}{2} + 2 .$$

Assume that we are able to choose the parameters $q_1, q_2, q_3, q', r_2, r_3, r'$ so that the norm $L^{q_2}L^{r_2}$ is Schrödinger admissible, $L^{q_3(p-1)}L^{r_3(p-1)}$ is σ-Schrödinger admissible, then we will meet the conditions in the construction of the X^σ norm and since $v_0 \in B_{X^\sigma}(\lambda)$ and $w \in B_{Y^\sigma_{t_0,\tau}}(\lambda)$ we will obtain

$$\|\Phi(w)\|_{Y^\sigma_{[t_0,t_0+\tau]}} \lesssim \left(\frac{\pi}{4} - t_0\right)^{-\alpha(p,d)} \tau^{\frac{1}{q_1}} \lambda^p \,,$$

hence (4.14).

In order to explain how to choose the parameters, we distinguish several cases.

Case 1. We assume $1 < p \leqslant \min\{1 + \frac{3}{d-2}, 1 + \frac{4}{d}\}$ so that $\sigma(p,d) = \frac{1}{2}$, and thus $0 < \sigma < \frac{1}{2}$. To control the $\mathcal{W}^{\sigma,r_2}$ norm of v_L we take $r_2 = \frac{2d}{d-1}$ We then choose $q_2 = 4$, so that (q_2, r_2) is Schrödinger admissible, hence

$$\|w\|_{L^4\mathcal{W}^{\sigma,\frac{2d}{d-1}}} \leqslant \|w\|_{Y^\sigma_{[t_0,t_0+\tau]}} \leqslant \lambda \,.$$

We also need that $r_3(p-1) \leqslant \frac{2d}{d-2}$, which is implied by r' satisfying $\frac{1}{r'} \geqslant p\left(\frac{1}{2} - \frac{1}{d}\right) + \frac{1}{2d}$. Note that in order to ensure that $q' \in [1,2]$ we need $\frac{1}{2} \leqslant \frac{1}{r'} \leqslant \frac{1}{2} + \frac{1}{d}$. The above restriction on r' makes it possible to choose such an r' if and only if

$$p\left(\frac{1}{2} - \frac{1}{d}\right) + \frac{1}{2d} \leqslant \frac{1}{2} + \frac{1}{d} \,,$$

which gives the condition $p \leqslant 1 + \frac{3}{d-2}$ satisfied by hypothesis. Then we can choose $\frac{1}{r'} = p\left(\frac{1}{2} - \frac{1}{d}\right) + \frac{1}{2d}$ (which fixes r_3 as desired, and satisfies $r' \in [1,2]$), and an associate q', which is also in the range $[1,2]$. To make sure that we can control the term $\|w\|_{L^{q_3(p-1)}L^{r_3(p-1)}}$ by the Strichartz norm $\|w\|_{Y^\sigma_{t_0,\tau}}$, we use the Sobolev embedding (which is applicable since $\frac{1}{r_3(p-1)} = \frac{2d}{d-2} \leqslant \frac{1}{2} \leqslant 1$, which reads

$$\|v_L\|_{L^{q_3(p-1)}L^{r_3(p-1)}} \lesssim \|v_L\|_{L^{q_3(p-1)}W^{\sigma,\tilde{r}_3(p-1)}}$$

where $\tilde{r}_3$ satisfies the Sobolev condition $\frac{1}{\tilde{r}_3(p-1)} \leqslant \frac{\sigma}{d} + \frac{1}{r_3(p-1)}$. If moreover $(q_3(p-1), \tilde{r}_3(p-1))$ is Schrödinger admissible, that is $\frac{2}{q_3(p-1)} + \frac{d}{\tilde{r}_3(p-1)} = \frac{d}{2}$ and $q_3(p-1), \tilde{r}_3(p-1) \geqslant 2$, then we can control $\|w\|_{L^{q_3(p-1)}L^{r_3(p-1)}}$ by the Strichartz norm $\|w\|_{Y^\sigma_{t_0,\tau}}$. We remark that the Sobolev condition can be written in the form

$$\frac{1}{r_3(p-1)} \geqslant -\frac{\sigma}{d} + \frac{1}{2} - \frac{2}{dq_3(p-1)} \,,$$

which can be written in variables q_1, p, d as

$$\frac{1}{q_1} \leqslant 1 - \frac{(p-1)(d-2\sigma)}{4} \,,$$

which is always satisfied for large q_1, since the quantity in the right-hand side is always bounded by below by $1 - \frac{\min\{\frac{3}{d-2}, \frac{4}{d}\}}{4}(d - 2\sigma) > 0$. Now we can choose freely q_3 such that $q_3(p-1) \geqslant 2$ large enough so that $\frac{1}{q_1} + \frac{1}{q_3} = \frac{1}{q'} - \frac{1}{4}$, which is possible since $\frac{1}{q'} > \frac{1}{4}$.

Case 2. We assume $d \geqslant 8$, $p \in (1+\frac{3}{d-2}, 1+\frac{4}{d})$ and thus $\sigma(p,d) = 2 - \frac{d-2}{2}(p-1)$. We set

$$q_2 := \frac{4}{(d-2)(p-1)-2} \geqslant 2 \text{ and } r_2 := \frac{2d}{d+2-(d-2)(p-1)} \geqslant 2\cdot$$

Because (q_2, r_2) is Schrödinger admissible we have $\|w\|_{L^{q_2}\mathcal{W}^{\sigma,r_2}} \leqslant \lambda$.

As above we need to make sure we can find r_3 such that $2 \leqslant r_3(p-1) \leqslant \frac{2d}{d-2}$ and

$$\frac{1}{r_3} + \frac{d+2-(d-2)(p-1)}{2d} = \frac{1}{r'}\cdot$$

The condition that $1 \leqslant q' \leqslant 2$ translates into $\frac{1}{2} \leqslant \frac{1}{r'} \leqslant \frac{1}{2} + \frac{1}{d}$. The condition on r_3 is written:

$$\frac{1}{2} + \frac{1}{d} + \frac{p-1}{d} \geqslant \frac{1}{r'} \geqslant \frac{(p-1)(d-2)}{2d} + \frac{d+2-(d-2)(p-1)}{2d} = \frac{1}{2} + \frac{1}{d}\cdot$$

hence we choose $\frac{1}{r'} = \frac{1}{2} + \frac{1}{d}$, satisfying both conditions, thus $q' = 2$. We now have $\frac{1}{r_3} = \frac{(d-2)(p-1)}{2d}$, so that $r_3(p-1) \geqslant 2$.

To finish the proof we need to control the norm $\|w\|_{L^{q_3(p-1)}L^{r_3(p-1)}}$. Arguing as in *Case 1.* we derive a condition on q_1, which is

$$\frac{1}{q_1} \leqslant 1 - \left(\frac{d}{4} - \frac{\sigma}{2}\right)(p-1)\,, \tag{4.17}$$

which is satisfied for large q_1 as soon as

$$1 - \left(\frac{d}{4} - \frac{\sigma(d)}{2}\right)(p-1) > 0\,,$$

that is $p - 1 \leqslant \frac{-d+4+\sqrt{d^2+8d-16}}{2(d-2)}$. This is satisfied since

$$p-1 \leqslant \frac{4}{d} \leqslant \frac{-d+4+\sqrt{d^2+8d-16}}{2(d-2)}\,.$$

Then we choose q_3 large enough so that $q_3(p-1) \geqslant 2$ and we conclude.

Proof of (4.15). We use the following, valid for any complex numbers a, b, c:

$$||a+b|^{p-1}(a+b) - |a+c|^{p-1}(a+c)| \lesssim_p |b-c|(|a|^{p-1} + |b|^{p-1} + |c|^{p-1})\,,$$

which implies that for any Schrödinger admisible pair (q,r), there holds

$$\|\Phi(w_1) - \Phi(w_2)\|_{Y^0_{t_0,\tau}} \lesssim_p \||w_1 - w_2|(|v_L|^{p-1} + |w_1|^{p-1} + |w_2|^{p-1})\|_{L^{q'}_{[t_0,t_0+\tau]}L^{r'}}\,.$$

With the use of the Hölder inequality we obtain

$$\|\Phi(w_1)-\Phi(w_2)\|_{Y^0_{t_0,\tau}} \lesssim_p \left(\frac{\pi}{4}-t_0\right)^{-\alpha(p,d)} \tau^{\frac{1}{q_1}} \|w_1-w_2\|_{L^{q_2}L^{r_2}} \times \left(\|v_L\|^{p-1}_{L^{q_3(p-1)}L^{r_3(p-1)}} + \|w_1\|^{p-1}_{L^{q_3(p-1)}L^{r_3(p-1)}} + \|w_2\|^{p-1}_{L^{q_3(p-1)}L^{r_3(p-1)}}\right) ,$$

where q_1, q_2, q_3, r_2, r_3, q', r' satisfy:

$$\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}=\frac{1}{q'}$$

and

$$\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r'} ,$$

which are the same conditions as in the proof of (4.14) thus with the same choices of q_1, q_2, q_3, r_2, r_3, q', r' we obtain (4.15).

Proof of (4.16). The proof follows is similar to that of (4.15), we omit the details. □

We give the details of how the estimates of Lemma 4.21 imply Proposition 4.3. Because we deal with some $1<p<2$, the fixed-point procedure seems not possible to work direcly in the space $Y^\sigma_{t_0,\tau}$, therefore we prove contraction in a weaker norm.

Proof of Proposition 4.3. From Lemma 4.21 we see that Φ stabilises the ball $B_{Y^\sigma_{t_0,\tau}}(\lambda)$ as soon as $\tau \leqslant C\left(\frac{\pi}{4}-t_0\right)^{\alpha(p,d)/\kappa}\lambda^{-(p-1)/\kappa}$, for some constant $C>0$.

Moreover, from (4.15) we see that on the ball $B_{Y^\sigma_{t_0,\tau}}(\lambda)$ (up to recucing the constant C), Φ is a contraction in the norm Y^0, thus has a unique fixed-point by the Picard fixed-point theorem.

To see that $w \in \mathcal{C}^0([t_0,t_0+\tau],\mathcal{H}^\sigma)$, we let $t_1<t_2\in[t_0,t_0+\tau]$ and write

$$w(t_1)-w(t_2)=\int_{t_1}^{t_2}\cos(2t')^{-\alpha(p,d)}\left(|v_L(t')+w(t')|^{p-1}(v_L(t')+w(t'))\right)\mathrm{d}t' ,$$

thus by the triangle inequality and (4.14) we have

$$\|w(t_1)-w(t_2)\|_{\mathcal{H}^\sigma} \leqslant_{p,d} \left(\frac{\pi}{4}-t_0\right)^{-\alpha(p,d)} |t_1-t_2|^\kappa \lambda^p ,$$

which implies the claimed continuity in time.

Continuity of the flow in the X^0 topology also follows from (4.16): to this end, let $v_0, \tilde{v}_0 \in B_{X^\sigma}(\lambda)$. From the local theory, we can construct associated solutions $v=v_L+w$, and $\tilde{v}=\tilde{v}_L+\tilde{w}$ on $[t_0,t_0+\tau]$ such that, thanks to (4.16) and the definition of τ, we have:

$$\|w-\tilde{w}\|_{Y^0_{t_0,\tau}} = \|\Phi(w)-\tilde{\Phi}(\tilde{w})\|_{Y^0_{t_0,\tau}} \lesssim_{p,d} \frac{1}{2}\left(\|v_0-\tilde{v}_0\|_{X^0}+\|w-\tilde{w}\|_{Y^0_{t_0,\tau}}\right) ,$$

which yields $\|w-\tilde{w}\|_{Y^0_{t_0,\tau}} \lesssim \|v_0-\tilde{v}_0\|_{X^0}$. Therefore, by the triangle inequality we obtain and Lemma 4.17 we obtain

$$\|v(t)-\tilde{v}(t)\|_{X^0} \lesssim \|v_0-\tilde{v}_0\|_{X^0}\,,$$

which proves the continuity in the X^0 norm. Interpolating with the fact that ϕ_t is bounded in the X^σ norm (which is a consequence of (4.14)) yields the continuity for all $\sigma' < \sigma$.

Finally, persistence of regularity (4.11) follows from the stabilisation estimates of Lemma (4.21) with σ' replaced by σ. (For details, see for example the proof of (8.7) in [BT20]. $\square$

3.2. Uniform approximation estimates. The construction of a global flow for (HNLS) will rely on a perturbation lemma.

Lemma 4.22 (Local approximation)**.** *Let $d \geqslant 2$, $p \in \left(1, 1+\frac{4}{d}\right)$, $\sigma \in (0, \sigma(p,d))$ and $t_0 \in (-\frac{\pi}{4}, \frac{\pi}{4})$. There exists $\tau > 0$ of the form (4.9) such that:*

(i) For any $v_0 \in B_{X^\sigma}(\lambda)$, there exists a unique solution $v = v_L + w$ (resp. $v_N = v_L + w_N$) in $e^{-i(t-t_0)}u_0 + Y^\sigma_{t_0,\tau}$ to (HNLS) (resp. (HNLS_N)).

(ii) For any $v_0, v_0^{(N)} \in B_{X^\sigma}(\lambda)$ such that $\|u_0^{(N)} - u_0\|_{X^{\sigma'}} \to 0$ for all $\sigma' < \sigma$, we have that for all $t \in [t_0, t_0+\tau]$, and all $\sigma' \in (0,\sigma)$, there holds

$$\|v(t) - v_N(t)\|_{X^{\sigma'}} \to 0\,.$$

(iii) Assume that $u_0^{(N)}, u_0 \in B_{X^{\sigma'}}(R) \cap B_{X^\sigma}(\lambda)$ for some $\sigma' \in (\sigma, \sigma(p,d))$, are that for all $0 \leqslant \delta < \sigma'$,

$$\|u_0 - u_0^{(N)}\|_{X^\delta} \leqslant C_1 N^{\delta-\sigma'}\,,$$

where C_1 does not depend on u_0. Then there exists a constant $C = C(t_0,\tau,R,\lambda) > 0$ such that for every $\delta < \sigma'$, there holds

$$\|w - w_N\|_{Y^\delta_{t_0,\tau}} \leqslant C N^{\delta-\sigma'}\,.$$

Proof. (i) directly follows from Proposition 4.3 and Remark 4.18, and provide solutions which we write $v(t) = e^{-itH}u_0 + w(t) = v_L(t) + w(t)$ and $v_N(t) = e^{-itH}u_0^{(N)} + w_N(t) = v_L^{(N)}(t) + w_N(t)$. In particular, for τ of the form (4.9) we have $\|w_N\|_{Y^\sigma_{t_0,\tau}}, \|w\|_{Y^\sigma_{t_0,\tau}} \leqslant \lambda$.

(ii) By the triangle inequality and Lemma 4.17 we have

$$\|v(t) - v_N(t)\|_{X^\sigma} \leqslant \|u_0 - u_0^{(N)}\|_{X^\sigma} + \|w - w_N\|_{Y^\sigma_{t_0,\tau}}\,.$$

Which is bounded (in N). Thus it is sufficient to prove the convergence for $\sigma' = 0$ only, using interpolation to obtain the result in its full generality. To this end, we observe that $w - w_N$ satisfies

$$(i\partial_t - H)(w - w_N) = \cos(2t)^{-\alpha(p,d)}\left(F(v_L + w) - \mathbf{S}_N F(\mathbf{S}_N v_L^{(N)} + \mathbf{S}_N w_N)\right)\,,$$

Let us set $z_N = \mathbf{S}_N v_L^{(N)} + \mathbf{S}_N w_N - (v_L + w)$, which satisfies

$$\begin{aligned}\|z_N(t)\|_{X^0} \leqslant \|(\mathrm{id} - \mathbf{S}_N) v_L(t)\|_{X^0} + \|\mathbf{S}_N(v_0 - v_0^{(N)})\|_{X^0} \\ + \|(id - \mathbf{S}_N) w(t)\|_{X^0} + \|\mathbf{S}_N(w(t) - w_N(t))\|_{X^0}\,,\end{aligned}$$

hence by the uniform continuity of $\mathbf{S}_N$ acting on X^0, and the assumption of the lemma we infer that

$$\|z_N(t)\|_{X^0} \leqslant C\|w - w_N\|_{Y^0_{t_0,\tau}} + \varepsilon(N)\,, \tag{4.18}$$

for some quantity $\varepsilon(N) \to 0$. We now observe that

$$\begin{aligned}(i\partial_t - H)(w - w_N) = \cos(2t)^{-\alpha(p,d)}(\mathrm{id} - \mathbf{S}_N) F(v_L + w) \\ + \cos(2t)^{\alpha(p,d)} \mathbf{S_N} \left(F(v_L + w) - F(v_L + w + z_N)\right)\,,\end{aligned}$$

hence by the Bernstein estimates and by the continuity of $\mathbf{S_N}$, we obtain:

$$\begin{aligned}\|w - w_N\|_{Y^0_{t_0,\tau}} \lesssim \cos(2t)^{-\alpha(p,d)} N^{-\sigma} \|F(v_L + w)\|_{\tilde{Y}^\sigma_{t_0,\tau}} \\ + \cos(2t)^{-\alpha(p,d)} \|\left(F(v_L + w) - F(v_L + w + z_N)\right)\|_{\tilde{Y}^0_{t_0,\tau}}\,.\end{aligned}$$

Therefore, using estimates (4.14) and (4.15), and up to choosing τ smaller (reducing the constant C in τ), but still of the form (4.9), we obtain

$$\|w - w_N\|_{Y^0_{t_0,\tau}} \leqslant C N^{-\sigma}\lambda + \frac{1}{2C}\|z_N\|_{X^0}\,,$$

where the constant C is as in (4.18), and therefore we obtain that

$$\|w(t) - w_N(t)\|_{X^0} \to 0\,,$$

hence the result for $\sigma' = 0$.

(iii) We readily know that $\|w - w_N\|_{X^{\sigma'}}$ is bounded, thus by interpolation it remains to prove that

$$\|w - w_N\|_{Y^0_{t_0,\tau}} \lesssim N^{-\sigma}\lambda\,. \tag{4.19}$$

The computations of (ii) imply that for a well-chosen τ of the form (4.9) we have

$$\|w - w_N\|_{Y^0_{t_0,\tau}} \leqslant C N^{-\sigma}\lambda + \frac{1}{2C}\|z_N\|_{X^{0,\sigma}}\,,$$

but thanks to the assumption we have

$$\|z_N\|_{Y^0_{t_0,\tau}} \leqslant C\|w - w_N\|_{Y^0_{t_0,\tau}} + 2N^{-\sigma}\lambda$$

which gives the claimed estimate. □

Iterating (iii) gives the following long-time estimate, provided a solution v to (HNLS) has been constructed on an interval $[0, T]$.

Corollary 4.23 (Long-time perturbation). *Let $d \geqslant 2$, $p \in \left(1, 1+\frac{4}{d}\right)$ and $0 < \sigma < \sigma' < \sigma(p,d))$. Let u_0 (resp. $u_0^{(N)}$) in $B_{X^{\sigma'}}(R) \cap B_{X^\sigma}(\lambda)$ and let $v = v_L + w$ (resp. $v_N = v_L + w_N$) be the associated solution to (HNLS) (resp. (HNLS_N)) and assume that v is well-defined on $[0,T]$. Then we have:*

$$\|w - w_N\|_{Y^\sigma_{[0,T]}} \leqslant CN^{\sigma-\sigma'} ,$$

where the constant C may depend on T, λ, R but not on u_0.

4. Quasi-invariant measures and their evolution

Our next task is to globalise the local statement of Proposition 4.3. In order to do so we need to keep track of the measures of the good sets of well-posedness, $i.e$ we need to estimate $\mu(\phi_t(A))$ where A is a μ-measurable set and ϕ_t stands for the flow of (HNLS).

4.1. Quasi-invariant measure evolution bounds. Due to explicit time dependence in the equation (HNLS) we do not expect the measure formally defined by

$$\nu_t(A) = \int_A e^{-\frac{1}{2}\|u\|^2_{\mathcal{H}^1} - \frac{\cos(2t)^{-\alpha(p,d)}}{p+1}\|u\|^{p+1}_{L^{p+1}}} \mathrm{d}u = \int_A e^{-\frac{\cos(2t)^{-\alpha(p,d)}}{p+1}\|u\|^{p+1}_{L^{p+1}}} \mathrm{d}\mu \,, \tag{4.20}$$

to be invariant, but only quasi-invariant. This is one of the main ideas in [BT20] and we will carry out the same program. The quasi-invariance will be obtained using a Liouville theorem on finite dimensional approximations of (HNLS) and a limiting argument.

The flow of the equation

$$\begin{cases} i\partial_t v - Hv = \cos(2t)^{p-3}\mathbf{S}_N(|\mathbf{S}_N v|^{p-1}\mathbf{S}_N v) \\ v(0) = u_0 \,, \end{cases} \tag{HNLS_N}$$

will be denoted by ϕ_t^N and can be decomposed as follows: writing $v = v_N + v^N$, a solution to (HNLS_N) with $v_N = \Pi_N v$ (where we recall that Π_N is the (non-smooth) projection on modes $0 \leqslant n \leqslant N$) we observe that from equation (HNLS_N), v_N satisfies

$$\begin{cases} i\partial_t v_N - Hv_N = \cos(2t)^{p-3}\mathbf{S}_N(|\mathbf{S}_N v_N|^{p-1}\mathbf{S}_N v_N) \\ v(0) = \mathbf{S}_N u_0 \,, \end{cases} \tag{4.21}$$

which is a finite dimensional ordinary differential equation in $\mathbb{C}^{N+1} \simeq \mathbb{R}^{2(N+1)}$ and denote by $\tilde{\phi}_t^N$ its flow.

From equation (HNLS_N), we observe that v^N satisfies

$$\begin{cases} i\partial_t v^N - Hv^N = 0 \\ v(0) = (\mathrm{id} - \Pi_N)u_0 \,, \end{cases} \tag{4.22}$$

so that if we identify v^N to the sequence $(v_n)_{n>N}$ such that $v^N = \sum_{n>N} v_n e_n$, we can explicitly solve (4.22) and find that for every $n \geqslant N+1$,

$$v_n(t) = e^{-it\lambda_n^2}(u_0)_n \,, \tag{4.23}$$

and denote by $\phi_t^{\perp,N}$ its flow.

We start with a lemma which is nothing but the Liouville theorem, which proof is recalled.

Lemma 4.24. *The measure* $\mathrm{d}\mu_N$ *is invariant under the flow* $\tilde{\phi}_t^N$ *of* (4.21).

Proof. First we recall that (4.21) is locally well-posed in $\mathcal{C}^1(\mathbb{R}, \mathbb{R}^{2(N+1)})$, thanks to the Cauchy theory for ordinary differential equations and globally well-posed since $t \mapsto \|u_N(t)\|_{L^2}$ is conserved. Moreover this equation admits a Hamiltonian structure. In order to see it, we write that for every n, $u_n = p_n + iq_n$ where p_n, q_n are real numbers. Then we claim that there exists a function $E_N = E_N(t, p_1, \dots, p_N, q_1, \dots, q_N)$ such that if $p = (p_1, \dots, p_n)$ and $q := (q_1, \dots, q_n)$, (4.21) takes the form

$$\begin{cases} p'(t) = & \partial_q E_N(t, p(t), q(t)) \\ q'(t) = & -\partial_p E_N(t, p(t), q(t)) \,. \end{cases} \tag{4.24}$$

In order to find E_N, write $u = p + iq$ and equate real and imaginary parts of (4.21) to obtain that a Hamiltonian satisfying (4.24) is given by

$$\begin{aligned} E_N(t, p_1, \dots, p_N, q_1, \dots, q_N) &:= \frac{1}{2} \sum_{n=0} \lambda_n^2 (a_n^2 + b_n^2) \\ &\quad + \frac{\cos(2t)^{\frac{d}{2}(p-1)-2}}{p+1} \left\| \sum_{n=0}^{N} (a_n + ib_n) e_n \right\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} . \end{aligned}$$

More details are given in [BTT13], Lemma 8.1. For a solution (p, q) to (4.24) we write $(p(t), q(t)) = \tilde{\phi}_t^N(p_0, q_0)$. Let

$$\mathrm{d}u_N := \bigotimes_{n=0}^{N} \mathrm{d}u_n = \bigotimes_{n=0}^{N} (\mathrm{d}p_n \otimes \mathrm{d}q_n) \,,$$

then for every smooth function f with compact support we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f(\tilde{\phi}_t^N(p_0, q_0)) \, \mathrm{d}u_N(p_0, q_0) = \int (\partial_p E_N \partial_q f - \partial_q E_N \, \partial_p f) \, \mathrm{d}p \, \mathrm{d}q = 0 \,,$$

where we integrated by parts in the last equality. Finally by density this shows that the measure $\mathrm{d}u_N$ is invariant under $\tilde{\phi}_t^N$, so is $\mathrm{d}\mu_N$. $\square$

The Hamiltonian we have found takes the form:

$$E_N(t) := \frac{1}{2} \|v(t)\|_{\mathcal{H}^1}^2 + \frac{\cos(2t)^{-\alpha(p,d)}}{p+1} \|\mathbf{S}_N v(t)\|_{L^{p+1}}^{p+1} .$$

It is *not conserved* under the flow ϕ_t^N and more precisely we have

$$E'_N(t) = \frac{d(p-1)-4}{p+1}\tan(2t)\cos(2t)^{-\alpha(p,d)}\|\mathbf{S}_N v(t)\|_{L^{p+1}}^{p+1}\,. \tag{4.25}$$

For $t \geqslant 0$ and $N \geqslant 0$ we define the finite measures, which are not necessarily probability measures, associated to the non-conserved energies E_N by:

$$\nu_t^{(N)}(A) := \int_A e^{-\frac{\cos(2t)^{-\alpha(p,d)}}{p+1}\|\mathbf{S}_N u\|_{L^{p+1}}^{p+1}}\,\mathrm{d}\mu(u)\,.$$

From the definition it follows that for all μ-measurable sets A one has $\nu_t^{(N)}(A) \leqslant \mu(A)$. Moreover we have the following convergence result.

Lemma 4.25. *Let $t \geqslant 0$. Then the measure ν_t is not trivial, i.e its density with respect to μ does not vanish almost surely. Moreover we have the strong convergence $\nu_t^{(N)} \to \nu_t$, that is for every measurable set A, $\nu_t^{(N)}(A) \to \nu_t(A)$.*

Proof. To see the first claim, just observe that for u in the support of the measure μ, since $p+1 \in \left(2, 2+\frac{4}{d}\right)$ we have $p+1 < \frac{2d}{d-2}$ and then $\|u\|_{L^{p+1}} < \infty$ thanks to Lemma 4.9. Moreover for such u in the support of μ we have $\mathbf{S}_N u \to u$ in L^{p+1} as $N \to \infty$. The domination $e^{-\frac{\cos(2t)^{-\alpha(p,d)}}{p+1}\|\mathbf{S}_N u\|_{L^{p+1}}^{p+1}} \leqslant 1$ and Lebesgue's convergence theorem ensures the strong convergence $\nu_t^{(N)} \to \nu_t$. $\square$

The quantitative quasi-invariance property which we are going so state and prove are exactly the same as in [BT20]. However for convenience we recall the proof.

Proposition 4.4 (Measure Evolution)**.** *Let $t \in (-\frac{\pi}{4}, \frac{\pi}{4})$, $N \geqslant 1$ and a μ-measurable set A.*

(i) $(\phi_t^N)_*\mu$ and μ are mutually absolutely continuous with respect to each other;

(ii) $\nu_t\left(\phi_t^N(A)\right) \leqslant \nu_0(A)^{\cos(2t)^{\alpha(p,d)}}$;

(iii) $\nu_0(A) \leqslant \nu_t\left(\phi_t^N(A)\right)^{\cos(2t)^{\alpha(p,d)}}$.

Proof. Note that once we have proved (ii) and (iii) then we immediately can conclude the proof for (i) since the two previous points give us that

$$(\phi_t^N)_*\nu_t \ll \nu_0 \ll (\phi_t^N)_*\nu_t\,.$$

As by definition $\mu \ll \nu_t$ and $\nu_t \ll \mu$, which proves (i).

To prove (ii) we start by studying the measure ν_t^N, writing

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\nu_t^{(N)}(\phi_t^N A) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{v\in\phi_t^N A} e^{-\frac{\cos(2t)^{-\alpha(p,d)}}{p+1}\|\mathbf{S}_N v\|_{L^{p+1}}^{p+1}}\,\mathrm{d}\mu(v)\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_A e^{-\frac{\cos(2t)^{\alpha(p,d)}}{p+1}\|\mathbf{S}_N\phi_t^{(N)}u_0\|_{L^{p+1}}^{p+1}}\,\mathrm{d}\mu(u_0)\right)\\ &= \frac{d(p-1)-4}{2}\tan(2t)\int_A \alpha_N(t,u)e^{-\alpha_N(t,u)}\,\mathrm{d}\mu(u_0)\,,\end{aligned}$$

with $\alpha_N(t,u)=\frac{\cos(2t)^{-\alpha(p,d)}}{p+1}\|\mathbf{S}_N u(t)\|_{L^{p+1}}^{p+1}$. In the first equality we have used the change of variable $v=\phi_t^N u_0$, which leaves $\mathrm{d}\mu$ invariant according to Lemma 4.24, indeed since $\mu=\mu_N\otimes\mu_N^\perp$ and $\phi_t^N=\tilde{\phi}_t^N\otimes e^{-itH}$, the invariance follows from the invariance of μ_N under $\tilde{\phi}_t^N$ (Application of Lemma 4.24 and conservation of $E_N(t)$) and invariance of $\mu_N^\perp$ under e^{-itH}, which is just invariance of complex Gaussian random variables under rotation.

Next we apply Hölder's inequality with a parameter $k\geqslant 1$ to be chosen later, and use that for all positive α we have $\alpha^k e^{-\alpha}\leqslant k^k e^{-k}$ which gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\nu}_t^{(N)}(\tilde{\phi}_t^N A) &\leqslant |d(p-1)-4|\tan(2t)\left(C_0\int_A \alpha^k(t,u(t))e^{-E_N(t)}\,\mathrm{d}u_0\right)^{\frac{1}{k}}\\ &\quad\times\left(\tilde{\nu}_t^{(N)}(\tilde{\phi}_t^N A)\right)^{1-\frac{1}{k}}\\ &= |d(p-1)-4|\tan(2t)\left(C_0\int_A \alpha^k(t,u(t))e^{-\alpha(t,u)-\frac{1}{2}\|\sqrt{H}u(t)\|_{L^2}^2}\,\mathrm{d}u_0\right)^{\frac{1}{k}}\\ &\quad\times\left(\tilde{\nu}_t^{(N)}(\tilde{\phi}_t^N A)\right)^{1-\frac{1}{k}}\\ &\leqslant |d(p-1)-4|\tan(2t)\frac{k}{e}\left(\tilde{\nu}_t^{(N)}(\tilde{\phi}_t^N A)\right)^{1-\frac{1}{k}}\,,\end{aligned}$$

where we used the backward change of variable that leaves the measure invariant again. Now we choose k to optimise this inequality, namely $k:=-\log\left(\nu_t^N(\phi_t^N A)\right)$ so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\nu_t^{(N)}(\phi_t^N A)\leqslant |d(p-1)-4|\tan(2t)\log\left(\nu_t^{(N)}(\phi_t^N A)\right)\nu_t^{(N)}(\phi_t^N A)\,.$$

We rewrite it as:

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\log\left(-\log\left(\nu_t^{(N)}(\phi_t^N A)\right)\right)\right) &\leqslant |d(p-1)-4|\tan(2t)\\ &= -\left|\frac{d}{2}(p-1)-2\right|\frac{\mathrm{d}}{\mathrm{d}t}\left(\log(\cos(2t))\right)\,,\end{aligned}$$

which after integration reads

$$-\log\left(\nu_t^{(N)}(\phi_t^N A)\right)\leqslant(\nu_0^{(N)}(A))^{\cos(2t)^{\alpha(p,d)}}\,.$$

Then we observe that for $M \geqslant N$ and for every μ-measurable set A, one has

$$\nu_t^{(N)}(\phi_t^N A) = \nu_t^{(M)}(\phi_t^N A) \to \nu_t(\phi_t^N A) \text{ as } M \to \infty$$

so that finally get the result passing to the limit.

The estimate (*iii*) is obtained by similar means, observing first that

$$\frac{\mathrm{d}}{\mathrm{d}t}\nu_t^{(N)}(\phi_t^N A) \geqslant -|d(p-1)-4|\tan(2t)\frac{k}{e}\left(\nu_t^{(N)}(\phi_t^N A)\right)^{1-\frac{1}{k}} ,$$

for every $k \geqslant 1$, optimising in k and integrating as before. □

4.2. Construction of the flow on a full probability set. The purpose of this section is simply to construct a set Σ such that $\mu_0(\Sigma) = 1$ and such that for any $u_0 \in \Sigma$ the flow of (HNLS) is well-defined.

Proposition 4.5 (Long-time probabilistic Cauchy theory)**.** *Let $d \geqslant 2$, $p \in \left(1, 1+\frac{4}{d}\right)$ and $\sigma \in (0, \sigma(p,d))$. For $i \geqslant i_0$ sufficiently large, the following holds. There exists a set $\Sigma^i \subset X^\sigma$, such that:*

(i) Σ^i is closed in the X^σ topology.

(ii) $\nu_0(X \setminus \Sigma^i) \leqslant e^{-ci}$ for some absolute constant $c > 0$.

(iii) For any $v_0 \in \Sigma^i$, there exists a solution $v = e^{-itH}v_0 + w = v_L + w \in v_L + \mathcal{C}^0((-\frac{\pi}{4}, \frac{\pi}{4}), \mathcal{H}^\sigma)$ which satisfies that for all $t \in (-\frac{\pi}{4}, \frac{\pi}{4})$, there holds

$$\|v(t)\|_{X^\sigma} \leqslant \sqrt{i}\left(\frac{\pi}{4} - |t|\right)^{-\frac{\alpha(p,d)}{2}} \left|\log\left(\frac{\pi}{4} - |t|\right)\right|^{\frac{1}{2}} , \tag{4.26}$$

and also

$$\|w(t)\|_{\mathcal{H}^\sigma} \leqslant \sqrt{i}\left(\frac{\pi}{4} - |t|\right)^{-\frac{\alpha(p,d)}{2}} \left|\log\left(\frac{\pi}{4} - |t|\right)\right|^{\frac{1}{2}} . \tag{4.27}$$

Remark 4.26. We will sometimes need a useful consequence of the proof of Proposition 4.5, that is, for an initial data $v_0 \in \Sigma^i$, $t_0 \in (0, \frac{\pi}{4})$, and $\varepsilon_0 > 0$ (arbitrarily small), if $\tau \sim \left(\frac{\pi}{4} - t_0\right)^{K+L(\frac{\alpha(p,d)}{2}+\varepsilon)}$ (the constants K, L being that of (4.9)), there holds

$$\|w\|_{Y^\sigma_{t_0,\tau}} \lesssim \sqrt{i}\left(\frac{\pi}{4} - t_0\right)^{-\frac{\alpha(p,d)}{2}-\varepsilon_0} .$$

We immediately infer the following consequence.

Corollary 4.27. *Let $d \geqslant 2$, $p \in \left(1, 1+\frac{4}{d}\right)$ and $\sigma \in (0, \sigma(p,d))$. There exists a set Σ of full ν_0 measure and such that for any $u_0 \in \Sigma$, a global solution*

$v = e^{-itH}u_0 + w = v_L + w$ (of the form given by Proposition 4.3) exists on $(-\frac{\pi}{4}, \frac{\pi}{4})$ and satisfies

$$\|v(t)\|_{X^\sigma}, \|w(t)\|_{\mathcal{H}^\sigma} \leqslant C(u_0)\left(\frac{\pi}{4}-|t|\right)^{-\frac{\alpha(p,d)}{2}}\left|\log\left(\frac{\pi}{4}-|t|\right)\right|^{\frac{1}{2}}, \tag{4.28}$$

where $\mu(C(u_0) > \lambda) \lesssim e^{-c\lambda^2}$.

Proof. This set is just $\Sigma := \bigcup_{i\geqslant i_0} \Sigma^i$ which satisfies the required properties and has measure

$$\nu_0(X \setminus \Sigma) \leqslant \inf_{i\geqslant 1} \nu_0(X \setminus \Sigma^i) = 0\,. \qquad \square$$

In order to prove this result we will need the following corresponding statement for (4.21).

Lemma 4.28 (Uniform in N long-time probabilistic Cauchy theory)**.** *Let $d \geqslant 2$, $p \in \left(1, 1+\frac{4}{d}\right)$ and $\sigma \in (0, \sigma(p,d))$. Let $i, j \geqslant 1$, and let $T_j = \frac{\pi}{4}(1-e^{-j})$. Let $N \geqslant 1$. Then there exists a set $\Sigma_N^{i,j}$ such that*

(1) $\nu_0(X \setminus \Sigma_N^{i,j}) \leqslant Ce^{-cij}$, at least for large enough i, j. The constants C, c do not depend on i, j.

(2) For any $u_0^{(N)} \in \Sigma_N^{i,j}$, the associated solution $v_N = v_L^{(N)} + w_N$ to (HNLS$_N$) satisfies, for all $t \in [-T_j, T_j]$,

$$\|v_N(t)\|_{X^\sigma}, \|w_N(t)\|_{\mathcal{H}^\sigma} \leqslant C\sqrt{i}e^{\frac{j\alpha(p,d)}{2}},$$

where the constant $C > 0$ is, in particular, independent of N.

Proof. By symmetry we deal only with positive times $[0, T_j]$. Let us define $\lambda := C_*\sqrt{ij}e^{\frac{j\alpha(p,d)}{2}}$, where C_* needs to be adjusted in the proof. Thanks to Proposition 4.3, there is a constant C_1 such that if we set $\tau = C_1 e^{-jK}\lambda^{-L}$ (the constants K, L) being that of (4.9), then the intervals $[n\tau, (n+1)\tau]$ for $n \in \left\{0, \dots, \frac{T_j}{\tau}\right\}$ are local well-posedness intervals for (HNLS$_N$) and initial data in $B_{X^\sigma}(\lambda)$, uniformly in N, thanks to Remark 4.18. Therefore, defining the set

$$\Sigma_N^{i,j} := \bigcap_{n=0}^{\lfloor T_j/\tau \rfloor} (\phi_{n\tau}^N)^{-1}(B_{X^\sigma}(\lambda))\,,$$

it follows that any initial data $u_0^{(N)}$ gives rise to an X^σ solution on $[-T_j, T_j]$ which matches the bounds (4.26) and (4.27). It remains to estimate $\nu_0(X \setminus \Sigma_N^{i,j})$, which by application of (4.4) gives:

$$\nu_0(X\setminus\Sigma_N^{i,j}) \leqslant \sum_{n=0}^{\lfloor T_j/\tau \rfloor} \nu_0((\phi_{n\tau}^N)^{-1}(X\setminus B_{X^\sigma}(\lambda))) \leqslant \sum_{n=0}^{\lfloor T_j/\tau \rfloor} \nu_{n\tau}(X\setminus B_{X^\sigma}(\lambda))^{\cos(2n\tau)^{\alpha(p,d)}}.$$

Then we use the crude bound $\nu_{n\tau}(X \setminus B_{X^\sigma}(\lambda)) \leqslant \mu_0(X \setminus B_{X^\sigma}(\lambda)) \leqslant e^{-c\lambda^2}$, so that finally

$$\nu_0(X \setminus \Sigma_N^{i,j}) \leqslant \frac{T_j}{\tau} e^{-c\lambda^2 \cos(2T_j)^{\alpha(p,d)}} \lesssim e^{jK}\lambda^L e^{-c\lambda^2 e^{-j\alpha(p,d)}},$$

so that by definition of λ this yields

$$\nu_0(X \setminus \Sigma_N^{i,j}) \lesssim e^{j(K+L\alpha(p,d)/2)} e^{-cC_1^2 ij},$$

which is less that Ce^{-cij} for large enough C_1 and for $i \geqslant i_0$, $j \geqslant j_0$. $\square$

Proof of Proposition 4.5. Let us set $\Sigma_N^i = \bigcap_{j \geqslant j_0} \Sigma_N^{i,j}$ which is such that

$$\nu_0(X \setminus \Sigma_N^i) \lesssim e^{-ci},$$

and such that any $u_{0,N} \in \Sigma_N^i$ gives rise to a global solution satisfying the bounds (4.27) and (4.26). We consider the set

$$\Sigma^i = \{u_0 \in X^\sigma, \exists N_k \to \infty \text{ and } u_0^{N_k} \in \Sigma_{N_k}^i, \|u_0 - u_0^{(N_k)}\|_{X^\sigma} \to 0\},$$

which is a closed subset of X^σ, proving (i).

To prove (ii), observe that $\limsup_N \Sigma_N^i \subset \Sigma_i$ thus by Fatou's lemma there holds

$$\nu_0(\Sigma^i) \geqslant \nu_0(\limsup_N \Sigma_N^i) \geqslant \limsup_N \nu_0(\Sigma_N^i) \geqslant \nu_0(X) - e^{-ci}.$$

To prove (iii), let $u_0 \in \Sigma^i$ and an associated sequence $u_0^{(N_k)} \in \Sigma_{N_k}^i$ such that

$$\|u_0 - u_0^{(N_k)}\|_{X^\sigma} \to 0.$$

We need to construct a solution $v = v_L + w = e^{-itH}v_0 + w$ to (HNLS) satisfying the bounds (4.27) and (4.26). To this end, let T such that $|T| < \frac{\pi}{4}$ and let τ be an associated local well-posedness time associated to u_0 and T. Without restriction, only considering large k, we can even assume that this τ is also a local well-posedness time for the $u_0^{(N_k)}$. By Proposition 4.22 we infer that $\|v(\tau) - v_{N_k}(\tau)\|_{X^{\sigma'}} \to 0$ for any $0 < \sigma' < \sigma$. We also know that by definition of $v_0^{(N_k)}$ we have the uniform bound $\|v_N(\tau)\|_{X^\sigma} \leqslant C(\tau)$, for some irrelevant constant C. Since v_N is bounded in $X^{0,\sigma}$ and converges to $v(\tau)$ in $X^{\sigma'}$, it follows that $v(\tau) \in X^\sigma$ and satisfies

$$\|v(\tau)\|_{X^\sigma} \leqslant C(\tau),$$

thus satisfying the same bound as the v_{N_k} and therefore we can iterate this argument for $n \leqslant \lfloor T/\tau \rfloor$. Since T is arbitrary, this yields (iii). $\square$

With this first global theory, we are able to pass to the limit $N \to \infty$ in Proposition 4.4.

Proposition 4.6. *Let $A \subset \Sigma$ be a measurable set, then there holds that for all $t \geqslant 0$,*

$$\nu_0(A) \leqslant \nu_t(\phi_t A)^{\cos(2t)^{\alpha(p,d)}} . \tag{4.29}$$

Proof. Let $0 < \sigma < \sigma(p,d)$. We recall that since $\nu_0(X \setminus X^\sigma) = 0$, and since X^σ is a complete space, ν_0 can be viewed as a regular measure on X^σ (this is Ulam's theorem, see for example Theorem 7.1.4 in [Dud02]). This remark will allow to reduce the analysis to compact sets. In our analysis we will need a $\sigma' \in (\sigma, \sigma(p,d))$, which we fix. We also further restrict to proving (4.29) for all $t \in [0,T]$, T being arbitrary in $(0, \frac{\pi}{4})$.

Let us first consider a compact set $K \subset X^\sigma$. Using that $\Sigma = \bigcup_{i \geqslant 1} \Sigma^i$, and $\nu_0(X^{\sigma'}) = 1$ we can write that

$$\nu_0(K) = \lim_{R\to\infty} \lim_{i\to\infty} \nu_0(K \cap \Sigma^i \cap B_{X^{\sigma'}}(R)) .$$

Since $B_{X^{\sigma'}}(R)$ is closed in $X^{0,\sigma}$ and Σ^i is closed in X^σ, the set $K_{R,i} := K \cap \Sigma^i \cap B_R$ is a compact of X^σ.

Let ε, and by Corollary 4.23 we can fix $N_0(\varepsilon) > 0$ such that for all $N \geqslant N_0(\varepsilon)$ there holds

$$\phi_t^N(K_{i,R}) \subset \phi_t K_{R,i} + B_{X^\sigma}(\varepsilon) .$$

Let us consider a local existence time τ for v in X^σ, associated to the initial data in $K_{R,i}$. We recall that such a local well-posedness time depend only on T and $\|u_0\|_{X^{0,\sigma}}$ which is bounded by R. But thanks to Proposition 4.3 (*iii*) we also know that v exists on $[0,\tau]$ as a $X^{\sigma'}$ solution, satisfying $\|u\|_{L^\infty_{[0,\tau]} X^{\sigma'}} \leqslant CR$. Therefore $\phi_\tau K_{i,R}$ is bounded in $X^{\sigma'}$ thus compact in X^σ. Since $K_{i,R} \subset \Sigma^i$, then for all $t \in [0,T]$ we have $\|v(t)\|_{X^\sigma} \leqslant C\left(\frac{\pi}{4} - T\right)^{-\frac{\alpha}{2}}$. We can iterate this argument on $[\tau, 2\tau]$, $\ldots$, $[k\tau, (k+1)\tau]$ (recall that τ depends on the bound $\|v(k\tau)\|_{X^\sigma}$ which is controlled) and therefore obtain that $\phi_t(K_{i,R})$ is compact in X^σ and in particular a closed set.

We are now ready to give the proof. We start by applying Proposition 4.4 to bound

$$\nu_0(K_{R,i}) \leqslant \nu_t(\phi_t^N(K_{R,i}))^{\cos(2t)^{\alpha(p,d)}} .$$

Observe that for $N \geqslant N(\varepsilon)$ we have

$$\nu_t(\phi_t^N(K_{R,i})) \leqslant \nu_t(\phi_t K_{R,i} + B_{X^\sigma}(\varepsilon)) ,$$

so that passing to the limit $\varepsilon \to 0$ gives

$$\nu_0(K_{R,i}) \leqslant \nu_t(\phi_t K_{R,i})^{\cos(2t)^{\alpha(p,d)}} \leqslant \nu_t(\phi_t K)^{\cos(2t)^{\alpha(p,d)}} ,$$

because $\phi_t K_{i,R}$ is closed in X^σ.

Passing to the limit $R \to \infty$ and $i \to \infty$ gives (4.29) for any compact set $K \subset \Sigma$. To recover the full result, let $A \subset \Sigma^i$ be a measurable set. Then by regularity of the measure ν_0 there exists a sequence of compact sets $K_n \subset A$ such that $\nu_0(K_n) \to \nu_0(A)$. Applying (4.29) to K_n yields that

$$\nu_0(K_n) \leqslant \nu_t(\phi_t K_n)^{\cos(2t)^{\alpha(p,d)}} \leqslant \nu_t(A)^{\cos(2t)^{\alpha(p,d)}},$$

and passing to the limit $n \to \infty$ yields the result. $\square$

5. Scattering

5.1. Growth of the L^{p+1} norms. In order to estimate the L^{p+1} norms of a solution v to (HNLS) we first state a result valid for a given time.

Lemma 4.29. *Let us define, for any $\lambda > 0$ and any $t \in \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ the set*

$$A_{t,\lambda} := \{u_0 \in X^0,\ \|\phi_t u_0\|_{L^{p+1}} > \lambda\}\,. \tag{4.30}$$

There holds:

$$\nu_0(A_t) \leqslant e^{-\frac{\lambda^{p+1}}{p+1}}\,. \tag{4.31}$$

Remark 4.30. The estimate (4.31) is better than the bound given by $\nu_0(A) \leqslant \mu(A)^{\cos(2t)^{\alpha(p,d)}}$, which amounts to saying that $\nu_0(A) \leqslant 1$ when $p \sim 1 + \frac{4}{d}$.

Proof. Let us prove (4.31). Using (4.29), and remarking that by definition of A_t we have $\|\phi_t u_0\| > \lambda$ for every $u_0 \in A_t$ so that:

$$\begin{aligned}\nu_0(A) &\leqslant \nu_t(\phi_t A)^{\cos(2t)^{\alpha(p,d)}}\\ &\leqslant \left(e^{-\cos(2t)^{-\alpha(p,d)}\frac{\lambda^{p+1}}{p+1}}\mu(\phi_t A)\right)^{\cos(2t)^{\alpha(p,d)}}\\ &\leqslant e^{-\frac{\lambda^{p+1}}{p+1}}\,,\end{aligned}$$

where we used that $\mu(\phi_t A) \leqslant 1$ in the last line. $\square$

Next we want to estimate the L^{p+1} norm of the solutions for all times. We will distinguish two cases and introduce a real number $p_{\max}$ defined by

$$p_{\max}(d) := \frac{5 - d + \sqrt{9d^2 - 2d + 9}}{2(d-1)} < 1 + \frac{3}{d-2} \tag{4.32}$$

if $d \leqslant 7$ and if $d \geqslant 8$ then

$$p_{\max}(d) := \min\{x > 0, P_d(x) = 0\}\,, \tag{4.33}$$

where

$$P_d := (d-2)X^3 + (d-4)X^2 - 6X - 2d - 4\,,$$

is a polynomial with only one real root.

Note that for $d \geqslant 8$ one has

$$p_{\max}(d) \geqslant \min\left\{\frac{5 - d + \sqrt{9d^2 - 2d + 9}}{2(d-1)}, 1 + \frac{3}{d-2}\right\}\,.$$

To prove this fact, we observe that the discriminant of P_d is negative, at least for $d \geqslant 8$ and thus P_d has a unique real root. Note that

$$\min\{\frac{5-d+\sqrt{9d^2-2d+9}}{2(d-1)}, 1+\frac{3}{d-2}\} = 1+\frac{3}{d-2}$$

as soon as $d \geqslant 9$. To conclude we need to show that $P_d(1+\frac{3}{d-2}) < 0$ which is equivalent to $d^2 - 10d + 7 > 0$, satisfied for $d \geqslant 9$. Similarly one has $P_d(\frac{5-d+\sqrt{9d^2-2d+9}}{2(d-1)}) < 0$ for $d = 8$. We also have $p_{\max}(d) \geqslant 1 + \frac{4}{d}$ if $d \leqslant 7$ with similar computations.

Proposition 4.7. *Let $d \geqslant 2$, $p \in (1, p_{\max}(d))$ and $\sigma \in (0, \sigma(p,d))$. Let also $\eta > 0$ and $|t| < \frac{\pi}{4}$. There exists a set $G_{\eta,t} \subset X$ such that $\nu_0(X^0 \setminus G_{\eta,t}) \leqslant \eta$ and such that for all $u_0 \in G_{\eta,t}$, there exists a unique solution to (HNLS) with initial data u_0 which writes $v(t') = e^{-itH}u_0 + w(t')$ where $w \in Y^{\sigma}_{[-t,t]}$. Furthermore this solution satisfies for all $t' \in [-t,t]$ the estimate*

$$\|w(t')\|_{L^{p+1}} \lesssim \log^{\frac{1}{2}}\left(\frac{1}{\eta}\right)\left|\log\left(\frac{\pi}{4} - |t|\right)\right|^{\frac{1}{2}}, \tag{4.34}$$

where the implicit constant only depending on p and d.

From there we can deduce exactly as in Corollary 4.27 the following global estimate.

Corollary 4.31. *Let $d \geqslant 2$ and $p \in (1, p_{max}(d))$. There exists $\varepsilon_0 > 0$ and a set G of full ν_0 measure, such that for all $u_0 \in G$ and all $\varepsilon \in [0, \varepsilon_0]$ there exists a unique global solution $v(t) = e^{-itH}u_0 + w(t)$ to (HNLS) with initial data u_0 which furthermore satisfies*

$$\|w(t)\|_{\mathcal{W}^{\varepsilon,p+1}} \lesssim C(u_0)\left|\log\left(\frac{\pi}{4} - |t|\right)\right|^{\frac{1}{2}}, \tag{4.35}$$

for every $|t| < \frac{\pi}{4}$. Moreover there exist $c, C > 0$ such that

$$\mu(C(u_0) > \lambda) \lesssim e^{-c\lambda^2}.$$

Proof. We only rapidly explain how we obtain the $\mathcal{W}^{\varepsilon,p+1}$ estimates. This follows from the proof of Lemma 4.7, where we can multiply equation (4.36) by $H^{\frac{\varepsilon}{2}}$, then all the estimates are essentially the same provided $\varepsilon > 0$ is small enough. We use in particular that we use non-endpoints estimates, for which there is always room for an ε modification of the parameters.

Let us set $G := \limsup_{n\to\infty} G_{2^{-n},\frac{\pi}{4}(1-2^{-n})}$. Then any $u_0 \in G$ belongs to infinitely many $G_{2^{-n},\frac{\pi}{4}(1-2^{-n})}$ therefore producing global solutions with required estimats. Finally observe that

$$\nu_0(G) = \nu_0\left(\limsup_{n\to\infty} G_{2^{-n},\frac{\pi}{4}(1-2^{-n})}\right) \geqslant \limsup_{n\to\infty} \nu_0(G_{2^{-n},\frac{\pi}{4}(1-2^{-n})}) = \nu_0(X) \quad \square$$

Proof of Proposition 4.7. Again we only deal with the forward in time part of the estimate, by time reversibility. Let $0 < t < \frac{\pi}{4}$, $\eta > 0$ and p as in the lemma. Let also $\tau > 0$ be a local existence time of the form (4.9), given by Proposition 4.3, which we choose uniform in $[0, t]$, and which we may shrink in the sequel. We also claim that we can prove the result on v rather than w since estimates of this kind on the linear part v_L are granted by Lemma 4.14. We may also assume, without restriction that we only consider initial data in $u_0 \in \Sigma$ so that we have access to estimate (4.28).

We start with the case $p > 1 + \frac{2}{d-1}$. We set $t_n := n\tau$ for $n = 0, \dots, \lfloor \frac{t}{\tau} \rfloor$. Let $t' \in [t_n, t_{n+1})$ and write $v(t')$ in terms of $v(t_n)$ as the solution of the initial value problem at t_n using the Duhamel formula:

(4.36)
$$v(t') = e^{-i(t'-t_n)H} v(t_n) - i \int_{t_n}^{t'} e^{-i(t'-s)H} \Big(\cos(2s)^{-\alpha(p,d)} |v(s)|^{p-1} v(s) \Big) \, \mathrm{d}s \, .$$

The triangle inequality and the dispersion estimates $L^{p+1} \to L^{\frac{p+1}{p}}$ given by Proposition 4.2 yield

$$\|v(t')\|_{L^{p+1}} \leqslant \|v(t_n)\|_{L^{p+1}} + \underbrace{\|(e^{-i(t'-t_n)} - \mathrm{id}) v(t_n)\|_{L^{p+1}}}_{I} + \underbrace{\int_{t_n}^{t'} \frac{1}{|t-s|^{d\left(\frac{1}{2} - \frac{1}{p+1}\right)}} \cos(2s)^{-\alpha(p,d)} \|v(s)\|_{L^{p+1}}^p \, \mathrm{d}s}_{II} \, . \tag{4.37}$$

In the sequel we estimate the terms I and II differently.

For II we use that
$$\cos(2s) \gtrsim \left(\frac{\pi}{4} - s \right)$$
for $s \in [t_n, t_{n+1}]$. Then for parameters $\gamma, \gamma' \geqslant 1$ satisfying $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$ and Hölder's inequality:

$$\begin{aligned} II &\lesssim \left(\frac{\pi}{4} - t \right)^{-\alpha(p,d)} \int_{t_n}^{t'} \frac{1}{|t'-s|^{d\left(\frac{1}{2} - \frac{1}{p+1}\right)}} \|v(s)\|_{L^{p+1}}^p \, \mathrm{d}s \\ &\lesssim \left(\frac{\pi}{4} - t \right)^{-\alpha(p,d)} \left(\int_{t_n}^{t'} |t'-s|^{-d\gamma\left(\frac{1}{2} - \frac{1}{p+1}\right)} \, \mathrm{d}s \right)^{\frac{1}{\gamma}} \left(\int_{t_n}^{t'} \|v(s)\|_{L^{p+1}}^{p\gamma'} \, \mathrm{d}s \right)^{\frac{1}{\gamma'}} \\ &\lesssim \left(\frac{\pi}{4} - t \right)^{-\alpha(p,d)} \tau^{\frac{1}{\gamma} - d\left(\frac{1}{2} - \frac{1}{p+1}\right)} \|v\|_{L^{p\gamma'}_{[t_n, t_{n+1}]} L^{p+1}}^p \, , \end{aligned}$$

provided the integrability condition $\gamma d \left(\frac{1}{2} - \frac{1}{p+1} \right) < 1$ which we write conveniently in the form

$$\frac{1}{\gamma} > d \left(\frac{1}{2} - \frac{1}{p+1} \right) . \tag{4.38}$$

Now we deal with the case $d \leqslant 8$ and $d \geqslant 8$ separately.

Case 1. Assume that $d \leqslant 8$ so that $\sigma(p,d) = \frac{1}{2}$. Recall that by (4.28) we have

$$\|v\|_{L^q_{[t_n,t_{n+1}]}L^r} \lesssim \left(\frac{\pi}{4} - t_n\right)^{-K}$$

for some $K > 0$ which only depend on d and p. Indeed such a bound can be obtained for $w = v - w_L$ associated to initial data $v(t_n)$ in $B_{X^\sigma}(\lambda)$, and the same estimates are obtained for v_L as soon as $r < 2 + \frac{4}{d}$.

Thus the norm $L^{p\gamma'}L^{p+1}$ is $\frac{1}{2}$-Schrödinger admissible (note that $p+1 < 2+\frac{4}{d}$) if and only if

$$p\gamma' > 2\,, \tag{4.39}$$

$$\frac{1}{p\gamma'} > \frac{d-1}{4} - \frac{d}{2(p+1)}\,, \tag{4.40}$$

and

$$\frac{d-1}{4} - \frac{d}{2(p+1)} > 0\,. \tag{4.41}$$

Condition (4.41) is equivalent to $p \geqslant 1 + \frac{2}{d-1}$ which is satisfied, since $p \geqslant 1 + \frac{2}{d}$ by hypothesis. Now observe that once conditions (4.38), (4.39) and (4.40) are satisfied, we obtain

$$II \leqslant \left(\frac{\pi}{4} - t\right)^{-K_1} \tau^{\beta_1}\,, \tag{4.42}$$

for some $K_1 > 0$ and $\beta_1 > 0$.

We remark that Conditions (4.38) and (4.39) are satisfied as soon as

$$\begin{cases} p & < & \frac{d+3}{d-3} \\ p & \leqslant & \frac{5-d+\sqrt{9d^2-2d+9}}{2(d-1)} \end{cases}. \tag{4.43}$$

Then one has $\frac{5-d+\sqrt{9d^2-2d+9}}{2(d-1)} \leqslant \frac{d+3}{d-3}$. We obtain that the conditions (4.38), (4.39) and (4.40) are satisfied for $p < p_{\max}$, which is satisfied by hypothesis.

Case 2. Assume that $d \geqslant 8$ and $p \geqslant 1 + \frac{3}{d-2}$. With $\sigma = \sigma(p,d)^-$, conditions (4.38), (4.39) and (4.40) become:

$$\begin{cases} \frac{1}{\gamma} & > & d\left(\frac{1}{2} - \frac{1}{p+1}\right) \\ p\gamma' & > & 2 \\ \frac{1}{\gamma'} & > & \frac{p}{2}\left(\frac{d}{2} - \sigma(p,d) - \frac{d}{p+1}\right) \end{cases}, \tag{4.44}$$

which is equivalent to

(4.45)
$$\begin{cases} (d-2)p^2+(d-6)p-2d-4<0 \\ (d-2)p^3+(d-4)p^2-6p-2d-4<0(d-2)p^2+(d-4)p-2d-2>0\,. \end{cases}$$

We observe that the second condition is precisely $p<p_{\max}(d)$ and that the first is equivalent to $p\leqslant \frac{d+2}{d-2}=1+\frac{4}{d-2}$ which is satisfied. The last condition is equivalent to $p>\frac{4-d+\sqrt{9d^2-16d}}{2(d-2)}$ which is smaller that $1+\frac{3}{d-2}$, so that the previous conditions are satisfied if and only if $p<p_{\max}(d)$.

We turn to estimating I. In order to do so, applying a Sobolev embedding in time and switching derivatives from time to space and provided we fix $\varepsilon>0$ sufficiently small, gives the existence of constants $C\coloneqq C_\varepsilon$, $\beta_2\coloneqq\beta_2(\varepsilon)$ and an integer $q\coloneqq q(\varepsilon)$ satisfying

$$I\leqslant C\tau^{\beta_2}\|S(\cdot-t_n)v(t_n)\|_{L^q_{\langle\cdot\rangle^{-2}\,\mathrm{d}t}(\mathbb{R},\mathcal{W}^{\varepsilon,p+1})}\,.$$

A complete proof ot this claim is postponed to Lemma 4.33 which we refer to for the details.

We introduce the sets

$$B_{n,\lambda}\coloneqq\left\{u_0\in X,\|S(\cdot-t_n)\phi_{t_n}v_0\|_{L^q_{\langle\cdot\rangle^{-2}\,\mathrm{d}t}(\mathbb{R},\mathcal{W}^{\varepsilon,p+1})}>\lambda\left(\frac{\pi}{4}-t_n\right)^{-\frac{\alpha(p,d)}{2}}\right\}$$

Let us also define

$$B'_{n,\delta}\coloneqq\{v_0\in X,\|S(\cdot-t_n)v_0\|_{L^q_{\langle\cdot\rangle^{-2}\,\mathrm{d}t}(\mathbb{R},\mathcal{W}^{\varepsilon,p+1})}>\delta\}\,.$$

Then applying Proposition 4.4 gives

$$\begin{aligned}\nu_0(B_{n,\lambda})&\leqslant\nu_{t_n}\left(\phi_{t_n}B_{n,\lambda}\right)^{\cos(2t_n)^{\alpha(p,d)}}\\&\leqslant\mu\left(\phi_{t_n}B_{n,\lambda}\right)^{\cos(2t_n)^{\alpha(p,d)}}\\&\leqslant\mu\left(B'_{n,\lambda(\frac{\pi}{4}-t_n)^{-\alpha(p,d)}}\right)^{\cos(2t_n)^{\alpha(p,d)}}\,.\end{aligned}$$

Now we claim that there exists constants $C,c>0$ only depending on d,p such that that

$$\mu(B'_{n,\delta})\leqslant e^{-c\delta^2}\,. \tag{4.46}$$

Assuming (4.46), we conclude that:

$$\nu_0(B_{n,\lambda})\leqslant e^{-c\lambda^2}\,. \tag{4.47}$$

For

$$u_0\in(\phi_{t_n})^{-1}\left(B_{X^\sigma}\left(\lambda(\frac{\pi}{4}-t_n)^{-\alpha(p,d)/2}\right)\right)\cap(X\setminus B_{n,\lambda})\cap(X\setminus A_{t_n,\lambda})\,,$$

where we recall that $A_{t_n,\lambda}$ is defined by (4.30), we have

$$\phi_{t_n} u_0 \in B_{X^\sigma}\left(\lambda(\frac{\pi}{4}-t_n)^{-\alpha(p,d)/2}\right),$$

and then:

$$I \lesssim \tau^{\beta_2}\left(\frac{\pi}{4}-t\right)^{-\frac{\alpha(p,d)}{2}}\lambda\,. \tag{4.48}$$

Taking into account (4.48) and (4.42) in (4.37) imply that we can adjust $K_2, K_3 >$ such that

$$\tau \sim \lambda^{-K_2}\left(\frac{\pi}{4}-t\right)^{K_3} \text{ and } \|v(t')\|_{L^{p+1}} \leqslant 2\lambda \text{ for } t' \in [t_n, t_{n+1}]\,, \tag{4.49}$$

The estimate (4.46) follows from

$$\|S(\cdot)f\|_{L^{p_0}_\Omega L^q_{\langle\cdot\rangle^{-2}\,\mathrm{d}t}(\mathbb{R},\mathcal{W}^{\varepsilon,p+1})} \lesssim \sqrt{p_0}\,, \tag{4.50}$$

at least for $p_0 \geqslant \max\{q,r\}$. Assuming such a bound gives the claim using the Markov inequality, and we skip the details, see Appendix A of [BT08b] for instance.

It remains to prove (4.50). Let $p_0 \geqslant \max\{q,r\}$. By Minkowski's inequality and Theorem 4.8 we get

$$\begin{aligned}\|S(\cdot)f\|_{L^{p_0}_\Omega L^q_{\langle t\rangle^{-2}\mathrm{d}t}(\mathbb{R},\mathcal{W}^{\varepsilon,p+1})} &\lesssim \left\|\left\|\sum_{n\geqslant 0}\frac{e^{it\lambda_n^2}}{\lambda_n^{1-\varepsilon}}g_n e_n\right\|_{L^{p_0}_\Omega}\right\|_{L^q_{\langle t\rangle^{-2}\mathrm{d}t}(\mathbb{R},\mathcal{W}^{\varepsilon,p+1})}\\ &\lesssim \sqrt{p_0}\|(\lambda_n^{-1+\varepsilon}e_n)\|_{L^{p+1}_x\ell^2_{\mathbb{N}}}\,.\end{aligned}$$

Remark that we used the finiteness of the measure $\langle t\rangle^{-2}\,\mathrm{d}t$. Another use of the Minkowski inequality, recalling that $p+1 \in \left(2, \frac{2d}{d-2}\right)$ and Lemma 4.13 finally prove the claim.

To conclude the proof, we repeat again the usual argument. The set of *good* initial data G_λ is defined as the set of initial data $u_0 \in X^0$ which give rise to solutions defined on $[0,t]$ and such that $\|v(t')\|_{L^{p+1}} \leqslant 2\lambda$ for every $t' \leqslant t$. We then observe that the analysis developed to get the bound (4.49) leads to the inclusion:

$$\bigcap_{n=0}^{\lfloor\frac{t}{\tau}\rfloor}\left((\phi_{n\tau})^{-1}\left(B_{X^\sigma}\left(\lambda(\frac{\pi}{4}-t_n)^{-\alpha(p,d)/2}\right)\right)\cap(X\setminus B_{n,\lambda})\cap(X\setminus A_{t_n,\lambda})\right)\subset G_\lambda\,.$$

The set of *bad* initial data $B_\lambda := X\setminus G_\lambda$ satisfies

$$\nu_0(B_\lambda) \lesssim \lambda^{K_2}\left(\frac{\pi}{4}-t\right)^{-K_3}e^{-c\lambda^2}\,,$$

where we used Lemma 4.4 to bound

$$\nu_0\left(X \setminus (\phi_{n\tau})^{-1}\left(B_{X^\sigma}\left(\lambda(\frac{\pi}{4}-t_n)^{-\alpha(p,d)/2}\right)\right)\right) \leqslant e^{-c\lambda^2},$$

and the expression of τ. Finally the measure of this set is made smaller than η by taking $\lambda \sim \log^{\frac{1}{2}}\left(\frac{1}{\eta}\right)\left|\log\left(\frac{\pi}{4}-|t|\right)\right|^{\frac{1}{2}}$, which ends the proof.

Finally we treat the case $p < 1+\frac{2}{d-1}$, which is much simpler, and sufficient for proving the lemma in dimension $d=2$. Then for $\sigma < \frac{1}{2}$ close enough to $\frac{1}{2}$ we have $\mathcal{H}^\sigma \hookrightarrow L^{p+1}$. Then

$$\|v(t_n) - v(t)\|_{L^{p+1}} \lesssim \|v(t_n) - v(t)\|_{\mathcal{H}^\sigma}\,. \tag{4.51}$$

Then we adjust τ such that if $v(t_n) \in B_{X^\sigma}(\lambda)$ and $|t-t_n| \leqslant \tau$ then by the local Cauchy theory of Proposition 4.3, there exists $K, \beta > 0$ depending on p, d such that $\|v(t_n)-v(t)\|_{\mathcal{H}^\sigma} \lesssim \lambda\tau^\beta\lambda\left(\frac{\pi}{4}-t_n\right)^{-K}$, which gives the control needed on the variation of $\|v(t)\|_{L^{p+1}}$. Then the same globalising argument as above is used to finish the proof in this case. □

5.2. Scattering for (NLS) and end of the proof. In order to prove Theorem 4.3 we need an $\mathcal{H}^\varepsilon$ scattering result for (HNLS). The main ingredient in the proof is the following proposition.

Proposition 4.8. *Let $d \in \{2,\dots,10\}$ and $p \in \left(1+\frac{2}{d}, 1+\frac{4}{d}\right)$. For almost every $u_0 \in X$ the unique global associated solution to (HNLS) constructed by Corollary 4.27 and Corollary 4.31 satisfies the following.*

(i) There exist $\sigma, \delta > 0$ only depending on p and d, and $u_\pm \in \mathcal{H}^{-\sigma}$ such that

$$\|v(t) - e^{-itH}(u_0 - u_\pm)\|_{\mathcal{H}^{-\sigma}} \lesssim C(u_0)\left(\frac{\pi}{4}-t\right)^\delta \underset{t\to\pm\frac{\pi}{4}}{\longrightarrow} 0\,. \tag{4.52}$$

(ii) There exists $\varepsilon > 0$ such that for all $t \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$:

$$\|v(t)\|_{\mathcal{H}^\varepsilon} \lesssim_\varepsilon C(u_0)\,. \tag{4.53}$$

In both cases there exist numerical constants $c, C > 0$ such that $\mu(C(u_0) > \lambda) \leqslant Ce^{-c\lambda}$.

Proof. Again we present the proof for the forward scattering part. We work with initial data in a set of full measure G which may be taken as the intersection of the full measure sets constructed in Corollary 4.27 and Corollary 4.31. Then every $u_0 \in G$ gives rise to a global solution to (HNLS),

$$v(t) = u_L(t) + w(t) = e^{-itH}u_0 + w(t)$$

satisfying the bounds (4.28) and (4.34). Up taking the intersection with another set of full measure we can always assume that $\|u_L\|_{L^q\mathcal{W}^{s,r}} < \infty$ as long as (q,r) satisfies the hypothesis of Lemma 4.13, and more precisely

$$\mu(u_0,\ \|u_L\|_{L^q\mathcal{W}^{s,r}} > \lambda) \leqslant Ce^{-c\lambda^2}\,.$$

We start the proof with the case $p < p_{\max}(d)$ and we will use Corollary 4.31. To start the proof we write

(4.54)
$$e^{itH}w(t) = -i\int_0^t e^{isH}\left(\cos(2s)^{-\alpha(p,d)}|u_L(s)+w(s)|^{p-1}(u_L(s)+w(s))\right)\,\mathrm{d}s\,.$$

By the Sobolev embedding set $\sigma > 0$ such that $L^{\frac{p+1}{p}} \hookrightarrow \mathcal{H}^{-\sigma}$, one can choose for example $\sigma := d\left(\frac{1}{2} - \frac{1}{p+1}\right)$. We claim that there exists $\delta > 0$ such that

(4.55)
$$\int_t^{\frac{\pi}{4}} |\cos(2s)|^{-\alpha(p,d)}\|(u_L(s)+w(s))|u_L(s)+w(s)|^{p-1}\|_{\mathcal{H}^{-\sigma}}\,\mathrm{d}s \lesssim \left(\frac{\pi}{4}-t\right)^{\delta}\,.$$

Assuming (4.55) proves that the integral in (4.54) converges absolutely in $\mathcal{H}^{-\sigma}$ and thus convergent to some $u_+ \in \mathcal{H}^{-\sigma}$ and proves (4.52).

In order to prove (4.55) we use the Sobolev embedding and the bound $\cos(2s) \gtrsim \left(\frac{\pi}{4} - s\right)$ to get

$$\begin{aligned}
&\int_t^{\frac{\pi}{4}} |\cos(2s)|^{-\alpha(p,d)}\|(u_L(s)+w(s))|u_L(s)+w(s)|^{p-1}\|_{\mathcal{H}^{-\sigma}}\,\mathrm{d}s\\
&\qquad\lesssim \int_t^{\frac{\pi}{4}} |\cos(2s)|^{-\alpha(p,d)}\|u_L(s)+w(s)\|^p_{L^{p+1}}\,\mathrm{d}s\\
&\qquad\lesssim \int_t^{\frac{\pi}{4}}\left(\frac{\pi}{4}-s\right)^{-\alpha(p,d)}\|u_L(s)\|^p_{L^{p+1}}\,\mathrm{d}s\\
&\qquad+\int_t^{\frac{\pi}{4}}\left(\frac{\pi}{4}-s\right)^{-\alpha(p,d)}\|w(s)\|^p_{L^{p+1}}\,\mathrm{d}s\\
&\qquad\lesssim \|u_L\|^p_{L^{pr}L^{p+1}}\left(\int_t^{\frac{\pi}{4}}\left(\frac{\pi}{4}-t\right)^{r'(-\alpha(p,d))}\,\mathrm{d}s\right)^{\frac{1}{r'}}\\
&\qquad+C(u_0)^p\int_t^{\frac{\pi}{4}}\left(\frac{\pi}{4}-t\right)^{-\alpha(p,d)}\left|\log\left(\frac{\pi}{4}-t\right)\right|^{\frac{p}{2}}\,\mathrm{d}s
\end{aligned}$$

where in the last line we used Hölder's inequality with $\frac{1}{r}+\frac{1}{r'} = 1$ and the bounds (4.34). As explained above, without loss of generality we can asume $\|u_L\|_{L^{pr}L^{p+1}}$ to be finite. It remains to choose r' such that $r'\alpha(p,d) < 1$ which ensures that the previous integrals are absolutely convergent and that they can be bounded by a positive power of $\frac{\pi}{4} - t$. This gives (4.55).

It remains to prove (4.53). First observe that w satisfies

$$i\partial_t w(t) - Hw(t) = \cos(2t)^{-\alpha(p,d)}(u_L(t)+w(t))|u_L(t)+w(t)|^{p-1}$$

then apply $H^{\frac{\varepsilon}{2}}$, multiply by $\overline{H^{\frac{\varepsilon}{2}}w(t)}$ and integrate in space to obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w(t)\|_{\mathcal{H}^\varepsilon}^2\right) &= 2\cos(2t)^{-\alpha(p,d)}\\ &\times \operatorname{Im}\left(\int_{\mathbb{R}^2} H^{\frac{\varepsilon}{2}}\left(|u_L(t)+w(t)|^{p-1}(u_L(t)+w(t))\right)\overline{H^{\frac{\varepsilon}{2}}w(t)}\,\mathrm{d}x\right)\\ &\lesssim \cos(2t)^{-\alpha(p,d)}\|(u_L(t)+w(t))|u_L(t)+w(t)|^{p-1}\|_{\mathcal{W}^{\varepsilon,\frac{p+1}{p}}}\|w(t)\|_{\mathcal{W}^{\varepsilon,p+1}}\,,\end{aligned}$$

where we used the Hölder inequality. By Sobolev's product laws and the minoration of the cos function we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w(t)\|_{\mathcal{H}^\varepsilon}^2\right) \lesssim \left(\frac{\pi}{4}-t\right)^{-\alpha(p,d)}&\left(\|u_L(t)\|_{L^{p+1}}^{p-1}+\|w(t)\|_{L^{p+1}}^{p-1}\right)\\ &\times\left(\|u_L(t)\|_{\mathcal{W}^{\varepsilon,p+1}}+\|w(t)\|_{\mathcal{W}^{\varepsilon,p+1}}\right)\|w(t)\|_{\mathcal{W}^{\varepsilon,p+1}}\,.\end{aligned}$$

Then we bound the L^{p+1} norms by the $\mathcal{W}^{\varepsilon,p+1}$ norms and also use $w(t) = v(t)-u_L(t)$, which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w(t)\|_{\mathcal{H}^\varepsilon}^2\right) \lesssim \left(\frac{\pi}{4}-t\right)^{-\alpha(p,d)}\left(\|u_L(t)\|_{\mathcal{W}^{\varepsilon,p+1}}^{p+1}+\|v(t)\|_{\mathcal{W}^{\varepsilon,p+1}}^{p+1}\right)\,.$$

After integration we have

$$\begin{aligned}\|w(t)\|_{\mathcal{H}^\varepsilon}^2 &\leqslant \underbrace{\|w_0\|_{\mathcal{H}^\varepsilon}^2}_{=0}+C\int_0^t\left(\frac{\pi}{4}-s\right)^{-\alpha(p,d)}\left(\|u_L(t)\|_{\mathcal{W}^{\varepsilon,p+1}}^{p+1}+\|v(t)\|_{\mathcal{W}^{\varepsilon,p+1}}^{p+1}\right)\mathrm{d}s\\ &\lesssim \|u_L\|_{L^{(p+1)r}\mathcal{W}^{\varepsilon,p+1}}^{p+1}\left(\int_0^{\frac{\pi}{4}}\left(\frac{\pi}{4}-t\right)^{r'(-\alpha(p,d))}\mathrm{d}s\right)^{\frac{1}{r'}}\\ &\quad + C(u_0)^{p+1}\int_0^{\frac{\pi}{4}}\left(\frac{\pi}{4}-t\right)^{-\alpha(p,d)}\left|\log\left(\frac{\pi}{4}-t\right)\right|^{\frac{p+1}{2}}\mathrm{d}s\,,\end{aligned}$$

where we used Hölder's inequality with $\frac{1}{r}+\frac{1}{r'}=1$ and (4.34). As above without loss of generality we can assume $\|u_L\|_{L^{pr}\mathcal{W}^{\varepsilon,p+1}}<\infty$. Then choosing r' such that $r'\alpha(p,d)<1$ all the above integrals are convergent, which proves (4.53).

Now we explain the case $p\geqslant p_{\max}(d)$. As remarked before this case is only necessary when $d\geqslant 8$. Let $p\in\left[p_{\max}(d),1+\frac{4}{d}\right)$. In order to prove (4.52) we need to prove that there exists $\delta>0$ such that (4.55) holds. Recalling the bounds on u_L all which is needed is the existence of $\delta>0$ such that

$$\int_t^{\frac{\pi}{4}}|\cos(2s)|^{-\alpha(p,d)}\|w(s)\|_{L^{p+1}}^p\,\mathrm{d}s\lesssim\left(\frac{\pi}{4}-t\right)^{\delta}\,. \tag{4.56}$$

Similarly, in order to prove (4.53), an examination of the above proof shows that one only needs to prove that for sufficiently small $\varepsilon>0$ the integral

$$\int_0^{\frac{\pi}{4}}|\cos(2s)|^{-\alpha(p,d)}\|w(s)\|_{\mathcal{W}^{\varepsilon,p+1}}^{p+1}\,\mathrm{d}s\,, \tag{4.57}$$

is finite. The proof of (4.56) and (4.57) are essentially the same, thus the proof of (4.56) is carried out in details and we only explain the modifications for (4.57). We know, thanks to 4.27 that there is a constant $C > 0$ such that

$$\|w(s)\|_{\mathcal{H}^\sigma} \leqslant C\left(\frac{\pi}{4}-s\right)^{-\frac{\alpha(p,d)}{2}-\varepsilon_0},$$

for all time, and where $\varepsilon_0 > 0$ is arbitrarily small. Therefore using the notation of Lemma 4.21 the local well-posedness time $\tau(t_0)$ at time t_0 has the form $\tau(t_0) \sim \left(\frac{\pi}{4}-t_0\right)^{\frac{(p+1)\alpha q_1}{2}}$, and we can verify that we can take q_1 such that $\frac{(p+1)\alpha q_1}{2} = 1$, which can be satisfied as soon as

$$\frac{(p+1)\alpha}{2} < 1 - \left(\frac{d}{4}-\frac{\sigma}{2}\right)(p-1)\,,$$

which is the case for $p \geqslant \frac{-d+4+\sqrt{d^2+32}}{4}$. One can check that $P_d(\frac{-d+4+\sqrt{d^2+32}}{4}) < 0$ for $d \in \{8, 9, 10\}$, meaning that for $p \geqslant p_{\max}(d)$ we can set $\tau(t_0) \sim \left(\frac{\pi}{4}-t_0\right)$.

We can consider a sequence of times $t_n = \frac{\pi}{4}(1-2^{-n})$ such that the $[t_n, t_{n+1}]$ are local well-posedness intervals, on which we have the bound

$$\|w\|_{Y^{\sigma(p,d)-}_{[t_n,t_{n+1}]}} \leqslant \left(\frac{\pi}{4}-t_n\right)^{-\frac{\alpha}{2}-\varepsilon_0} \tag{4.58}$$

We claim that the real sequence $(u_n)_{n\geqslant 0}$ defined by

$$u_n := \int_0^{t_n} |\cos(2s)|^{-\alpha(p,d)}\|w(s)\|^p_{L^{p+1}}\,\mathrm{d}s$$

is a Cauchy sequence. In order to prove this claim, we use the bound $\cos(2s) \gtrsim \left(\frac{\pi}{4}-s\right)$ and Hölder's inequality with $\frac{1}{\gamma}+\frac{1}{\gamma'} = 1$, where γ such that there exists $\sigma \in [0, \sigma(p,d))$ such that $(p\gamma', p+1)$ is $(\sigma(p,d))$-Schrödinger admissible, that is $p\gamma' \geqslant 2$ and $\frac{2}{p\gamma'}+\frac{d}{p+1} = \frac{d}{2}-\sigma$. Note that the condition $p\gamma' \geqslant 2$ can be written as $\frac{d}{2}-\frac{d}{p+1}-\sigma(p,d) < 1$ which is equivalent to $(d-2)p^2+(d-6)p-2d-4 < 0$, that is $p < \frac{d+2}{d-2}$. This last inequality is always satisfied in our case. We also mention that the $\sigma(p,d)$ admissibility also requires that $\frac{d}{2}-\frac{d}{p+1}-\sigma(p,d) > 0$, which is equivalent to $p > \frac{1}{-d+4+\sqrt{9d^2-16d}}2(d-2)$. Since $p > 1+\frac{3}{d-2} > \frac{1}{-d+4+\sqrt{9d^2-16d}}2(d-2)$, this last inequality is, again, always satisfied.

We can summarise the above discussion: provided $-\alpha(p,d)+\frac{1}{\gamma}>0$, which will be checked later, we obtain:

$$\begin{aligned}|u_{n+1}-u_n| &\leqslant \int_{t_n}^{t_{n+1}}\left(\frac{\pi}{4}-s\right)^{-\alpha(p,d)}\|w(s)\|_{L^{p+1}}^p\,\mathrm{d}s\\ &\lesssim \left(\frac{\pi}{4}-t_n\right)^{-\alpha(p,d)+\frac{1}{\gamma}}\|w\|^p_{L^{p\gamma'}_{[t_n,t_{n+1}]}L^{p+1}}\\ &\lesssim \left(\frac{\pi}{4}-t_n\right)^{-\alpha(p,d)+\frac{1}{\gamma}}\|w\|^p_{Y^{\sigma(d,p)-}_{[t_n,t_{n+1}]}}\,.\end{aligned}$$

Using (4.58) we infer

$$\begin{aligned}|u_{n+1}-u_n| &\lesssim \left(\frac{\pi}{4}-t_n\right)^{-\alpha(p,d)+\frac{1}{\gamma}}\left(\frac{\pi}{4}-t_{n+1}\right)^{-\frac{p(\alpha(p,d)+\varepsilon_0)}{2}}\\ &\lesssim 2^{-n(\delta+\varepsilon_0)}\,,\end{aligned}$$

with $\delta:=\frac{p+2}{2}\alpha(p,d)+\frac{1}{\gamma}$ and ε_0 arbitrarily small. Assume that $\delta>0$, then we conclude that $(u_n)_{n\geqslant 0}$ is a Cauchy sequence and moreover if we choose n such that $\frac{\pi}{4}-t\in[2^{-(n+1)},2^{-n}]$ then we can bound:

$$\int_t^{\frac{\pi}{4}}|\cos(2s)|^{-\alpha(p,d)}\|w(s)\|^p_{L^{p+1}}\,\mathrm{d}s\lesssim\sum_{k\geqslant n}2^{-k\delta}\lesssim 2^{-n\delta}\lesssim\left(\frac{\pi}{4}-t\right)^{\delta}.$$

This proves (4.56), provided we show that $\delta>0$. Before checking this fact we run the same analysis for (4.57) in order to obtain the inequalities that the parameters must satisfy. This time we set

$$z_n:=\int_0^{t_n}(\cos(2s))^{-\alpha(p,d)}\|w(s)\|^{p+1}_{\mathcal{W}^{\varepsilon,p+1}}\,\mathrm{d}s\,,$$

and we only need to prove that this sequence forms a Cauchy sequence. Again we use Hölder's inequality with $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$ such that there exists $\sigma\in[0,\sigma(p,d))$ such that $(\gamma'(p+1),p+1)$ is $\sigma(p,d)$-Schrödinger admissible, that is $\gamma'(p+1)\geqslant 2$ and $\frac{2}{(p+1)\gamma'}+\frac{d}{p+1}=\frac{d}{2}-\sigma$. The condition $\gamma'(p+1)\geqslant 2$ is equivalent to $p\leqslant\frac{d+2}{d-2}$ as above, which is satisfied. Then, with the same estimates as above and provided $-\alpha(p,d)+\frac{1}{\gamma}>0$ we arrive at

$$|z_{n+1}-z_n|\lesssim\left(\frac{\pi}{4}-t_n\right)^{-\alpha(p,d)+\frac{1}{\gamma}}\|v\|^{p+1}_{Y^{\sigma(p,d)-}_{[t_n,t_{n+1}]}}\,.$$

Using (4.58) and the fact that $\left(\frac{\pi}{4}-t_n\right)\sim C2^{-n}$, in order to get

$$|z_{n+1}-z_n|\lesssim 2^{-n(\tilde{\delta}+\varepsilon_0)}$$

where $\tilde{\delta}:=\dfrac{(p+3)\alpha(p,d)}{2}+\dfrac{1}{\gamma}$. Assuming that $\tilde{\delta}>0$ proves that $(u_n)_{n\geqslant 0}$ is a Cauchy sequence, and ends the proof of (4.57)

We need to find the range of p (depending on d) for which $\delta, \tilde{\delta} > 0$. Note that would immediately imply $\alpha(p,d) + \frac{1}{\gamma} > 0$ which was a necessary condition to bound $|u_{n+1} - u_n|$ and $|z_{n+1} - z_n|$.

The claim $\delta > 0$ is equivalent to:

$$\frac{p+2}{2}(-\alpha(p,d)) + 1 - \frac{p}{2}\left(\frac{d}{2} - \sigma(d) - \frac{d}{p+1}\right),$$

which after simplification reads

$$Q_d(p) := 2p^3 + dp^2 + (d-6)p - 2d - 4 > 0\,.$$

Then we can compute that, as a polynomial in p, $\mathrm{discrim}(Q_d) = 9d^4 - 76d^3 + 36d^2 - 1728d$ thus Q_d has exactly one real root if $d \leqslant 9$ and exactly three for $d \geqslant 10$. In any cases we check that Q_d has exactly one positive real root for $d \geqslant 8$. Let us denote by p_d this root and we claim that $p_{\max}(d) > p_d$. To verify this statement we remark that both P_d and Q_d are increasing $[1, 1 + \frac{4}{d}]$ thus we only need to check that $P_d \leqslant Q_d$ in that region, which is equivalent to $(d-4)p^2 - 4p - d < 0$ which is satisfied for $p \geqslant \frac{d}{d-4}$ (which is greater than $1 + \frac{4}{d}$)) and $d \geqslant 8$.

In a similar fashion, $\tilde{\delta} > 0$ is equivalent to:

$$\frac{p+3}{2}(-\alpha(p,d)) - \frac{p+1}{2}\left(\frac{d}{2} - \sigma(p,d) - \frac{d}{p+1}\right) > 0\,,$$

which after simplification reads

$$R_d(p) := p^2 + \frac{d}{2}p - \frac{d}{2} - 3 > 0\,,$$

that is $p > -\frac{d}{4} + \frac{1}{4}\sqrt{d^2 + 8d + 48}$. We claim that

$$p_{\max}(d) > -\frac{d}{4} + \frac{1}{4}\sqrt{d^2 + 8d + 48} := p'_d$$

if and only if $d \leqslant 24$. This claim is equivalent to $P_{24}(p'_{24})P_{25}(p'_{25}) < 0$. This reduces to $(813\sqrt{51} - 5806)(44181\sqrt{97} - 435131) < 0$ which is the case. $\square$

5.3. Proof of the main theorems.

Proof of Theorem 4.2. This is the content of Corollary 4.27 and an application of the *lens transform*. $\square$

Proof of Theorem 4.3. In order to end the proof of the theorem we assume that there exist $\delta, \sigma, \varepsilon > 0$ such as constructed in Proposition 4.8. This immediately proves part (i) of Theorem 4.3.

We deduce that $u_+ \in \mathcal{H}^\varepsilon$. In fact, $e^{itH}w(t)$ being bounded in the Hilbert space $\mathcal{H}^\varepsilon$ we can extract a subsequence, weakly converging to some $\tilde{u}_+ \in \mathcal{H}^\varepsilon$ but this convergence also holds in $\mathcal{H}^{-\sigma}$ where $e^{itH}w(t) \to u_+$ thus by uniqueness of the weak limit, $u_+ = \tilde{u}_+ \in \mathcal{H}^{\varepsilon}$.

We claim that the bound

$$\|e^{itH}w(t) - u_+\|_{\mathcal{H}^{\varepsilon_0}} \lesssim \left(\frac{\pi}{4} - t\right)^{(2d+1)\varepsilon_0} \tag{4.59}$$

implies the estimate (4.5). Indeed, let u be the solution to Schrödinger equation (NLS) associated to u_0. We denote by $s = \frac{\tan t}{2}$ the time variable of u where t is the time variable of $v := \mathcal{L}u$. We refer to Appendix Appendix B for details. Then from Lemma 4.34 we have

$$\begin{aligned}
\|u(s) - e^{is\Delta_y}(u_0 + u_+)\|_{\mathcal{H}^{\varepsilon_0}} &\lesssim \left(\frac{\pi}{4} - t(s)\right)^{-2d\varepsilon_0} \|\mathcal{L}u - \mathcal{L}(e^{is\Delta_y}(u_0 + u_+))\|_{\mathcal{H}^{\varepsilon_0}} \\
&\lesssim \left(\frac{\pi}{4} - t(s)\right)^{-2d\varepsilon_0} \|w(t(s)) - e^{-it(s)H}u_+\|_{\mathcal{H}^{\varepsilon_0}} \\
&\lesssim \left(\frac{\pi}{4} - t(s)\right)^{-2d\varepsilon_0} \|e^{it(s)H}w(t(s)) - u_+\|_{\mathcal{H}^{\varepsilon_0}} \\
&\underset{s\to\infty}{\longrightarrow} 0\,,
\end{aligned}$$

where we used that $v(t(s)) = w(t(s)) + e^{-it(s)H}u_0$ and (4.59).

In order to prove (4.59), let $\theta \in [0,1]$ and introduce $\sigma(\theta) := -\sigma\theta + (1-\theta)\varepsilon$. Interpolating between (4.52) and (4.53) we have

$$\begin{aligned}
\|e^{itH}w(t) - u_+\|_{\mathcal{H}^{\sigma(\theta)}} &\leqslant \|e^{itH}w(t) - u_+\|_{\mathcal{H}^{\varepsilon}}^{1-\theta} \|e^{itH}w(t) - u_+\|_{\mathcal{H}^{-\sigma}}^{\theta} \\
&\lesssim C(\varepsilon)^{1-\theta} \|e^{itH}w(t) - u_+\|_{\mathcal{H}^{-\sigma}}^{\theta} \\
&\lesssim \left(\frac{\pi}{4} - t\right)^{\theta\delta}\,.
\end{aligned}$$

We claim that we can find $\varepsilon_0 \in (0,\varepsilon)$ satisfying $\sigma(\theta) = \varepsilon_0$ and $\delta\theta = (2d+1)\varepsilon_0$. Indeed one can take

$$\varepsilon_0 := \frac{\varepsilon}{1 + \frac{2d+1}{\delta}(\sigma + \varepsilon)}\,.$$

Finally in order to prove (4.6), observe that by properties of the lens transform we have:

$$\begin{aligned}
\|e^{-is\Delta_y}u(s) - (u_0 + u_+)\|_{\mathcal{H}^{\varepsilon_0}} &= \|e^{it(s)H}(u(s) - e^{is\Delta_y}(u_0 + u_+))\|_{\mathcal{H}^{\varepsilon_0}} \\
&\lesssim \left(\frac{\pi}{4} - t(s)\right)^{-2d\varepsilon_0} \|u(s) - e^{is\Delta_y}(u_0 + u_+))\|_{\mathcal{H}^{\varepsilon_0}}
\end{aligned}$$

which converges to zero thanks to (4.5), up to taking $\varepsilon > 0$ small enough.

We have thus proven the convergences (4.5) and (4.6) at a rate which is $\left(\frac{\pi}{4} - t(s)\right)^{\varepsilon_0}$ as $s \to \infty$. To conclude it suffices to remark that:

$$\frac{\pi}{4} - t(s) = \frac{1}{2}\arctan\left(\frac{1}{2s}\right) \sim \frac{C}{s} \text{ as } s \to \infty \qquad \square$$

Appendix A. Technical estimates in harmonic Sobolev spaces

In thie appendix we recall some well-known facts concerning the harmonic oscillator-based Sobolev spaces.

Lemma 4.32 (Harmonic Sobolev Spaces)**.** *The following propertes hold.*

(i) For $\sigma \in \mathbb{R}$ *and* $p \in (1,\infty)$, $\|u\|_{\mathcal{W}^{\sigma,p}} = \|H^{\frac{\sigma}{2}}u\|_{L^p} \sim \|\langle\nabla\rangle^\sigma u\|_{L^p} + \|\langle x\rangle^\sigma u\|_{L^p}$.

(ii) Sobolev embedding: let $p_1, p_2 \geqslant 1$*, then in dimension* d, $\mathcal{W}^{\sigma_1,p_1} \hookrightarrow \mathcal{W}^{\sigma_2,p_2}$ *as soon as*

$$\frac{1}{p_1} - \frac{\sigma_1}{d} \leqslant \frac{1}{p_2} - \frac{\sigma_2}{d}\,.$$

(iii) For $\sigma \geqslant 0$, $q \in (1,\infty)$ *and* $q_1, q_2, q_1' q_2' \in (1,\infty]$ *one has*

$$\|uv\|_{\mathcal{W}^{\sigma,q}} \lesssim \|u\|_{L^{q_1}}\|v\|_{\mathcal{W}^{\sigma,q_1'}} + \|u\|_{\mathcal{W}^{\sigma,q_2}}\|v\|_{L^{q_2'}}\,,$$

as soon as $\dfrac{1}{q} = \dfrac{1}{q_1} + \dfrac{1}{q_1'} = \dfrac{1}{q_2} + \dfrac{1}{q_2'}$.

(iv) (Chain Rule) For $s \in (0,1)$, $p \in (1,\infty)$ *and a function* $F \in \mathcal{C}^1(\mathbb{R})$ *such that* $F(0) = 0$ *and such that there is a* $\mu \in L^1([0,1])$ *such that for every* $\theta \in [0,1]$*:*

$$|F'(\theta v + (1-\theta)w)| \leqslant \mu(\theta)\,(G(v) + G(w))$$

where $G > 0$*; we have*

$$\|F \circ u\|_{\mathcal{W}^{s,p}} \lesssim \|u\|_{\mathcal{W}^{s,p_0}}\|G \circ u\|_{L^{p_1}}\,. \tag{4.60}$$

Proof. (i) is proved in [BTT13]. The other statements are proven for usual Sobolev spaces in [Tay07] and their readaptation to harmonic Sobolev spaces results from the use of (i). We also refer to [Den12]. For usual Sobolev spaces, (iii) and (iv) can be found in [Tay07], Chapter 2. □

Our next lemma is a technical estimate which aims at decoupling the norm in time.

Lemma 4.33. *Let* $\varepsilon > 0$, $\alpha \in (0,1)$ *and* $q, r \geqslant 1$ *be such that* $W^{\frac{\varepsilon}{2},q}(\mathbb{R}) \hookrightarrow C^{0,\alpha}(\mathbb{R})$*. Then for every* $f \in \mathcal{W}^{\varepsilon,r}$ *holds*

$$\|(e^{-i(t-t_0)H} - \mathrm{id})f\|_{L^\infty_{[t_0,t_0+\tau]}L^r} \lesssim \tau^\alpha \|e^{itH}f\|_{L^q_{\langle t\rangle^{-2}\,\mathrm{d}t}(\mathbb{R},\mathcal{W}^{\varepsilon,r})}\,.$$

where the implicit constant depends on ε, q *and* α.

Proof. Let $\chi(t)$ a smooth function such that for $|t| \leqslant \pi$, $\chi(t) = \langle 2\pi\rangle^{-2}$ and for $|t| \geqslant 2\pi$ one has $\chi(t) = \langle t\rangle^{-2}$. Set $F(t) := e^{-i(t-t_0)H}f$ for convenience. Then we use the definition of the $C^{0,\alpha}$ norm:

$$\begin{aligned}\|(e^{-i(t-t_0)H}-\mathrm{id})f\|_{L^\infty_{[t_0,t_0+\tau]}L^r} &\leqslant |t-t_0|^\alpha \|F\|_{C^{0,\alpha}([t_0,t_0+\tau],L^r)}\\ &\leqslant \tau^\alpha \|F\|_{C^{0,\alpha}([-\pi,\pi],L^r)}\,.\end{aligned}$$

Now we use that $\|F\|_{C^{0,\alpha}([-\pi,\pi],L^r)} \leqslant C\|\chi(\cdot - t_0)F\|_{C^{0,\alpha}(\mathbb{R},L^r)}$ with a constant C which does not depend on τ. Combined with the Sobolev embedding $W^{\frac{\varepsilon}{2},q}(\mathbb{R}) \hookrightarrow C^{0,\alpha}(\mathbb{R})$ we have:

$$\begin{aligned}\|(e^{-i(t-t_0)H}-\mathrm{id})f\|_{L^\infty_{[t_0,t_0+\tau]}L^r} &\lesssim \tau^\alpha \|\chi(\cdot - t_0)F\|_{W^{\frac{\varepsilon}{2},q}(\mathbb{R},L^r)}\\ &\lesssim \tau^\alpha \|\chi(t)e^{-itH}f\|_{W^{\frac{\varepsilon}{2},q}(\mathbb{R},L^r)}\,.\end{aligned}$$

Now observe that in order to transfer derivatives from time to space we need to commute χ and $\langle D_t\rangle^{\frac{\varepsilon}{2}}$, which follows from the following observation:

$$\begin{aligned}\langle D_t\rangle^{\frac{\varepsilon}{2}}\chi &= \left(1+\left[\langle D_t\rangle^{\frac{\varepsilon}{2}},\chi\right]\langle D_t\rangle^{\frac{-\varepsilon}{2}}\chi^{-1}\right)\chi\langle D_t\rangle^{\frac{\varepsilon}{2}}\\ &= (\mathrm{id}+A)\chi\langle D_t\rangle^{\frac{\varepsilon}{2}}\,,\end{aligned}$$

with $A := \left[\langle D_t\rangle^{\frac{\varepsilon}{2}},\chi\right]\langle D_t\rangle^{-\frac{\varepsilon}{2}}\chi^{-1}$. Since χ, χ^{-1} are zero order pseudodifferential operators and that $\langle D_t\rangle^{\pm\frac{\varepsilon}{2}}$ are pseudodifferential operators of order $\pm\frac{\varepsilon}{2}$, the pseudodifferential calculus implies that A is of order -1 which is regularising and finally $(\mathrm{id}+A)$ is of order zero, thus continuous on all L^p spaces, as soon as $p\in(1,\infty)$, see [Hör65] for instance, and the continuity constant does not depend on τ. This gives

$$\begin{aligned}\|e^{-itH}f\|_{W^{\frac{\varepsilon}{2},q}(\mathbb{R},L^r)} &\lesssim \|\langle D_t\rangle^{\frac{\varepsilon}{2}}\chi(t)e^{-itH}f\|_{L^q(\mathbb{R},L^r)}\\ &\lesssim \|\chi(t)\langle D_t\rangle^{\frac{\varepsilon}{2}}e^{-itH}f\|_{L^q(\mathbb{R},L^r)}\\ &= \|\langle D_t\rangle^{\frac{\varepsilon}{2}}e^{-itH}f\|_{L^q_{\langle t\rangle^{-2}\,\mathrm{d}t}(\mathbb{R},L^r)}\,.\end{aligned}$$

Since by definition $D_t(e^{itH}f) = H(e^{itH}f)$ we deduce by the usual functional calculus that $D_t^{\frac{\varepsilon}{2}}(e^{itH}f) = H^{\frac{\varepsilon}{2}}(e^{itH}f)$, which ends the proof. □

Appendix B. The lens transform

The lens transform $\mathcal{L}: \mathcal{S}'(\mathbb{R}\times\mathbb{R}^d) \to \mathcal{S}'(\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\times\mathbb{R}^d)$ and its inverse $\mathcal{L}^{-1}$ are defined by the following formulae

$$v(t,x) := \mathcal{L}u(t,x) = \frac{1}{\cos(2t)^{\frac{d}{2}}}u\left(\frac{\tan(2t)}{2},\frac{x}{\cos(2t)}\right)\exp\left(-\frac{i|x|^2\tan(2t)}{2}\right),$$

$$u(s,y) = \mathcal{L}^{-1}v(s,y) = (1-4s^2)^{-\frac{d}{4}}v\left(\frac{\arctan(2s)}{2},\frac{y}{\sqrt{1-4s^2}}\right)\exp\left(\frac{i|y|^2s}{\sqrt{1-4s^2}}\right).$$

Formal computations show that u solves

$$i\partial_s u + \Delta_y u = 0 \text{ on } \mathbb{R}_s\times\mathbb{R}^d_y \text{ and } u(0)=u_0$$

if and only if $v = \mathcal{L}u$ solves

$$i\partial_t v - Hv = 0 \text{ on } \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \times \mathbb{R}^d_x \text{ and } v(0) = u_0 \,.$$

With the variables $s = \frac{\tan(2t)}{2}$, or equivalently $t = t(s) = \frac{\arctan(2s)}{2}$, and $y = \frac{x}{\cos(2t)}$ an elementary computation shows that $\mathcal{L}$ maps solution of (NLS) to solution of (HNLS) with the same initial data. In particular

$$\mathcal{L}(e^{is\Delta_y} v_0) = e^{-it(s)H} v_0 \,.$$

In the proof of Theorem 4.3 it is needed to compare the H^σ norms of u and the $\mathcal{H}^\sigma$ norms of v. This will be possible thanks to the following lemma.

Lemma 4.34 (Lens Transform)**.** *If u and U are related by*

$$u(x) = \frac{1}{\cos^{\frac{d}{2}}(2t)} U\left(\frac{x}{\cos(2t)^d}\right) \exp\left(-i\frac{x^2\tan(2t)}{2}\right)$$

then for any $\sigma \in [0,1]$ and $t \in \left[0, \frac{\pi}{4}\right]$,

$$\|U\|_{H^\sigma} \lesssim \|u\|_{\mathcal{H}^\sigma} \,,$$

$$\|\langle\cdot\rangle^\sigma U\|_{L^2(\mathbb{R}^2)} \lesssim \left(\frac{\pi}{4} - t\right)^{-2d\sigma} \|u\|_{\mathcal{H}^\sigma(\mathbb{R}^2)} \,,$$

and

$$\|U\|_{L^q} \leqslant \cos(2t)^{d\left(\frac{1}{2}-\frac{1}{q}\right)} \|u\|_{L^q} \,.$$

Proof. See [BT20], Lemma A.2 with only minor modification to the d dimensional case. □

Remark 4.35. This result shows that estimates at regularity $\mathcal{H}^\sigma$ for u transfers into estimates in $\mathcal{H}^\sigma$ for U with a loss $\left(\frac{\pi}{4} - t\right)^{-2d\sigma}$. This explains that in the proof of Theorem 4.3 we needed explicit decay rates on the scattering estimates for (HNLS) in order to be transferred into a scattering result for (NLS) .

Remark 4.36. The last inequality of Lemma 4.34 applied to solutions u of (NLS) and the corresponding solution v to (HNLS) implies $\|u(s)\|_{L^q} \lesssim \|v(t(s))\|_{L^q}$ as soon as $q > 1$.

Remark 4.37. The "lens transform" may look surprising and somehow unexpected. We briefly explain how one can come up with such a transformation, only using basic insight regarding the symmetries of (NLS). In fact this heuristic is useful to derive other symmetries or pseudo-symmetries to the Schrödinger equations, see [DR15] for other instances of pseudo-symmetries and [MR19] for an application.

To simplify, take $p = 1 + \frac{4}{d}$. Our starting point is the scaling symmetry

$$u(s,y) \mapsto u_\lambda(s,y) := \frac{1}{\lambda^{\frac{d}{2}}} u\left(\frac{s}{\lambda^2}, \frac{y}{\lambda}\right) .$$

In order to derive more general symmetries we seek for a scaling, depending on time. We change variables $(t,x) \leftrightarrow (s,y)$ imposing a local scaling symmetry $\mathrm{d}s = \frac{\mathrm{d}t}{\lambda(t)^2}$ and $\mathrm{d}y = \frac{\mathrm{d}x}{\lambda(t)}$ which can be integrated as

$$y = \frac{x}{\lambda(s)} \text{ and } s = \int_0^t \frac{\mathrm{d}t'}{\lambda(t')^2} \cdot$$

Thus one may seek for a change of variable of the form:

$$v(t,x) = \lambda(t)^{-d/2} u\left(\int_0^t \frac{\mathrm{d}t'}{\lambda(t')^2}, \frac{x}{\lambda(t)}\right) .$$

This change of variable is not sufficient. Indeed, writing the equation satisfied by v in variables t, x writes at the first derivatives order:

$$i\partial_t v + \Delta_x v - i\frac{\lambda'(t)}{\lambda(t)} x \cdot \nabla_x v + \cdots$$

which is not the linear Schrödinger equation. However it is possible to multiply by the correct exponential to eliminate the term $x \cdot \nabla_x v$ just as in the method of variation of constants. This leads to the choice

$$v(t,x) = \lambda(t)^{-d/2} u\left(\int_0^t \frac{\mathrm{d}t'}{\lambda(t')^2}, \frac{x}{\lambda(t)}\right) \exp\left(\frac{i|x|^2\lambda'(t)}{4\lambda(t)}\right) .$$

Now if one wants to compactify time, a usual way to do so is to chose $t = \frac{1}{2}\arctan(2s)$ which maps $\mathbb{R}$ to $[-\frac{\pi}{4}, \frac{\pi}{4}]$. Hence we arrive at the given form.

Chapitre 5

Construction of High Regularity Invariant Measures for the Euler Equations and Remarks on the Growth of the Solutions

Abstract. We consider the Euler equations on the two-dimensional torus and construct invariant measures for the dynamics of these equations, concentrated on sufficiently regular Sobolev spaces so that strong solutions are also known to exist. The proof follows the method of Kuksin in [Kuk04] and we obtain in particular that these measures do not have atoms, excluding trivial invariant measures. Then we prove that almost every initial data with respect to the constructed measures give rise to global solutions for which the growth of the Sobolev norms are at most polynomial. We rely on an argument of Bourgain. Such a combination of Kuksin's and Bourgain's arguments already appear in the work of Sy [Sy19a]. We point out that up to the knowledge of the author, the only general upper-bound for the growth of the Sobolev norm to the $2d$ Euler equations is double exponential. The argument also works for the three-dimensional case, except that in this case we cannot exclude the trivial invariant measure δ_0: we obtain the alternative that either there exists an invariant measure without any atom, or the measure is trivial.

1. Introduction

1.1. The Euler equations. This article is concerned with the incompressible Euler equations posed on the torus $\mathbb{T}^d$ of dimension 2 and 3:

$$\text{(E)} \qquad \begin{cases} \partial_t u + u \cdot \nabla u + \nabla p &= 0 \\ \nabla \cdot u &= 0 \\ u(0) &= u_0 \in H^s(\mathbb{T}^d, \mathbb{R}^d)\,, \end{cases}$$

where $s > 0$, H^s stands for the usual Sobolev space and the unknowns are the velocity field $u(t) : \mathbb{T}^d \to \mathbb{R}^d$ and the pressure $p : \mathbb{T}^d \to \mathbb{R}$. We will emphasis the dimension by referring to (E^d) as the Euler equation (E) in dimension d.

We recall that the pressure p can be recovered from u by solving the elliptic problem

$$-\Delta p = \nabla \cdot (u \cdot \nabla u) \text{ on } \mathbb{T}^d\,.$$

In dimension 2, the *vorticity* defined as $\xi \coloneqq \nabla \wedge u$ is a more convenient variable. Taking the rotational in (E^2), the latter can be recast as:

$$\left\{ \begin{array}{ccc} \partial_t \xi + u \cdot \nabla \xi & = & 0 \\ \xi(0) & = & \nabla \wedge u_0 \, , \end{array} \right. \tag{5.1}$$

where u is recovered from ξ *via* the Biot-Savart law:

$$\mathcal{K}(\xi)(t,x) = \frac{1}{2\pi} \int_{\mathbb{T}^2} \frac{\xi(t,y)(x-y)^\perp}{|x-y|^2} \, \mathrm{d}y \, , \tag{5.2}$$

where we recall that for any $y = (y_1, y_2) \in \mathbb{T}^2$ we define $y^\perp = (-y_2, y_1)$.

The Cauchy problem for (E^2) has been studied intensively during the last century. In particular, in dimension 2 global well-posedness is known.

Theorem 5.1 (Wolibner, [Wol33])**.** *Let $s > 2$. The Cauchy problem for* (E^2) *is globally well-posed in $\mathcal{C}^0(\mathbb{R}_+, H^s(\mathbb{T}^2, \mathbb{R}^2))$.*

In dimension 3, the Cauchy problem for (E^3) is locally well-posed in the space $\mathcal{C}([0,T], H^s(\mathbb{T}^3, \mathbb{R}^3))$ but no global well-posedness result is known at this regularity. For a detailed account on the Cauchy problem for the Euler equations we refer to the monograph [Che90].

1.2. Main results.

1.2.1. *Invariant measures for* (E)*.* In this article we first construct invariant measure for (E^2) supported at high regularity.

Theorem 5.2 (Dimension 2)**.** *Let $s > 2$. There exists a measure μ concentrated on $H^s(\mathbb{T}^2, \mathbb{R}^2)$ such that:*

- *(i) The equation* (E^2) *is μ-almost-surely globally well-posed in time.*
- *(ii) μ is an invariant measure for* (E^2)*.*
- *(iii) μ does not have any atom and $\mathbb{E}_\mu[\|u\|^2_{H^1}] > 0$*
- *(iv) μ charges large norm data; that is, for any $R > 0$,*

$$\mu(u \in H^s, \ \|u\|_{H^s} > R) > 0 \, .$$

Remark 5.3. (*iv*) proves that this theorem may be viewed as a *large data* global well-posedness result.

Moreover, the measure constructed by Theorem 5.2 satisfies the following properties.

Theorem 5.4 (Properties of the measure in dimension 2)**.** *Let $s > 2$ and μ the measure obtained in Theorem 5.2. There exists a continuous increasing function $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that $p(0) = 0$ and for every Borel subset $\Gamma \subset \mathbb{R}_+$ there holds:*

$$\mu\left(u \in H^s, \|u\|_{L^2} \in \Gamma\right) \leqslant p(|\Gamma|) \, .$$

In dimension 3, most of the argument works using the same techniques, except that the trivial invariant measure δ_0 cannot be excluded. However, in the case where the invariant measure is not δ_0 we are able to prove that such an invariant measure can be produced without any atom. More precisely, the result in dimension 3 reads as follows.

Theorem 5.5 (Remarks in dimension 3)**.** *Let $s > \frac{7}{2}$. There exists a measure μ concentrated on $H^s(\mathbb{T}^3, \mathbb{R}^3)$ such that:*

(i) The equation (E^3) is μ-almost-surely globally well-posed in time.

(ii) μ is an invariant measure for (E^3).

(iii) There is an alternative: either $\mu = \delta_0$, or μ does not possess any atom.

Statement (*iii*) of Theorem 5.5 is a consequence of the following result.

Theorem 5.6 (Properties of the measure in dimension 3)**.** *Let $s > \frac{7}{2}$. The measure μ constructed by Theorem 5.5 satisfies the following property: for any $\varepsilon > 0$, there exists a continuous increasing function $p_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that $p_\varepsilon(0) = 0$ and for every Borel subset $\Gamma \subset \mathbb{R}_+$ satisfying $\mathrm{dist}(\Gamma, 0) \geqslant \varepsilon$ there holds*

$$\mu\left(u \in H^s, \|u\|_{L^2} \in \Gamma\right) \leqslant p_\varepsilon(|\Gamma|)\,.$$

1.2.2. *Remarks on the growth of the Sobolev norms for solutions to* (E)*.* This work was originally motivated by the growth of the Sobolev norms and the L^∞ norm of the vorticity gradient in dimension 2. Let us recall some known results. Up to the knowledge of the author, the only known technique to establish such bounds is to estimate $\|\mathcal{K}\|_{L^p \to L^p}$, with $\mathcal{K}$ being defined by (5.2). For example, in order to estimate $\|\xi\|_{H^s}$, one simply applies ∇^s to (5.1) and obtain bounds of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^s \xi(t)\|_{L^2} \lesssim \|K\|_{L^p \to L^p}\|\nabla^s\xi(t)\|_{L^2}^{1+\frac{2}{p}}\,,$$

which after optimisation in p leads to the following estimates.

Theorem 5.7 (Bound on the growth of Sobolev norms, [KŠ14], Section 2)**.** *Let u be a smooth, global solution to (E^2). There exists a constant $C = C(u_0) > 0$ such that for any $t > 0$,*

(i) $\|u(t)\|_{H^s} \leqslant Ce^{e^{Ct}}$.

(ii) $\|\nabla\xi(t)\|_{L^\infty} \leqslant Ce^{e^{Ct}}$.

The question is then to estimate how much these norm can *really* grow. Some specific initial data which produce infinite time norm growth in $\mathbb{T}^2$ were exhibited by Bahouri and Chemin in [BC94]. More recently, the work of Kiselev and Šverák achieved the double exponential growth on a disk. On the torus however, and up to the knowledge of the author, no such result is known. Only initial data producing exponential growth are known. This is the content of the result of Zlatoš in [Zla15]. We summarise these two results in the following theorem.

Theorem 5.8 (Examples of growth estimates, [KŠ14, Zla15])**.** *Let U denote either the unit disc $D = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 \leqslant 1\}$ or the torus $\mathbb{T}^2$. There exists an initial data u_0 which lies in $\mathcal{C}^\infty$ in the case of $U = D$ and $\mathcal{C}^{1,\alpha}$ (for some $\alpha \in (0, 1)$) if $U = \mathbb{T}^2$ such that the unique associate global solution to the Euler equation (E^2) constructed in Theorem 5.1 satisfies:*

(i) If $U = D$, then there exists $C > 0$ such that for all $t \geqslant 0$,

$$\|\nabla\xi(t)\|_{L^\infty} \geqslant C \exp\left(Ce^{Ct}\right) .$$

(ii) If $U = \mathbb{T}^2$, then there exists $t_0 > 0$ such that for $t \geqslant t_0$ there holds:

$$\sup_{t' \leqslant t} \|\nabla\xi(t')\|_{L^\infty} \geqslant e^t .$$

The next question is then to quantify how likely is it for an initial data to produce such growth. Our result in this direction is the following.

Theorem 5.9 (Growth estimates in dimension 2)**.** *Let $s > 2$ and μ being the associate invariant measure for (E^2) constructed by Theorem 5.2 on $H^s(\mathbb{T}^2, \mathbb{R}^2)$. Then we have the following estimates.*

(i) For μ-almost every $u_0 \in H^s$ the associate unique global solution $u \in \mathcal{C}^0(\mathbb{R}_+, H^s(\mathbb{T}^2, \mathbb{R}^2))$ from Theorem 5.1 obeys the following growth estimate:

$$\|u(t)\|_{H^\sigma} \leqslant C(u_0)t^\alpha , \tag{5.3}$$

for all $t \geqslant 1$, $\sigma \in (1, s]$ and any $\alpha < \frac{\sigma-1}{s+1-\sigma}$.

(ii) If $s > 3$ then we have the following corollary:

$$\|\nabla\xi(t)\|_{L^\infty} \leqslant C(u_0)t^\alpha ,$$

for all $t \geqslant 1$, $\sigma \in (1, s]$ and any $\alpha < \frac{\sigma-1}{s+1-\sigma}$.

Remark 5.10. It should be noted that in dimension 2, the gradient of the vorticity $X := \nabla\xi$ satisfies the equation 4

$$\partial_t X + u \cdot X = -\nabla u \cdot \nabla X ,$$

which is very similar to the equation satisfied by the vorticity in dimension 3. Thus the problem of the growth of the vorticity gradient in dimension 2 may be seen as related model to the growth of the vorticity in dimension 3. The setting of dimension 2 is convenient since global solutions are already known to exist.

Theorem 5.11 (Growth estimates in dimension 3)**.** *Let $s > \frac{7}{2}$ and μ being the associate invariant measure for* (E^3) *constructed by Theorem 5.5 on $H^s(\mathbb{T}^3, \mathbb{R}^3)$. Then μ-almost every initial data $u_0 \in H^s(\mathbb{T}^3)$ produces a global solution to* (E^3) *obeying the growth estimate:*

$$\|u(t)\|_{H^\sigma} \leqslant C(u_0)t^\alpha \,, \tag{5.4}$$

for all $t \geqslant 1$, $\sigma \in (0, s-1)$ and any $\alpha < \frac{\sigma}{2s-\sigma}$.

Remark 5.12. The condition $s > \frac{7}{2}$ is needed in the proof of Lemma 5.40. We did not try to prove the most optimal result: it is possible that a refinement of the techniques used in this paper may lead to results for $s > \frac{5}{2}$ but we did not pursue this end.

1.2.3. *Main limitation of the results.* Let μ be a measure constructed by Theorem 5.2 (resp. Theorem 5.5) in dimension 2 (resp. 3). The main limitation of our result is the following: if $u \in H^s \setminus \{0\}$ is a stationary solution, then Theorem 5.4 (resp. Theorem 5.6) implies that $\mu(\{u\}) = 0$, but this does not prevent the measure μ to be supported *exclusively* by stationary solutions.

If the measure μ does not concentrate on stationary solutions, then Theorem 5.4 (resp. Theorem 5.6) produces non-trivial slowly growing global solutions.

In the eventuality of μ concentrating on stationary solutions, this would exhibit a stability property of the set of stationary solutions with respect to the approximation procedure and shows that our compactness procedure does not extract measures with non-trivial dynamical properties.

This question seems both fundamental and non-trivial. We highlight that this has been raised in several works [BCZGH16, GHŠV15] and also very recently in [FS20]. To the knowledge of the author, there are no available works which are able to rule out the case of a measure supported by stationary solutions only. For example this is a limitation in [FS20].

Finally, let us mention that in finite dimension there exists a positive result obtained in [MP14], where a finite dimensional Euler system is considered.

1.3. Existing results pertaining to the construction of invariant measures for the Euler equations. Theorems 5.2 and 5.5 assert the existence of a measure μ, invariant under the dynamics of (E^2) (resp. (E^3)). We recall the main methods which produce invariant measures for partial differential equations. To the knowledge of the author there exist at least three such techniques.

(*i*) The Gibbs-measure invariant technique, [Bou94, Bou96] and many authors after him. This technique is adapted for Hamiltonian PDE's. In the context of the two-dimensional Euler equations some results have been obtained by Flandoli in [Fla18].

(*ii*) Propagation of Gaussian initial data, initiated by Burq-Tzvetkov in [BT08a, BT14] in the context of wave equations. It consists in solving the equation with initial data taking the form $u_0 = \sum_{n\in\mathbb{Z}} g_n u_n e_n(x)$ where $(g_n)_{n\in\mathbb{Z}}$ are identically distributed independent Gaussian random variables.

(*iii*) The *fluctuation-dissipation method* of Kuksin. It consists of approximating the considered equation with a dissipation term and a fluctuating (random) term and in constructing invariant measures for these approximations. Then the basic idea is to take the vanishing viscosity limit, and retain some properties of the measures. Kuksin obtained the following result:

Theorem 5.13 (Kuksin, [Kuk04, KS15])**.** *There exists a topological space $X \subset H^2$ and a measure μ supported on X such that μ is invariant under the dynamics of* (E^2)*. Moreover the measure μ is such that:*

(i) There exists a continuous function p, increasing and such that $p(0) = 0$ satisfying the following: for any Borelian $A \subset \mathbb{R}$ there holds,

$$\mu(\|u\|_{L^2} \in \Gamma) + \mu(\|\nabla u\|_{L^2} \in \Gamma) \leqslant p(|\Gamma|)\,,$$

where $|\Gamma|$ denotes the Lebesgue measure of Γ. In particular μ does not possess any atom.

(ii) For any $A \subset H^2$ of finite Hausdorff dimension, $\mu(A) = 0$.

Let us make a few comments. As we want to construct measures in higher regularity spaces, the method (*i*) does not seem well adapted. Indeed, an invariant measure has been constructed in the space H^{-1} by Flandoli in [Fla18], which is too low in regularity for our purposes. We also refer to [AC90].

The method (*ii*) seems to be difficult to apply in our situation. However this method has the advantage to produce a measure whose support is dense in Sobolev spaces, and a very precise description of the measures.

Finally, the method (*iii*) appears to be more flexible than the others in the context of fluid mechanics or even dispersive PDEs. We refer to [KS04] for applications of this method to dispersive equations and the work of Sy [Sy19a, Sy18, Sy19b] and Sy-Yu [SY20a, SY20b]. The draw-back of the *fluctuation-dissipation* method is that since the invariant measures are constructed by a compactness technique, the nature of the invariant measures is not as well-understood as compared to the Gaussian measures of method (*ii*) or (*i*). Nevertheless, by a suitable analysis one can often prove some good features of these measures. We refer to [KS15] for more details.

1.4. Structure of the proof of the main results. The proof essentially contains two main ingredients: the construction of invariant measures for (E) at regularity H^s following the original argument of Kuksin in [Kuk04], and a globalisation argument of Bourgain in [Bou94]. This combination of techniques has already appeared in the work of Sy, see [Sy19a] for example.

We start by explaining how a global invariant measure with good properties is used to globalise the solutions controling their growth.

1.4.1. *The globalisation argument.* Let us recall an argument contained in [Bou94] which aims at extending a local Cauchy theory to a global theory with additional bounds on the growth of the Sobolev norms. Note that for our purposes, a deterministic local theory (even global in dimension 2) is already known and we only need to estimate the growth of the solutions.

Using a Borel-Cantelli argument, we reduce the almost-sure growth estimate of Theorem 5.9 and Theorem 5.11 to the following estimate: for any $\varepsilon > 0$ and any $T > 0$ there is a set $G_{\varepsilon,T} \subset H^s$ such that $\mu(G_{\varepsilon,T}) \geqslant 1 - \varepsilon$ and for $u \in G_{\varepsilon,T}$ a solution u exists on $[0, T]$ such that there holds

$$\|u(t)\|_{H^s} \leqslant CT^\alpha \text{ for any } t \leqslant T\,.$$

In order to prove such a bound, the key ingredients are:

(i) the existence of a *formal* invariant measure μ for the considered equation, enjoying nice decay estimates, for example subgaussian estimates of the form $\mu(\|u\|_{H^s} > \lambda) \leqslant Ce^{-\lambda^2}$ or weaker decay estimates; meaning that initial data are not likely to be of large norm. As the measure is invariant this implies that at any time, the solution is not likely to be of large norm.

(ii) a *nice* local well-posedness theory.

Then, picking $0 = t_0 < \cdots < t_N = T$, we can ensure that $\|u(t_k)\|_{H^s}$ is small for any k thanks to (i). We control the growth of $\|u(t)\|_{H^s}$ for $t \in [t_k, t_{k+1}]$ thanks to the local well-posedness theory (ii). We refer to Section 5 for details.

1.4.2. *Producing an invariant measure.* We explain the general strategy for producing invariant measures for (E) at regularity H^s, following [Kuk04].

Dimension 2. In the following we write $s = 2 + \delta$ where $\delta > 0$. The case $\delta = 0$ is precisely the content of [Kuk04]. We rewrite (E^2) taking the Leray projection (which we denote by $\mathbf{P}$, see Section 2 for a definition). The Euler equation (E^2) now takes the form:

$$\partial_t u + B(u, u) = 0\,, \tag{E^2}$$

where $B(u, u) \coloneqq \mathbf{P}(u \cdot \nabla u)$.

We introduce a random forcing η (see (5.6) for a precise definition) and a dissipative operator $L \coloneqq (-\Delta)^{1+\delta}$. Then, the Euler equation (E^2) is approximated by a randomly forced hyper-viscous equation:

$$\partial_t u_\nu + \nu L u_\nu + B(u_\nu, u_\nu) = \sqrt{\nu}\eta \text{ and } \nabla \cdot u_\nu = 0\,, \tag{5.5}$$

with initial condition $u_\nu(0) = u_0 \in H^{2+\delta}(\mathbb{T}^2, \mathbb{R}^2)$.

With only slight modification of the argument the method in [KS15], Chapter 2 we will construct invariant measures μ_ν to (5.5) concentrated on $H^{2+\delta}$. Then we will prove compactness of the family $(\mu_\nu)_{\nu>0}$ in order to obtain an invariant measure μ for (E^2).

Dimension 3. In the case of dimension three we will adapt this scheme to construct, for any $N \geqslant 0$ global solutions u_N and an invariant measure μ_N to the finite dimensional Euler equation:

$$(\mathrm{E}^3_N) \qquad \partial_t u_N + B_N(u,u) = 0\,,$$

where $u_N \in V_N$, a finite dimensional space. See Section 3 for precise definitions.

Then we will prove some estimates for μ_N of the form $\mathbb{E}_{\mu_N}[\|u\|^2_{H^s}] \leqslant C$, uniformly in N and deduce, by compactness arguments an invariant measure μ for the Euler equation in dimension 3, which we will write:

$$(\mathrm{E}^3) \qquad \partial_t u + B(u,u) = 0\,.$$

1.4.3. *Organisation of the paper.* Section 2 recalls some basic results that will be used in Section 3 to construct global solutions to approximate equations. In Section 4, invariant measures are constructed and in Section 5 and the proof of the main theorems is given. Section 6 is devoted to the proof of Theorem 5.9 and Theorem 5.11. Some important results and computations are postponed to the Appendix for convenience.

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2. Notation and preliminary results

2.1. Notation. We use the notation $\mathbb{Z}^2_0 = \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$.

We write $A \lesssim_c B$ when there is a constant $C(c)$ depending on c such that $A \leqslant C(c)B$.

For a function F we write $\mathrm{d}F(u;v)$ the differential of F at u evaluated at v and $\mathrm{d}^2(u;\cdot,\cdot)$ for the second order differential of F at u.

2.1.1. *Fuctional spaces.* H^s stands for the usual Sobolev spaces. In this article we use some variants of theses spaces. We still denote the complete space

$$\left\{u \in H^s(\mathbb{T}^d, \mathbb{R}^d)\,,\ \text{such that}\ \int_{\mathbb{T}^d} u = 0\right\}\,,$$

by H^s for convenience, and refer to it as the space of H^s functions with zero-mean. We endow this space with the norms

$$\|u\|_{\dot{H}^s}^2 = \sum_{n\in\mathbb{Z}^2} |n|^{2s}|\hat{u}(n)|^2 \text{ and } \|u\|_{H^s}^2 = \sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2s}|\hat{u}(n)|^2\,.$$

Recall that these two norms are equivalent on the space of zero mean H^s functions:

$$\|\cdot\|_{\dot{H}^s} \leqslant \|\cdot\|_{H^s} \leqslant 2\|\cdot\|_{\dot{H}^s}\,.$$

Given a space X, the space $X_{\mathbf{div}}$ refers to the space of functions $u \in X$ such that $\nabla \cdot u = 0$, endowed with the norm of X.

We introduce the Leray projector $\mathbf{P} : L^2(\mathbb{T}^d, \mathbb{R}^d) \to L^d_{\text{div}}(\mathbb{T}^d, \mathbb{R}^d)$ defined as the Fourier multiplier with coefficients $M_{ij}(n) = \delta_{ij} - \frac{n_i n_j}{|n|^2}$, where $n = (n_i)_{1\leqslant i\leqslant n} \in \mathbb{Z}^d$. We define $\mathbf{P}_{\leqslant N}$ to be the projection on frequencies $|n| \leqslant N$. We recall that with this definition $\mathbf{P}_{\leqslant N}$ is not continuous on L^p when $p \neq 2$, but we will not use estimates in L^p.

We set $B(u, v) := \mathbf{P}(u \cdot \nabla v)$ defined for sufficiently smooth u, v. B is then extended to $L^2_{\mathbf{div}} \times L^2_{\mathbf{div}}$ by duality by $\langle B(u, v), \varphi\rangle := -\langle u \otimes v : \nabla\varphi\rangle$ for such that $\nabla\varphi \in L^\infty$.

Let $(X, \|\cdot\|_X)$ be a normed space and I an interval. The Sobolev space $W^{s,p}(I, X)$ is endowed with the norm

$$\|u\|^p_{W^{s,p}(I,X)} := \|u\|^p_{L^p(I,X)} + \iint_{I\times I} \frac{\|u(t) - u(t')\|^p_X}{|t - t'|^{1+sp}}\,\mathrm{d}t\,\mathrm{d}t'\,.$$

We also sometime use $L^p_T X$ as a shorthand for $L^p((0, T), X)$.

2.1.2. *Probability theoretic notation.* If E is topological space then $\mathcal{P}(E)$ stands for the set of probability measures on E and $\mathcal{B}(E)$ stands for the set of its Borelians.

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and further assume $\mathcal{F}$ to be complete, that is $\mathcal{F}$ contains all the sets contained in zero probability measure measurable sets. Let $(\mathcal{F}_t)_{t\geqslant 0}$ be a filtration of $\mathcal{F}$, that is, $t \mapsto \mathcal{F}_t$ is non-increasing. The quadruple $(\Omega, \mathcal{F}, F_t, \mathbb{P})$ is called a filtered probability space.

A process $(x(t))_{t\geqslant 0}$ is said to be *progressively measurable* with respect to a filtration $(\mathcal{F}_t)_{t\geqslant 0}$ if for any $T > 0$, $x_{|[0,T]}$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$ measurable.

The expectation with respect to the probability $\mathbb{P}$ will be denoted $\mathbb{E}$ and when the expectation will be taken with respect to a measure μ we will write $\mathbb{E}_\mu$.

2.1.3. *The noise.* We let $(\mathcal{G}_t)_{t\geqslant 0}$ be the completed filtration associated to identically distributed independent Brownian motions $(\beta_n(t))_{n\in\mathbb{Z}}$.

In this text we will use a Hilbert basis of the space of $\{u \in L^2_{\mathrm{div}}, \fint_{\mathbb{T}^2} u = 0\}$:

(1) In dimension 2 we can take $(e_n)_{n\in\mathbb{Z}^2_0}$, where

$$e_n(x) := \frac{(-n_2, n_1)^T}{\sqrt{2}\pi|n|} \begin{cases} \sin(n\cdot x) & \text{if } n_1 > 0 \text{ or } (n_1 = 0 \text{ and } n_2 > 0) \\ \cos(n\cdot x) & \text{otherwise,} \end{cases}$$

and for $|n| = \sqrt{n_1^2 + n_2^2} \neq 0$. Then for any $n \in \mathbb{Z}^2_0$ we have $(-\Delta)e_n = |n|^2 e_n$.

(2) In dimension 3 we let $(e_j)_{j\geqslant 0}$ be a Hilbert basis formed of divergence-free eigenfunctions of $(-\Delta)$ and denote by λ_j^2 the corresponding eigenvalues.

We set $\eta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\zeta(t)$ where

$$\zeta(t) = \sum_{n\in\mathbb{Z}^2_0} \phi_n \beta_n(t) e_n\,, \tag{5.6}$$

in dimension 2, (resp. $\zeta(t) = \sum_j \phi_j\beta_j(t)e_j$ in dimension 3), for some numbers $(\phi_n)_{n\in\mathbb{Z}^2_0}$ (resp. $(\phi_j)_{j\geqslant 1}$) to be chosen later. We introduce the

$$\mathcal{B}_k := \sum_{n\in\mathbb{Z}^2_0} |n|^{2k}|\phi_n|^2\,,$$

in dimension 2 (resp. $\mathcal{B}_k := \sum_{j\geqslant 0}\lambda_j^{2k}|\phi_j|^2$ in dimension 3).

2.2. Deterministic preliminaries. In the following we gather some standard results that we will use on many occasions.

We start with some basic estimates of the bilinear form in dimension $d = 2$.

Lemma 5.14 (Properties of the bilinear form in dimension 2)**.** *Let $u \in H^1_{\mathrm{div}}(\mathbb{T}^2)$ and $v, w \in H^1(\mathbb{T}^2)$. Then we have the following.*

(i) $\langle B(u,v), v\rangle_{L^2} = 0$.

(ii) $\langle B(u,u), u\rangle_{H^1} = 0$.

(iii) $\|B(u,v)\|_{H^{-1}} \lesssim \|u\|_{H^{\frac{1}{2}}}\|v\|_{H^{\frac{1}{2}}}$.

Proof. The fact (i) comes from the fact that $\nabla\cdot u = 0$ and an integration by parts. The fact (ii) is very specific to dimension 2 and false in dimension 3. It is proven by a direct computation:

$$(\nabla(u\cdot\nabla u), \nabla u)_{L^2} = (u\cdot\nabla\xi, \xi)_{L^2} = 0\,,$$

where $\xi = \nabla\wedge u$. For details, see [KS15], Lemma 2.1.16.

Let us prove (*iii*). For any $\varphi \in H^1$ we have use the definition of $B(u,v)$, so that
$$\langle u \cdot \nabla v, \varphi \rangle_{L^2} = -\langle u \otimes v, \nabla \varphi \rangle_{L^2} .$$
Using the Hölder inequality and the Sobolev embedding $H^{\frac{1}{2}} \hookrightarrow L^4$, we get
$$\langle B(u,v), \varphi \rangle_{L^2} \leqslant \|u\|_{L^4} \|v\|_{L^4} \|\varphi\|_{H^1} \lesssim \|u\|_{H^{\frac{1}{2}}} \|v\|_{H^{\frac{1}{2}}} \|\varphi\|_{H^1} ,$$
which is equivalent to (*iii*) by duality. □

Remark 5.15. The property (*ii*) will appear to be a crucial algebraic cancellation which will be heavily used through the rest of the paper, and is very specific to the dimension 2.

In dimension 3 we have the following corresponding estimates of the bilinear term. Note that the bilinear cancellation only holds in L^2.

Lemma 5.16 (Properties of the bilinear form in dimension 3)**.** *Let $u \in H^1_{\mathrm{div}}(\mathbb{T}^3)$ and $v, w \in H^1(\mathbb{T}^3)$. Then we have the following.*

(i) $\langle B(u,v), v \rangle_{L^2} = 0$.

(ii) If $u, v \in H^{\frac{3}{4}}_{\mathrm{div}}$ then $\|B(u,v)\|_{H^{-1}} \lesssim \|u\|_{H^{\frac{3}{4}}} \|v\|_{H^{\frac{3}{4}}}$.

The proof is similar, the only modification is in (*ii*) where we use of the continuous embedding $H^{\frac{3}{4}}(\mathbb{T}^3) \hookrightarrow L^4(\mathbb{T}^3)$.

We will also need a basic heat kernel estimate, which writes the same in dimension 2 and 3.

Lemma 5.17. *Consider the operator $L = (-\Delta)^{1+\delta}$ for $\delta > 0$. Let $s_0 \in \mathbb{R}$, $\alpha \geqslant 0$ and $F \in H^{s_0}(\mathbb{T}^d)$, where $d \in \{2,3\}$ with mean zero. For any $s < s_0 + \alpha(1+\delta)$ and $t > 0$ there holds*
$$\left\| e^{-tL} F \right\|_{H^s} \lesssim t^{-\frac{\alpha}{2}} \|F\|_{H^{s_0}} , \tag{5.7}$$
where the implicit constant only depends on α.

Proof. We take the Fourier transform and use the mean zero condition to write
$$\begin{aligned} \left\| e^{-tL} F \right\|^2_{H^s} &= \sum_{n \in \mathbb{Z}^2} \langle n \rangle^{2s} e^{-2t|n|^{2(1+\delta)}} |\hat{F}(n)|^2 \\ &= \sum_{n \in \mathbb{Z}^2_0} \langle n \rangle^{2s} \frac{e^{-2t|n|^{2(1+\delta)}} t^{\alpha} |n|^{2\alpha(1+\delta)}}{t^{\alpha} |n|^{2\alpha(1+\delta)}} |\hat{F}(n)|^2 . \end{aligned}$$
Then observe that the function $x \mapsto x^{\alpha} e^{-x}$ is bounded so that
$$\left\| e^{-tL} F \right\|^2_{H^s} \leqslant C t^{-\alpha} \sum_{n \in \mathbb{Z}^2_0} \langle n \rangle^{2(s-s_0) - 2\alpha(1+\delta)} \langle n \rangle^{2s_0} |\hat{F}(n)|^2 ,$$
which is bounded by $t^{-\alpha} \|F\|^2_{H^{s_0}}$ as soon as $s < s_0 + \alpha(1+\delta)$. □

2.3. Probabilistic preliminaries. We will use the Itô isometry in the following form:

Theorem 5.18 (Itô isometry, [DPZ92]). *Let $F(t)$ be an $\mathcal{G}_t$-adapted process. Then, for any $t > 0$ there holds*

$$\left\| \int_0^t F(t') \, \mathrm{d}\beta(t') \right\|_{L^2(\Omega)} = \|F\|_{L^2((0,t)} . \tag{5.8}$$

Moreover we will often use the following facts:

(i) When F is a deterministic function, then $\int_0^t F(t') \, \mathrm{d}\beta_n(t')$ is a Gaussian random variable of mean 0 and variance $\|F\|^2_{L^2((0,t)}$, see [DPZ92].

(ii) If Y is a Gaussian random variable on a Hilbert space H (that is, $(Y, a)_H$ is a real Gaussian for any $a \in H$), then one has

$$\|Y\|_{L^p_\omega} \lesssim \sqrt{p} \|Y\|_{L^2_\omega} , \tag{5.9}$$

for any $p \geqslant 1$.

We also use the following well-known regularity criterion.

Lemma 5.19 (Kolmogorov,[DPZ92]). *Let $(X(t))_{t \geqslant 0}$ be a stochastic process with values in a Banach space endowed with a norm $\| \cdot \|$. Assume that there exist $p \geqslant 1$ and $\alpha > 0$ such that*

$$\mathbb{E}[\|X(t) - X(s)\|^p] \lesssim |t - s|^{1+\alpha} .$$

Then for any $\varepsilon > 0$ small enough, then $(X(t))_{t \geqslant 0}$ admits a modification (i.e., there is $(\tilde{X}(t))_{t \geqslant 0}$ such that for all $t \geqslant 0$, $\tilde{X}(t) = X(t)$ almost surely) that is almost-surely $(\frac{\alpha}{p} - \varepsilon)$ Hölder continuous.

3. Constructions of solutions to approximate equations

In this section, we study the well-posedness of the following equations. The first is the stochastic hyper-viscous equation in dimension 2:

$$\left\{ \begin{array}{rcl} \partial_t u_\nu + \nu L u_\nu + B(u_\nu, u_\nu) & = & \sqrt{\nu}\eta \\ u_\nu(0) & = & u_0 \in H^1_{\mathrm{div}}(\mathbb{T}^2) , \end{array} \right. \tag{HVE$^2_\nu$}$$

where we recall that $L = (-\Delta)^{1+\delta}$ for some $\delta > 0$. $\eta = \frac{\mathrm{d}}{\mathrm{d}t}\zeta(t)$ where $\zeta(t)$ is defined by (5.6) and satisfies $\mathcal{B}_{3+\delta} < \infty$.

The second equation considered is a three-dimensional finite-dimensional hyper-viscous equation which writes:

$$\left\{ \begin{array}{rcl} \partial_t u_{\nu,N} + \nu L u_{\nu,N} + B_N(u_{\nu,N}, u_{\nu,N}) & = & \sqrt{\nu}\eta_N \\ u_{\nu,N}(0) & = & \mathbf{P}_{\leqslant N} u_0 \in V_N , \end{array} \right. \tag{HVE$^3_{\nu,N}$}$$

where $B_N = \mathbf{P}_{\leqslant N} B$, and $\eta_N \coloneqq \mathbf{P}_{\leqslant N}\eta$. $\eta = \frac{\mathrm{d}}{\mathrm{d}t}\zeta(t)$ where $\zeta(t)$ is defined by (5.6) and satisfies $\mathcal{B}_{3+\delta} < \infty$. We recall that $L = (-\Delta)^s$ where $s \coloneqq 1+\delta$ with $\delta > \frac{3}{2}$. The space V_N is defined as:

$$V_N = \mathrm{Span}\{e_n, |n| \leqslant N\}\,.$$

The main results of this section are the following. We let $(\Omega, \mathbb{P}, \mathcal{F})$ be a probability space with the Brownian motion completed filtration $(\mathcal{G}_t)_{t\geqslant 0}$. We start with the construction of global solutions for (HVE^2_ν)

Proposition 5.1 (Solutions to (HVE^2_ν))**.** *Let $\nu > 0$ and u_0 a $\mathcal{G}_0$-measurable random variable such that $u_0 \in H^1_{\mathrm{div}}$ almost surely. We also assume that the noise η is such that $\mathcal{B}_{3+2\delta} < \infty$. Then there exists a set Ω_1 such that $\mathbb{P}(\Omega_1) = 1$ and such that for any $\omega \in \Omega_1$:*

- *(i) (HVE^2_ν) has a unique global solution $t \mapsto u^\omega_\nu(t)$ associated to u^ω_0, in the space $\mathcal{C}^0(\mathbb{R}_+, H^1_{\mathrm{div}})$.*
- *(ii) Moreover $u^\omega_\nu \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H^{2+\delta})$.*

Furthermore, the process u_ν satisfies the following properties.

- *(iii) u_ν may be written in the form*

$$u_\nu(t) = u_0 + \int_0^t f(s)\,\mathrm{d}s + \sqrt{\nu}\zeta(t)\,,$$

where equality holds in $H^{-\delta}$ and where $f(s)$ is a $\mathcal{G}_s$-progressively measurable process satisfying that almost surely $f \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H^{-\delta})$. In particular $u(t)$ is a $\mathcal{F}_t$-progressively measurable process.

- *(iv) There exists a measurable map $U : H^1 \times \mathcal{C}^0(\mathbb{R}_+, H^1) \to \mathcal{C}^0(\mathbb{R}_+, H^1)$, continuous in its first variable, such that for any $\omega \in \Omega_1$, $u^\omega_\nu = U(u_0, \zeta)$.*

Remark 5.20. The reader already aware of the methods used in [Kuk04] can skip the proof provided in this section, as the proof follows the same lines. However we recall the proof giving full details.

In dimension 3, the equation ($\mathrm{HVE}^3_{\nu,N}$) being finite dimensional we can prove the following.

Proposition 5.2 (Solutions to ($\mathrm{HVE}^3_{\nu,N}$))**.** *Let $\nu, N > 0$ and u_0 a $\mathcal{G}_0$-measurable random variable such that for all $N \geqslant 0$, $u_0 \in V_N$ almost surely. We also assume that the noise η is such that $\mathcal{B}_{3+2\delta} < \infty$. Then there exists a set Ω_1 such that $\mathbb{P}(\Omega_1) = 1$ and such that for any $\omega \in \Omega_1$:*

- *(i) ($\mathrm{HVE}^3_{\nu,N}$) has a unique global solution $t \mapsto u^\omega_{\nu,N}(t)$ associated to the initial data u^ω_0, in the space $\mathcal{C}^0(\mathbb{R}_+, V_N)$.*
- *(ii) Moreover $u^\omega_{\nu,N} \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H^{1+\delta})$.*

Furthermore, the processes $u_{\nu,N}$ satisfy the following properties.

(iii) $u_{\nu,N}$ *may be written in the form*

$$u_{\nu,N}(t) = u_0 + \int_0^t f(s)\,\mathrm{d}s + \sqrt{\nu}\zeta_N(t)\,,$$

with $\zeta_N \coloneqq \mathbf{P}_{\leqslant N}\zeta$ *and where equality holds in* $H^{-(1+\delta)}$*;* $f(s)$ *is a* $\mathcal{G}_s$*-progressively measurable process satisfying that almost surely* $f \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H^{-(1+\delta)})$*. In particular* $u_{\nu,N}(t)$ *is a* $\mathcal{F}_t$*-progressively measurable process.*

(iv) There exists a measurable map $U_N : H^1 \times \mathcal{C}^0(\mathbb{R}_+, V_N) \to \mathcal{C}^0(\mathbb{R}_+, V_N)$*, continuous in its first variable, such that for any* $\omega \in \Omega_1$*,* $u^\omega_{\nu,N} = U(u_0, \zeta_N)$*.*

A general strategy for solving a nonlinear stochastic partial differential equation is to seek for solutions which have a particular structure, designed to eliminate the randomness and apply a fixed-point argument.

We explain the general scheme for proving Proposition 5.1. In order to solve (HVE$^2_\nu$) we first look for solutions z_ν to the following equation:

$$\left\{\begin{array}{rcl} \partial_t z_\nu(t) + \nu L z_\nu(t) & = & \sqrt{\nu}\eta(t) \\ z_\nu(0) & = & 0\,. \end{array}\right. \tag{5.10}$$

Then we seek for solutions to (HVE$^2_\nu$) taking the form $u_\nu = z_\nu + v_\nu$ where v_ν formally satisfies:

$$\left\{\begin{array}{rcl} \partial_t v_\nu + \nu L v_\nu + B(z_\nu + v_\nu, z_\nu + v_\nu) & = & 0 \\ v_\nu(0) & = & u_0\,, \end{array}\right. \tag{5.11}$$

This is a deterministic equation and can be solved in the space $\mathcal{C}^0(\mathbb{R}_+, H^1_{\mathrm{div}})$.

Remark 5.21. We should emphasise that the point of this construction is fundamentally different in nature to the work of Burq-Tzvetkov, or Bourgain or Kuksin-Shirikyan [KS15] where the emphasis was put on lowering the regularity threshold at which one can find a solution. This is why in our construction we did not try to lower the regularity threshold: the initial condition for the stochastic part is 0.

In the three-dimensional case we replace equations (5.10) and (5.11) respectively with

$$\left\{\begin{array}{rcl} \partial_t z_{\nu,N}(t) + \nu L z_{\nu,N}(t) & = & \sqrt{\nu}\eta_N(t) \\ z_{\nu,N}(0) & = & 0\,, \end{array}\right. \tag{5.12}$$

and

$$\left\{\begin{array}{rcl} \partial_t v_{\nu,N} + \nu L v_{\nu,N} + B_N(z_{\nu,N} + v_{\nu,N}, z_{\nu,N} + v_{\nu,N}) & = & 0 \\ v_\nu(0) & = & u_0\,. \end{array}\right. \tag{5.13}$$

A solution to (5.10) (resp. (5.12)) is explicitly given by

$$z_\nu(t) = \sqrt{\nu} \sum_{n\in\mathbb{Z}_0^2} \phi_n \left(\int_0^t e^{-\nu(t-t')L} \mathrm{d}\beta_n(t') \right) e_n \in H^1_{\mathrm{div}}(\mathbb{T}^2)\,, \tag{5.14}$$

respectively by

$$z_\nu(t) = \sqrt{\nu} \sum_{j=1}^{N} \phi_j \left(\int_0^t e^{-\nu(t-t')L} \mathrm{d}\beta_j(t') \right) e_j \in V_N\,, \tag{5.15}$$

as this can be checked by the Itô formula for example, see Appendix A. Moreover, the processes $z_\nu, z_{\nu,N}$ satisfy the following properties.

Lemma 5.22. *Under the assumptions of Proposition 5.1 (resp. Proposition 5.2), there exists a set Ω_1 such that $\mathbb{P}(\Omega_1) = 1$ and which satisfies the following properties for any $\omega \in \Omega_1$. Let us denote by z the solution z_ν in (5.14) (resp. $z_{\nu,N}$ in (5.15)). Then:*

(i) $z \in \mathcal{C}^0(\mathbb{R}_+, H^1_{\mathrm{div}}) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, H^{2+\delta})$.

(ii) There exists $\alpha > 0$ such that z almost surely belongs to the space $\mathcal{C}^\alpha_{\mathrm{loc}}(\mathbb{R}_+, W^{2,4})$.

(iii) $\mathbb{E}[\|z(t)\|^2_{\dot{H}^1}] \leqslant \frac{\mathcal{B}_1}{2}\nu t$, *for any $t \geqslant 0$.*

We postpone the proof of this Lemma. Then, (5.11) can be globally solved using the following lemma.

Lemma 5.23. *Let u_0 a random variable such that $u_0 \in H^1_{\mathrm{div}}$ almost surely. Then there exists a set Ω_2 of probability 1 such that for any $\omega \in \Omega_2$, the associate Cauchy problem to (5.11) is globally well-posed in $\mathcal{C}(\mathbb{R}_+, H^1_{\mathrm{div}}(\mathbb{T}^2))$. Furthermore:*

(i) For any $T > 0$, the flow map $H^1 \to \mathcal{C}^0([0,T], H^1_{\mathrm{div}})$ defined by $u_0 \mapsto v_\nu$ is locally Lipschitz.

(ii) $v_\nu \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H^{2+\delta})$.

Assuming Lemma 5.22 and Lemma 5.23 we are now ready to prove Proposition 5.1.

Proof of Proposition 5.1. Part (i) follows from Lemma 5.22 and Lemma 5.23 and part (ii) follows from Lemma 5.23.

We prove (iv). In fact, Lemma 5.23 proves that v_ν is a continuous function of z_ν and u_0 as a corollary of the local well-posedness theory, furthermore, there exists a measurable function $Z : \mathcal{C}^0(\mathbb{R}_+, H^1) \to \mathcal{C}^0(\mathbb{R}_+, H^1)$ such that $z_\nu = Z(\zeta)$, where $Z(\zeta)$ is given by (5.14). The measurability comes from the fact that the maps Z_M defined by

$$Z_M(\zeta)(t) = \sqrt{\nu} \sum_{|n|\leqslant M} \phi_n \left(\int_0^t e^{-\nu(t-t')L} \, \mathrm{d}\beta_n(t') \right) e_n \,,$$

are continuous, hence measurable; and that $Z = \lim_{M\to\infty} Z_M$ almost surely. For details, see [KS15], Remark 2.4.3.

In order to prove (iii) we write

$$u_\nu(t) = u_0 + \int_0^t f(s) \, \mathrm{d}s + \sqrt{\nu}\zeta(t) \,,$$

where $f(s) = -\nu L u_\nu(s) + B(u_\nu(s), u_\nu(s))$ and $\zeta(t) := \sum_{n\in\mathbb{Z}^2} \phi_n \beta_n(t) e_n$. Then $f(s)$ is $\mathcal{G}_s$ progressively measurable. Indeed, we know that $s \mapsto z_\nu(s)$ is progressively measurable by construction and properties of the stochastic integral, and we also know that $s \mapsto v_\nu(s)$ is progressively measurable thanks to (iv). Finally we explain why $f \in L^2_{\mathrm{loc}} H^{-\delta}$. First, since u_ν lies in $L^2_{\mathrm{loc}} H^{2+\delta}$ thanks to Lemma 5.23 we obtain $L u_\nu \in L^2_{\mathrm{loc}} H^{-\delta}$. For the bilinear term we use Lemma 5.14 to bound

$$\|B(u_\nu, u_\nu)\|_{H^{-\delta}} \leqslant \|B(u_\nu, u_\nu)\|_{L^2} \leqslant \|u_\nu\|^2_{H^1} \,.$$

Since $u_\nu \in \mathcal{C}^0(\mathbb{R}, H^1_{\mathrm{div}})$, it follows that $B(u_\nu, u_\nu) \in L^2_{\mathrm{loc}}(\mathbb{R}_+, H^{-\delta})$ and thus the result. $\square$

The next subsections give the proof of Lemma 5.22, Lemma 5.23 and Proposition 5.2.

3.1. Properties of the stochastic objects. We will prove Lemma 5.22 in dimension 2 only. Indeed, in the case of dimension 3 a proof can be written following the same lines, but even simpler since L^2 or H^k functions are the same spaces when restricted to frequencies in V_N.

Proof of Lemma 5.22. In the following we will use that $-\Delta e_n = |n|^2 e_n$. Let $s \geqslant 0$.

We first prove that for any $t > 0$ we have that almost surely $z_\nu(t) \in H^s$ which results from proving that there exists a $p \geqslant 1$ such that

$$\left\| \|\langle\nabla\rangle^s z_\nu(t)\|_{L^r_x} \right\|_{L^p_\Omega} < \infty \,. \tag{5.16}$$

In order to prove (5.16), remark that using the Minkowski inequality for $p \geqslant 2$ and the Gaussian bound (5.9) we have:

$$\begin{aligned} \left\| \|\langle\nabla\rangle^s z_\nu(t)\|_{L^2_x} \right\|_{L^p_\Omega} &\leqslant \left\| \|\langle\nabla\rangle^s z_\nu(t)(x)\|_{L^p_\Omega} \right\|_{L^2_x} \\ &\lesssim \sqrt{p} \left\| \|\langle\nabla\rangle^s z_\nu(t)(x)\|_{L^2_\Omega} \right\|_{L^2_x} \,. \end{aligned}$$

Then we use the Itô isometry (5.8) to get that for any $t > 0$ and $x \in \mathbb{T}^2$,

$$\mathbb{E}\left[|\langle\nabla\rangle^s z_\nu(t)(x)|^2\right] = \nu \sum_{n\in\mathbb{Z}_0^2} \langle n\rangle^{2s}|\phi_n|^2 \int_0^t e^{-2\nu(t-t')|n|^{2(1+\delta)}}\,\mathrm{d}t'$$

$$= \sum_{n\in\mathbb{Z}_0^2} \langle n\rangle^{2s}\frac{|\phi_n|^2}{|n|^{2(1+\delta)}}\left(e^{-2t|n|^{2(1+\delta)}} - 1\right).$$

Then

$$\|\langle\nabla\rangle^s z_\nu(t)(x)\|_{L^2_\Omega}^2 \lesssim \sum_{n\in\mathbb{Z}_0^2} |\phi_n|^2\langle n\rangle^{2s-2(1+\delta)}$$

$$\lesssim \mathcal{B}_{s-(1+\delta)}\,,$$

which implies (5.16) after integration in space, since the torus $\mathbb{T}^2$ has finite measure.

Now we apply the Kolmogorov criterion to prove the continuity in time. A similar computation yields, for $t_2 > t_1$,

$$|z_\nu(t_1)(x) - z_\nu(t_2)(x)|^2 = \sum_{n,n'\in\mathbb{Z}_0^2} \phi_n\phi_{n'}e^{i(n-n')x}A_{n,n'}(t_1,t_2)\,,$$

where $A_{n,n'}(t_1,t_2) = a_n(t_1)\bar{a}_{n'}(t_1) + a_n(t_2)\bar{a}_{n'}(t_2) - a_n(t_1)\bar{a}_{n'}(t_2) - a_n(t_2)\bar{a}_{n'}(t_1)$ and

$$a_n(t) = \int_0^t e^{-\nu(t-t')L}\,\mathrm{d}\beta_n(t')\,.$$

Taking the expectation and using the Itô isometry (5.8) gives

$$\begin{aligned}&\mathbb{E}[|\langle\nabla\rangle^s(z_\nu(t_1)(x) - z_\nu(t_2)(x))|^2]\\ &\qquad= \sum_{n\in\mathbb{Z}_0^2} \langle n\rangle^{2s}|\phi_n|^2\left(b_n(t_2,t_1) + b_n(t_2,t_2) - b_n(t_1,t_2) - b_n(t_2,t_1)\right),\end{aligned}$$

where we compute $b_n(t_1,t_2) := \left(e^{(t_1-t_2)|n|^{2(1+\delta)}} - e^{-(t_1+t_2)|n|^{2(1+\delta)}}\right)$. Let $\alpha \in [0,1]$. Thanks to the elementary bound, that for any $a, b > 0$:

$$|e^{-a} - e^{-b}| \lesssim |a-b|^\alpha\,,$$

we have

$$|b_n(t_1,t_2) - b_n(t_1,t_1)| \lesssim |t_1 - t_2|^\alpha|n|^{2\alpha(1+\delta)}\,.$$

Gathering all these estimates we obtain

$$\|\langle\nabla\rangle^s z_\nu(t_1)(x) - z_\nu(t_2)(x)\|_{L^2(\Omega)}^2 \lesssim |t_1 - t_2|^\alpha\mathcal{B}_{s+\alpha(1+\delta)}\,.$$

Then we obtain, for $p \geqslant r$,

$$\|\|z_\nu(t_1) - z_\nu(t_2)\|_{W^{s,r}}\|_{L^p_\Omega} \lesssim \sqrt{p}|t_1 - t_2|^{\frac{\alpha}{2}}\mathcal{B}_{s+\alpha(1+\delta)}\,.$$

We finish the proof using Theorem 5.19 and conclude that almost surely $u_\nu \in \mathcal{C}^{\frac{\alpha}{2}-\varepsilon}_{\mathrm{loc}}(\mathbb{R}_+, W^{2,r})$, for small enough $\varepsilon > 0$.

Now we can consider the set
$$\Omega_1^{(T)} := \left\{\omega \in \Omega, \|z_\nu\|_{L^\infty([0,T],H^1)} + \|z_\nu\|_{L^2([0,T],H^{2+\delta})} + \|z_\nu\|_{L^2([0,T],W^{2,4}\cap W^{1,\infty})} < \infty\right\},$$
and claim that this set is of probability 1 (which is a direct consequence of the previous estimates). Then, in order to finish the proof, it suffices to set
$$\Omega_1 := \bigcap_{N>0} \Omega_1^{(N)},$$
which is of probability 1.

The fact that z_ν is divergence free can be seen directly on the exact formula for z_ν. Finally, in order to prove (iii) we apply the Itô formula to $F(u) = \|u\|_{\dot H^1}^2$, see Appendix A for more details. We obtain:
$$\mathbb{E}\left[\|z_\nu(t)\|_{\dot H^1}^2\right] + \nu\int_0^t \left(2\mathbb{E}\left[\|z_\nu(t)\|_{\dot H^{2+\delta}}^2\right] - \mathcal{B}_1\right) \mathrm{d}t' = 0,$$
leading to (iii). □

3.2. Proof of Lemma 5.23. The proof of Lemma 5.23 is well-known in the context of Navier-Stokes equations. However we recall its proof, since there is the term z_ν to handle in the equation.

We introduce the space $X_T := \mathcal{C}^0([0,T], H^1_{\mathrm{div}})$ and the map:
$$\Psi : \left\{ \begin{array}{ccc} X_T & \longrightarrow & X_T \\ u & \longmapsto & \Psi(u), \end{array}\right.$$
where
$$\Psi(u)(t) := e^{-\nu t L}u_0 + \int_0^t e^{-\nu(t-t')L} B(u(t') + z_\nu(t'), u(t') + z_\nu(t'))\,\mathrm{d}t'. \tag{5.17}$$

We note that thanks to the definition of Ψ, $\Psi(u)$ is divergence free if u and u_0 are. The fact that $\Psi(u)$ belongs to X_T will be a consequence of the following lemmata and direct computations. The proof of Lemma 5.23 relies on the following results.

Lemma 5.24. *There exist constants $c, \kappa > 0$ such that if we take $T = \frac{c}{\|u_0\|_{H^1}^\kappa}$, then the map Ψ defined by (5.17) satisfies:*

(i) $\Psi : B(0, 2\|u_0\|_{H^1}) \to B(0, 2\|u_0\|_{H^1})$

(ii) *Ψ is a contraction on $B(0, 2\|u_0\|_{H^1})$.*

Lemma 5.25. *For any solution $u_\nu \in \mathcal{C}^0([0,T), H^1_{\mathrm{div}})$ we have:*
$$\|u_\nu\|_{L^\infty([0,T),H^1)} < \infty, \text{ and } \|u_\nu\|_{L^2([0,T],H^{2+\delta})} < \infty.$$

Proof of Lemma 5.23. By Lemma 5.22 and Lemma 5.24, for any fixed $\omega \in \Omega_1$ fixed, $u_0^\omega \in H^1$ gives rise to a solution $u_\nu := z_\nu + v_\nu$ to (HVE$^2_\nu$). Indeed, Ψ is a contraction in the Banach space $\mathcal{C}^0([0,T],H^1)$, where $T = \frac{c}{\|u_0\|_{H^1}^\kappa}$ for some $\kappa > 0$, which provides a unique local solution on $[0,T]$.

Iterating the local well-posedness gives us a maximal existence time interval $[0,T^*)$, and the following blowup criterion: if $T^* < \infty$ then $\|u_\nu(t)\|_{H^1} \underset{t\to T^*}{\longrightarrow} \infty$. By contradiction, if $T^* < \infty$, then Lemma 5.25 proves that

$$\sup_{t\in[0,T^*)} \|u_\nu(t)\|_{H^1} < \infty\,,$$

which is a contradiction. Hence u_ν is global and belongs to $L^2_{\mathrm{loc}}(\mathbb{R}_+, H^{2+\delta})$. □

Proof of Lemma 5.24. We only prove the first claim as the second will follow using similar estimates. We also omit ν indices, as they are not relevant to the proof. Using the definition of Ψ we have:

$$\|\Psi(v)(t)\|_{H^1} \leqslant \|e^{-\nu tL}u_0\|_{H^1} + \left\|\int_0^t e^{-\nu(t-t')L}B(v+z,v+z)\,\mathrm{d}t'\right\|_{H^1}\,.$$

First, we note that $\|e^{-\nu tL}u_0\|_{H^1} \leqslant \|u_0\|_{H^1}$. Next, we define the function $F(t) := \langle\nabla\rangle B(v(t)+z(t), v(t)+z(t))$ and write

$$\left\|\int_0^t e^{-\nu(t-t')L}B(v+z,v+z)\,\mathrm{d}t'\right\|_{H^1} \leqslant \int_0^t \|e^{-\nu(t-t')L}F(t')\|_{L^2}\,\mathrm{d}t'\,.$$

Then we use Lemma 5.17 with $\alpha := \frac{2}{1+\delta} + \varepsilon < 2$ (this is possible for sufficiently small $\varepsilon > 0$). We arrive at

$$\|\Psi(v)(t)\|_{H^1} \leqslant \|u_0\|_{H^1} + \int_0^t \frac{1}{|t-t'|^{\frac{\alpha}{2}}}\|F(t')\|_{H^{-2}}\,\mathrm{d}t'\,.$$

We observe that

$$\begin{aligned}\|F(t')\|_{H^{-2}} &\leqslant \|B(v(t')+z(t'), v(t')+z(t'))\|_{H^{-1}}\\ &\lesssim \|v(t')+z(t')\|^2_{H^{\frac12}}\\ &\lesssim \|v(t')\|^2_{H^1} + \|z(t')\|^2_{H^1}\,.\end{aligned}$$

Since $\frac{\alpha}{2} < 1$, we can integrate in time to get

$$\|\Psi(v)\|_{X_T} \leqslant \|u_0\|_{H^1} + C(\alpha)T^{1-\frac{\alpha}{2}}\left(\|v\|^2_{X_T} + \|z\|^2_{X_T}\right)\,,$$

which implies the result for sufficiently small $T > 0$ of the required form taking into account Lemma 5.22 (iii). □

Proof of Lemma 5.25. Let $T > 0$ and u as local solution to (5.11) on $[0,T)$ constructed as before.

Let us write an energy estimate at regularity H^1 and use Lemma 5.14 (ii):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{\dot H^1} + \nu\|v(t)\|^2_{\dot H^{2+\delta}} &\lesssim \langle B(v(t)+z(t), v(t)+z(t)), v(t)\rangle_{\dot H^1}\\ &= \langle B(v(t),z(t)) + B(z(t),v(t)) + B(z(t),z(t)), v(t)\rangle_{\dot H^1}\,.\end{aligned}$$

Then we can use product laws in Sobolev spaces to estimate two terms (and where $\varepsilon > 0$ is arbitrarily small):

$$\begin{aligned}|\langle B(v(t),z(t))+ \ & B(z(t),z(t)),v(t)\rangle_{\dot H^1}| \leqslant (\|v\cdot\nabla z\|_{H^1}+\|z\cdot\nabla z\|_{H^1})\,\|v\|_{H^1}\\ &\leqslant \|\nabla z(t)\|_{W^{1,4}}\|v(t)\|_{H^1}^2+\|z(t)\|_{H^{1+\varepsilon}}\|\nabla z(t)\|_{H^{1+\varepsilon}}\|v(t)\|_{H^1}\\ &\lesssim \|z(t)\|_{W^{2,4}}\|v(t)\|_{H^1}^2+\|z(t)\|_{H^{2+\delta}}^2\|v(t)\|_{H^1}\,,\end{aligned}$$

where we used the Young inequality in the last line.

For the term $\langle B(z(t),v(t)),v(t)\rangle_{\dot H^1} = \langle z(t)\cdot\nabla v(t),v(t)\rangle_{\dot H^1}$ (due to the incompressibility of $v(t)$), we have to observe that the leading order term vanishes. Indeed, we can check that

$$\nabla(z\cdot\nabla v) = (z\cdot\nabla)(\nabla v)+\nabla z\cdot\nabla v$$

so that thanks to Lemma 5.14 (ii) we have

$$|\langle z(t)\cdot\nabla v(t),v(t)\rangle_{\dot H^1}| \lesssim \|\nabla z(t)\cdot\nabla v(t):\nabla v(t)\|_{L^1}\lesssim \|z(t)\|_{W^{1,\infty}}\|v(t)\|_{\dot H^1}^2\,.$$

Gathering all these estimates gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{\dot H^1}^2+\nu\|v(t)\|_{\dot H^{2+\delta}}^2 \lesssim \|z(t)\|_{W^{2,4}}\|v(t)\|_{H^1}^2+\|z(t)\|_{H^{2+\delta}}^2\|v(t)\|_{H^1}\,. \tag{5.18}$$

Then, recalling that $z_\nu\in L^2([0,T],H^{2+\delta})$ and dividing (5.18) by $\|v(t)\|_{\dot H^1}^2$ (up to some omitted regularisation procedure) and integrating on $[0,t]$ implies that

$$\|v(t)\|_{H^1}\leqslant C(z,\|u_0\|_{H^1})+\int_0^t\|z(t')\|_{W^{2,4}}\|v(t')\|_{H^1}\,\mathrm{d}t'\,.$$

Since $z\in L^2_{\mathrm{loc}}(\mathbb{R}_+,W^{2,4})\subset L^1_{\mathrm{loc}}(\mathbb{R}_+,W^{2,4})$, the Grönwall lemma provides us with

$$\sup_{t\in[0,T)}\|v(t)\|_{H^1}<\infty\,.$$

Finally, the fact that $v\in L^2_{\mathrm{loc}}(\mathbb{R}_+,H^{2+\delta})$ now follows from integrating (5.18). □

3.3. Proof of Proposition 5.2. In this paragraph we prove Proposition 5.2 using that V_N is finite dimensional.

In the following we use the observation that $\mathbf{P}_{\leqslant N}u_{\nu,N}=u_{\nu,N}$. Since $\mathbf{P}_{\leqslant N}$ is self adjoint, Lemma 5.16 (i) implies the key cancellation:

$$(B_N(u_{\nu,N},u_{\nu,N}),u_{\nu,N})_{L^2}=0\,. \tag{5.19}$$

Proof of Proposition 5.2. We only prove existence and uniqueness of the $u_{\nu,N}$ as the other properties follow from arguments similar to the proof of Proposition 5.1. Lemma 5.22 gives the existence of a global linear solution $z_{\nu,N}\in V_N$, hence the proof boils down to proving that ($\mathrm{HVE}^3_{\nu,N}$) is globally well-posed in V_N.

We remark that L and B_N are locally Lipschitz functionals on the finite dimensional space V_N thus the Cauchy-Lipschitz provides us with a unique local solution $v_{\nu,N}$ to ($\mathrm{HVE}^3_{\nu,N}$) in $\mathcal{C}^0([0,T),V_N)$. Then we prove that the norm $\|u_{\nu,N}(t)\|_{L^2}$ does not blow-up as $t\to T$.

In order to do so, we use an energy estimate in L^2, the cancellation (5.19) and the Bernstein estimates (frequencies of functions on V_N are bounded by N) to get:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{\nu,N}\|_{L^2} &\lesssim \|\nabla v_{\nu,N}\|_{L^2}\|z_{\nu,N}\|_{L^\infty} + \|\nabla z_{\nu,N}\|_{L^\infty}\|v_{\nu,N}\|_{L^2} + \|\nabla z_{\nu,N}\|_{L^4}\|z_{\nu,N}\|_{L^4}\\ &\lesssim N\|z\|_{W^{1,\infty}}\|v(t)\|_{L^2} + \|z_{\nu,N}\|^2_{W^{1,4}}\,.\end{aligned}$$

Then the Grönwall lemma and the estimates on $z_{\nu,N}$ of Lemma 5.22 give that for any $T>0$ $\|v\|_{L^\infty([0,T),L^2)}<\infty$ thus the solutions are global in V_N. □

Remark 5.26. We emphasis that this global well-posedness theory is not uniform in N.

4. Construction of invariant measures

We study the invariance properties of the solutions constructed in Section 3 by Proposition 5.1. In order to do so, we remark that the processes constructed by Proposition 5.1 enjoy a Markovian structure. We explain how in the dimension 2 setting: enlarge the probability set defining $\tilde{\Omega} := H^1_{\mathrm{div}}\times\Omega$ endowed with the σ-algebra $\mathcal{B}(H^1)\otimes\mathcal{F}$ and the filtration $\tilde{\mathcal{F}}_t := \mathcal{B}(H^1)\otimes\mathcal{G}_t$ and denote $\tilde{\omega}=(v,\omega)$ for elements in $\tilde{\Omega}$. We let $\tilde{u}^{\tilde{\omega}}(t) := u^\omega_\nu(t,v)$, standing for the solution constructed by Proposition 5.1 with initial data $u_0=v\in H^1$. Let $\mathbb{P}_v := \delta_v\otimes\mathbb{P}$. Then $(u_\nu(t),\mathbb{P}_v)$ forms a Markovian system, as it satisfies the *Markov property*:

$$\mathbb{P}_v(u_\nu(t+s)\in\Gamma|\mathcal{F}_s) = \mathbb{P}_{u_\nu(s)}(u_\nu(t)\in\Gamma)\,,$$

which comes from Proposition 5.1, (iv). Let us remark that if U_t denotes the restriction at time t of the map U of Proposition 5.1, then $\mathcal{L}(u_\nu(t)) = (U_t)_*(\delta_v\otimes m_{\zeta,T})$, where $m_{\zeta,T}=\mathcal{L}(\zeta_{|[0,T]})$. In particular we have $\mathcal{L}(u_0)=(U_0)_*\mathcal{L}(u_\nu)$.

Similarly, in dimension 3 the solutions $u_{\nu,N}$ to ($\mathrm{HVE}^3_{\nu,N}$) admit a Markovian structure $(u_{\nu,N},\mathbb{P}_v)$ for $v\in V_N$.

4.1. Existence of an invariant measure. In order to construct an invariant measure for (HVE^2_ν) and ($\mathrm{HVE}^3_{\nu,N}$) we follow the strategy in [KS15] which consists in applying the Krylov-Bogolioubov argument, which is recalled below.

Theorem 5.27 (Krylov–Bogoliubov, [Dud02])**.** *Let X be a Polish space. If there exists a point $x\in X$ for which the family of probability measures $\{P_t(x,\cdot), t>0\}$ is uniformly tight and the semigroup $(P_t)_{t>0}$ satisfies the Feller property and has pathwise continuous trajectories, then there exists at least one stationary measure for $(P_t)_{t>0}$, i.e., a probability measure μ on X such that $(P_t)_*\mu=\mu$ for all $t>0$.*

In this paragraph we omit the reference to ν and denote by u (resp. u_N) the solutions u_ν (resp. $u_{\nu,N}$).

Let us introduce some more notation, let us denote $P_t(u,\Gamma) := \mathbb{P}_v(u(t)\in\Gamma)$, and define the semi-group $\mathfrak{B}_t : L^\infty \to L^\infty$ defined, for any $f\in L^\infty$ by $\mathfrak{B}_t(f) = z \mapsto \int_{H^1} f(z)P_t(v,\mathrm{d}z)$.

We define the dual $\mathfrak{B}_t^* : \mathcal{P}(H^1)\to\mathcal{P}(H^1)$ by $\mathfrak{B}_t^*(\mu) := \Gamma \mapsto \int_{H^1} P_t(v,\Gamma)\mu(\mathrm{d}v)$, and observe that $\mathcal{L}(u(t)) = \mathfrak{B}_t^*(\mathcal{L}(u_0))$.

We start with some higher Sobolev estimates for processes such that $\mathcal{L}(u_0) = \delta_0$.

Lemma 5.28 (Dimension 2)**.** *Assume that $\mathcal{L}(u_0) = \delta_0$ and let u_ν being the corresponding process produced by Proposition 5.1. Then there exists a constant $C>0$ independent of ν such that for any $t>0$,*

$$\mathbb{E}\left[\int_0^t \|u_\nu(t')\|^2_{\dot H^{2+\delta}}\,\mathrm{d}t'\right] \leqslant Ct\,.$$

Proof. We apply the Itô formula (see Proposition 5.12 and the subsequent discussion) to the functional $F(u) := \|u\|^2_{\dot H^1}$ and we find that for all $t\in\mathbb{R}$,

$$\mathbb{E}\left[\|u(t)\|^2_{\dot H^1}\right] - \mathbb{E}\left[\|u_0\|^2_{\dot H^1}\right] + 2\nu\mathbb{E}\left[\int_0^t \|u(t')\|^2_{\dot H^{2+\delta}}\,\mathrm{d}t'\right] = \nu\mathcal{B}_1 t\,.$$

Then the result follows from $\mathbb{E}[\|u_0\|^2_{\dot H^1}] = 0$, since $\mathcal{L}(u_0) = \delta_0$. □

Lemma 5.29 (Dimension 3)**.** *Assume that $\mathcal{L}(u_0) = \delta_0$ and let $u_{\nu,N}$ being the corresponding process produced by Proposition 5.2. Then there exists a constant $C>0$ independent of ν, such that for any $t>0$,*

$$\mathbb{E}\left[\int_0^t \|u_{\nu,N}(t')\|^2_{\dot H^{1+\delta}}\,\mathrm{d}t'\right] \leqslant Ct\,.$$

Proof. The proof is similar to that of Lemma 5.28 except that we use the Itô formula based on the L^2 cancellation of the bilinear form and obtain:

$$\mathbb{E}\left[\|u(t)\|^2_{L^2}\right] - \mathbb{E}\left[\|u_0\|^2_{L^2}\right] + 2\nu\mathbb{E}\left[\int_0^t \|u(t')\|^2_{\dot H^{1+\delta}}\,\mathrm{d}t'\right] = \nu\mathcal{B}_0^{(N)} t\,,$$

and conclude as before. We also refer to Proposition 5.12 and the subsequent discussion for the application of the Itô formula. □

We can now state the main results of the section.

Proposition 5.3 (Dimension 2)**.** *The Markov system $(u_\nu(t),\mathbb{P}_v)_{v\in H^1}$ admits a stationary measure. Moreover, for any stationary measure $\mu_\nu\in\mathcal{P}(H^1)$, the following properties hold.*

(i) $\mathbb{E}_{\mu_\nu}\left[\|u\|^2_{\dot H^{1+\delta}}\right] = \frac{\mathcal{B}_0}{2}$.

(ii) $\mathbb{E}_{\mu_\nu}\left[\|u\|^2_{\dot H^{2+\delta}}\right] = \frac{\mathcal{B}_1}{2}$.

(iii) There exists $\gamma > 0$, and $C < \infty$, only depending on the noise parameters $(\phi_n)_{n\in\mathbb{Z}^2_0}$ such that one has $\mathbb{E}_{\mu_\nu}\left[e^{\gamma\|u\|^2_{\dot H^1}}\right] \leqslant C$.

Proof. Let us denote by u the solution starting at u_0 with law $\mathcal{L}(u_0) = \delta_0$, and let $\lambda_t := \mathfrak{B}_t^*\delta_0$. We introduce $\bar\lambda_t := \frac{1}{t}\int_0^t \lambda_{t'}\,\mathrm{d}t'$. In order to apply the Krylov-Bogolioubov theorem we need to show that the family $(\bar\lambda_t)_{t>0}$ is tight in H^1. Since the embedding $H^{2+\delta} \hookrightarrow H^1$ is compact it is sufficient to prove that

$$\sup_{t>0} \bar\lambda_t\left(H^1 \setminus B_{H^{2+\delta}}(0,R)\right) \underset{R\to\infty}{\longrightarrow} 0\,.$$

Observe that thanks to Lemma 5.28, we have

$$\begin{aligned}\bar\lambda_t\left(H^1 \setminus B_{H^{2+\delta}}(0,R)\right) &\leqslant \frac{1}{t}\int_0^t \mathbb{P}(\|u_\nu(t')\|_{H^{2+\delta}} > R)\,\mathrm{d}t' \\ &\leqslant \frac{1}{tR^2}\int_0^t \mathbb{E}\left[\|u(t')\|^2_{H^{2+\delta}}\right] \\ &\leqslant \frac{C}{R^2}\,,\end{aligned}$$

which goes to zero uniformly in $t > 0$. Then the Krylov-Bogolioubov theorem ensures the existence of stationary measures.

Let μ_ν be such a stationary measure and let us prove the required estimates.

(i) and (ii) are proven using the same argument. Let us only prove (ii). The Itô formula just as in the proof of Lemma 5.28: we take $u_\nu(t)$ a process solving (HVE$^2_\nu$) with stationary measure μ_ν. We can write, for any $t \geqslant 0$, thanks that the Itô formula:

$$\mathbb{E}\left[\|u_\nu(t)\|^2_{\dot H^1}\right] - \mathbb{E}\left[\|u_0\|^2_{\dot H^1}\right] + 2\nu\mathbb{E}\left[\int_0^t \|u_\nu(t')\|^2_{\dot H^{2+\delta}}\,\mathrm{d}t'\right] = \nu\mathcal{B}_1 t\,,$$

for any $t > 0$. Using invariance, we obtain

$$\int_0^t \left(2\mathbb{E}\left[\|u_\nu(t)\|^2_{\dot H^{2+\delta}}\right] - \mathcal{B}_1\right) = 0$$

for all $t \geqslant 0$ so that $\mathbb{E}\left[\|u_\nu(t)\|^2_{\dot H^{2+\delta}}\right] = \frac{\mathcal{B}_1}{2}$, for almost any $t > 0$. However by invariance again (and since $u \mapsto \|u\|^2_{H^{2+\delta}}$ is a Borelian function of H^1, see Lemma 5.45) we know that the quantity $\mathbb{E}[\|u(t)\|^2_{\dot H^{2+\delta}}]$ is time-invariant, thus finite for *all* t, thus identically equal to $\mathcal{B}_1$.

The same argument applied to $\|u_\nu(t)\|^2_{L^2}$ instead of $\|u_\nu(t)\|^2_{\dot H^1}$ applies in order to prove (i), as the main observation being the cancellation $(B(u,u),u)_{L^2} = 0$.

(iii) also comes from the Itô formula and we refer to Appendix A for more details of the computations. The Itô formula applied to the functional defined by $G(u) = e^{\gamma\|u\|^2_{\dot H^1}}$ yields that for any $\gamma > 0$ and any $t > 0$:

$$\mathbb{E}\left[e^{\gamma\|u_\nu(t)\|^2_{\dot H^1}}\right] = \mathbb{E}\left[e^{\gamma\|u_0\|^2_{\dot H^1}}\right]$$
$$+ 2\gamma\nu\mathbb{E}\left[\int_0^t e^{\gamma\|u_\nu(t')\|^2_{\dot H^1}}\left(\frac{\mathcal{B}_1}{2} - \|u_\nu(t')\|^2_{\dot H^{2+\delta}} + \gamma\sum_{n\in\mathbb{Z}^2}|n|^2|\phi_n|^2|u_n(t')|^2\right)\mathrm{d}t'\right],$$

where $u_n(t) = (u_\nu(t), e_n)_{L^2}$.

Since $\max\{|\phi_n^2|, n\in\mathbb{Z}^2\} < \infty$, choosing γ such that $\gamma\sup|\phi_n|^2 \simeq \frac{1}{2}$ leads to the inequality

$$\mathbb{E}\left[e^{\gamma\|u_\nu(t)\|^2_{\dot H^1}}\right] - \mathbb{E}\left[e^{\gamma\|u_0\|^2_{\dot H^1}}\right] \leqslant C\nu\mathbb{E}\left[\int_0^t A(t')\,\mathrm{d}t'\right],$$

where $A(t') := e^{\gamma\|u_\nu(t')\|^2_{\dot H^1}}\left(C - \|u_\nu(t')\|^2_{\dot H^{2+\delta}}\right)$ and $C > 0$ does not depend on ν.

Remark that if $\|u_\nu(t)\|^2_{\dot H^{2+\delta}} > 2C$ then

$$e^{\gamma\|u_\nu(t')\|^2_{\dot H^1}}\left(2C - \|u_\nu(t')\|^2_{\dot H^{2+\delta}}\right) \leqslant 0,$$

and if $\|u_\nu(t)\|^2_{\dot H^{2+\delta}} \leqslant 2C$ then

$$e^{\gamma\|u_\nu(t')\|^2_{\dot H^1}}\left(2C - \|u_\nu(t')\|^2_{\dot H^{2+\delta}}\right) \leqslant 2Ce^{2\gamma C},$$

so that in any case

$$e^{\gamma\|u_\nu(t')\|^2_{\dot H^1}}\left(2C - \|u_\nu(t')\|^2_{\dot H^{2+\delta}}\right) \leqslant \underbrace{2Ce^{2\gamma C}}_{C_1},$$

which implies

$$A(t') \leqslant C_1 - Ce^{\gamma\|u_\nu(t')\|^2_{\dot H^1}},$$

and therefore introducing $Y(t) := \mathbb{E}\left[e^{\gamma\|u_\nu(t)\|^2_{\dot H^1}}\right]$, we have obtained

$$Y(t) + C\nu\int_0^t Y(t')\,\mathrm{d}t' \leqslant Y(0) + C_1\nu t,$$

where C and C_1 do not depend on ν and the Grönwall lemma gives

$$Y(t) \leqslant e^{-C\nu t}Y(0) + C_2, \tag{5.20}$$

where C_2 is a function of C and C_1, and provided $Y(0) < \infty$. Then recalling that invariance implies $Y(t) = Y(0)$ and letting $t\to\infty$ this yields $Y(0) \leqslant C_2$, a finite constant independent of ν.

In the general case, where we do not assume $Y(0) < 0$ we consider the random time $\tau_N = \min\{t \geqslant 0, \|u(t)\|_{\dot H^1} > N\}$ and $Y_N(t) := Y(\min\{t, \tau_N\})$, for which we can write (see Theorem 7.7.5 in [KS15]):

$$\mathbb{E}\left[Y_N(t)\right] - \mathbb{E}\left[Y_N(0)\right] \leqslant C\nu\mathbb{E}\left[\int_0^{\min\{t,\tau_N\}} A(t')\,\mathrm{d}t'\right],$$

and the previous analysis provides us with

$$Y_N(t) + C\nu\mathbb{E}\left[\int_0^{\min\{t,\tau_N\}} A(t')\,\mathrm{d}t'\right] \leqslant Y_N(0) + C_1\nu t\,,$$

and by finiteness of $Y_N(0) = Y_N(t)$ we simplify this as this as

$$\mathbb{E}\left[\int_0^{\min\{t,\tau_N\}} A(t')\,\mathrm{d}t'\right] \leqslant C_2 t\,,$$

and by Fatou's theorem, letting $N \to \infty$ leads to

$$\mathbb{E}\left[\int_0^{t} A(t')\,\mathrm{d}t'\right] \leqslant C_2 t\,,$$

which is $tY(0) \leqslant C_2 t$, *i.e.*, $Y(0) \leqslant C_2$. □

The counterpart for the three-dimensional case is the following.

Proposition 5.4. *The Markov system $(u_{\nu,N}(t), \mathbb{P}_v)_{v\in L^2}$ admits a stationary measure. Moreover, for any stationary measure $\mu_{\nu,N} \in \mathcal{P}(L^2)$ the following properties hold.*

(i) There holds $\mathbb{E}_{\mu_{\nu,N}}\left[\|u\|^2_{\dot{H}^{1+\delta}}\right] = \frac{\mathcal{B}_0^{(N)}}{2} \leqslant \frac{\mathcal{B}_0}{2}$.

(ii) There exists $\gamma > 0$, and $C < \infty$, only depending on the noise parameters $(\phi_n)_{n\in\mathbb{Z}^3}$ such that one has $\mathbb{E}_{\mu_{\nu,N}}\left[e^{\gamma\|u\|^2_{L^2}}\right] \leqslant C$.

Proof. The proof of similar to that of Proposition 5.3 thus omitted. We only recall that the main point of all these uniform estimates, especially for (ii) is, again, the Itô formula:

$$\mathbb{E}\left[\|u_{\nu,N}(t)\|^2_{L^2}\right] + \nu\int_0^t \left(2\mathbb{E}\left[\|u_{\nu,N}(t')\|^2_{\dot{H}^{1+\delta}}\right] - \mathcal{B}_0^{(N)}\right)\mathrm{d}t' = \mathbb{E}\left[\|\mathbf{P}_{\leqslant N}u_0\|^2_{L^2}\right]\,. \;\square$$

4.2. Tightness and limit. Once and for all we fix some stationary measures μ_ν (resp. $\mu_{\nu,N}$). Our next task is to pass to the limit $\nu \to 0$. In order to do so, we need to prove several compactness estimates. We denote by $\bar{\mu}_\nu$ (resp. $\bar{\mu}_{\nu,N}$) the law of $u_\nu(\cdot)$ (resp. $u_{\nu,N}(\cdot)$).

We will use compactness arguments based on the Aubin-Lions-Simon creterion, which we recall.

Theorem 5.30 (Aubin-Lions-Simon compactness theorem)**.** *Let I be a compact interval. Let B_0, B, B_1 be three separable complete spaces. Assume that $B_0 \hookrightarrow B \hookrightarrow B_1$ continuously, the embedding $B_0 \hookrightarrow B$ being compact.*

(i) Let $W := \{u \in L^p(I, B_0)$ and $\partial_t u \in L^q(I, B_1)\}$. Then for $p < \infty$, W is compactly embedded into $L^p(I, B)$. If $p = \infty$ and $q > 1$ then W is compactly embedded into $\mathcal{C}^0(I, B)$.

(ii) *Let* $s_0, s_1 \in \mathbb{R}$ *and* $1 \leqslant r_0, r_1 \leqslant \infty$. *Let* $W := W^{s_0,r_0}(I, B_0) \cap W^{s_1,r_1}(I, B_1)$. *Assume that there exists* $\theta \in (0,1)$ *such that for all* $v \in B_0 \cap B_1$ *one has*

$$\|v\|_B \lesssim \|v\|_{B_0}^{1-\theta} \|v\|_{B_1}^{\theta} .$$

Let

$$s_\theta := (1-\theta)s_0 + \theta s_1 \text{ and } \frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} ,$$

then if $s_\theta > \frac{1}{r_\theta}$, W *is compactly embedded into* $\mathcal{C}^0(I, B)$.

Assertion (i) is the classical Aubin-Lions theorem, and assertion (ii) is the content of Corollary 9 in [Sim87].

4.2.1. *Case* $d = 2$. Proposition 5.3 already asserts that $\bar{\mu}_\nu(L^2_{\text{loc}}(\mathbb{R}_+, H^{2+\delta}) \cap \mathcal{C}^0(\mathbb{R}_+, H^1)) = 1$.

We will prove compactness estimates on $\bar{\mu}_\nu$ rather than simply on μ_ν, because not only do we want to make the measures μ_ν converge to some measure μ, but we want the processes u_ν to converge to a process u solving the Euler equations.

We want to prove that the family of measures $(\bar{\mu}_\nu)_{\nu>0}$ is tight in

$$Y := \mathcal{C}^0(\mathbb{R}_+, H^{1-\varepsilon}) \cap L^2_{\text{loc}}(\mathbb{R}_+, H^{2+\delta-\varepsilon}) ,$$

for a fixed $\varepsilon > 0$ that we may take arbitrarily small. It is sufficient to prove the tightness on every time interval $I_n = [0, n]$, namely that for all $n \geqslant 1$, $(\bar{\mu}_\nu)_{\nu>0}$ is tight in

$$Y_n := \mathcal{C}^0(I_n, H^{1-\varepsilon}) \cap L^2_{\text{loc}}(I_n, H^{2+\delta-\varepsilon}) .$$

Since the time interval does not play any specific role, we will prove tightness in

$$Y_1 := \mathcal{C}^0(I, H^{1-\varepsilon}) \cap L^2_{\text{loc}}(I, H^{2+\delta-\varepsilon}) ,$$

where $I = [0, 1]$.

In order to do so, we introduce the space

$$\begin{aligned} X &:= L^2(I, H^{2+\delta}) \cap \left(H^1(I, H^{-\delta}) + W^{\frac{3}{8},4}(I, H^{1-2\varepsilon}) \right) \\ &= \underbrace{L^2(I, H^{2+\delta}) \cap H^1(I, H^{-\delta})}_{X_1} + \underbrace{L^2(I, H^{2+\delta}) \cap W^{\frac{3}{8},4}(I, H^{1-2\varepsilon})}_{X_2} . \end{aligned}$$

We will prove that following lemmata.

Lemma 5.31. *Both spaces* X_1 *and* X_2 *are compactly embedded into* Y_1.

Lemma 5.32. *The sequence* $(u_\nu)_{\nu>0}$ *is bounded in* $L^2(\Omega, X)$.

These results then imply that the family $(\bar{\mu}_\nu)_{\nu>0}$ is tight in Y.

Proof of Lemma 5.31. Since $H^{2+\delta-\varepsilon} \hookrightarrow H^{2+\delta}$ compactly, we have compact embeddings $X_1 \hookrightarrow L^2(I, H^{2+\delta})$ and $X_2 \hookrightarrow L^2(I, H^{2+\delta})$ thanks to Theorem 5.30, (i).

The compactness of the embedding $X_1 \hookrightarrow \mathcal{C}^0(I, H^{1-\varepsilon})$ follows from Theorem 5.30, (ii). Indeed, take $s_0 = 0$, $s_1 = 1$, $r_0 = r_1 = 2$ and observe that $1-\varepsilon = (1-\theta)(2+\delta-\varepsilon) + \theta(-\delta)$ with $\theta > \frac{1}{2}$, so that $s_\theta > \frac{1}{r_\theta}$.

The embedding $X_2 \hookrightarrow \mathcal{C}^0(I, H^{1-\varepsilon})$ is compact thanks to Theorem 5.30, (ii). This time take $s_0 = 0$, $s_1 = \frac{3}{8}$, $r_0 = 2$ and $r_1 = 4$. We can compute that $s_\theta - \frac{1}{r_\theta} = \frac{5\theta}{8} > 0$, since $\theta = \frac{1+\delta-\varepsilon}{1+\delta+\varepsilon}$ is arbitrarily close to 1. □

Proof of Lemma 5.32. We recall that $\|u\|_X = \inf\{\|u_1\|_{X_1} + \|u_2\|_{X_2}\}$ where the infimum runs over decompositions $u = u_1 + u_2$ with $u_1 \in X_1$, and $u_2 \in X_2$. We write

$$u_\nu(t) = u_\nu(0) + \int_0^t B(u_\nu(t'), u_\nu(t'))\,\mathrm{d}t' - \nu\int_0^t Lu_\nu(t')\,\mathrm{d}t' + \sqrt{\nu}\zeta(t)\,.$$

Then we set $u_\nu^{(1)} := u_\nu(0) + \displaystyle\int_0^t B(u_{\nu(t')}, u_\nu(t'))\,\mathrm{d}t' - \nu\int_0^t Lu_\nu(t')\,\mathrm{d}t'$ and $u_\nu^{(2)}(t) = \sqrt{\nu}\zeta(t)$.

The bounds in $L^2(\Omega, L^2([0,1], H^{2+\delta}))$ follow from Proposition 5.3, (ii). For the other bounds we proceed as follows.

We observe that $\partial_t u_\nu^{(1)}(t) = B(u_\nu(t), u_\nu(t)) + Lu_\nu(t)$ thus u_ν is bounded in $L^2(\Omega, L^2(I, H^{-\delta}))$ thanks to Proposition 5.3, (ii).

It remains to prove that $u_\nu^{(2)}$ is bounded un $L^2(\Omega, W^{\frac{3}{8},4}([0,1], H^{1-2\varepsilon}))$, which in facts will result from proving that $\zeta \in L^2(\Omega, W^{\frac{3}{8},4}([0,1], H^{1-2\varepsilon})$. We compute:

$$\mathbb{E}[\|\zeta(t)\|^4_{H^{1-2\varepsilon}}] = \mathbb{E}\left[\left(\sum_{n\in\mathbb{Z}^2_0}\langle n\rangle^{2(1-2\varepsilon)}|\phi_n|^2\beta_n(t)^2\right)^2\right] \leqslant \sum_{n\in\mathbb{Z}^2_0}\langle n\rangle^4|\phi_n|^4\mathbb{E}[\beta_n(t)^4]\,.$$

Now recall that by the properties of Gaussian variables, $\mathbb{E}[\beta_n(t)^4] = 3t^2 \leqslant 3$ on the time-inverval I so that finally $\mathbb{E}[\|\zeta\|^4_{L^4(I,H^1)}] < \infty$, since ϕ is a *standard noise*. By the same techniques we obtain

$$\mathbb{E}\left[\int_{[0,1]\times[0,1]}\frac{\|\zeta(t)-\zeta(r)\|^4_{H^1}}{|t-r|^{1+4\alpha}}\,\mathrm{d}r\,\mathrm{d}t\right] < \infty\,,$$

and combined with the previous estimate we finally have $\mathbb{E}\left[\|u_\nu^{(2)}\|^2_{W^{\frac{3}{8},4}}\right] \lesssim \nu$. □

4.2.2. *Case $d = 3$.* In this setting, we can prove the following.

Proposition 5.5. *The sequence $(\bar{\mu}_{\nu,N})_{\nu>0}$ is tight in $\mathcal{C}^0(\mathbb{R}_+, V_N)$.*

Proof. We use the same argument as above: we can prove that $(u_{\nu,N})_{\nu>0}$ is bounded in $L^2(\Omega, X)$ with

$$X \coloneqq L^2(I, H^{1+\delta}) \cap \left(H^1(I, H^{-(1+\delta)}) + W^{\frac{3}{8},4}(I, H^1)\right) ,$$

which is compactly embedded into $\mathcal{C}^0(I, H^{-\varepsilon})$ for some $\varepsilon < 0$. Then, since on V_N we have $H^{-\varepsilon} = L^2$ by finite dimensionality, this gives the result. $\square$

4.3. Passing to the limit. Thanks to the previous subsection, we can assume that the sequences $\bar{\mu}_\nu$ and $\bar{\mu}_{\nu,N}$ converge as $\nu \to 0$. Indeed we have:

- (Case $d = 2$) The Prokhorov theorem (see Theorem 11.5.4 in [Dud02]), there exists a sequence $\nu_j \to 0$ and $\bar{\mu} \in \mathcal{P}\left(\mathcal{C}^0(\mathbb{R}, H^{1-\varepsilon}) \cap L^{2+\delta-\varepsilon}\right)$ such that $\bar{\mu}_{\nu_j} \to \bar{\mu}$ in law. This means that $\mathbb{E}_{\bar{\mu}_{\nu_j}}[f] \to \mathbb{E}_{\bar{\mu}}[f]$ for any bounded Lipschitz function.

  Note that the measures $(\mu_{\nu_j})_{j\geqslant 0}$ also satisfy Prokhorov's theorem in $H^{2+\delta-\varepsilon}$, due to the uniform estimates in $H^{2+\delta}$, hence there is no loss of generality in assuming that $\mu_{\nu_j} \to \mu$ weakly in $\mathcal{P}(H^{2+\delta})$, where μ denotes the restriction at time $t = 0$ of $\bar{\mu}$.
- (Case $d = 3$) Similarly by the Prokhorov theorem and the tightness from Proposition 5.32 we may assume that $\bar{\mu}_{\nu,N} \to \bar{\mu}_N$ weakly in $\mathcal{C}^0(\mathbb{R}_+, V_N)$ and $\mu_{\nu,N} \to \mu_N$ weakly in V_N.

By the Skorokhod theorem (see [Dud02], Theorem 11.7.2) there exists a sequence of $\bar{\mu}_\nu$-stationary processes $\tilde{u}_\nu$ (resp. a sequence of $\bar{\mu}_{\nu,N}$-stationary processes $\tilde{u}_{\nu,N}$) and an Y-valued process u defined on the same probability space (resp. u_N), such that

$$\begin{cases} \mathcal{L}(\tilde{u}_{\nu_j}) = \mu_{\nu_j} \text{ and } \mathcal{L}(u) = \mu \\ \mathbb{P}\left(\tilde{u}_{\nu_j} \to u \text{ in } Y \text{ as } j \to \infty\right) = 1 \,, \end{cases}$$

where $Y = \mathcal{C}^0(\mathbb{R}, H^{1-\varepsilon}) \cap L^{2+\delta-\varepsilon}$; respectively,

$$\begin{cases} \mathcal{L}(\tilde{u}_{\nu_j,N}) = \mu_{\nu_j,N} \text{ and } \mathcal{L}(u_N) = \mu_N \\ \mathbb{P}\left(\tilde{u}_{\nu_j,N} \to u_N \text{ in } \mathcal{C}^0(\mathbb{R}_+, V_N) \text{ as } j \to \infty\right) = 1 \,. \end{cases}$$

We can further assume that the $\tilde{u}_{\nu_j}$ satisfy the same equation as u_{ν_j}. More precisely, the u_{ν_j} satisfy an equation of the general form $F(u_{\nu_j}) = \sqrt{\nu}\zeta$, where F is a deterministic measurable functional and ζ is the random variable defined by (5.6). Since $\mathcal{L}(u_{\nu_j}) = \mathcal{L}(\tilde{u}_{\nu_j})$, then $\mathcal{L}(F(\tilde{u}_{\nu_j})) = \mathcal{L}(F(u_{\nu_j})) = \mathcal{L}(\sqrt{nu}\zeta)$ thus there exists a process $\tilde{\zeta}$ equal in law to $\tilde{\zeta}$ (and in particular share the same numbers ϕ_n, that is there exists independent brownian motions $\tilde{\beta}_n$ such that $\tilde{\zeta}(t) = \sum_{n\in\mathbb{Z}_0^2} \phi_n\tilde{\beta}_n(t)e_n)$ such that $F(\tilde{u}_{\nu_j}) = \sqrt{\nu}\tilde{\zeta}$. In the rest of this section we will drop the $\tilde{}$ and write u_{ν_j}, ζ.

We are ready to state the main results of this section.

Proposition 5.6 (Dimension 2)**.** *Let $\bar{\mu}, \mu$ and u as defined above. Then we have,*

(i) $\bar{\mu}(L^2_{\mathrm{loc}}(\mathbb{R}_+, H^{2+\delta})) = 1$ *and* $\mu(H^{2+\delta}) = 1$.

(ii) $\mathbb{E}_\mu \left[\|u\|^2_{\dot{H}^{2+\delta}}\right] \leqslant \mathcal{B}_1$ *and* $\mathbb{E}_\mu \left[e^{\gamma\|u\|^2_{\dot{H}^1}}\right] < \infty$.

(iii) The process u is stationary for the measure μ and satisfies the Euler equation (E^2).

(iv) The constant C defined by $\mathbb{E}_\mu \left[\|u\|^2_{\dot{H}^1}\right] = C$ *satisfies:*

$$C_\delta \coloneqq \frac{1}{2}\left(\frac{\mathcal{B}_0}{\mathcal{B}_1^{\frac{\delta}{1+\delta}}}\right)^{1+\delta} \leqslant C \leqslant \frac{\mathcal{B}_0}{2}\,. \tag{5.21}$$

Remark 5.33. Similarly, one can prove that there exists a constant $C'_\delta > 0$ such that $\mathbb{E}_\mu[\|u\|^2_{L^2}] \geqslant C'_\delta > 0$.

Proof. (i) Let $T > 0$. Thanks to Proposition 5.3 we have $\mathbb{E}_{\mu_\nu}[\|u\|^2_{L^2([0,T],H^{2+\delta})}] \leqslant C$ independently of ν. Note that we also have, uniformly in ν, M and K:

$$\mathbb{E}_{\mu_{\nu_j}}\left[\min\left\{\|\mathbf{P}_{\leqslant M}u\|^2_{L^2([0,T],H^{2+\delta})}, K\right\}\right] \leqslant C\,.$$

Since the map map $u \mapsto \min\left\{\|\mathbf{P}_{\leqslant M}u\|^2_{H^{2+\delta}}, K\right\}$ is continuous and bounded as a map $H^{2+\delta-\varepsilon} \to \mathbb{R}$ we can pass to the limit $j \to \infty$ by weak convergence of the measures and get

$$\mathbb{E}_{\mu}\left[\min\left\{\|\mathbf{P}_{\leqslant M}u\|^2_{L^2([0,T],H^{2+\delta})}, K\right\}\right] \leqslant C\,,$$

and conclude by monotone convergence as $K \to \infty$ and then $M \to \infty$.

(ii) follows by a similar argument.

(iv) First, we claim that for any j the following uniform inequalities hold:

$$C_\delta \leqslant \mathbb{E}_{\mu_{\nu_j}}\left[\|u\|^2_{\dot{H}^1}\right] \leqslant \frac{\mathcal{B}_0}{2} \tag{5.22}$$

In order to prove (5.22), we observe that the right-hand side inequality is contained in Proposition 5.3 (i). For the lower bound, we start by using the interpolation inequality

$$\|u\|^2_{\dot{H}^{1+\delta}} \leqslant \|u\|^{\frac{2}{1+\delta}}_{\dot{H}^1}\|u\|^{\frac{2\delta}{1+\delta}}_{\dot{H}^{2+\delta}}\,,$$

and by the Hölder inequality we infer

$$\mathbb{E}_{\mu_{\nu_j}}[\|u\|^2_{\dot{H}^{1+\delta}}] \leqslant \mathbb{E}_{\mu_{\nu_j}}\left[\|u\|^2_{\dot{H}^1}\right]^{\frac{1}{1+\delta}}\mathbb{E}_{\mu_{\nu_j}}\left[\|u\|^2_{\dot{H}^{2+\delta}}\right]^{\frac{\delta}{1+\delta}}\,.$$

The proof of the claim now follows from Proposition 5.3 (i) and (ii).

We claim that there exists a constant $C > 0$ such that

$$\limsup_{j\geqslant 0} \int_{\|u\|_{\dot H^1}>R} \|u\|_{\dot H^1}^2 \,\mu_{\nu_j}(\mathrm{d}u) \leqslant \frac{C}{R}\,. \tag{5.23}$$

Assuming this claim we can finish the proof of (iii). To this end, let χ be a smooth bump function, such that $\chi = 1$ on $[-1,1]$, vanishing outside $[-2,2]$ and let $\chi_R = \chi(\cdot/R)$. Let $\varepsilon > 0$ and $R > 0$ be fixed. Then by (5.23) there exists $J = J(R) > 0$ large enough such that for any $j \geqslant J$ there holds

$$\int_{\|u\|_{\dot H^1}\leqslant R} \|u\|_{\dot H^1}^2 \,\mu_{\nu_j}(\mathrm{d}u) \geqslant C_\delta - \varepsilon\,.$$

Now observe that $\chi_{\frac{R}{2}}(\|u\|_{\dot H^1}) \leqslant \mathbf{1}_{\|u\|_{\dot H^1}>R}$ hence

$$\int \chi_{\frac{R}{2}}(\|u\|_{\dot H^1})\|u\|_{\dot H^1}^2 \,\mu_{\nu_j}(\mathrm{d}u) \geqslant C_\delta - \varepsilon\,.$$

Then passing to the limit $j \to \infty$, and because $u \mapsto \chi_{\frac{R}{2}}(\|u\|_{\dot H^1})\|u\|_{\dot H^1}^2$ is a continuous bounded map $\dot H^1 \to \mathbb{R}$, we obtain

$$\int_{\|u\|_{\dot H^1}\leqslant R} \|u\|_{\dot H^1}^2 \,\mu(\mathrm{d}u) \geqslant C_\delta - \varepsilon\,,$$

and finally the monotone convergence theorem in the limit $R \to \infty$ proves (5.21).

It remains to establish (5.23). In order to do so, we remark that the Cauchy-Schwarz inequality followed by the Markov inequality yields:

$$\begin{aligned}\int_{\|u\|_{\dot H^1}>R} \|u\|_{\dot H^1}^2 \,\mu_{\nu_j}(\mathrm{d}u) &\leqslant \mu_{\nu_j}(\|u\|_{\dot H^1} > R)^{\frac12}\left(\int_{\|u\|_{\dot H^1}>R} \|u\|_{\dot H^1}^4 \,\mu_{\nu_j}(\mathrm{d}u)\right)^{\frac12}\\ &\leqslant \frac{1}{R}\mathbb{E}_{\mu_{\nu_j}}\left[\|u\|_{\dot H^1}^2\right]^{\frac12}\mathbb{E}_{\mu_{\nu_j}}\left[\|u\|_{\dot H^1}^4\right]^{\frac12}\,.\end{aligned}$$

Now remark that there is a constant $C = C(\gamma) > 0$ such that for any real numbers x there holds $x^4 \leqslant Ce^{\gamma x^2}$ so that we obtain

$$\begin{aligned}\int_{\|u\|_{\dot H^1}>R} \|u\|_{\dot H^1}^2 \,\mu_{\nu_j}(\mathrm{d}u) &\leqslant \frac{C}{R}\mathbb{E}_{\mu_{\nu_j}}\left[\|u\|_{\dot H^1}^2\right]^{\frac12}\mathbb{E}_{\mu_{\nu_j}}\left[e^{\gamma\|u\|_{\dot H^1}^2}\right]^{\frac12}\\ &\leqslant \frac{C}{R}\,,\end{aligned}$$

where in the last line we used the uniform estimates of Proposition 5.3 (iii). This establishes (5.23).

(iii) Invariance follows from the weak convergence of the measures. It remains to prove that u satisfies the Euler equation on any time-interval $[0,T]$. We start by writing that

$$u_\nu(t) = e^{-\nu tL}u_0 + \int_0^t e^{-\nu(t-t')L}\left(B(u_\nu(t'),u_\nu(t')) + \sqrt{\nu}\zeta(t')\right)\mathrm{d}t'\,. \tag{5.24}$$

As we want to pass to the limit in L^2, we use the previous uniform bounds. First, we have $e^{-t\nu L}u_0 \to u_0$ in $L^\infty((0,T),L^2)$ and also $u_\nu \to u$ in $L^\infty((0,T),L^2)$. Next, we use the triangle inequality to bound:

$$\sqrt{\nu}\left\|\int_0^t e^{-\nu(t-t')L}\zeta(t')\,\mathrm{d}t'\right\|_{L^2} \leqslant \sqrt{\nu}\int_0^t \|\zeta(t')\|_{L^2}\,\mathrm{d}t'\,,$$

We have seen in the previous subsection that $\mathbb{E}[\|\zeta\|_{L^4((0,T),H^{1-2\varepsilon}}] < \infty$, which implies in particular that $\sqrt{\nu}\zeta \to 0$ in probability, in the space $L^4((0,T),L^2)$, thus up to passing to a sub-sequence this yields

$$\mathbb{P}\left(\lim_{j\to\infty}\sqrt{\nu_j}\zeta = 0 \text{ in } L^4((0,T),L^2)\right) = 1\,,$$

and then for $t \in [0,T]$:

$$\sqrt{\nu_j}\left\|\int_0^t e^{-\nu_j(t-t')L}\zeta(t')\,\mathrm{d}t'\right\|_{L^2} \leqslant T^{3/4}\|\sqrt{\nu_j}\zeta\|_{L^4((0,T),L^2)} \underset{j\to\infty}{\longrightarrow} 0\,.$$

To bound the remaining term, we use Lemma 5.17 and write:

$$\begin{aligned}
\Big\|\int_0^t e^{-\nu(t-t')L}&\left(B(u_\nu(t'),u_\nu(t'))- B(u(t'),u(t'))\right)\mathrm{d}t'\Big\|_{L^2} \\
&\lesssim \int_0^T \frac{1}{\sqrt{t'}}\|B(u_\nu(t'),u_\nu(t')) - B(u(t'),u(t'))\|_{H^{-1}}\,\mathrm{d}t' \\
&\lesssim \|B(u_\nu,u_\nu) - B(u,u)\|_{L^\infty((0,T),H^{-1})} \\
&\lesssim \|u_\nu - u\|_{L^\infty((0,T),H^{\frac12})}\|u_\nu\|_{L^\infty((0,T),H^{\frac12})}\,.
\end{aligned}$$

Since $(u_{\nu_j})_{j\geqslant 0}$ is convergent to u (and therefore bounded) in $L^\infty((0,T),H^{1-\varepsilon})$we obtain that for any $t \in [0,T]$:

$$\left\|\int_0^t e^{-\nu_j(t-t')L}\left(B(u_{\nu_j}(t'),u_{\nu_j}(t')) - B(u(t'),u(t'))\right)\mathrm{d}t'\right\|_{L^2} \underset{j\to\infty}{\longrightarrow} 0\,.$$

Finally, the process u almost surely satisfies $u \in \mathcal{C}^0(\mathbb{R}_+,H^1_{\mathrm{div}}) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+,H^{2+\delta})$ and obeys:

$$u(t) = u_0 + \int_0^t \mathbf{P}\left(u(t')\cdot\nabla u(t')\right)\mathrm{d}t'\,,$$

where the equality holds in $\mathcal{C}^0(\mathbb{R}_+,L^2)$, hence u is a solution to the Euler equation (E). □

Again the proofs of this section may be adapted to the three dimensional cases. More precisely, we can prove the following.

Proposition 5.7 (Dimension $d=3$)**.** *Let $\bar\mu_N, \mu_N$ and u_N as defined above. We have the following properties.*

(i) $\mathbb{E}_{\mu_N}\left[e^{\gamma\|u\|^2_{L^2}}\right] \leqslant C$*, where C does not depend on N.*

(ii) $\mathbb{E}_{\mu_N}\left[\|u\|^2_{\dot H^{1+\delta}}\right] = \mathcal{B}_0^{(N)} \leqslant \mathcal{B}_0$*, where γ does not depend on N.*

(iii) The process u_N is stationary for the measure μ_N and satisfies the approximate Euler equation (E^3_N).

Proof. The proof of (i) and (ii) follow the same arguments as used in the proof of Proposition 5.6 and is based on the *a priori* estimates on $\mu_{\nu,N}$ uniform in $\nu > 0$, namely that of Proposition 5.4.

(iii) is simpler than in dimension 3, as differential operators are continuous on V_N. Therefore, since we may assume $u_{\nu,N} \underset{\nu\to 0}{\longrightarrow} u_N$ in $L^\infty_T L^2$ we also have $u_{\nu,N} \underset{\nu\to 0}{\longrightarrow} u_N$ in $L^\infty_T H^s$ for any $s \to 0$ and then the limiting argument becomes straightforward. $\square$

5. Proof of the main results

5.1. About the classical local well-posedness theory for the Euler equations. In dimension 2 and 3 we will use that the local well-posedness time only depends on the size of the initial data. In dimension 2 we have the following well-known result.

Lemma 5.34 (Diemsion 2 local well-posedness)**.** *Let $s > 2$. Then the Euler equation* (E^2) *is globally well-posed in $\mathcal{C}^0(\mathbb{R}_+, H^s(\mathbb{T}^2))$. Moreover, there exists $c > 0$ such that if $u_0 \in H^s(\mathbb{T}^2)$ and $\tau := \frac{c}{\|u_0\|_{H^s}}$, then there holds*

$$\|u(t)\|_{H^s} \leqslant 2\|u_0\|_{H^s}\,,$$

for any $t \in [0, \tau]$.

Proof. The local well-posedness statement follows from standard approximation arguments based on the following *a priori* estimate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{H^s} &\lesssim \|\nabla u(t)\|_{L^\infty}\|u(t)\|^2_{H^s}\\ &\lesssim \|u(t)\|^3_{H^s}\,,\end{aligned}$$

where we used the Sobolev embedding $H^{s-1} \hookrightarrow L^\infty$. Integrating this inequality, there exists $C > 0$ such that

$$\|u(t)\|_{H^s} \leqslant \frac{\|u_0\|_{H^s}}{1 - Ct\|u_0\|_{H^s}}\,,$$

as long as $1 - Ct\|u_0\|_{H^s} > 0$. Setting $\tau := \frac{1}{2C\|u_0\|_{H^s}}$ yields the result. $\square$

Finally the global well-posedness can be obtained using the blow-up criterion in terms of vorticity (of Beale-Kato-Majda type), which states that if we denote T^* the maximal existence time, then:

$$T^* < \infty \Longrightarrow \int_0^{T^*} \|\xi(t)\|_{L^\infty}\,\mathrm{d}t = \infty\,. \tag{5.25}$$

For a proof, see [BKM84]. Since the vorticity satisfies (5.1), we see that ξ is transported by the flow, thus its L^∞ norm is conserved, so that if we assume that $T^* < \infty$, then

$$\int_0^{T^*} \|\xi(t)\|_{L^\infty} \,\mathrm{d}t = T^* \|\xi_0\|_{L^\infty} < \infty$$

which is a contradiction, thus $T^* = \infty$.

In dimension 3 we know that (E^3_N) is globally well-posed (thanks to finite-dimensionality), but no global theory in H^s, $s > \frac{5}{2}$ is known for (E^3). However the local theory for both (E^3_N) and (E^3) is uniform in N. More precisely we have the following.

Lemma 5.35 (Local theory for (E^3))**.** *Let $s > \frac{5}{2}$ and $u_0 \in H^s(\mathbb{T}^3)$. There exists $c > 0$ such that if we let $\tau := \frac{c}{\|u_0\|_{H^s}}$ then there exists a solution $u \in \mathcal{C}^0([0,\tau], H^s(\mathbb{T}^3))$ to (E^3) and also solutions $u_N \in \mathcal{C}^0([0,\tau], H^s(\mathbb{T}^3))$ to (E^3_N) such that there holds*

$$\|u_N(t)\|_{H^s} \leqslant 2\|u_0\|_{H^s} \text{ and } \|u(t)\|_{H^s} \leqslant 2\|u_0\|_{H^s}\,,$$

for all $t \in [0,\tau]$.

Remark 5.36. In the following it will be crucial that τ can be taken independent of N, and only depend on the size of $\|u_0\|_{H^s}$. In the following we will say that τ is a *uniform local well-posedness time.*

Proof. We provide the proof for the u_N and explain the uniform statement. As before we have the *a priori* estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_N(t)\|_{H^s} \leqslant \|\nabla u_N(t)\|_{L^\infty}\|u_N(t)\|_{H^s} \leqslant C\|u_N(t)\|^3_{H^s}\,,$$

where we used that $W^{1,\infty} \hookrightarrow H^s$ because $s > \frac{5}{2}$. Note that C is independent of N. Using the discussion in the proof of 5.34 we can set $\tau_N := \frac{1}{2C\|u_N(0)\|_{H^s}}$ for which the required estimates hold.

Then, we set $\tau := \frac{1}{2C\|u_0\|_{H^s}}$, which is independent of N and since

$$\|\mathbf{P}_{\leqslant N} u_0(0)\|_{H^s} \leqslant \|u_0\|_{H^s}\,,$$

we get $\tau_N \geqslant \tau$ hence the result. $\square$

5.2. The two-dimensional case. First we remark that thanks to the global well-posedness result for two-dimensional Euler equations we have the following result.

Proposition 5.8. *The measure μ is invariant under the well-defined flow $\Phi_t : H^{2+\delta} \to H^{2+\delta}$ of (E^2). More precisely, for any $A \in \mathcal{B}(H^{2+\delta})$ and every $t > 0$ there holds:*

$$\mu(\Phi_{-t}(A)) = \mu(A)\,.$$

Proof. This result will follow from the stationarity of the process u with respect to μ once we prove that μ almost-surely, $u(t) = \Phi_t(u_0)$. It is a weak-strong uniqueness statement. Let $u_0 \in H^1$. As we have already seen it, μ-almost surely, $u_0 \in H^{2+\delta}$, so that there is no loss of generality assuming that $u_0 \in H^{2+\delta}$. Then, let $v = \Phi_t(u_0)$ the global solution to (E^2) associated to u_0 which is constructed by Lemma 5.34. Let us denote by u the solution constructed by Proposition 5.3, associated to the initial data u_0 and solution to (E) in the class $\mathcal{C}^0 H^{1-\varepsilon} \cap L^2_{\mathrm{loc}} H^{2+\delta-\varepsilon}$. Let $T > 0$. We want to prove that $u = v$ on $[0, T]$. We introduce $w := u - v$, and then we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{L^2}^2 = -2(w \cdot \nabla u, w)_{L^2} \leqslant 2\|w\|_{L^2}^2 \|u\|_{H^{2+\delta-\varepsilon}} ,$$

where we used the properties of the bilinear form and the Sobolev embedding. Since we have $u \in L^2([0,T], H^{2+\delta-\varepsilon}) \subset L^1([0,T], H^{2+\delta-\varepsilon})$, and $w(0) = 0$, the Grönwall lemma implies that $w = 0$ on $[0, T]$, hence the conclusion. □

Before proving Theorem 5.9, let us remark that from the bounds on $\|u\|_{\dot{H}^1}$ and $\|u\|_{\dot{H}^{2+\delta}}$ we can infer the following interpolation inequality.

Lemma 5.37. *Let $\sigma \in (1, 2+\delta)$, then:*

$$\mathbb{P}\left(\|u\|_{H^\sigma} > \lambda\right) \lesssim \lambda^{-\frac{2(1+\delta)}{\sigma-1}} \log^{\beta(s)}(\lambda) ,$$

with $\beta(\sigma) = \frac{2+\delta-\sigma}{1+\delta}$ and where the implicit constant depends only on the constants C_1, C_2 and σ.

We postpone the proof of this technical estimate to the end of the section and proceed to the proof of the main theorems. First, note that Theorem 5.2 (i), (ii) in the two-dimensional case is already proven since we have constructed an invariant measure μ in Section 4 and its properties will be studied in Section 6. It remains to prove Theorem 5.9 and Theorem 5.2 (iii).

Proof of Theorem 5.9 (i). Given that μ is an invariant measure for (E), we are going to prove the theorem using the same argument as in [Bou94], explained in the introduction. Let $\sigma < s$ where $s = 2 + \delta$. Let $\tau \sim \frac{c}{\lambda}$ denote the local well-posedness time given by the local Cauchy theorem for (E^2), which says that if $\|u_0\|_{H^\sigma} \leqslant \lambda$, then there is a unique local solution $u(t)$ to (E) in $\mathcal{C}^0([0,\tau], H^\sigma)$ on $[0, \tau]$ obeying $\|u(t)\|_{H^\sigma} \leqslant 2\lambda$ for any $t \in [0, \tau]$.

Let $T > 0$. Thanks to the local theory we now that if the initial data u_0 is such that $\|u(n\tau)\|_{H^\sigma} \leqslant \lambda$ for any $n = 0, \dots, \lfloor \frac{T}{\tau} \rfloor$ then we can solve on the time intervals $[n\tau, (n+1)\tau]$ and extend the local solution up until time T, with the bound $\|u(t)\|_{H^\sigma} \leqslant 2\lambda$ for any $t \in [0, T]$. Thanks to this observation we let

$$G_{\lambda,T} := \bigcap_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \{u_0 \in H^\sigma \, , \Phi_{n\tau}(u_0) \in B_\lambda\} ,$$

where $B_\lambda = B_{H^\sigma}(0,\lambda)$.

Thus for any $u_0 \in G_{\lambda,T}$ there exists a solution on $[0,T]$ to (E) with initial data $\|u_0\|_{H^\sigma} \leqslant \lambda$ and moreover $\|u(t)\|_{H^\sigma} \leqslant 2\lambda$ for $t \in [0,T]$.

Then we compute, using the invariance of the measure on H^s:

$$\begin{aligned}
\mu\left(H^\sigma \setminus G_{\lambda,T}\right) &\leqslant \sum_{n=0}^{\left\lfloor \frac{T}{\tau}\right\rfloor} \mu\left(\Phi_{n\tau}\left(H^\sigma \setminus B_\lambda\right)\right)\\
&\leqslant \sum_{n=0}^{\left\lfloor \frac{T}{\tau}\right\rfloor} \mu\left(H^\sigma \setminus B_\lambda\right)\\
&\lesssim T\tau^{-1}\mu\left(H^\sigma \setminus B_\lambda\right) .
\end{aligned}$$

Recall that $\tau^{-1} \sim \lambda$, and incorporate Lemma 5.37:

$$\mu\left(H^\sigma \setminus B_\lambda\right) \leqslant \lambda^{\frac{\sigma-3-2\delta}{\sigma-1}} \log^{\beta(\sigma)} ,$$

so that $\mu(H^\sigma \setminus G_{\lambda,T}) \lesssim T\lambda^{\frac{\sigma-3-2\delta}{\sigma-1}} \log^{\beta(\sigma)}$ and then we have $\mu(H^\sigma \setminus G_{\lambda,T}) \leqslant \varepsilon$ if $\lambda \sim (T\varepsilon^{-1})^{\frac{\sigma-1}{3-\sigma+2\delta}+\varepsilon}$ for arbitrarily small $\varepsilon > 0$.

Finally, we provide the Borel-Cantelli argument, set $T_n \coloneqq 2^n$ and $\varepsilon_n \coloneqq \frac{1}{n^2}$ then consider

$$G \coloneqq \bigcup_{n\geqslant 1} \bigcap_{k\geqslant n} G_{\varepsilon_k, T_k} .$$

Then we see that $\mu(G) = 1$ and for any $u_0 \in G$ there is a n_0 such that we have $u_0 \in G_{n_0}$, which gives

$$\|u(t)\|_{H^\sigma} \lesssim (T_n\varepsilon_n^{-1})^{\frac{\sigma-1}{3-\sigma+2\delta}+\varepsilon} \lesssim T_n^{\frac{\sigma-1}{3-\sigma+2\delta}+\varepsilon} \log^{\frac{2(\sigma-1)}{3-\sigma+2\delta}+2\varepsilon} T_n ,$$

which concludes the proof. □

Proof of Theorem 5.9 (ii). (ii) It is a direct consequence of (i) and the use of the Sobolev embedding $H^\sigma \mapsto W^{2,\infty}$ as soon as $\sigma > 3$:

$$\|\nabla\xi(t)\|_{L^\infty} \lesssim \|\xi(t)\|_{H^{\sigma-1}} \lesssim \|u(t)\|_{H^\sigma} .$$ □

Proof of Theorem 5.2. Part (i) and (ii) have already been proven.

(iii) Theorem 5.4 implies in particular that such measures cannot have any atom: indeed, if u^0 is an atom, and $c \coloneqq \|u^0\|_{L^2}$ then

$$\mu\left(u \in H^{2+\delta} \text{ such that } \|u\|_{L^2} = c\right) > 0 ,$$

but also

$$\mu\left(u \in H^{2+\delta} \text{ such that } \|u\|_{L^2} \in (c-\varepsilon, c+\varepsilon)\right) \leqslant p(2\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0 ,$$

which is a contradiction.

(iv) Let $k>0$. We claim that there exists a choice of the noise parameters $(\phi_n)_{n\in\mathbb{Z}_0^2}$ such that the associated invariant measure $\mu^{(k)}$ satisfies

$$\mathbb{E}_{\mu^{(k)}}[\|u\|_{\dot{H}^1}^2]\sim k^{\alpha(\delta)}\,, \tag{5.26}$$

for some $\alpha(\delta)>0$.

Assuming (5.26), one can finish the proof by letting

$$\mu\coloneqq\sum_{k\geqslant 1}\frac{\mu^{(k)}}{2^k}\,,$$

which is an invariant probability measure by construction. Moreover, observe that for any $R>0$ we have $\mathbb{P}(\|u\|_{\dot{H}^{2+\delta}}^2>R)>0$, completing the proof of (iv). Indeed, if there exists $R_0>0$ such that $\mathbb{P}(\|u\|_{\dot{H}^{2+\delta}}^2\leqslant R_0)=1$ then for $k>R_0$ we have

$$R_0^2\geqslant\mathbb{E}_{\mu^k}[\|u\|_{\dot{H}^{2+\delta}}^2]\sim k^{\alpha(\delta)}\to\infty\,,$$

as $k\to\infty$, which is a contradiction.

It remains to prove (5.26). In view of (5.21) the matter reduces to choosing the ϕ_n such that

$$\frac{\mathcal{B}_0}{\mathcal{B}_1^{\frac{\delta}{1+\delta}}}\gtrsim_\delta K^\varepsilon\,,$$

for some $\varepsilon>0$. Take $\alpha\geqslant\frac{\varepsilon(1-\delta)}{2}+\delta$ and define

$$\phi_n=\begin{cases} e^{-n} & \text{if } |n|<k\\ \frac{k^\alpha}{n} & \text{if } k\leqslant|n|\leqslant 2k\\ e^{-(n-2k)} & \text{if } |n|>2k\,.\end{cases}$$

observe that

$$\begin{aligned}\mathcal{B}_0&=\sum_{|n|<k}|\phi_n|^2+\sum_{k\leqslant|n|\leqslant 2k}|\phi_n|^2+\sum_{|n|>2k}|\phi_n|^2\\ &=\mathcal{O}(1)+k^{2\alpha}\mathcal{O}(1)+\mathcal{O}(1)\\ &\sim k^{2\alpha}\,,\end{aligned}$$

and similarly $\mathcal{B}_1\sim k^{2\alpha+2}$. Therefore, thanks to the assumption on α we deduce (5.26). □

To complete this paragraph we give a proof of Lemma 5.37.

Proof of Lemma 5.37. We use an elementary argument : let $A>0$ be a parameter to be chosen later. Since for $\sigma\in(1,2+\delta)$ we have the interpolation inequality

$$\|u\|_{H^\sigma}\leqslant\|u\|_{H^1}^{\theta(\sigma)}\|u\|_{H^{2+\delta}}^{1-\theta(\sigma)}$$

where $\theta(\sigma)\coloneqq\frac{2+\delta-s}{1+\delta}$, we write that for any $\lambda>0$ and $A>0$:

$$\{\|u\|_{H^\sigma}>\lambda\}\subset\{\|u\|_{H^1}>A^{-\frac{1}{\theta(\sigma)}}\lambda\}\cup\{\|u\|_{H^2}>A^{\frac{1}{1-\theta(\sigma)}}\lambda\}\,,$$

so that the Markov inequality implies the bound

$$\begin{aligned}\mathbb{P}(\|u\|_{H^\sigma} > \lambda) &\leqslant \mathbb{P}\left(e^{\gamma\|u\|^2_{H^1}} > \exp\left(\gamma A^{-\frac{2}{\theta}}\lambda^2\right)\right) + \mathbb{P}\left(\|u\|^2_{H^{2+\delta}} > A^{\frac{2}{1-\theta}}\lambda^2\right)\\ &\lesssim \exp\left(-A^{-\frac{2}{\theta}}\lambda^2\right) + A^{-\frac{2}{1-\theta}}\lambda^{-2}\,.\end{aligned}$$

Next, we want to optimise in A in the right-hand side so that both terms have the same size, *i.e.*, $\exp\left(-A^{-\frac{2}{\theta}}\lambda^2\right) \simeq A^{-\frac{2}{1-\theta}}\lambda^{-2}$, or taking the logarithm twice in this relation gives $\frac{2}{\theta}\log A \sim 2\log\lambda$ and re-plugging in the previous relation suggests the choice $A := C(\sigma)\lambda^{-\frac{1}{\theta}}\log^{\frac{\theta}{2}}(\lambda)$ leading to

$$\mathbb{P}\left(\|u\|_{H^s} > \lambda\right) \lesssim \lambda^{-\frac{2}{1-\theta}}\log^{\frac{\theta}{1-\theta}}(\lambda)\,,$$

which concludes the proof recalling the definition of θ. □

5.3. The three-dimensional case. We recall that in Section 4 we have constructed a measure μ_N, stationary for the limit process u_N, solving (E^3_N).

Corollary 5.38. *Let $s = 1+\delta > \frac{5}{2}$. Then the measure μ_N constructed by Proposition 5.4 on H^s is invariant under the global flow $\Phi^{(N)}_t$ of (E^3_N) acting on H^s.*

Proof. This comes from the fact that almost surely $u_N(t) = \Phi^N_t(t)$ using a weak-strong uniqueness statement which proof is identical to that of Proposition 5.8. □

We define $\tilde{\mu}_N$ a measure on $L^2(\mathbb{T}^3)$ defined for any Borelian set A by:

$$\tilde{\mu}_N := \mu_N(\mathbf{P}_{\leqslant N}A)\,.$$

We remark that $\tilde{\mu}_N$ is invariant under $\Phi^{(N)}_t$ and satisfies the same estimates as μ_N. In the following we will write μ_N for $\tilde{\mu}_N$. Our next result follows from the uniform estimates of Proposition 5.4 and allow to extract a limiting measure μ from the $(\mu_N)_{N\geqslant 0}$. More precisely we have the following.

Lemma 5.39. *Let $(\mu_N)_{N\geqslant 0}$ be the above sequence of invariant measures for (E^3_N) at regularity $s = 1+\delta$. Then up to extraction, μ_N weakly converges to a measure μ on any $H^{1+\varepsilon}$, $\varepsilon < \delta$. We also have:*

(i) $\mathbb{E}_\mu[\|u\|^2_{\dot{H}^{1+\delta}}] \leqslant \mathcal{B}_0$.

(ii) $\mathbb{E}_\mu\left[e^{\gamma\|u\|^2_{L^2}}\right] < \infty$ *for some* $\gamma > 0$.

Proof. By Proposition 5.4 we have

$$\mathbb{E}_{\mu_N}\left[\|u\|^2_{H^{1+\delta}}\right] \leqslant C\,,$$

uniformly in N. Let $\varepsilon < \delta$. Since the embedding $H^{1+\varepsilon} \hookrightarrow H^{1+\delta}$ is compact, it is the Prokhorov theorem which ensures that, up to extraction, we may assume that $(\mu_N)_{N\geqslant 0}$ weakly converges to a measure μ on all $H^{1+\varepsilon}$.

Finally, (i) and (ii) follow from the same monotone convergence argument that we used in the proof of Proposition 5.3. $\square$

Before we proceed to the proof of Theorem 5.2 and Theorem 5.11, we state and prove two approximations lemmata. In this section we will write $B_{H^s}(R)$ the ball $B(0,R)$ in the space H^s.

Lemma 5.40 (Approximation)**.** *Let $s > \frac{7}{2}$, $T > 0$ and $K > 0$. Let $u_0 \in H^s(\mathbb{T}^3)$ such that $\|u_0\|_{H^s} \leqslant K$. Let $s' < s-1$ and and such that there exists N_0 satisfying*

$$N_0^{s-1-s'} \geqslant 16K(1+KT)e^{KT}2^{\frac{2KT}{c}}\,,$$

where c is the constant appearing in Lemma 5.35; and for which $\|u_{N_0}(t)\|_{H^s} \leqslant K$ holds for all $t \leqslant T$. Then:

(i) There exists a solution $u \in \mathcal{C}([0,T], H^{s'}(\mathbb{T}^3))$ to (E^3).

(ii) For all $N \geqslant N_0$ we have $\|u_N - u\|_{L^\infty_T H^{s'}} \underset{N\to\infty}{\longrightarrow} 0$.

(iii) The solution u satisfies $\|u\|_{L^\infty_T H^{s'}} \leqslant 2K$.

Proof. Let $\tau = \frac{c}{2K}$ be a uniform local well-posedness time given by Lemma 5.35 associated to initial data with H^s norm lesser than $2K$. In the proof we write $s' = s-1-\varepsilon$ and $n := \lfloor \frac{T}{\tau} \rfloor = \frac{2KT}{c}$.

The proof consists in two steps: first we prove (ii) on the time-interval $[0,\tau]$ and then we iterate the argument.

Step 1. *Short time quantitative estimates.* On $[0,\tau]$, both u and all the u_N are defined and enjoy the bound

$$\|u\|_{L^\infty_\tau H^{s'}},\ \|u_N\|_{L^\infty_\tau H^{s'}} \leqslant 4K\,.$$

We set $w_N := u - u_N$ and observe that w_N satisfies $w_N(0) = \mathbf{P}_{>N}u_0$ and the equation

$$\partial_t w_N = \mathbf{P}\left(B(w_N,w_N) - B(u_N,w_N) - B(w_N,u_N)\right) - \mathbf{P}_{>N}B(u_N,u_N)\,.$$

We use standard energy estimates on $\langle\nabla\rangle^{s'}w_N$ (which makes use of the bilinear cancellation) to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w_N(t)\|_{H^{s'}} &\leqslant \left(\|u(t)\|_{H^{s'}} + \|u_N\|_{H^{s'}}\right)\|w_N(t)\|_{H^{s'}} + N^{-\varepsilon}\|u_N\|^2_{H^s}\\ &\leqslant 8K\|w_N(t)\|_{H^{s'}} + 16N^{-\varepsilon}K^2\,,\end{aligned}$$

where in the last line we used the *a priori* estimates on u and u_N. Since $\|w_N(0)\|_{H^{s'}} \leqslant 2N^{-1-\varepsilon}K$, the Grönwall lemma implies the bound

$$\|w_N(t)\|_{H^{s'}} \leqslant C(N^{-1-\varepsilon}K + N^{-\varepsilon}K^2t)e^{Kt}\,,$$

where $C > 0$ is a universal constant. Letting $N \to \infty$ implies (ii) on $[0,\tau]$ and thus (iii).

Step 2. *Iteration of the local estimate.* We explain how we can iterate on the time interval $[\tau, 2\tau]$, the statement will then follow from an induction argument. The analysis on $[0, \tau]$ implies that

$$\|u(\tau)\|_{H^{s'}} \leqslant K + \|u_{N_0}(\tau) - u(\tau)\|_{H^{s'}} \leqslant 2K\,,$$

by definition of N_0. Thus we can solve the equation satisfied by u in $H^{s'}$ on the time interval $[\tau, 2\tau]$ by definition of τ.

Moreover for $N \geqslant N_0$ we have the analysis on $[0, \tau]$ gives the more precise estimate:

$$\|u_N(\tau) - u(\tau)\|_{H^{s'}} \leqslant 2 \times 2^{-n} K\,,$$

thus we can repeat the argument with the same estimates (requiring the use of the uniform estimate $\|u_{N_0}\|_{H^s} \leqslant K$) leading to the bound:

$$\|u_N(t) - u(t)\|_{H^{s'}} \leqslant 2\|u_N(\tau) - u(\tau)\|_{H^{s'}}\,,$$

for any $t \in [\tau, 2\tau]$. Furthermore, letting $N \geqslant N_0$ proves (ii) on $[\tau, 2\tau]$. Then, we observe that we have the bound

$$\|u(2\tau)\|_{H^{s'}} \leqslant K + \|u_{N_0}(2\tau) - u(2\tau)\|_{H^{s'}} \leqslant 2K\,,$$

which allow for iteration on $[2\tau, 3\tau]$ and so on. After an induction argument we obtain the required bounds and also the fact that $\|u - u_{N_0}\|_{L^\infty H^{s'}} \leqslant K$, which proves (iii). □

A useful consequence of this lemma is the following.

Corollary 5.41 (Approximation variant)**.** *Let $s > \frac{7}{2}$ and $R > 0$. Let τ be defined Lemma 5.35 which works for any $u_0 \in B_{H^s}(R)$. Then we have, for all $s' < s - 1$:*

$$\sup_{u_0 \in B_{H^s}(R)} \|u - u_N\|_{L^\infty_T H^{s'}} \xrightarrow[N\to\infty]{} 0\,. \tag{5.27}$$

Another direct consequence is the global well-posedness in the space H^s.

Corollary 5.42 (Propagation of regularity)**.** *For u_0 such as in Lemma 5.40, the solution u constructed in $H^{s'}$ lies in $\mathcal{C}^0([0,T], H^s(\mathbb{T}^3))$, thus $u(t) = \Phi_t(u_0)$.*

Proof. We already know that $u \in \mathcal{C}^0([0,T], H^{s'})$ and that u_0 gives rise to a local solution $u \in \mathcal{C}^0([0,T^*), H^s)$. In order to prove that $T^* \geqslant T$, from the proof of Lemma 5.35 we see that it is sufficient to prove that

$$\int_0^T \|\nabla u(t)\|_{L^\infty}\,\mathrm{d}t < \infty\,,$$

which follows from the Sobolev embedding (recall that $s' > \frac{3}{2}$) and the fact that for $t \in [0,T]$ there holds $\|u(t)\|_{H^{s'}} \leqslant K$. □

The proof of Theorem 5.11 uses the following interpolation estimate, which proof is identical to that of Lemma 5.37.

Lemma 5.43. *Let $s' \in (0, 1+\delta)$, then:*

$$\mathbb{P}\left(\|u_N\|_{H^{s'}} > \lambda\right) \lesssim \lambda^{-\frac{2(1+\delta)}{s'}} \log^{\beta(s)}(\lambda)\,,$$

with $\beta(s') = \frac{1+\delta-s'}{1+\delta}$ and where the implicit constant depends only on the constants C_1, C_2 and s' but not N.

Proof of Theorem 5.2 and Theorem 5.11. We focus on constructing a global dynamics for (E^3) with μ-almost sure growth estimates.

Global theory. Let $s' < s - 1$ as in Theorem 5.11, and $\alpha \coloneqq \alpha(s') = \frac{s'}{2s-s'}$. As in the proof of Theorem 5.9 we use a Borel-Cantelli argument to reduce the proof to the following fact: for any given $T, \varepsilon > 0$ there exists a set $G_{T,\varepsilon}$ such that $\mu(H^s \setminus G_{T,\varepsilon}) \leqslant \varepsilon$ and for any $u_0 \in G_{T,\varepsilon}$, the associate local solution to (E^3) exists on $[0,T]$ satisfies the growth estimates

$$\|u(t)\|_{H^{s'}} \leqslant 2CT^{\alpha}\,, \tag{5.28}$$

Let us denote by G_ε the set of initial data such that inequality (5.28) holds, and where we define $C \coloneqq \varepsilon^{-\frac{1}{\alpha}}$.

Let τ be a *uniform local well-posedness time*, associated to initial data of size lesser than $\leqslant CT^{\alpha} =: 2\lambda$. We define $K \coloneqq CT^{\alpha}$ and let N_0 associated to such K and N for which for any $N \geqslant N_0$ there holds

$$\|u_N(t) - u_{N_0}(t)\|_{H^s} \leqslant CT^{\alpha}\,,$$

for any $t \leqslant T$.

For $N \geqslant N_0$ we define the sets

$$G_N \coloneqq \bigcap_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} (\Phi_{n\tau}^{(N)})^{-1}(\overline{B}_\lambda)\,,$$

whre $\overline{B}_\lambda$ is the closed ball of $H^{s'}$ centered at 0 of radius λ. Let $u_0 \in G_N$. Then, by the definition of τ and G_N we see that there exists a solution u_N to (E^3_N) on $[0,T]$ obeying the bound $\|u_N(t)\|_{H^{s'}} \leqslant 2\lambda$. Then Lemma 5.40 with $K \coloneqq \lambda$ and $N \geqslant N_0$ gives rise to a solution u to (E^3) on $[0,T]$ obeying the bound $\|u(t)\|_{H^{s'}} \leqslant 4\lambda$. It follows that $G_N \subset G$ for any $N \geqslant N_0$.

Since ϕ_t is act continuously on H^s we see that G is a closed set, hence by the Portmanteau theorem we deduce:

$$\begin{aligned}
\mu(H^s \setminus G) &\leqslant \liminf_{N\to\infty} \mu_N(H^s \setminus G) \\
&\leqslant \liminf_{N\to\infty} \mu_N\left(\bigcup_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} (\Phi_{n\tau}^{(N)})^{-1}(H^s \setminus \overline{B}_{H^s}(\lambda))\right)\,.
\end{aligned}$$

Using the invariance of μ_N under $\Phi_t^{(N)}$ and the uniform estimates for the measures μ_N of Proposition 5.4 we have:

$$\mu(H^s \setminus G) \leqslant T\tau^{-1}\lambda^{-\frac{2s}{s'}}\,.$$

Recalling that $\tau \sim c\lambda^{-1}$ and $\lambda = CT^{\alpha(s')}$, we see that the choice of C gives $\mu(G) \geqslant 1-\varepsilon$.

Moreover we remark that the solutions being constructed on $\mathcal{C}^0([0,T],H^{s'})$ are in fact in $\mathcal{C}^0([0,T],H^s)$ since $s' > \frac{5}{2}$ and Lemma 5.42.

Invariance of the measure. We follow the proof in [Sy19a], Theorem 7.1. Let $A \subset H^s$ be a Borelian subset. We want to prove that $\mu(A) = \mu(\phi_t A)$ for any $t > 0$. Since μ is a regular measure, it follows from Ulam's theorem (see Theorem 7.1.4. in [Dud02]) that μ is inner regular. More precisely, for any Borelian $A \subset H^s$ we have

$$\mu(A) = \sup\{\mu(F),\ F \subset A \ \text{ closed}\}\,.$$

A standard procedure reduces the proof to the verification of the invariance on compact sets. Up to a limiting argument, it is then sufficient to prove that, for any compactly supported Lipschitz function $f : \bigcap_{s'<s} H^{s'}(\mathbb{T}^3) \to \mathbb{R}$ there holds

$$\mathbb{E}_\mu[(\Phi_t)^* f] = \mathbb{E}_\mu[f]\,. \tag{5.29}$$

We fix such a function f. Since the support of this function is compact, it is in particular bounded by a constant R, that is for any $v \in \operatorname{supp}(f)$, $\|v\|_{H^{s'}} \leqslant R$ for all $s' < s$. Since μ almost-surely we have $v \in H^s$, taking the limit $s' \to s$ implies that f has support contained in $B_{H^s}(R)$. We also fix τ a *uniform local well-posedness time* for (E_N^3) in H^s associated to size R. Up to iterating the invariance results it is sufficient to can prove the invariance only for times $t \in [0,\tau]$.

Since by weak convergence and μ_N-invariance we have:

$$\mathbb{E}_{\mu_N}[(\Phi_t^{(N)})^* f] = \mathbb{E}_{\mu_N}[f] \underset{N\to\infty}{\longrightarrow} \mathbb{E}_\mu[f]\,,$$

and since

$$\mathbb{E}_{\mu_N}[(\Phi_t)^* f] \underset{N\to\infty}{\longrightarrow} \mathbb{E}_\mu[(\Phi_t)^* f]\,,$$

the proof of (5.29) boils down to the proof of

$$\left|\mathbb{E}_{\mu_N}\left[\Phi_t^{(N)} f - \Phi_t f\right]\right| \underset{N\to\infty}{\longrightarrow} 0\,. \tag{5.30}$$

Using that f is Lipschitz and Lemma 5.41 we have

$$\begin{aligned}\left|\mathbb{E}_{\mu_N}\left[\Phi_t^{(N)}f-\Phi_t f\right]\right| &\lesssim \sup_{u_0\in B_{H^s}(R)}\|\Phi_t^{(N)}u_0-\Phi_t u_0\|_{H^s}\mu_N(B_R)\\ &\leqslant \sup_{u_0\in B_{H^s}(R)}\|u_N(t)-u(t)\|_{H^s}\\ &\underset{N\to\infty}{\longrightarrow} 0\,,\end{aligned}$$

which is (5.30).

Part (iii) of Theorem 5.5. Assume that $a:=\mu(\{0\})<1$. By Theorem 5.11 we now that 0 is the only possible atom for μ. Let $\tilde{\mu}:=\mu-a\delta_0$, and since δ_0 is invariant for (E), so is $\tilde{\mu}$ with no atom at zero. ☐

6. Properties of the measures

The goal of this section is to prove Theorem 5.4 and Theorem 5.6.

6.1. The two-dimensional case. Note that thanks to the Portmanteau theorem and the interior regularity of the Lebesgue measure, it is sufficient to prove the following proposition for the measures μ_ν. The proof of (ii) of Theorem 5.4 is postponed to the end of the section.

Proposition 5.9. *The measures $(\mu_\nu)_{\nu>0}$ satisfy the following properties.*

(i) There exists a constant $C>0$ independent of ν such that for all $\varepsilon>0$ there hods:

$$\mu_\nu\left(\|u\|_{L^2}<\varepsilon\right)\leqslant C\varepsilon^{\frac{2(1+\delta)}{2+\delta}}\,.$$

(ii) There exists a continuous increasing function $p:\mathbb{R}_+\to\mathbb{R}_+$ and a constant $C>0$ which does not depend on ν such that for every Borel set $\Gamma\subset\mathbb{R}_+$ there holds

$$\mu_\nu\left(\|u\|_{L^2}\in\Gamma\right)\leqslant p(|\Gamma|)\,.$$

The proof will heavily rely on Proposition 5.13 whose proof is given in Appendix A. The proof is taken from [KS15] with some minor modifications designed to adapt to our case.

Proof of Proposition 5.9. We start with (i). First, we prove that

$$\mu\left(\left\{u\in L^2,\ 0<\|u\|_{L^2}\leqslant\varepsilon\right\}\right)\lesssim\varepsilon^{\frac{2(1+\delta)}{2+\delta}}\,. \tag{5.31}$$

We let $u(t)=u_\nu(t)$ be a stationary process for μ_ν satisfying (HVE_ν^2). We apply Proposition 5.13 to $\Gamma=[\alpha,\beta]$ where $\alpha>0$ and $g\in\mathcal{C}^2(\mathbb{R})$ is a function such that $g(x) = x^{\frac{1+\delta}{2+\delta}}$ for $x \geqslant \alpha$ and vanishes for $x \leqslant 0$. This results in

$$\mathbb{E}\left[\int_\alpha^\beta \mathbf{1}_{(a,\infty)}\left(\|u\|_{L^2}^{\frac{2(1+\delta)}{2+\delta}}\right)\left(\frac{1+\delta}{2+\delta}\frac{\mathcal{B}_0/2 - \|\nabla^{1+\delta}u\|_{L^2}^2}{\|u\|_{L^2}^{\frac{2}{2+\delta}}} - \frac{(1+\delta)\sum_{n\in\mathbb{Z}_0^2}|\phi_n|^2|u_n|^2}{(2+\delta)^2\|u\|_{L^2}^{\frac{2(3+\delta)}{2+\delta}}}\right)\mathrm{d}a\right]$$
$$+ C(\delta)\sum_{n\in\mathbb{Z}_0^2}|\phi_n|^2\mathbb{E}\left[\mathbf{1}_{[\alpha,\beta]}\left(\|u\|_{L^2}^{\frac{2(1+\delta)}{2+\delta}}\right)\|u\|_{L^2}^{-\frac{4}{2+\delta}}u_n^2\right] = 0\,.$$

In particular this gives

$$\mathbb{E}\left[\int_\alpha^\beta \mathbf{1}_{(a,\infty)}\left(\|u\|_{L^2}^{\frac{2(1+\delta)}{2+\delta}}\right)\left(\frac{1+\delta}{2+\delta}\frac{\mathcal{B}_0/2 - \|\nabla^{1+\delta}u\|_{L^2}^2}{\|u\|_{L^2}^{\frac{2}{2+\delta}}} - \frac{(1+\delta)\sum_{n\in\mathbb{Z}_0^2}|\phi_n^2||u_n^2|}{(2+\delta)^2\|u\|_{L^2}^{\frac{2(3+\delta)}{2+\delta}}}\right)\mathrm{d}a\right] \leqslant 0\,,$$

which gives

$$\mathbb{E}\left[\int_\alpha^\beta \frac{\mathbf{1}_{(a,\infty)}\left(\|u\|_{L^2}^{\frac{2(1+\delta)}{2+\delta}}\right)}{\|u\|_{L^2}^{\frac{2(3+\delta)}{2+\delta}}}\left(\mathcal{B}_0(1+\frac{\delta}{2})\|u\|_{L^2}^2 - \sum_{n\in\mathbb{Z}_0^2}|\phi_n|^2u_n^2\right)\mathrm{d}a\right] \lesssim (\beta-\alpha)\mathbb{E}\left[\frac{\|u\|_{\dot H^{1+\delta}}^2}{\|u\|_{L^2}^{\frac{2}{2+\delta}}}\right]. \tag{5.32}$$

Observe that by interpolation

$$\mathbb{E}\left[\frac{\|u\|_{\dot H^{1+\delta}}^2}{\|u\|_{L^2}^{\frac{2}{2+\delta}}}\right] \lesssim \mathbb{E}\left[\|u\|_{\dot H^{2+\delta}}^{\frac{2(1+\delta)}{2+\delta}}\right]$$

and

$$\mathcal{B}_0(1+\frac{\delta}{2})\|u\|_{L^2}^2 - \sum_{n\in\mathbb{Z}_0^2}|\phi_n|^2u_n^2 \geqslant \frac{\delta\mathcal{B}_0}{2}\|u\|_{L^2}\,.$$

Plugging it into (5.32) provides:

$$\mathbb{E}\left[\int_\alpha^\beta \mathbf{1}_{(a,\infty)}\left(\|u\|_{L^2}^{\frac{2(1+\delta)}{2+\delta}}\right)\|u\|_{L^2}^{-\frac{2}{2+\delta}}\mathrm{d}a\right] \lesssim (\beta-\alpha) \lesssim \beta\,,$$

with constants independent of ν. Then for any $\varepsilon > \beta$ one has

$$\mathbb{E}\left[\int_\alpha^\beta \mathbf{1}_{(a,\infty)}\left(\|u\|_{L^2}^{\frac{2(1+\delta)}{2+\delta}}\right)\|u\|_{L^2}^{-\frac{2}{2+\delta}}\mathrm{d}a\right] \geqslant \varepsilon^{-\frac{2}{2+\delta}}\mathbb{E}\left[\int_0^\beta \mathbf{1}_{(a,\varepsilon)}\left(\|u\|_{L^2}^{\frac{2(1+\delta)}{2+\delta}}\right)\mathrm{d}a\right]$$
$$= \varepsilon^{-\frac{2}{2+\delta}}\int_0^\beta \mathbb{P}\left(a^{\frac{2+\delta}{2(1+\delta)}} < \|u\|_{L^2} < \varepsilon^{\frac{2+\delta}{2(1+\delta)}}\right)\mathrm{d}a\,.$$

so that finally,

$$\frac{1}{\beta}\int_0^\beta \mathbb{P}\left(a^{\frac{2+\delta}{2+2\delta}} < \|u\|_{L^2} < \varepsilon^{\frac{2+\delta}{2+2\delta}}\right)\,\mathrm{d}a \lesssim \varepsilon^{\frac{2}{2+\delta}}\,,$$

and then passing to the limit $\beta \to 0$, one gets (5.31).

In order to complete the proof of (i) we need to prove that μ_ν does not have any atom at 0. We refer to [Shi11] for details of this proof and also [KS15] for an alternative proof. Here the dependence on ν is harmless, hence we drop the subscripts. Let μ_j be the law of the random variable $u_j(t) \coloneqq (u(t), e_j)_{L^2}$. We will prove that μ_j has no atom at zero, for any j, thus proving the desired fact. The Itô formula reads:

$$u_j(t) - u_j(0) = \int_0^t h_j(s)\,\mathrm{d}s + \sqrt{\nu}\sum_{n\in\mathbb{Z}_0^2}\phi_n \mathrm{d}\beta_n\,,$$

where $h_j(s) = (e_j, -B(u(s), u(s)) + \nu L u(s))_{L^2}$. Now observe that an application of Proposition 5.13 gives the existence of a random time $\Lambda_t(a)$ satisfying

$$\begin{aligned}\Lambda_t(a) = |u_j(t) - a| - |u_j(0) - a| - \int_0^t \mathbf{1}_{(a,\infty)}(u_j(s))h_j(s)\,\mathrm{d}s \\ - \sqrt{\nu}\sum_{n\in\mathbb{Z}_0^2}\phi_n\int_0^t \mathbf{1}_{(a,\infty)}(u_j(s))\,\mathrm{d}\beta_n(s)\,.\end{aligned} \tag{5.33}$$

Taking the expectation and using invariance yields

$$\mathbb{E}[\Lambda_t(a)] = -t\mathbb{E}[\mathbf{1}(a,\infty)(u_j(0))h_j(0)]\,.$$

Theorem 5.44 implies

$$2\int_\Gamma \mathbb{E}\left[\Lambda_t(a)\right]\,\mathrm{d}a = \sqrt{\nu}\mathcal{B}_0\int_0^t \mathbb{E}\left[\mathbf{1}_\Gamma(u_j(s))\right]\,\mathrm{d}s = \sqrt{\nu}t\mathcal{B}_0\mathbb{P}((u, e_j)_{L^2}\in\Gamma)\,.$$

Combining these two identities and remarking that

$$|h_j(0)| \lesssim |j|\left(\|u\|^2_{\dot H^{2+\delta}} + 1\right)\,,$$

we see that $h_j(0)$ is of finite expectation thus

$$\mathbb{P}(\langle u, e_j\rangle_{L^2}\in\Gamma) \lesssim \nu^{-1/2}\ell(\Gamma)\mathbb{E}[|h_j(0)|]\,.$$

This finishes the proof that there is no atom at zero for the measure μ_ν. Combined with the first part this gives statement (i) for μ_ν.

It remains to prove (ii). Applying Proposition 5.13 to $g(x) = x$ and taking the expectation immediately gives

$$\mathbb{E}\left[\mathbf{1}_\Gamma(\|u\|^2_{L^2})\sum_{n\in\mathbb{Z}_0^2}|\phi_n|^2|u_n|^2\right] \leqslant \int_\Gamma \mathbb{E}\left[\mathbf{1}_{(a,\infty)}(\|u\|^2_{\dot H^{1+\delta}})\|u\|^2_{L^2}\right]\,\mathrm{d}a\,,$$

and observe that using the bound for $\mathbb{E}_{\mu_\nu}[\|u\|^2_{\dot H^{2+\delta}}] = \mathcal{B}_1$ we infer the bound:

$$\int_\Gamma \mathbb{E}\left[\mathbf{1}_{(a,\infty)}(\|u\|^2_{L^2})\|\nabla^{1+\delta}u\|^2_{L^2}\right]\,\mathrm{d}a \lesssim |\Gamma|\,.$$

For any $N \geqslant 1$, we introduce $\tilde\phi_N := \min\{|\phi_n|, |n| \leqslant N\}$. Then we have:

$$\begin{aligned}\sum_{n\in\mathbb{Z}^2_0}|\phi_n|^2|u_n|^2 &\geqslant \tilde\phi_N^2 \sum_{0<|n|\leqslant N}|u_n|^2\\ &= \tilde\phi_N^2\left(\|u\|^2_{L^2} - \sum_{|n|>N}|u_n|^2\right)\\ &\geqslant \tilde\phi_N^2\left(\|u\|^2_{L^2} - N^{-2(2+\delta)}\|u\|^2_{\dot H^{2+\delta}}\right)\,,\end{aligned}$$

Let $\varepsilon > 0$ and remark that if $\|u\|_{L^2} \geqslant \varepsilon$ and $\|u\|_{\dot H^{2+\delta}} \leqslant \varepsilon^{-1/2}$ we have

$$\sum_{n\in\mathbb{Z}^2_0}|\phi_n|^2|u_n|^2 \geqslant \tilde\phi_N^2(\varepsilon^2 - N^{-2(2+\delta)}\varepsilon^{-1})\,.$$

Let us choose $N := N(\varepsilon) := \left(\frac{\varepsilon^3}{2}\right)^{-\frac{1}{2(2+\delta)}}$ to get

$$\sum_{n\in\mathbb{Z}^2_0}|\phi_n|^2|u_n|^2 \geqslant \frac{1}{2}\varepsilon\tilde\phi^2_{N(\varepsilon)} =: \kappa(\varepsilon)\,,$$

where $\kappa(\varepsilon)$ goes to 0 as $\varepsilon \to 0$. Indeed $N(\varepsilon) \to \infty$ since $(\phi_n)_{n\in\mathbb{Z}^2}$ is summable.

We introduce the set

$$\Omega_\varepsilon := \{v \in L^2, \text{such that } \|v\|_{L^2} \leqslant \varepsilon \text{ or } \|v\|_{\dot H^{2+\delta}} \geqslant \varepsilon^{-\frac{1}{2}}\}\,.$$

Remark that we have

$$\mathbb{P}(\Omega_\varepsilon) \leqslant \mathbb{P}(\|u\|_{L^2} \leqslant \varepsilon) + \mathbb{P}(\|\nabla^{2+\delta}u\|_{L^2} > \varepsilon^{-1/2}) \lesssim \varepsilon\,,$$

thanks to the Markov inequality and (i).

Then we decompose the set $\{\|u\|_{L^2} \in \Gamma\}$ on Ω_ε and Ω_ε^c to obtain the estimate

$$\begin{aligned}\mathbb{P}\left(\|u\|_{L^2} \in \Gamma\right) &= \mathbb{P}\left(\{\|u\|_{L^2} \in \Gamma\} \cap \Omega_\varepsilon\right) + \mathbb{P}\left(\{\|u\|_{L^2} \in \Gamma\} \cap \Omega_\varepsilon^c\right)\\ &\lesssim \mathbb{P}(\Omega_\varepsilon) + \kappa(\varepsilon)^{-1}\mathbb{E}\left[\mathbf{1}_\Gamma\left(\|u\|_{L^2}\sum_{n\in\mathbb{Z}^2_0}|\phi_n|^2|u_n|^2\right)\right]\\ &\lesssim \varepsilon + \kappa(\varepsilon)^{-1}|\Gamma|\,.\end{aligned}$$

Newt, we take $\varepsilon := \ell(\Gamma)$ if $\ell(\Gamma) \neq 0$ so that

$$\mathbb{P}\left(\|u\|_{L^2} \in \Gamma\right) \leqslant p(|\Gamma|)\,,$$

where $p(r) = C\left(r + \kappa(r)^{-1}r\right)$. Extend p by $p(0) = 0$ and remark that if $|\Gamma| = 0$ then we have

$$\mathbb{P}\left(\|u\|_{L^2} \in \Gamma\right) \lesssim \varepsilon \to 0\,,$$

which is coherent with the definition of p at zero. The proof will be complete when we check that p defines indeed a continuous increasing function. It is sufficient to prove that κ defines a decreasing function, which can be seen directly on the definition of κ and can be made continuous up to some minor modification. □

6.2. The three-dimensional case. In dimension 3, Theorem 5.6 will result from the Portmanteau theorem, the weak convergence $\mu_{\nu,N} \to \mu_N$ and $\mu_N \to \mu$ and the following result.

Proposition 5.10. *Let $r > 0$. The measures $(\mu_{N,\nu})_{\nu>0,N>0}$ satisfy the following property: there exists a continuous increasing function $p_r : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $C > 0$ which does not depend on ν nor N such that for every Borel set $\Gamma \subset \mathbb{R}_+$ satisfying $\operatorname{dist}(\Gamma, 0) \geqslant r$, there holds*

$$\mu_{\nu,N}\left(\|u\|_{L^2} \in \Gamma\right) \leqslant p_r(|\Gamma|)\,.$$

Proof. The proof is very similar to the proof of Proposition 5.9 part (ii). We write $u = u_{\nu,N}$ a process satisfying ($\mathrm{HVE}^3_{\nu,N}$) with stationary law $\mu_{\nu,N}$. Applying Proposition 5.13 to $g(x) = x$, taking the expectation and using the bound $\mathbb{E}_{\mu_{\nu,N}}[\|u\|^2_{\dot{H}^{1+\delta}}] \leqslant C$ where C does not depend on ν nor N (see Proposition 5.4), it follows that:

$$\mathbb{E}\left[\mathbf{1}_\Gamma(\|u\|^2_{L^2}) \sum_{1\leqslant j\leqslant N} |\phi_j|^2 |u_j|^2\right] \lesssim |\Gamma|\,,$$

where again $u_j(t) = (u(t), e_j)_{L^2}$. We define

$$\Omega_\varepsilon := \{v \in L^2, \text{such that } \|v\|_{L^2} \leqslant \varepsilon \text{ or } \|v\|_{\dot{H}^{1+\delta}} \geqslant \varepsilon^{-\frac{1}{2}}\}\,,$$

and observe that similarly to the proof of Proposition 5.9, for $u \in \Omega^c_\varepsilon$ we have

$$\sum_{j\geqslant 1} |\phi_j|^2 |u_j|^2 \geqslant \kappa(\varepsilon)\,,$$

for some $\kappa(\varepsilon)$ which goes to 0 as $\varepsilon \to 0$.

Now, for $\varepsilon < r$ we have $\{\|u\|_{L^2} \in \Gamma\} \cap \Omega_\varepsilon \subset \{\|u\|_{\dot{H}^{1+\delta}} > \varepsilon^{-1/2}\}$ hence

$$\mathbb{P}(\{\|u\|_{L^2} \in \Gamma\} \cap \Omega_\varepsilon) \leqslant \mathbb{P}(\|\nabla^{1+\delta} u\|_{L^2} > \varepsilon^{-1/2}) \lesssim \varepsilon\,,$$

thanks to the Markov inequality and Proposition 5.4 which provides estimates uniform in ν and N. Then the proof procceds exactly as the proof of Proposition 5.9. □

Appendix A. Tools from stochastic analysis

This appendix gathers some details about Itô formulas and local times for martingales.

Appendix A.1. About Itô formulas in infinite dimension. In this section we explain how we have applied the Itô formula without mentioning the hypotheses in the previous sections. The version of the Itô formula that we use is due to Shirikyan [Shi02], see also [KS15], Chapter 7 for a textbook presentation.

Definition 5.11. Let us consider a Gelfand triple (V^*, H, V) and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_t)_t$ be the filtration associated to identically distributed independent Brownian motions $(\beta_n(t))_{n\in\mathbb{Z}}$. Let $(e_n)_{n\in\mathbb{Z}}$ be a Hilbertian basis of H and $\phi : H \to H$ a linear map. Let $y(t)$ be a $\mathcal{F}_t$ progressivly measurable process which writes

$$y(t) = y(0) + \int_0^t x(s)\,\mathrm{d}s + \sum_{n\in\mathbb{Z}} \phi(e_n)\beta_n(t)\,.$$

We assume that $u \in \mathcal{C}^0(\mathbb{R}_+, H) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+, V)$, x is almost surely in $L^2_{\mathrm{loc}}(\mathbb{R}_+, V^*)$ and

$$\sum_{n\in\mathbb{Z}} \|\phi(e_n)\|_H^2 < \infty\,.$$

Such a process is called a *standard Itô process.*

Proposition 5.12 (Itô formula)**.** *Let $y(t)$ be a standar Itô process as in Definition 5.11. Let $F : H \to \mathbb{R}$ be twice differentiable and uniformly continuous on bounded subsets. Assume also that:*

(i) Let $T > 0$ and assume that there exists a continuous function K_T such that for all $u \in V, v \in V^*$ there holds

$$|\mathrm{d}F(u; v)| \leqslant K_T(\|u\|_H)\|u\|_V\|v\|_{V^*}\,.$$

(ii) If $w_k \to w$ in V and $v \in V^*$ then $\mathrm{d}F(w_k; v) \to \mathrm{d}F(w; v)$.

Then there holds:

$$\begin{aligned}\mathbb{E}[F(y(t))] &= \mathbb{E}[F(y(0))]\\ &\quad + \int_0^t \mathbb{E}\left[\mathrm{d}F(y(s); x(s)) + \frac{1}{2}\sum_{n\in\mathbb{Z}^2} \mathrm{d}^2F(y(t); \phi(e_n), \phi(e_n))\right]\mathrm{d}t\,.\end{aligned}$$

Proof. This is a combination of Theorem 7.7.5 and the proof of Corollary 7.7.6 in [KS15]. ☐

We are now able to explain the computations that we used previously:

- (Dimension 2). The Itô formula can be applied to the process $u_\nu(t)$ of Section 3 as a Markovian system with Gelfand triplet $(V^*, H, V) := (H^{-\delta}, H^1, H^{2+\delta})$. The functional $F(u) = \|u\|^2_{\dot{H}^1}$ is such that $\mathrm{d}F(\cdot;\cdot)$ is continuous on $V^* \times V$. We also compute $\mathrm{d}^2F(u; v, v) = 2\|v\|^2_{\dot{H}^1}$, and taking into account that $(B(u,u),u)_{H^1} = 0$; which is the key cancellation in dimension 2, at the H^1 regularity, the Itô formula reads

$$\mathbb{E}\left[\|u_\nu(t)\|_{\dot H^1}^2\right] - \mathbb{E}\left[\|u_\nu(0)\|_{\dot H^1}^2\right] = \nu \int_0^t \left(\mathcal{B}_1 - 2\mathbb{E}\left[\|u_\nu(t')\|_{\dot H^{2+\delta}}^2\right]\right) \mathrm{d}t' .$$

- (Dimension 3). We apply the Itô formula to the process $u_{\nu,N}(t)$, as a Markovian system with Gelfand triple defined by $(V^*, H, V) := (H^{-(1+\delta)}, L^2, H^{1+\delta})$. The functional $F(u) := \|u\|_{L^2}^2$ satisfies all the above hypotheses, hence with the fact that $(B_N(u_{\nu,N}, u_{\nu,N}), u_{\nu,N}) = 0$; which is the key cancellation in dimension 3, only at the L^2 level, we obtain:

$$\mathbb{E}\left[\|u_{\nu,N}(t)\|_{L^2}^2\right] - \mathbb{E}\left[\|\mathbf{P}_{\leqslant N} u_0\|_{L^2}^2\right] = \nu \int_0^t \left(\mathcal{B}_0^{(N)} - 2\mathbb{E}\left[\|u_{\nu,N}(t')\|_{\dot H^{1+\delta}}^2\right]\right) \mathrm{d}t' .$$

- (Other cases). Similarly we can apply the Itô formula to $G(u) := e^{\gamma\|u\|_{\dot H^1}^2}$ (resp. $G(u) := e^{\gamma\|u\|_{L^2}^2}$) in dimension 2 (resp. dimension 3). In dimension 2, we have $\mathrm{d}G(u;v) = 2\gamma\langle u, v\rangle_{H^1} e^{\gamma\|u\|_{\dot H^1}^2}$ and the second derivative is

$$\mathrm{d}^2 G(u; v, v) = 2\gamma \left(\|v\|_{\dot H^1}^2 + 2\gamma\langle u, v\rangle_{\dot H^1}^2\right) e^{\gamma\|u\|_{\dot H^1}^2} ,$$

so that all hypotheses of Proposition 5.12 are satisfied. Keeping in mind that we have the cancellation $(u, B(u,u))_{\dot H^1} = 0$, the Itô formula now reads:

$$\mathbb{E}\left[e^{\gamma\|u(t)\|_{\dot H^1}^2}\right] = \mathbb{E}\left[e^{\gamma\|u_0\|_{\dot H^1}^2}\right] + 2\gamma\nu\mathbb{E}\left[\int_0^t e^{\gamma\|u(t')\|_{\dot H^1}^2} \left(\frac{\mathcal{B}_1}{2} - \|u(t')\|_{\dot H^{2+\delta}}^2 + \gamma \sum_{n\in\mathbb{Z}^2} |n|^2|\phi_n|^2|u_n(t')|^2\right) \mathrm{d}t'\right] ,$$

where $u_n(t) = (u(t), e_n)_{L^2}$.

Appendix A.2. Local times for martingales. This appendix gathers some preliminary material used in Section 6. We start with the main abstract result on local times for martingales and explain how it applies to our purposes.

Theorem 5.44 (See [KS91] Theorem 7.1 in Chapter 3)**.** *Let $y(t)$ be a standard Itô process of the form*

$$y(t) = y(0) + \int_0^t x(s)\,\mathrm{d}s + \sum_{n\in\mathbb{Z}} \int_0^t \theta_n(s)\,\mathrm{d}\beta_n(s) ,$$

where $x(t), \theta_n(t)$ are $\mathcal{F}_t$-adapted processes such that there holds

$$\mathbb{E}\left[\int_0^t \left(|x(s)| + \sum_{n\in\mathbb{Z}} |\theta_n(s)|^2\right) \mathrm{d}s\right] < \infty \text{ for any } t > 0 .$$

Then there exists a random field that we denote by $\Lambda_t(a,\omega)$, for $t \geqslant 0$, $a \in \mathbb{R}$, $\omega \in \Omega$ such that the following properties hold.

(i) $(t, a, \omega) \mapsto \Lambda_t(a, \omega)$ *is measurable and for any* $a \in \mathbb{R}$ *the process* $t \mapsto \Lambda_t(a, \cdot)$ *is* $\mathcal{F}_t$*-adapted continuous and non-decreasing. For any* $t \geqslant 0$*, and almost every* $\omega \in \Omega$ *the function* $a \mapsto \Lambda_t(a, \omega)$ *is right-continuous.*

(ii) For any non-negative Borel function $g : \mathbb{R} \to \mathbb{R}$ *and with probability* 1 *we have for any* $t \geqslant 0$

$$\int_0^t g(y(s)) \left(\sum_{n \in \mathbb{Z}} |\theta_n(s)|^2 \right) \mathrm{d}s = 2 \int_{\mathbb{R}} \Lambda_t(a, \omega) \, \mathrm{d}a \,. \tag{5.34}$$

(iii) For any convex function $f : \mathbb{R} \to \mathbb{R}$ *and with probability* 1 *holds*

$$\begin{aligned} f(y(t)) = f(y(0)) + \int_0^t \partial^- f(y(s)) x(s) \, \mathrm{d}s + \int_{\mathbb{R}} \Lambda_t(a, \omega) \partial^2 f(\mathrm{d}a) \\ + \sum_{n \in \mathbb{Z}^2} \int_0^t \partial^- f(y(s)) \theta_n(s) \, \mathrm{d}\beta_n(s) \,, \end{aligned} \tag{5.35}$$

and where $\partial^- f$ *stands for the subdifferential of* f*.*

In order to be applied in our context, we make the following remarks:

- When (5.35) is applied to the convex function $f : x \mapsto (x - a)_+$ we obtain

$$\begin{aligned} (y(t) - a)_+ - (y(0) - a)_+ = \int_0^t \mathbf{1}_{[a, \infty)}(y(s)) x(s) \, \mathrm{d}s + \Lambda_t(a, \omega) \\ + \sum_{n \in \mathbb{Z}} \int_0^t \mathbf{1}_{[a, \infty)}(y(s)) \theta_n(s) \, \mathrm{d}\beta_n(s) \,. \end{aligned}$$

  Then, assuming that both $y(s), \theta_n(s)$ and $x(s)$ are stationary processes we deduce:

$$\mathbb{E} \left[\Lambda_t(a, \omega) \right] = -t \mathbb{E} \left[\mathbf{1}_{[a, \infty)}(y(0)) x(0) \right] \,. \tag{5.36}$$

- Let $\Gamma \subset \mathbb{R}$ be a Borelian, apply (5.34) to $g = \mathbf{1}_\Gamma$ and take the expectation. It writes:

$$\int_\Gamma \mathbb{E} \left[\Lambda_t(a) \right] \, \mathrm{d}a = \frac{t}{2} \mathbb{E} \left[\mathbf{1}_\Gamma(y(0)) \left(\sum_{n \in \mathbb{Z}} |\theta_n(0)|^2 \right) \right] \,. \tag{5.37}$$

Then we can prove the following result.

Proposition 5.13. *Let* μ_ν *be a stationary measure for* (HVE^2_ν) *constructed in Section 4. For any borel set* $\Gamma \subset \mathbb{R}_+$ *and any function* $g \in \mathcal{C}^2(\mathbb{R})$ *whose second* *derivative has at most polynomial growth at infinity we have*

$$\mathbb{E}_{\mu_\nu}\left[\int_\Gamma \mathbf{1}_{(a,\infty)}(g(\|u\|_{L^2}^2))\left(g'(\|u\|_{L^2}^2)\left(\frac{\mathcal{B}_0}{2} - \|\nabla^{1+\delta}u\|_{L^2}^2\right) + g''(\|u\|_{L^2}^2)\sum_{n\in\mathbb{Z}^2}|\phi_n|^2|u_n|^2\right)\mathrm{d}a\right]$$
$$+\sum_{n\in\mathbb{Z}^2}|\phi_n|^2\mathbb{E}_{\mu_\nu}\left[\mathbf{1}_\Gamma(g(\|u\|_{L^2}^2))(g'(\|u\|_{L^2}^2)|u_n|)^2\right] = 0\,,$$

where $u_n = (u, e_n)_{L^2}$.

Proof. The proof of this proposition comes from the previous identities. Indeed, let the functional $F : H^1 \to \mathbb{R}$ defined by $F(u) := g(\|u\|_{L^2}^2)$ so that

$$F'(u; v) = 2g'(\|u\|_{L^2}^2)(u, v)_{L^2}$$

and

$$F''(u; v, v) = 2g'(\|u\|_{L^2}^2)\|v\|_{L^2}^2 + 4g''(\|u\|_{L^2}^2)(u, v)_{L^2}^2\,.$$

With such a function g and the process $f(t) := g(\|u(t)\|_{L^2}^2)$, the Itô formula now reads

$$f(t) = f(0) + \nu\int_0^t A(s)\,\mathrm{d}s + 2\sqrt{\nu}\sum_{n\in\mathbb{Z}^2}\phi_n\int_0^t g'(\|u\|_{L^2}^2)u_n\,\mathrm{d}\beta_n(s)\,, \tag{5.38}$$

where after integration by parts,

$$A(t) := \left(g'(\|u\|_{L^2}^2)\left(\mathcal{B}_0 - 2\|u\|_{\dot{H}^{1+\delta}}^2\right) + 4g''(\|u\|_{L^2}^2)\sum_{n\in\mathbb{Z}^2}|\phi_n|^2u_n^2\right).$$

Remark that for each $n \in \mathbb{Z}^2$, the process $t \mapsto g'(\|u(t)\|_{L^2})(u(t), e_n)_{L^2}$ is stationnary. Indeed, $u(t)$ is an H^1 stationary process and the map $G : u \mapsto g'(\|u\|_{L^2}^2)(u, e_n)_{L^2}$ is continuous from H^1 to $\mathbb{C}$, thus Borelian.

Then by Theorem 5.44, and more precisely, equation (5.37) we have

$$\int_\Gamma \mathbb{E}[\Lambda_t(a)]\,\mathrm{d}a = 2\nu t\sum_{n\in\mathbb{Z}^2}|\phi_n|^2\mathbb{E}\left[\mathbf{1}_\Gamma(f)(g'(\|u\|_{L^2}^2)|u_n|)^2\right]\,,$$

and (5.36) yields

$$\mathbb{E}[\Lambda_t(a)] = -\nu t\mathbb{E}[\mathbf{1}_{(a,\infty)}(f(0))A(0)]\,,$$

which concludes the proof. □

Appendix A.3. Questions of measurability. Finally, we give an easy result, which is used through the text, sometimes without mentioning it.

Lemma 5.45 (Measurability)**.** *Let* $s_1 > s_2 \geqslant 0$. *Then any Borelian subset of* H^{s_1} *(with respect to the* H^{s_1} *topology) is a Borelian subset of* H^{s_2} *(with respect to the* H^{s_2} *topology). Consequently, the map* $F : H^{s_2} \to \mathbb{R}$ *defined by* $F(u) = \|u\|_{H^{s_1}}$ *is Borelian.*

Proof. It is sufficient to prove that the closed ball $\bar{B}_R := \{\|u\|_{H^{s_1}} \leqslant R\}$ is a Borelian subset of H^{s_2}. Then we can write:

$$\bar{B}_R = \bigcap_{N \geqslant 1} \{u \in H^{s_2},\ \|\mathbf{P}_{\leqslant N} u\|_{H^{s_1}} \leqslant R\} \ .$$

It remains to explain why the set $\{u \in H^{s_2},\ \|\mathbf{P}_{\leqslant N} u\|_{H^{s_1}} \leqslant R\}$ is a Borelian subset of H^{s_2}. This comes from the fact that the map $\mathbf{P}_{\leqslant N} : H^{s_2} \to H^{s_1}$ is continuous. $\square$

CHAPITRE 6

Probabilistic Local Well-Posedness for the Schrödinger Equation Posed for the Grushin Laplacian

In collaboration with Louise Gassot

ABSTRACT. We study the local well-posedness of the nonlinear Schrödinger equation associated to the Grushin operator with random initial data. To the best of our knowledge, no well-posedness result is known in the Sobolev spaces H^k when $k \leqslant \frac{3}{2}$. In this article, we prove that there exists a large family of initial data such that, with respect to a suitable randomization in H^k, $k \in (1, \frac{3}{2}]$, almost-sure local well-posedness holds. The proof relies on bilinear and trilinear estimates.

1. Introduction

1.1. The Schrödinger equation on the Heisenberg group and the Grushin equation. We consider the Grushin-Schrödinger equation

$$i\partial_t u - \Delta_G u = |u|^2 u\,, \tag{NLS-G}$$

where $(t, x, y) \in \mathbb{R} \times \mathbb{R}^2$ and $\Delta_G = \partial_{xx} + x^2\partial_{yy}$. The natural associated Sobolev spaces in this case are the Grushin Sobolev spaces H^k_G on $\mathbb{R}^2$, defined by replacing powers of the usual operator $\sqrt{-\Delta}$ by powers of $\sqrt{-\Delta_G}$.

This equation is a simplification of the semilinear Schrödinger equation on the Heisenberg group in the radial case

$$i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u\,, \tag{NLS-$\mathbb{H}^1$}$$

where $(t, x, y, s) \in \mathbb{R} \times \mathbb{H}^1$. In the radial case, the solution u only depends on t, $|x+iy|$ and s and the sub-Laplacian is written $\Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_{xx}+\partial_{yy})+(x^2+y^2)\partial_{ss}$. Our simplification of this equation consists in removing one of the two variables x, y since they play the same role, leading to (NLS-G).

When $k > k_C$ where $k_C = 2$ (resp. $k_C = \frac{3}{2}$ for (NLS-G)), one can use the algebra property of the spaces $H^k(\mathbb{H}^1)$ (resp. H^k_G)) and solve the Cauchy problem associated to (NLS-$\mathbb{H}^1$) (resp. (NLS-G)) locally in time, see Appendix Appendix B for details. However, the conservation of energy only controls the H^k norm when $k = 1$, and since no conservation law is known for $k > 1$, we have no information about global existence of maximal solutions in the range of Sobolev exponents $k > k_C$.

For Sobolev exponents below the critical exponent k_C, existence and uniqueness of general weak solutions is an open problem. To go further, the Schrödinger equation on the Heisenberg group displays a total lack of dispersion [BGX00], implying that the flow map for (NLS-$\mathbb{H}^1$) (resp. (NLS-G)) cannot be smooth in the Sobolev spaces H^k when $k < k_C$. We refer to the introduction of [GG10] and Remark 2.12 in [BGT04] for details.

1.2. Main results. In this subsection we will only introduce the needed notations to state our main result, we refer to Section 2 for precise definitions.

Fix u_0 in some Sobolev space H^k_G for $k > 0$. Then u decomposes as a sum

$$u_0 = \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} u_{I,m}\,, \tag{6.1}$$

where the $u_{I,m}$ will be defined by (6.8).

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. We consider a sequence $(X_{I,m})_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}}$ of independent and identically distributed Gaussian random variables and define the measure μ_{u_0} as the image measure of $\mathbb{P}$ under the *randomization map*

$$\omega \in \Omega \mapsto u_0^\omega := \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} X_{I,m}(\omega) u_{I,m}\,. \tag{6.2}$$

For $k \geqslant 0$ and $\rho \geqslant 0$, we introduce the subspace $\mathcal{X}^k_\rho$ of H^k_G in which we will prove almost-sure local well-posedness. Denoting $\langle x\rangle = \sqrt{1+x^2}$, the space $\mathcal{X}^k_\rho$ corresponds to the norm

$$\|u_0\|^2_{\mathcal{X}^k_\rho} := \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} (1+(2m+1)I)^k \langle I\rangle^\rho \|u_{I,m}\|^2_{L^2_G}\,. \tag{6.3}$$

The powers $(1+(2m+1)I)^k$ refer to the Sobolev regularity: for instance, when $\rho = 0$, then $\|u_0\|_{\mathcal{X}^k_0} \sim \|u_0\|_{H^k_G}$. However, the powers $\langle I\rangle^\rho$ only corresponds to partial regularity with respect to the last variable, see the precise definition of decomposition (6.1) in Section 2 and Remark 6.11 for details.

Our main result is the following.

Theorem 1 (Local Cauchy Theory for (NLS-G))**.** *Let $k \in (1, \frac{3}{2}]$ and $u_0 \in \mathcal{X}^k_1 \subset H^k_G$.*

(i) For any $\ell \in (\frac{3}{2}, k+\frac{1}{2})$, for almost-every $\omega \in \Omega$, there exists $T > 0$ and a unique local solution with initial data u_0^ω to (NLS-G) in the space

$$e^{it\Delta_G} u_0^\omega + \mathcal{C}^0([0,T), H^\ell_G) \subset \mathcal{C}^0([0,T), H^k_G)\,.$$

More precisely, there exists $c > 0$ such that for all $R \geqslant 1$, outside a set of probability at most e^{-cR^2}, one can choose $T \geqslant (R\|u_0\|_{\mathcal{X}^k_1})^{-2}$.

(ii) (Non-smoothing under randomization) If $u_0 \in H^k_G \setminus (\bigcup_{\varepsilon>0} H^{k+\varepsilon}_G)$*, then*

$$\mathrm{supp}(\mu_{u_0}) \subset H^k_G \setminus (\bigcup_{\varepsilon>0} H^{k+\varepsilon}_G)\,.$$

(iii) (Density of measures with rough potentials) Let $\varepsilon > 0$*, then there exists* $v_0 \in \mathcal{X}^k_1 \setminus (\bigcup_{\varepsilon'>0} H^{k+\varepsilon'}_G)$ *such that*

$$\mathrm{supp}(\mu_{v_0}) \cap B_{H^k_G}(u_0, \varepsilon) \neq \varnothing\,.$$

Remark 6.1 (Consequences of (i)). For every $k > 1+2\varepsilon > 1$ and $u_0 \in \mathcal{X}^k_{1+2\varepsilon+1-k}$, the continuous embedding $\mathcal{X}^k_{1+2\varepsilon+1-k} \hookrightarrow \mathcal{X}^{1+2\varepsilon}_1$ implies that for almost-every $\omega \in \Omega$, the initial data u^ω_0 gives rise to a unique local solution

$$u \in e^{it\Delta_G} u^\omega_0 + \mathcal{C}^0([0,T), H^{\frac{3}{2}+\varepsilon}_G)\,,$$

where we check that $\ell = \frac{3}{2} + \varepsilon \in (\frac{3}{2}, 1 + 2\varepsilon + \frac{1}{2})$.

Therefore, in the case $k = \frac{3}{2}$, we observe that the limiting almost-sure well-posedness space is $\bigcap_{\varepsilon>0} \mathcal{X}^{\frac{3}{2}}_{\frac{1}{2}+\varepsilon}$. We recall that for $k = \frac{3}{2} + \varepsilon$, local well-posedness is known to hold in $H^{\frac{3}{2}+\varepsilon}_G = \mathcal{X}^{\frac{3}{2}+\varepsilon}_0$. It is interesting to note that our approach looses an exponent $\rho = \frac{1}{2}$, since we do not recover the same limit space in the limit $k \to \frac{3}{2}$.

Remark 6.2 (Decomposition of the solution). In (i), we claim that it is possible to construct local solutions to (NLS-G) in the space $\mathcal{C}([0,T), H^k_G)$ for small values of k. However, uniqueness holds only on a smaller subset, as a consequence of an *a priori* decomposition of the solution as sum of the solution to the linear equation (LS-G) with initial data u^ω_0 and a smoother part. This decomposition can be interpreted as a more *nonlinear* decomposition of the solution than seeking $v \in H^k_G$, as we seek for u in an affine space $e^{it\Delta_G} u^\omega_0 + H^\ell_G$ instead of a vector space. Such a decomposition is the simplest nonlinear decomposition, akin to [BT08a], and is a key feature in most random data well-posedness works. More recently, even more nonlinear decompositions for the solutions to some dispersive equations have been exhibited in [OTW20, DNT19] (see also [GIP15, Hai13, Hai14] in the context of stochastic equations).

Remark 6.3 (Regularity and density of the measures). Parts (ii) and (iii) give regularity properties of the measures μ_{u_0}. In fact, (ii) ensures that the measure μ_{u_0} does not charge solutions which are more regular than u_0. Actually, the measure μ_{u_0} charges solutions that have regularity $W^{k+1/4,4}_G$ on L^p based Sobolev spaces, but no better regularity bound is expected to hold (see Proposition 6.3 and Remark 6.23), so this estimate alone is not enough to establish Theorem 1.

Statement (iii) goes even further since we prove that

$$\bigcup \Big\{ \operatorname{supp}(\mu_{v_0}) \mid v_0 \in \mathcal{X}_1^k \setminus (\bigcup_{\varepsilon>0} H_G^{k+\varepsilon}) \Big\}$$

is dense in H_G^k. This result is related to the support density in the Euclidean case. Indeed, for the nonlinear Schrödinger equation on the torus, probabilistic local well-posedness holds with respect to measures which are dense in Sobolev spaces, see for instance Appendix B of [BT08a] and [BT14].

Remark 6.4 (Admissible initial data). When u_0 only has a finite number of modes m, the assumption $u_0 \in \mathcal{X}_1^k$ is equivalent to the condition $u_0 \in H_G^{k+1}$, but since $k+1 > k_C = \frac{3}{2}$, the result is void. For this reason, our result does not extend to the nonlinear half-wave equation $\partial_t u \pm \sqrt{-\Delta} u = |u|^2 u$, which also admits a similar decomposition to (6.1) but only with a finite number of modes m. However, we will see in Remark 6.11 that the condition $u_0 \in \mathcal{X}_1^k$ still allows a general set of low regularity initial data in our context.

Remark 6.5 (Defocusing case). One can replace equation (NLS-G) by its defocusing variant and get the exact same local well-posedness theory. Indeed, we only address local well-posedness, which mainly depends on the order of magnitude of the nonlinearity and not its sign.

Remark 6.6 (Randomization). The measures μ_{u_0} are defined in (6.2) with Gaussian random variables. However, most of the results in this article are stated for more general subgaussian random variables (see Definition 6.2), except when using the Wiener chaos estimates from Corollary 6.20, which is stated only for Gaussian random variables.

Note that the randomization along on a unit scale in the variable m is quite classical, as this is the variable along which we establish our bilinear estimates. However, the variable I plays a different role which allows us to only take a randomization on a dyadic scale. One could compare this choice with the construction of adapted dilated cubes in [BOP15b].

1.3. Further work. We expect that for the Schrödinger equation (NLS-$\mathbb{H}^1$) on the Heisenberg group, the local well-posedness theory for the randomized Cauchy problem holds with a same gain of almost $\frac{1}{2}$ derivative compared to the critical exponent $k_C = 2$. More precisely, for $u_0 \in H^k(\mathbb{H}^1)$, $k \in (\frac{3}{2}, 2]$, there holds a decomposition similar to (6.1). Assuming that u_0 belongs to a space similar to (6.3), we conjecture that there exists a unique local solution with random initial data u_0^ω in the space $e^{it\Delta_{\mathbb{H}^1}} u_0^\omega + \mathcal{C}^0([0,T), H^\ell(\mathbb{H}^1)) \subset \mathcal{C}^0([0,T), H^k(\mathbb{H}^1))$ with $\ell \in (2, k+\frac{1}{2})$. This will be the object of a subsequent work. In the non radial case, we would have to tackle the additional terms in the expression of the sub-Laplacian on the Heisenberg group $\mathcal{L} = \Delta_{\mathbb{H}^1} + (y\partial_x - x\partial_y)\partial_s$.

1.4. Deterministic and probabilistic Cauchy theory for (NLS-$\mathbb{H}^1$). As mentioned at the beginning of this introduction, the nonlinear Schrödinger equation on the Heisenberg group lacks dispersion, therefore the dispersive paradigm cannot be applied for lowering the critical well-posedness exponent below $k_C = 2$ (resp. $k_C = \frac{3}{2}$ for (NLS-G)) given by the Sobolev embedding. More precisely, the lack of dispersion precludes the usual way in which Strichartz estimates are proven, that is a combination of a dispersive estimate and a duality TT^* argument. The result in [BGX00] goes even further, as the non smoothness of the flow map for (NLS-$\mathbb{H}^1$) in H^k for $k < k_C$ makes it impossible to implement a fixed point argument.

The lack of dispersion and the lack of Strichartz estimates for the Schrödinger equation on the Heisenberg group have been recently investigated in [BG20] and [BBG19]. In these works, the authors prove that there exist anisotropic Strichartz estimates [BBG19], local in space versions of the dispersive estimates (Theorem 1 in [BG20]) and local version of Strichartz estimates (Theorem 3 in [BG20]). These results follow the general strategy of Fourier restriction methods for proving Strichartz estimates, dating back to Strichartz [Str77], and use the Fourier analysis on the Heisenberg group [BCD18, BCD19]. We also refer to [Mül90] for restriction theorems on the Heisenberg group.

Probabilistic methods have proven to be very useful to break the scaling barrier in the context of dispersive equations. Such a study has been pioneered in [Bou94]: the purpose is to construct global solutions for nonlinear Schrödinger equations posed on the torus, using invariant measures and a probabilistic local Cauchy theory. In [BT08a, BT08b], the authors extend these results to other dispersive equations, opening the way to a very active area of research and leading to an immense body of results.

Invariant measure methods mostly reduce their scope to compact spaces, the setting of the torus being used on many works. For non-compact spaces, the probabilistic method of [BT08a] remains largely adaptable through the use of Gaussian random initial data. We refer for example to [BOP15b, BOP15a] where probabilistic local well-posedness is obtained for the nonlinear Schrödinger equations on $\mathbb{R}^d$, and to [OP16, Poc17] for similar results with the wave equation.

Several works go beyond the Euclidean Laplacian. For instance, in [BTT13] the authors replace the standard Laplacian $-\Delta$ with a harmonic oscillator $-\Delta + x^2$ and study the local Cauchy theory for the associated nonlinear Schrödinger equation. Our work is partly inspired from this work, and also subsequent works [Den12, BT20, Lat20a]. Indeed, in our case, rescaled harmonic oscillators parameterized by one of the variables appear when one considers a partial Fourier transform of the equation.

We point out that although no progress had been obtained in the direction of local well-posedness up to now, traveling waves and their stability have been studied in [Gas21, Gas20].

1.5. Outline of the proof and main arguments. In this section, we briefly review the main ideas leading to the proof of Theorem 1.

1.5.1. *General strategy for almost-sure local well-posedness.* We follow the probabilistic approach to the local well-posedness problem from [BT08a] in order to study (NLS-G). We fix $u_0 \in H^k_G$, where $0 < k < k_C = \frac{3}{2}$, and consider the randomization u_0^ω defined in (6.2). We seek for solutions to (NLS-G) under the form

$$u(t) = e^{it\Delta_G} u_0^\omega + v(t)$$

where $v(t)$ belongs to some space $H_G^{\frac{3}{2}+\varepsilon}$, $\varepsilon > 0$, on which a deterministic local well-posedness theory is known to hold. Plugging this ansatz in the Duhamel representation of (NLS-G) leads to

$$u(t) = e^{it\Delta_G} u_0^\omega - \int_0^t e^{i(t-t')\Delta_G} \left(|e^{it'\Delta_G} u_0^\omega + v(t')|^2 (e^{it'\Delta_G} u_0^\omega + v(t')) \right) \mathrm{d}t' ,$$

so that with the notation $z^\omega(t) = e^{it\Delta_G} u_0^\omega$, we expect that $v(t) = \Phi v(t) \in H_G^{\frac{3}{2}+\varepsilon}$, where

$$\Phi v(t) = -i \int_0^t e^{i(t-t')\Delta_G} \left(|z^\omega(t') + v(t')|^2 (z^\omega(t') + v(t')) \right) \mathrm{d}t' .$$

In view of the lack of Strichartz estimates, the best known bounds on Φv are the trivial estimates (we take $t \leqslant T$ and forget time estimates, as we only give heuristic arguments)

$$\begin{aligned} \|\Phi v\|_{H_G^{\frac{3}{2}+\varepsilon}} &\lesssim \|(v + z^\omega)^3\|_{H_G^{\frac{3}{2}+\varepsilon}} \\ &\lesssim \|v^3\|_{H_G^{\frac{3}{2}+\varepsilon}} + \|(z^\omega)^3\|_{H_G^{\frac{3}{2}+\varepsilon}} + \|z^\omega v^2\|_{H_G^{\frac{3}{2}+\varepsilon}} + \|(z^\omega)^2 v\|_{H_G^{\frac{3}{2}+\varepsilon}} . \end{aligned}$$

The term v^3 is handled using the algebra property of the space $H_G^{\frac{3}{2}+\varepsilon}$, since v has high regularity. The terms involving z^ω are mode difficult because $z^\omega \in H^k_G \setminus H_G^{\frac{3}{2}+\varepsilon}$ only has H^k regularity since the randomization does not gain derivatives, as stated in Theorem 1 (*ii*) and proven in Section 3. A first approach would be to estimate

$$\|(z^\omega)^3\|_{H_G^{\frac{3}{2}+\varepsilon}} \sim \|\langle -\Delta_G\rangle^{\frac{3}{4}+\frac{\varepsilon}{2}} z^\omega (z^\omega)^2\|_{L^2_G} \lesssim \|z^\omega\|_{W_G^{\frac{3}{2},\infty}} \|z^\omega\|^2_{L^4_G} ,$$

thus it is important to study the effect of the randomization on u_0 in terms of regularity in L^p based Sobolev spaces. As proven in Proposition 6.3, linear estimates in $W_G^{k,p}$ spaces gain up to $\frac{1}{4}$ derivatives. However, the linear estimates alone are not sufficient to gain the $\frac{1}{2}$ derivatives in regularity and therefore deal with low values of k in Theorem 1.

In order to improve our estimates, we establish bilinear estimates: we prove that given the random solutions z^ω with initial data in H^k_G to the linear Schrödinger equation

$$i\partial_t z - \Delta_G z = 0\,, \tag{LS-G}$$

then almost-surely we have $(z^\omega)^2 \in H_G^{k+\frac{1}{2}}$. In this article, we prove the following bilinear and trilinear estimates in the spaces $\mathcal{X}^k_\rho$, which could be of independent interest.

Theorem 2 (Bilinear and trilinear estimates for random solutions)**.** *Let $k \in (1, \frac{3}{2}]$ and $u_0 \in \mathcal{X}^k_1 \subset H^k_G$ (see (6.3)). Let u_0^ω as in (6.2), and let us denote by $z^\omega = e^{it\Delta_G}u_0^\omega \in \mathcal{C}^0(\mathbb{R}, H^k_G)$ the solution to (LS-G) associated to u_0^ω. Then there exists $c > 0$ such that the following statements hold. Fix $q \in [2, \infty)$. For $T > 0$, denote $L^q_T := L^q([0, T])$.*

(i) For all $R \geqslant 1$ and $T > 0$, outside a set of probability at most e^{-cR^2}, one has

$$\|(z^\omega)^2\|_{L^q_T H_G^{k+\frac{1}{2}}} + \||z^\omega|^2\|_{L^q_T H_G^{k+\frac{1}{2}}} \leqslant R^2 T^{\frac{1}{q}} \|u_0\|^2_{\mathcal{X}^k_1}\,, \tag{6.4}$$

$$\||z^\omega|^2 z^\omega\|_{L^q_T H_G^{k+\frac{1}{2}}} \leqslant R^3 T^{\frac{1}{q}} \|u_0\|^3_{\mathcal{X}^k_1}\,. \tag{6.5}$$

(ii) We further require that $u_0 \in \mathcal{X}^k_{1+\varepsilon_0}$ for some $\varepsilon_0 > 0$. Let $\ell < k + \frac{1}{2}$. For all $R \geqslant 1$ and $T > 0$, there exists a set $E_{R,T}$ of probability at least $1 - e^{-cR^2}$ such that the following holds. Fix $\omega \in E_{R,T}$ and $v, w \in L^\infty_T H^\ell_G$, then

$$\|z^\omega v w\|_{L^q_T H^\ell_G} \leqslant R T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}^k_{1+\varepsilon_0}} \|v\|_{L^\infty_T H^\ell_G} \|w\|_{L^\infty_T H^\ell_G}\,. \tag{6.6}$$

Note that v and w may depend on ω.

Remark 6.7. The time variable does not play an important role. Indeed, in the course of the proof, we establish deterministic pointwise estimates in the time variable, and the L^q_T norm instead of L^∞_T only appears in order to apply the Khinchine inequality and Wiener chaos estimates from part 2.4. In comparison, bilinear smoothing estimates for the nonlinear Schrödinger equation on $\mathbb{R}^2$ crucially exploit the time variable, as the smoothing occurs in time averages.

The heuristic explained above for proving Theorem 1 by using Theorem 2 is implemented rigorously in Section 8.

1.5.2. *Mltilinear random estimates.* The bulk of this paper aims at establishing Theorem 2, thus we briefly outline the main aspects of the proof.

First, because of the random nature of the z^ω, we use random decoupling in order to reduce the estimates to “building block estimates”, that is estimating products $\|u_a v_b w_c\|_{H_G^{\frac{3}{2}+\varepsilon}}$ for u_a, v_b and w_c obtained by restricting the Sobolev frequencies of u, v and w around the values a, b and c. This reduction is a consequence of Corollary 6.20.

In the Euclidean setting, the Bernstein estimates and the Littlewood-Paley decomposition would justify the heuristics $\nabla(u_a v_b w_c) \simeq \nabla(u_a) v_b w_c + u_a \nabla(v_b) w_c + u_a v_b \nabla(w_c)$ and thus reduce the analysis of $\|u_a v_b w_c\|_{H_G^{\frac{3}{2}+\varepsilon}}$ to that of $\|u_a v_b w_c\|_{L_G^2}$. In our case, this results still holds, but rigorous justification is more intricate and is the content of Section 4.

The purpose of Section 5 is to prove "building-block" estimates of the form

$$\|u_a v_b\|_{L_G^2} \leqslant \frac{C}{\max\{a,b\}^{\frac{1}{2}}} \|u_a\|_{L_G^2} \|v_b\|_{L_G^2} \,.$$

To give the main idea of the proof, it is instructing to see that in partial Fourier transform along the last variable, we can think of u_a and v_b as

$$\mathcal{F}_{y\to\eta}(u_a)(x,\eta) = f(\eta) h_m(|\eta|^{\frac{1}{2}} x) \text{ and } \mathcal{F}_{y\to\eta}(v_b)(x,\eta) = g(\eta) h_n(|\eta|^{\frac{1}{2}} x)\,,$$

where h_m, h_n are Hermite functions. Thus we can see that estimating $u_a u_b$ involves estimating convolution products of rescaled Hermite functions. The key fact is now that Hermite functions, due to their localization and normalization, enjoy bilinear estimates that are better than trivial Hölder bounds, see [BTT13], relying on pointwise estimates from [KT05], see also [KTZ07].

1.5.3. *Bilinear random-deterministic estimates.* It turns out to be more difficult to prove a multilinear estimate on probabilistic-detemrinistic products such as $u^\omega v w$, where v and w are deterministic, which is the content of (6.6). In this case, one should pay attention that the required set of ω constructed in Theorem 2 (ii) does not depend on v and w. This precludes a direct use of decoupling offered by Corollary 6.20 as exploited in the proof of Theorem B (i). To understand the difference between the treatment of $|z^\omega|^2 z^\omega$ and $z^\omega v w$, remark that for example in (6.27) the set of ω which is removed depends on the $z_{I,m}$, that is on z, which is fixed in Theorem 2. In the case of $z^\omega v^2$ this would remove a set of ω depending on v and w.

In order to circumvent such a difficulty, the idea is to apply probabilistic decoupling only on terms involving the random part z^ω. The implementation of this strategy is carried out in Section 7 and relies on a preparatory step introduced in Section 7.1 aiming at splitting the analysis of $z^\omega v w$ into a deterministic part $\mathbf{K} := \mathbf{K}(v,w)$ and a probabilistic part $\mathbf{J}^\omega := \mathbf{J}(z^\omega)$, which are treated in Section 7.2 and Section 7.3.

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2. Notation and preliminary estimates

The purpose of this section is twofold. First we introduce decomposition (6.1), which is due to the structure of the Grushin operator. Then, we recall some useful estimates, such as Sobolev embeddings, product laws, eigenfunction estimates and probabilistic decoupling estimates.

We will use the notation $f \lesssim g$ to denote that there exists $C > 0$ such that $f \leqslant Cg$.

2.1. Decomposition along Hermite functions for the Grushin operator.

In this subsection we give an explicit description of the Grushin operator $\Delta_G = \partial_x^2 + x^2\partial_y^2$, acting on $L^2(\mathbb{R}^2)$.

Let us consider the orthonormal basis of $L^2(\mathbb{R})$ given by the Hermite functions $(h_m)_{m\geqslant 0}$. By definition, the Hermite functions are eigenfunctions of the harmonic oscillator: for all $m \geqslant 0$, we have

$$(-\partial_x^2 + x^2)h_m = (2m+1)h_m\,.$$

Taking the Fourier transform in y, with Fourier variable η, we observe that for all $\eta \in \mathbb{R}$, we have

$$(-\partial_x^2 + x^2\eta^2)h_m(|\eta|^{\frac{1}{2}}x) = (2m+1)|\eta| h_m(|\eta|^{\frac{1}{2}}x)\,.$$

Therefore, one can decompose the Fourier transform $\mathcal{F}_{y\to\eta}(u)(\cdot,\eta)$ of $u \in H_G^k$ along the basis $(h_m(|\eta|^{\frac{1}{2}}\cdot))_{m\geqslant 0}$, so u becomes a sum

$$\mathcal{F}_{y\to\eta}(u)(x,\eta) = \sum_{m\geqslant 0} f_m(\eta)h_m(|\eta|^{\frac{1}{2}}x)\,.$$

Moreover, this decomposition is invariant by the action of the Grushin operator $-\Delta_G$, so that we can explicitly write the H_G^k norm as

$$\|u\|_{H_G^k}^2 := \|(\mathrm{Id} - \Delta_G)^{\frac{k}{2}}\|_{L_G^2}^2 = \sum_{m\geqslant 0}\int_{\mathbb{R}} (1+(2m+1)|\eta|)^k |f_m(\eta)|^2 |\eta|^{-\frac{1}{2}}\,\mathrm{d}\eta\,. \tag{6.7}$$

Remark 6.8. The quantity $(2m+1)|\eta|$ plays the role of "taking two" derivatives, that is a similar role as the Fourier variable $|\xi|^2$ in the context of the Euclidean Sobolev spaces. Keep in mind that however this quantity mixes the Hermite modes m with the partial Fourier variable η.

Remark 6.9. In (6.7), the extra factor $|\eta|^{-\frac{1}{2}}$ should be understood as a normalization factor. Indeed, the L_x^2 norm of the function $x \mapsto h_m(|\eta|^{\frac{1}{2}}x)$ is $|\eta|^{-\frac{1}{4}}$.

In order to deal with the Sobolev norms, we further decompose u according to its regularity in the y variable as (6.1)

$$u = \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} u_{I,m}\,.$$

The definition for the $u_{I,m}$ is the following. Taking the Fourier transform along the y variable, the support of $\mathcal{F}_{y\to\eta}(u_{I,m})(x,\eta)$ satisfies the condition $|\eta|\in[I,2I]$ for some dyadic relative integer $I\in 2^{\mathbb{Z}}$:

$$\mathcal{F}_{y\to\eta}(u_{I,m})(x,\eta)=f_m(\eta)h_m(|\eta|^{\frac{1}{2}}x)\mathbf{1}_{|\eta|\in[I,2I]}\,. \tag{6.8}$$

When using the bilinear estimates, it will be useful to regroup the global frequencies $1+(2m+1)|\eta|$ in dyadic blocs $1+(2m+1)I\in[A,2A]$ where $A\in 2^{\mathbb{N}}$ is a dyadic integer. For the shortness of notation, we will write $(m+1)I\sim A$ instead of $1+(2m+1)I\in[A,2A]$. Therefore, we denote

$$u_A:=\sum_{\substack{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}\\ (m+1)I\sim A}} u_{I,m} \tag{6.9}$$

so that

$$u=\sum_{A\in 2^{\mathbb{N}}} u_A\,.$$

It is useful to note that, writing $\langle I\rangle=\sqrt{1+I^2}$, we have $\frac{1}{2m+1}\lesssim\frac{\langle I\rangle}{1+(2m+1)I}$.

Because of the orthogonality of the h_m, we have the following useful identities, which we will refer to as using *orthogonality*.

Lemma 6.10. *For all $k\in\mathbb{R}$, there holds*

$$\|u\|_{H^k_G}^2=\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}}\|u_{I,m}\|_{H^k_G}^2=\sum_{A\in 2^{\mathbb{N}}}\|u_A\|_{H^k_G}^2\,.$$

Observe that on the support of $u_{I,m}$, we have

$$1+(2m+1)|\eta|\in[1+(2m+1)I,1+(2m+1)2I]\subset[A,4A]\,,$$

so that

$$\|u_{I,m}\|_{H^k_G}\sim(1+(2m+1)I)^{\frac{k}{2}}\|u_{I,m}\|_{L^2_G}\sim A^{\frac{k}{2}}\|u_{I,m}\|_{L^2_G}\,.$$

Using orthogonality, one also has: for any $A\in 2^{\mathbb{N}}$,

$$\|u_A\|_{H^k_G}\sim A^{\frac{k}{2}}\|u_A\|_{L^2_G}\,.$$

Remark 6.11. With the above notation, one can interpret the norm (6.3) in $\mathcal{X}^k_\rho$

$$\|u\|_{\mathcal{X}^k_\rho}^2:=\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}}(1+(2m+1)I)^k\langle I\rangle^\rho\|u_{I,m}\|_{L^2_G}^2$$

as

$$\|u\|_{\mathcal{X}^k_\rho}^2\sim\sum_{m\geqslant 0}\int_{\mathbb{R}}(1+(2m+1)|\eta|)^k(1+|\eta|)^\rho|f_m(\eta)|^2|\eta|^{-\frac{1}{2}}\,\mathrm{d}\eta\,.$$

In particular, every function $u\in H^k_G$ with additional partial regularity of $\frac{\rho}{2}$ in the y variable belongs to $\mathcal{X}^k_\rho$.

2.2. Hermite functions. Let us first recall some pointwise bounds for the Hermite functions $(h_m)_{m\geqslant 0}$. We denote by $\lambda_m = \sqrt{2m+1}$ the square root of the m-th eigenvalue for the harmonic oscillator.

Theorem 6.12 (Pointwise estimates for Hermite functions [KT05], Lemma 5.1)**.** *For any $m \geqslant 0$ and $x \in \mathbb{R}$, there holds*

$$|h_m(x)| \lesssim \begin{cases} \frac{1}{|\lambda_m^2 - x^2|^{1/4}} & \text{if} \quad |x| < \lambda_m - \lambda_m^{-1/3} \\ \lambda_m^{-1/6} & \text{if} \quad ||x| - \lambda_m| \leqslant \lambda_m^{-1/3} \\ \frac{e^{-s_m(x)}}{|\lambda_m^2 - x^2|^{1/4}} & \text{if} \quad |x| > \lambda_m + \lambda_m^{-1/3}\,, \end{cases}$$

where

$$s_m(x) = \int_{\lambda_m}^{x} \sqrt{t^2 - \lambda_m^2}\,\mathrm{d}t\,.$$

Remark 6.13. In order to understand how bilinear estimates on $h_m h_n$ are proven (see [BTT13]), one may roughly picture h_m to be concentrated on $[-\sqrt{2m+1}, \sqrt{2m+1}]$, and work with models of the form

$$h_m(x) \sim \frac{1}{\sqrt{2}}(2m+1)^{-\frac{1}{4}}\mathbf{1}_{[-\sqrt{2m+1},\sqrt{2m+1}]}(x)\,,$$

as long as one does not take pointwise estimates, and as long as one does not consider L^p norms for p too big.

The pointwise estimates imply the following lemma on the L^p norm of the Hermite functions.

Lemma 6.14 (L^p norms for Hermite functions [KT05], Corollary 3.2)**.** *For any $p \geqslant 2$ there holds uniformly in m*

$$\|h_m\|_{L^p(\mathbb{R})} \lesssim \frac{1}{\lambda_m^{\zeta(p)}},$$

where $\lambda_m = \sqrt{2m+1}$ and

$$\zeta(p) = \begin{cases} \frac{1}{2} - \frac{1}{p} & \text{if } 2 \leqslant p \leqslant 4 \\ \frac{1}{6} + \frac{1}{3p} & \text{if } 4 < p \leqslant +\infty. \end{cases} \tag{6.10}$$

However, these L^p norm estimates will not be sufficient for our purpose, and we will rather make use of the following simplified pointwise estimates.

Corollary 6.15 (Rough pointwise estimates for Hermite functions)**.** *There exists $c > 0$ such that for any $m \geqslant 0$ and $x \in \mathbb{R}$,*

$$|h_m(x)| \lesssim \begin{cases} \lambda_m^{-\frac{1}{2}} & \text{if} \quad |x| \leqslant \frac{\lambda_m}{2} \\ \left(\lambda_m^{\frac{2}{3}} + |x^2 - \lambda_m^2|\right)^{-\frac{1}{4}} & \text{if} \quad \frac{\lambda_m}{2} \leqslant |x| \leqslant 2\lambda_m \\ e^{-\frac{1}{8}x^2} & \text{if} \quad |x| \geqslant 2\lambda_m. \end{cases}$$

For a proof of Corollary 6.15 based on Theorem 6.12, see Appendix A.

2.3. Sobolev spaces. We will use on several occasions Sobolev embeddings for the Grushin operator, which correspond to the Folland-Stein embedding for the Sobolev spaces on the Heisenberg group [FS74].

Theorem 6.16 (Folland-Stein embedding)**.** *Let* $p \in [2, \infty)$*.*

(i) For $k > \frac{3}{2}$*, then* $H^k_G \hookrightarrow L^\infty_G$*.*

(ii) For $k \leqslant \frac{3}{2}$ *, if* $\frac{1}{p} \geqslant \frac{1}{2} - \frac{k}{3}$ *, one has* $H^k_G \hookrightarrow L^p_G$*.*

Proposition 6.1. *Let* $k > 0$ *and* $u, v \in H^k_G$*. Then*

(i) (Product rule) $\|uv\|_{H^k_G} \lesssim \|u\|_{H^k_G}\|v\|_{L^\infty_G} + \|u\|_{L^\infty_G}\|v\|_{H^k_G}$*;*

(ii) (Algebra property) if $k > \frac{3}{2}$*,* $\|uv\|_{H^k_G} \lesssim \|u\|_{H^k_G}\|v\|_{H^k_G}$*;*

(iii) (Chain rule) if $k > \frac{3}{2}$*, for every* $p \in \mathbb{N}^*$*,* $\|u^p\|_{H^k_G} \lesssim \|u\|^p_{H^k_G}$*.*

More details about the proof of Proposition 6.1 can be found in Appendix B.

2.4. Probabilistic preliminaries. Only basic probability notions will be used in this article. Recall that we have fixed once and for all a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and denote $\omega \in \Omega$.

Our main probabilistic tool is the Khinchine inequality for subgaussian random variables, and a is a multilinear version for Gaussian random variables.

Definition 6.2 (Subgaussian random variables)**.** We say that the family of independent and identically distributed complex-valued random variables $(X_{I,m})_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}}$ is *subgaussian* if there exists $c > 0$ such that for all $\gamma > 0$,

$$\mathbb{E}\left[e^{\gamma X_{I,m}}\right] \leqslant e^{c\gamma^2}\,.$$

Theorem 6.17 (Khinchine inequality / Kolmogorov-Paley-Zygmund [BT08a], Lemma 3.1)**.** *Let* $\mathcal{I}$ *be a countable set, and let* $(X_n)_{n\in\mathcal{I}}$ *be a sequence of independent and identically distributed complex-valued subgaussian random variables (centred and of variance* 1*). Then there exists* $C > 0$ *such that for every complex-valued sequence* $(\Psi_n)_{n\in\mathcal{I}} \in \ell^2(\mathcal{I})$ *and all* $r \in [2, \infty)$*, one has:*

$$\left\|\sum_{n\in\mathcal{I}} \Psi_n X_n\right\|_{L^r_\Omega} \leqslant C\sqrt{r}\left(\sum_{n\in\mathcal{I}} |\Psi_n|^2\right)^{\frac{1}{2}}\,.$$

Corollary 6.18 (Probabilistic decoupling)**.** *Let* $\mathcal{I}$ *be a countable set, and let* $(X_n)_{n\in\mathcal{I}}$ *be a sequence of independent and identically distributed complex-valued* *subgaussian random variables (centred and of variance* 1*). Then there exists* $c > 0$ *such that the following holds.*

Fix a sequence $(\Psi_n)_{n\in\mathcal{I}}$ *of functions of the variable* $\psi \in \mathbb{R}^d$ *in*[1] L^p_ψ, $p = (p_1, \dots, p_d) \in [2,\infty)^d$. *Fix a countable set* $\mathcal{P}$ *and a partition of* $\mathcal{I}$ *denoted* $(\mathcal{I}_k)_{k\in\mathcal{P}}$. *Then there exists* $R_0(p)$ *large enough such that for* $R \geqslant R_0(p)$, *outside a set of probability less that* e^{-cR^2}, *there holds:*

$$\sum_{k\in\mathcal{P}} \left\| \sum_{n\in\mathcal{I}_k} \Psi_n X_n(\omega) \right\|^2_{L^p_\psi} \leqslant R^2 \sum_{n\in\mathcal{I}} \|\Psi_n\|^2_{L^p_\psi} .$$

Proof. First, from the triangle inequality, we have

$$\left\| \sum_{k\in\mathcal{P}} \left\| \sum_{n\in\mathcal{I}_k} \Psi_n X_n(\omega) \right\|^2_{L^p_\psi} \right\|_{L^r_\Omega} \leqslant \sum_{k\in\mathcal{P}} \left\| \sum_{n\in\mathcal{I}_k} \Psi_n X_n(\omega) \right\|^2_{L^{2r}_\Omega L^p_\psi} .$$

Now, by the Minkowski inequality and Theorem 6.17 we have for $2r \geqslant \max\{p_1, \dots, p_d\}$, for all $k \in \mathcal{P}$,

$$\left\| \sum_{n\in\mathcal{I}_k} \Psi_n X_n \right\|_{L^{2r}_\Omega L^p_\psi} \leqslant C\sqrt{r} \left\| \left(\sum_{n\in\mathcal{I}_k} |\Psi_n|^2 \right)^{\frac{1}{2}} \right\|_{L^p_\psi} ,$$

and applying the Minkowski inequality again,

$$\left\| \sum_{n\in\mathcal{I}_k} \Psi_n X_n \right\|_{L^{2r}_\Omega L^p_\psi} \leqslant C\sqrt{r} \left(\sum_{n\in\mathcal{I}_k} \|\Psi_n\|^2_{L^p_\psi} \right)^{\frac{1}{2}} .$$

Thus by the Markov inequality, there exists $C > 0$ such that

$$\mathbb{P}\left(\sum_{k\in\mathcal{P}} \left\| \sum_{n\in\mathcal{I}_k} \Psi_n X_n \right\|^2_{L^p_\psi} > R^2 \sum_{n\in\mathcal{I}} \|\Psi_n\|^2_{L^p_\psi} \right) \leqslant \left(\frac{C\sqrt{r}}{R} \right)^r ,$$

and the conclusion follows by optimizing in r, which leads to the choice $r = \frac{R^2}{4C}$. □

We have the following multilinear version for Gaussian variables.

Theorem 6.19 (Wiener Chaos estimates [OT18], Lemma 2.6 and [Sim74], Lemma I.18-I.22)**.** *Let* $\mathcal{I}$ *be a countable set,* $\ell \geqslant 1$ *an integer, and let* $\Psi : \mathcal{I}^\ell \to \mathbb{R}$.

[1] We defined $\|u\|_{L^p_\psi} := \|u\|_{L^{p_1}_{\psi_1} \cdots L^{p_d}_{\psi_d}}$

Let $(g_n)_{n\in\mathcal{I}}$ be independent and identically distributed standard real-valued Gaussian variables. Then there exists $C(\ell)$ such that the following holds. Let

$$F^\omega := \sum_{(n_1,\dots,n_\ell)\in\mathcal{I}^\ell} \Psi_{n_1,\dots,n_\ell} g_1^\omega \cdots g_\ell^\omega\,,$$

and assume that $F^\omega \in L^2_\Omega$. Then one has that for any $r \geqslant 2$,

$$\|F^\omega\|_{L^r_\Omega} \leqslant C(\ell) r^{\frac{\ell}{2}} \|F^\omega\|_{L^2_\Omega}\,.$$

We now state the main consequences of this theorem and of the Markov inequality that we will use in this article.

Corollary 6.20 (Probabilistic decoupling)**.** *Let $\mathcal{I}$ be a countable set. Let $X_n = g_n + ih_n$ complex Gaussian random variables, where the $\{g_n, h_n\}_{n\in\mathcal{I}}$ are independent and identically distributed real-valued centred Gaussian variables of variance 1. There exists $c>0$ such that the following holds. Let $p = (p_1,\dots,p_d) \in [2,\infty)^d$ and $(\Psi_{n,n'})_{n,n'\in\mathcal{I}}$ and $(\Psi_{n,n',n''})_{n,n',n''\in\mathcal{I}}$ be functions of the variable $\psi \in \mathbb{R}^d$ belonging to $L^p_\psi = L^{p_1}_{\psi_1}\dots L^{p_d}_{\psi_d}$. Then for $R \geqslant R_0(p)$ large enough the following holds.*

(i) Outside a set of probability at most e^{-cR^2},

$$\Big\|\sum_{n,n'\in\mathcal{I}} \Psi_{n,n'} X_n \bar{X}_{n'}\Big\|_{L^p_\psi}^2 \leqslant R^4 \sum_{n,n'\in\mathcal{I}} \|\Psi_{n,n'}\|^2_{L^p_\psi} + R^4 \Big(\sum_{n\in\mathcal{I}} \|\Psi_{n,n}\|_{L^p_\psi}\Big)^2$$

$$\Big\|\sum_{n,n'\in\mathcal{I}} \Psi_{n,n'} X_n X_{n'}\Big\|_{L^p_\psi}^2 \leqslant R^4 \sum_{n,n'\in\mathcal{I}} \|\Psi_{n,n'}\|^2_{L^p_\psi}\,.$$

(ii) We assume that $\Psi_{n,n',n''} = \Psi_{n',n,n''}$ for every $n,n',n'' \in \mathcal{I}$. Outside a set of probability at most e^{-cR^2},

$$\Big\|\sum_{n,n',n''\in\mathcal{I}} \Psi_{n,n',n''} X_n X_{n'} \bar{X}_{n''}\Big\|_{L^p_\psi}^2 \leqslant R^6 \sum_{n,n',n''\in\mathcal{I}} \|\Psi_{n,n',n''}\|^2_{L^p_\psi} + R^6 \sum_{n\in\mathcal{I}} \Big(\sum_{n'\in\mathcal{I}} \|\Psi_{n,n,n'}\|_{L^p_\psi}\Big)^2\,.$$

(iii) We assume that $\psi = (\psi_-, \psi_+)$, $p = (p_-, p_+)$, and we relax the assumption on p as $p_- = (p_1,\dots,p_{d_-}) \in [1,\infty)^{d_-}$ and $p_+ = (p_{d_-+1},\dots,p_d) \in [2,\infty)^{d-d_-}$. Then outside a set of probability at most e^{-cR^2},

$$\Big\|\sum_{n,n'\in\mathcal{I}} \Psi_{n,n'} X_n \bar{X}_{n'}\Big\|_{L^p_\psi} \leqslant R^2 \Bigg\| \Big(\sum_{n,n'\in\mathcal{I}} \|\Psi_{n,n'}\|^2_{L^{p_+}_{\psi_+}}\Big)^{1/2} + \sum_{n\in\mathcal{I}} \|\Psi_{n,n}\|_{L^{p_+}_{\psi_+}} \Bigg\|_{L^{p_-}_{\psi_-}}\,.$$

Proof. The proof follows from Theorem 6.19 by expansion, writing that $\Psi_{n,n'} = b_{n,n'} + i c_{n,n'}$ and using the independence of g_n from h_n. We fix $r \geqslant \max\{p_1, \dots, p_d\}$.

(i) Applying the Minkowski inequality, Theorem 6.19 and the Markov inequality, we get that outside a set of probability at most e^{-cR^2} there holds

$$\Big\| \sum_{n,n' \in \mathcal{I}} \Psi_{n,n'} X_n \bar{X}_{n'} \Big\|_{L^p_\psi} \leqslant R^2 \Big\| \sum_{n,n' \in \mathcal{I}} \Psi_{n,n'} X_n \bar{X}_{n'} \Big\|_{L^p_\psi L^2_\Omega} .$$

But for every $x \in \mathbb{R}^d$, we expand

$$\begin{aligned} \Big\| \sum_{n,n' \in \mathcal{I}} \Psi_{n,n'}(\psi) X_n \bar{X}_{n'} \Big\|^2_{L^2_\Omega} =& \mathbb{E}\left[\Big| \sum_{n,n' \in \mathcal{I}} \Psi_{n,n'}(\psi) X_n \bar{X}_{n'} \Big|^2 \right] \\ =& \sum_{n_1, n_2, n_1', n_2' \in \mathcal{I}} \mathbb{E}[X_{n_1} \bar{X}_{n_1'} \bar{X}_{n_2} X_{n_2'}] \Psi_{n_1, n_1'}(\psi) \overline{\Psi_{n_2, n_2'}(\psi)} . \end{aligned}$$

The crucial observation is then the following, $\mathbb{E}[X_{n_1} \bar{X}_{n_1'} \bar{X}_{n_2} X_{n_2'}] = 0$ unless $\{n_1, n_2'\} = \{n_1', n_2\}$. Indeed, since $X := X_{I,m}$ is a complex gaussian, we have $0 = \mathbb{E}[X] = \mathbb{E}[X^2]$, and the same holds for $\bar{X}$. Therefore we are left with:

$$\Big\| \sum_{n,n' \in \mathcal{I}} \Psi_{n,n'}(\psi) X_n \bar{X}_{n'} \Big\|^2_{L^2_\Omega} = C \sum_{n,n' \in \mathcal{I}} |\Psi_{n,n'}(\psi)|^2 + C' \Big(\sum_{n \in \mathcal{I}} |\Psi_{n,n}(\psi)| \Big)^2 .$$

Taking the norm L^p_ψ of the square root of this term and using the Minkowski inequality lead to the first inequality. For the second inequality, the proof is the same except for the remark that $\mathbb{E}[X_{n_1} X_{n_1'} \bar{X}_{n_2} \bar{X}_{n_2'}] = 0$ unless $\{n_1, n_1'\} = \{n_2, n_2'\}$.

(ii) Applying the Minkowski inequality, Theorem 6.19 and the Markov inequality, we get that outside a set of probability at most e^{-cR^2} there holds

$$\Big\| \sum_{n,n',n'' \in \mathcal{I}} \Psi_{n,n',n''} X_n X_{n'} \bar{X}_{n''} \Big\|_{L^p_\psi} \leqslant R^3 \Big\| \sum_{n,n',n'' \in \mathcal{I}} \Psi_{n,n',n''} X_n X_{n'} \bar{X}_{n''} \Big\|_{L^p_\psi L^2_\Omega} .$$

We first fix $x \in \mathbb{R}^d$ and estimate $\big\| \sum_{n,n',n'' \in \mathcal{I}} \Psi_{n,n',n''}(\psi) X_n X_{n'} \bar{X}_{n''} \big\|_{L^2_\Omega}$ by expanding

$$\begin{aligned} &\Big\| \sum_{n,n',n'' \in \mathcal{I}} \Psi_{n,n',n''}(\psi) X_n X_{n'} \bar{X}_{n''} \Big\|^2_{L^2_\Omega} \\ &\quad = \sum_{n_1, n_1', n_1'', n_2, n_2', n_2'' \in \mathcal{I}} \mathbb{E}[X_{n_1} X_{n_1'} \bar{X}_{n_1''} \bar{X}_{n_2} \bar{X}_{n_2'} X_{n_2''}] \Psi_{n_1, n_1', n_1''}(\psi) \overline{\Psi_{n_2, n_2', n_2''}(\psi)} . \end{aligned}$$

Since $X := X_n$ is a complex gaussian, we have $0 = \mathbb{E}[X] = \mathbb{E}[X^2] = \mathbb{E}[X^3]$, and the same holds for $\bar{X}$. Therefore, if

$$\mathbb{E}[X_{n_1} X_{n_1'} \bar{X}_{n_1''} \bar{X}_{n_2} \bar{X}_{n_2'} X_{n_2''}] \neq 0,$$

then one can see by double inclusion that

$$\{n_1, n_1', n_2''\} = \{n_1'', n_2, n_2'\}\,.$$

More precisely, using that $\mathbb{E}[X^2\bar{X}] = 0$, we even have that the indices n_1, n_1', n_2'', are in bijection with the indices n_1'', n_2, n_2'. This implies that up to permutation of the pairs of indices playing the same role, namely the pair of indices n_1 and n_1' and the pair of indices n_2 and n_2', we are in one of the following two cases:

- $n_1 = n_2$, $n_1' = n_2'$, $n_1'' = n_2''$;
- $n_1 = n_1''$, $n_1' = n_2'$ and $n_2 = n_2''$.

Therefore, outside a set of probability at most e^{-cR^2}, there holds

$$\begin{aligned}\Big\| \sum_{n,n',n'' \in \mathcal{I}} \Psi_{n,n',n''} X_n X_{n'} \bar{X}_{n''} \Big\|_{L^p_\psi}^2 &\leqslant R^6 \Big\| \Big(\sum_{n,n',n'' \in \mathcal{I}} |\Psi_{n,n',n''}|^2 \Big)^{1/2} \Big\|_{L^p_\psi}^2 \\ &+ R^6 \Big\| \Big(\sum_{n' \in \mathcal{I}} \Big(\sum_{n \in \mathcal{I}} |\Psi_{n,n',n}| \Big)^2 \Big)^{1/2} \Big\|_{L^p_\psi}^2 ,\end{aligned}$$

so that from the Minkowski inequality on the sum over n, n', n'' resp. n',

$$\begin{aligned}\Big\| \sum_{n,n',n'' \in \mathcal{I}} \Psi_{n,n',n''} X_n X_{n'} \bar{X}_{n''} \Big\|_{L^p_\psi}^2 &\leqslant R^6 \sum_{n,n',n'' \in \mathcal{I}} \|\Psi_{n,n',n''}\|_{L^p_\psi}^2 \\ &+ R^6 \sum_{n' \in \mathcal{I}} \Big\| \sum_{n \in \mathcal{I}} |\Psi_{n,n',n}| \Big\|_{L^p_\psi}^2 ,\end{aligned}$$

and from the triangle inequality on the sum over n,

$$\begin{aligned}\Big\| \sum_{n,n',n'' \in \mathcal{I}} \Psi_{n,n',n''} X_n X_{n'} \bar{X}_{n''} \Big\|_{L^p_\psi}^2 &\leqslant R^6 \sum_{n,n',n'' \in \mathcal{I}} \|\Psi_{n,n',n''}\|_{L^p_\psi}^2 \\ &+ R^6 \sum_{n' \in \mathcal{I}} \Big(\sum_{n \in \mathcal{I}} \|\Psi_{n,n',n}\|_{L^p_\psi} \Big)^2 .\end{aligned}$$

(*iii*) Applying the Minkowski inequality, Theorem 6.19 and the Markow inequality, we have that outside a set of probability at most e^{-cR^2} there holds

$$\Big\|\sum_{n,n'\in\mathcal{I}}\Psi_{n,n'}X_n\bar{X}_{n'}\Big\|_{L^p_\psi}\leqslant R^2\Big\|\sum_{n,n'\in\mathcal{I}}\Psi_{n,n'}X_n\bar{X}_{n'}\Big\|_{L^p_\psi L^2_\Omega}.$$

For $\psi\in\mathbb{R}^d$, we expand

$$\begin{aligned}\Big\|\sum_{n,n'\in\mathcal{I}}\Psi_{n,n'}(\psi)X_n\bar{X}_{n'}\Big\|^2_{L^2_\Omega}&=\mathbb{E}\left[\Big|\sum_{n,n'\in\mathcal{I}}\Psi_{n,n'}(\psi)X_n\bar{X}_{n'}\Big|^2\right]\\&=\sum_{n_1,n_2,n_1',n_2'\in\mathcal{I}}\mathbb{E}[X_{n_1}\bar{X}_{n_1'}\bar{X}_{n_2}X_{n_2'}]\Psi_{n_1,n_1'}(\psi)\overline{\Psi_{n_2,n_2'}}(\psi)\,.\end{aligned}$$

Since $X:=X_n$ is a complex gaussian, we have $0=\mathbb{E}[X]=\mathbb{E}[X^2]$, and the same holds for $\bar{X}$. Therefore, if $\mathbb{E}[X_{n_1}\bar{X}_{n_1'}\bar{X}_{n_2}X_{n_2'}]\neq 0$, then one can see that either $\Big(n_1=n_1'$ and $n_2=n_2'\Big)$ or $\Big(n_1=n_2$ and $n_1'=n_2'\Big)$. Therefore, we have

$$\Big\|\sum_{n,n'\in\mathcal{I}}\Psi_{n,n'}X_n\bar{X}_{n'}\Big\|_{L^p_\psi}\leqslant R^2\left\|\Big(\sum_{n,n'\in\mathcal{I}}|\Psi_{n,n'}|^2\Big)^{1/2}+\sum_{n\in\mathcal{I}}|\Psi_{n,n}|\right\|_{L^p_\psi}.$$

Moreover, applying the Minkowski inequality for the admissible exponents $p_+\in[2,\infty)^{d-d_-}$, we obtain that outside a set of probability at most e^{-cR^2} there holds

$$\Big\|\sum_{n,n'\in\mathcal{I}}\Psi_{n,n'}X_n\bar{X}_{n'}\Big\|_{L^p_\psi}\leqslant R^2\left\|\Big(\sum_{n,n'\in\mathcal{I}}\|\Psi_{n,n'}\|^2_{L^{p_+}_{\psi_+}}\Big)^{1/2}+\sum_{n\in\mathcal{I}}\|\Psi_{n,n}\|_{L^{p_+}_{\psi_+}}\right\|_{L^{p_-}_{\psi_-}}.\ \square$$

3. Random data and linear random estimates

In this section, we study in detail how the multiplication of each mode by independent normal distributions improves the integrability of the potential in the L^p spaces, without changing the Sobolev regularity.

Let us recall the construction of the random linear solutions associated to some fixed initial data $u_0\in\mathcal{X}^k_\rho\subset H^k_G$. From (6.3), the $u_{I,m}$ defined by (6.8) satisfy

$$\|u_0\|^2_{\mathcal{X}^k_\rho}=\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}}(1+(2m+1)I)^k\langle I\rangle^\rho\|u_{I,m}\|^2_{L^2_G}\,.$$

Given a family of Gaussian independent and identically distributed random variables denoted $(X_{I,m})_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}}$, the probability measure μ_{u_0} is the push-forward of $\mathbb{P}$ under the map (6.2)

$$\omega\mapsto u_0^\omega:=\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}}X_{I,m}(\omega)u_{I,m}\,.$$

We denote $z^\omega(t) = e^{it\Delta_G} u_0^\omega$ the solution to the linear equation (LS-G) associated to the initial data u_0^ω. In particular, we have

$$z^\omega = \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} X_{I,m}(\omega) z_{I,m}\,, \tag{6.11}$$

with $z_{I,m}(t) = e^{it\Delta_G} u_{I,m}$.

3.1. Probabilistic integrability and smoothing estimates. In this part, we prove that the randomized potential z^ω belongs to the space $L^q_T L^p_G$ for every $p, q \in [2,\infty)$, and even to the space $L^q_T W^{k+\zeta(p),p}_G$, $\zeta(p) \leqslant \frac{1}{4}$ being defined in (6.10).

Proposition 6.3 (Integrability improvement)**.** *There exists $c > 0$ such that the following holds. Let $k \in \mathbb{R}$. Fix $u_0 \in \mathcal{X}^k_{\zeta(p)+\frac12-\frac1p}$, denote u_0^ω its randomization from (6.2), and write $z^\omega = e^{it\Delta_G} u_0^\omega$. Let $p, q \in [2,\infty)$ and $\varepsilon > 0$. Then outside a set of probability at most e^{-cR^2}, there holds*

$$\|z^\omega\|_{L^q_T W^{k+\zeta(p),p}_G} \leqslant R T^{\frac1q} \|u_0\|_{\mathcal{X}^k_{\zeta(p)+\frac32-\frac3p}}\,, \tag{6.12}$$

$$\sum_{A\in 2^{\mathbb{N}}} A^{k+\zeta(p)-\varepsilon} \|z^\omega_A\|^2_{L^q_T W^{\varepsilon,p}_G} \leqslant R^2 T^{\frac2q} \|u_0\|^2_{\mathcal{X}^k_{\zeta(p)+\frac32-\frac3p}}\,. \tag{6.13}$$

Note that we have $\zeta(p)+\frac32-\frac3p = 2-\frac4p$ when $p \leqslant 4$ and $\zeta(p)+\frac32-\frac{3}{2p} = \frac53-\frac{8}{3p}$ when $p > 4$. The derivative gain is maximal when $\zeta(p)$ is, that is when $p = 4$ leading to a $\frac14$ gain of derivatives.

Remark 6.21 (Case $p = \infty$). Since $\frac4p \times p = 4 > 3$ one can use the embedding $W^{\frac4p,p}_G \hookrightarrow L^\infty_G$ and obtain that oustide a set of probability at most e^{-cR^2}, there holds

$$\|z^\omega\|_{L^q_T W^{\varepsilon,\infty}_G} \lesssim_\varepsilon R T^{\frac1q} \|u_0\|_{\mathcal{X}^{\frac4p-\zeta(p)+\varepsilon}_{\zeta(p)+\frac32-\frac3p}}\,,$$

which is obtained by taking $k = \frac4p - \zeta(p) + \varepsilon$ in (6.12). Observe that $\mathcal{X}^1_1 \hookrightarrow \mathcal{X}^{\frac4p-\zeta(p)+\varepsilon}_{\zeta(p)+\frac32-\frac3p}$ when $\varepsilon + \frac12 + \frac1p \leqslant 1$, satisfied as soon as $\varepsilon < \frac12$ by choosing p large enough. Therefore, for $\varepsilon \in [0,\frac12)$, outside a set of probability at most e^{-cR^2}, there holds

$$\|z^\omega\|_{L^q_T W^{\varepsilon,\infty}_G} \lesssim_\varepsilon R T^{\frac1q} \|u_0\|_{\mathcal{X}^1_1}\,.$$

Similarly, one obtains that for $\varepsilon \in (0,\frac12)$, outside a set of probability at most e^{-cR^2}, there holds

$$\sum_{A\in 2^{\mathbb{N}}} A^\varepsilon \|z^\omega_A\|^2_{L^q_T L^\infty_G} \lesssim_\varepsilon R^2 T^{\frac2q} \|u_0\|^2_{\mathcal{X}^1_1}\,. \tag{6.14}$$

The proof of Proposition 6.3 relies on the following deterministic estimates.

Lemma 6.22. *With the notation form Proposition 6.3, let $p, q \in [2, \infty)$. Then for all $T > 0$, writing $L^q_T = L^q([0,T])$,*

$$\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} (1+(2m+1)I)^{k+\zeta(p)} \|z_{I,m}\|^2_{L^q_T L^p_G} \lesssim T^{\frac{2}{q}} \|u_0\|^2_{\mathcal{X}^k_{\zeta(p)+\frac{3}{2}-\frac{3}{p}}} . \tag{6.15}$$

Proof of Lemma 6.22. Let $p \in [2, \infty)$. We apply the Hausdorff-Young inequality in the y variable, so that if we denote p' the conjugate exponent of p, we obtain that for all $t \in [0,T]$,

$$\begin{aligned}
\|z_{I,m}(t)\|_{L^p_G} &\lesssim \|e^{it(2m+1)|\eta|} f_m(\eta) h_m(|\eta|^{\frac{1}{2}} x) \mathbf{1}_{|\eta|\in[I,2I]}\|_{L^p_x L^{p'}_\eta} \\
&= \|f_m(\eta) h_m(|\eta|^{\frac{1}{2}} x) \mathbf{1}_{|\eta|\in[I,2I]}\|_{L^p_x L^{p'}_\eta} .
\end{aligned}$$

Moreover, using the Minkowski inequality, since $p \geqslant p'$, we have

$$\|z_{I,m}(t)\|_{L^p_G} \lesssim \|f_m(\eta) h_m(|\eta|^{1/2} x) \mathbf{1}_{|\eta|\in[I,2I]}\|_{L^{p'}_\eta L^p_x} .$$

Since $z_{I,m}(0) = u_{I,m}$, this implies

$$\|z_{I,m}(t)\|_{L^p_G} \lesssim \|u_{I,m}\|_{L^{p'}_\eta L^p_x} .$$

Therefore, estimate (6.15) is a consequence of the inequality

$$\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} (1+(2m+1)I)^{k+\zeta(p)} \|u_{I,m}\|^2_{L^{p'}_\eta L^p_x} \lesssim \|u_0\|^2_{\mathcal{X}^k_{\zeta(p)+\frac{3}{2}-\frac{3}{p}}} , \tag{6.16}$$

which we will now establish.

Using the upper bounds in L^p on the Hermite functions h_m from Lemma 6.14, and the notation (6.10) for the exponent $\zeta(p)$, we have

$$\begin{aligned}
\|u_{I,m}\|_{L^{p'}_\eta L^p_x} &= \|f_m(\eta) h_m(|\eta|^{\frac{1}{2}} x) \mathbf{1}_{|\eta|\in[I,2I]}\|_{L^{p'}_\eta L^p_x} \\
&\lesssim (2m+1)^{-\frac{\zeta(p)}{2}} \left\| f_m(\eta) \eta^{-\frac{1}{4}} \mathbf{1}_{|\eta|\in[I,2I]} \eta^{\frac{1}{4}-\frac{1}{2p}} \right\|_{L^{p'}_\eta} \\
&\lesssim (2m+1)^{-\frac{\zeta(p)}{2}} I^{\frac{1}{4}-\frac{1}{2p}} \|f_m(\eta) \eta^{-\frac{1}{4}} \mathbf{1}_{|\eta|\in[I,2I]}\|_{L^{p'}_\eta} .
\end{aligned}$$

From Hölder's inequality in η (and because the interval $[I, 2I]$ has length I), we get

$$\begin{aligned}
\|u_{I,m}\|_{L^p_G} &\lesssim (2m+1)^{-\frac{\zeta(p)}{2}} I^{\frac{1}{4}-\frac{1}{2p}-\frac{1}{2}+\frac{1}{p'}} \|f_m(\eta) \eta^{-\frac{1}{4}} \mathbf{1}_{|\eta|\in[I,2I]}\|_{L^2_\eta} \\
&\lesssim (1+(2m+1)I)^{-\frac{\zeta(p)}{2}} \langle I \rangle^{\frac{\zeta(p)}{2}+\frac{1}{4}-\frac{1}{2p}-\frac{1}{2}+\frac{1}{p'}} \|u_{I,m}\|_{L^2_G} ,
\end{aligned}$$

which leads to (6.16) with $\zeta(p) + \frac{1}{2} - \frac{1}{p} - 1 + \frac{2}{p'} = \zeta(p) + \frac{3}{2} - \frac{3}{p}$. $\square$

Proof of Proposition 6.3. Let us first establish (6.12). We start with an application of the probabilistic decoupling from Corollary 6.18: outside a set or probability at most e^{-cR^2}, we have

$$\|z^\omega\|^2_{L^q_T W^{k+\zeta(p),p}_G} \lesssim R^2 \sum_{I,m} \|z_{I,m}\|^2_{L^q_T W^{k+\zeta(p),p}_G} \,.$$

Since for all $t \in \mathbb{R}$ and all $k' \in \mathbb{R}$, we have $\|(-\Delta_G)^{k'/2} z_{I,m}(t)\|^2_{L^p_G} \lesssim (1+(2m+1)I)^{k'/2}\|z_{I,m}(t)\|^2_{L^p_G}$, we deduce that

$$\|z^\omega\|_{L^q_T W^{k+\zeta(p),p}_G} \lesssim R\left(\sum_{I,m}(1+(2m+1)I)^{k+\zeta(p)}\|z_{I,m}\|^2_{L^q_T L^p_G}\right)^{\frac{1}{2}} .$$

Inequality (6.12) is now a consequence of (6.15).

Similarly, fix $\varepsilon > 0$. For $(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}$, we denote by A the dyadic integer such that $(m+1)I \sim A$. Then from Corollary 6.18 applied to the partition given by the $\{(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N} \mid (2m+1)I \sim A\}$ for $A \in 2^{\mathbb{N}}$. Therefore, outside a set or probability at most e^{-cR^2}, we have

$$\begin{aligned}\sum_{A\in 2^{\mathbb{N}}} A^{k+\zeta(p)-\varepsilon}\|z^\omega_A\|^2_{L^q_T W^{\varepsilon,p}_G} &\lesssim R^2 \sum_{I,m} A^{k+\zeta(p)-\varepsilon}\|z_{I,m}\|^2_{L^q_T W^{\varepsilon,p}_G}\\ &\lesssim R^2 \sum_{I,m}(1+(2m+1)I)^{k+\zeta(p)}\|z_{I,m}\|^2_{L^q_T L^p_G}\,,\end{aligned}$$

enabling us to conclude thanks to (6.15) again. □

3.2. Non-smoothing properties of the randomization. We now prove Theorem 1 (*ii*), stating that the randomization does not improve the Sobolev regularity in the H^k spaces, in a similar spirit as in [BT08a].

Proposition 6.4 (Non-smoothing for random initial data). *We assume that $X_{I,m}$ is not almost-surely equal to 0. Let $u_0 \in H^k_G \setminus (\bigcup_{\varepsilon>0} H^{k+\varepsilon}_G)$ and let u_0^ω defined by (6.2). Then, almost surely, we have $u_0^\omega \in H^k_G \setminus (\bigcup_{\varepsilon>0} H^{k+\varepsilon}_G)$.*

Proof. Without loss of generality, we assume that $k=0$. First, let us remark that $\mathbb{E}[u_0^\omega]=0$. Then, since by orthogonality one has

$$\|u_0^\omega\|^2_{L^2_G} = \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} |X_{I,m}(\omega)|^2\|u_{I,m}\|^2_{L^2_G}\,,$$

and since the $X_{I,m}$ have a finite variance, we conclude that

$$\mathbb{E}\left[\|u_0^\omega\|^2_{L^2_G}\right] \lesssim \|u_0\|^2_{L^2_G} < \infty\,.$$

Let $\varepsilon > 0$, it remains to prove that almost surely we have $u_0^\omega \notin H^\varepsilon_G$. In fact, one only needs to show that

$$\mathbb{E}\left[e^{-\|u_0^\omega\|^2_{H^\varepsilon_G}}\right] = 0\,. \tag{6.17}$$

In order to establish (6.17), we expand decomposition (6.2) using the independence of the random variables $X_{I,m}$:

$$\mathbb{E}\left[e^{-\|u_0^\omega\|^2_{H^\varepsilon_G}}\right] = \prod_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} \mathbb{E}\left[e^{-|X_{I,m}|^2\|u_{I,m}\|^2_{H^\varepsilon_G}}\right].$$

Since for all I, m, there holds $\|u_{I,m}\|^2_{H^\varepsilon_G} \geqslant (1+(2m+1)I)^\varepsilon\|u_{I,m}\|^2_{L^2_G}$, we get

$$\mathbb{E}\left[e^{-\|u_0^\omega\|^2_{H^\varepsilon_G}}\right] \leqslant \prod_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} \mathbb{E}\left[e^{-|X_{I,m}|^2(1+(2m+1)I)^\varepsilon\|u_{I,m}\|^2_{L^2_G}}\right].$$

Then we have the following alternative.

First case: Assume that the terms $(1+(2m+1)I)^\varepsilon\|u_{I,m}\|^2_{L^2_G}$ do not go to zero as $(I,m)\to\infty$. Then there exists $\gamma>0$ and an infinite set S of indices $(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}$ satisfying $(1+(2m+1)I)^\varepsilon\|u_{I,m}\|^2_{L^2_G}\geqslant\gamma$. This leads to the bound

$$\begin{aligned}\mathbb{E}\left[e^{-\|u_0^\omega\|^2_{H^\varepsilon_G}}\right] &\leqslant \prod_{(I,m)\in S} \mathbb{E}\left[e^{-|X_{I,m}|^2(1+(2m+1)I)^\varepsilon\|u_{I,m}\|^2_{L^2_G}}\right]\\ &\leqslant \prod_{(I,m)\in S} \mathbb{E}\left[e^{-|X_{I,m}|^2\gamma}\right] = 0\,,\end{aligned}$$

where in the last step we used that S is infinite, the fact that the $X_{I,m}$ are identically distributed and the assumption $\mathbb{P}(X_{I,m}=0)<1$.

Second case: Assume that $(1+(2m+1)I)^\varepsilon\|u_{I,m}\|^2_{L^2_G}\to 0$ as $(I,m)\to\infty$. In this case, we fix $R>0$ such that $\delta_R:=\mathbb{P}(|X_{I,m}|>R)>0$, which is possible thanks to the assumption $\mathbb{P}(X_{I,m}=0)<1$ on the $X_{I,m}$. Moreover, δ_R does not depend on (I,m) since the $X_{I,m}$ are identically distributed.

For every I, m, we have

$$\mathbb{E}\left[e^{-|X_{I,m}|^2(1+(2m+1)I)^\varepsilon\|u_{I,m}\|^2_{L^2_G}}\right] \leqslant (1-\delta_R)+\delta_R e^{-R^2(1+(2m+1)I)^\varepsilon\|u_{I,m}\|^2_{L^2_G}}.$$

Since the sequence $\left((1+(2m+1)I)^\varepsilon\|u_{I,m}\|^2_{L^2_G}\right)_{I,m}$ is convergent to zero as $(I,m)\to\infty$, this sequence is bounded by some constant $C>0$. But on the interval $[0,R^2C]$, there holds the inequality $e^{-x}\leqslant 1-c_Rx$ for some $c_R>0$, so that

$$\mathbb{E}\left[e^{-|X_{I,m}|^2(1+(2m+1)I)^\varepsilon\|u_{I,m}\|^2_{L^2_G}}\right] \lesssim 1-\delta_Rc_RR^2(1+(2m+1)I)^\varepsilon\|u_{I,m}\|^2_{L^2_G}\,,$$

and the upper bound is positive. Moreover, since

$$\sum_{I,m}\delta_Rc_RR^2(1+(2m+1)I)^\varepsilon\|u_{I,m}\|^2_{L^2_G} \gtrsim \delta_Rc_RR^2\|u_0\|^2_{H^\varepsilon_G}=\infty\,,$$

we conclude that the infinite product with general term $(1-\delta_R c_R R^2(1+(2m+1)I)^{\varepsilon}\|u_{I,m}\|^2_{L^2_G})$ is zero, so that

$$\prod_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} \mathbb{E}\left[e^{-|X_{I,m}|^2(1+(2m+1)I)^{\varepsilon}\|u_{I,m}\|^2_{L^2_G}}\right]=0\,.$$

In any case the above discussion proves (6.17). □

Remark 6.23. Note that using the fact that the decay rate for the L^p norms of h_m is optimal in Lemma 6.14 (see Lemma 5.1 in [KT05]) and the probabilistic decoupling argument from Corollary 6.20, it seems highly unlikely that u_0^{ω} belongs to $W_G^{\zeta(p),p}$ for $\varepsilon>0$ (see also [IRT16a]).

3.3. Density of the measure μ_{u_0}. In this part, we establish Theorem 1 (*iii*). Before we turn to the density properties of the measures μ_{u_0} associated to rough potentials u_0, we briefly justify that we can construct functions $u_0\in\mathcal{X}_1^k$ such that $u_0\in H^k_G\setminus(\bigcup_{\varepsilon>0}H_G^{k+\varepsilon})$.

Lemma 6.24 (Existence of rough potentials). *Let $k\geqslant 0$ and $\rho\geqslant 0$. There exists a function $u_0\in\mathcal{X}_\rho^k\subset H_G^k$ such that for all $\varepsilon>0$, $u_0\notin H_G^{k+\varepsilon}$.*

Remark 6.25. In fact, this lemma implies that there exists uncountably many such functions. Indeed, we can apply another randomization argument to the potential u_0 from the lemma. Take $v_0^{\omega}=\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}}\varepsilon_{I,m}(\omega)u_{I,m}$, where $\varepsilon_{I,m}$ are independent random signs, then the functions v_0^{ω} almost-surely satisfy the requirements.

Proof. Let $k,\rho\geqslant 0$. We consider $u_0:=\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} u_{I,m}$ defined in Fourier variable by $u_{I,m}=0$ if $I<1$, and if $I\geqslant 1$, we take $u_{I,m}$ "constant" by parts

$$\mathcal{F}_{y\to\eta}(u_{I,m})(x,\eta)=\|u_{I,m}\|_{L^2_G}|\eta|^{\frac14}h_m\left(|\eta|^{\frac12}x\right)\,.$$

where for any $(I,m)\in 2^{\mathbb{N}}\times\mathbb{N}$, we choose

$$\|u_{I,m}\|^2_{L^2_G}=\frac{1}{(1+(2m+1)I)^k\langle I\rangle^{\rho}\log(1+I)^2(m+1)\log(m+2)^2}\,.$$

First, we observe that for $\varepsilon\in\mathbb{R}$,

$$\|u_0\|^2_{H_G^{k+\varepsilon}}\sim\sum_{I\in 2^{\mathbb{N}}}\frac{1}{\langle I\rangle^{\rho}\log(1+I)^2}\sum_{m\in\mathbb{N}}\frac{(1+(2m+1)I)^{\varepsilon}}{(m+1)\log(m+2)^2}\,.$$

This series with positive general term is divergent when $\varepsilon>0$. For $\varepsilon\leqslant 0$, this series is bounded by $C\sum_{I\in 2^{\mathbb{N}}}\frac{1}{\langle I\rangle^{\rho}\log(1+I)^2}$ which is convergent.

It remains to prove that $u_0 \in \mathcal{X}^k_\rho$. In order to do so, we compute:

$$\|u_0\|^2_{\mathcal{X}^k_\rho} = \sum_{(I,m)\in 2^{\mathbb{N}}\times\mathbb{N}} \frac{1}{\log(1+I)^2(m+1)\log(m+2)^2}\,,$$

which is indeed finite. □

We now establish density properties of the support of the measures with rough potentials (Theorem 1 (*iii*)).

Proposition 6.5 (Density of measures with rough potentials)**.** *Let $k \geqslant 0$ and $\rho \geqslant 0$. We assume that for any $r > 0$, $\mathbb{P}(|X_{I,m} - 1| < r) > 0$. Let $u_0 \in H^k_G$ and $\varepsilon > 0$. Then there exists $v_0 \in \mathcal{X}^k_1 \setminus (\bigcup_{\varepsilon'>0} H^{k+\varepsilon'}_G)$ such that*

$$\mu_{v_0}\Big(B_{H^k_G}(u_0,\varepsilon)\Big) > 0\,.$$

Proof. Without loss of generality, we assume that $k = 0$, and we only deal with the Grushin case. Fix $\varepsilon > 0$ and $u_0 \in L^2_G$. We first construct $v_0 \in \mathcal{X}^0_2 \setminus (\bigcup_{\varepsilon'>0} H^{\varepsilon'}_G)$ satisfying the condition $\|u_0 - v_0\|_{L^2_G} \leqslant \varepsilon$, then we prove that v_0 meets the requirements from the proposition.

To construct v_0, let $K_0 > 0$ be such that

$$\sum_{|(I,m)|>K} \|u_{I,m}\|^2_{L^2_G} \leqslant \varepsilon^2\,,$$

where $|(I,m)| = \max\{|\log(1+I)|, m\}$. Then, let $\delta > 0$ to be determined later, and denote by

$$\widetilde{u}_0 = \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} \widetilde{u}_{I,m} \in \mathcal{X}^0_1 \setminus (\bigcup_{\varepsilon'>0} H^{\varepsilon'}_G)$$

the potential constructed in Lemma 6.24 for $k = 0$ and $\rho = 1$. Set

$$v_{I,m} := \begin{cases} u_{I,m} & \text{if } |(I,m)| \leqslant K_0 \\ \delta\widetilde{u}_{I,m} & \text{if } |(I,m)| > K_0\,. \end{cases}$$

Since the coefficients $v_{I,m}$ for $|(I,m)| > K_0$ are the ones from Lemma 6.24, we know that v belongs to $\mathcal{X}^0_1 \setminus (\bigcup_{\varepsilon'>0} H^{\varepsilon'}_G)$. Then we compute that when δ is small enough,

$$\|u_0 - v_0\|^2_{L^2_G} \leqslant \sum_{|(I,m)|>K_0} \|u_{I,m}\|^2_{L^2_G} + \delta^2 \sum_{|(I,m)|>K_0} \|\widetilde{u}_{I,m}\|^2_{L^2_G} \leqslant 2\varepsilon^2\,.$$

In the end of this proof, we establish that

$$\mu_{v_0}\Big(w \in L^2_G, \|w - v_0\|_{L^2_G} \leqslant \varepsilon\Big) > 0\,,$$

as this would imply

$$\mu_{v_0}\Big(w \in L^2_G, \|w - u_0\|_{L^2_G} \leqslant 2\varepsilon\Big) > 0\,.$$

For future occurrences, for $K > 0$ and $w = \sum_{I,m} w_{I,m} \in L^2_G$, let us denote

$$\Pi_K(w) := \sum_{|(I,m)|\leqslant K} w_{I,m} \,.$$

Because of the inclusion

$$\left\{w \in L^2_G, \|\Pi_K(w-v_0)\|_{L^2_G} \leqslant \frac{\varepsilon}{2}\right\} \cap \left\{w \in L^2_G, \|(\mathrm{Id}-\Pi_K)(w-v_0)\|_{L^2_G} \leqslant \frac{\varepsilon}{2}\right\} \subset \left\{w \in L^2_G, \|w-v_0\|_{L^2_G} \leqslant \varepsilon\right\}\,,$$

we know by independence that

$$\mu_{v_0}\left(w \in L^2_G, \|w-v_0\|_{L^2_G} \leqslant \varepsilon\right) \geqslant \mu_{v_0}\left(w \in L^2_G, \|\Pi_K(w-v_0)\|_{L^2_G} \leqslant \frac{\varepsilon}{2}\right) \times \mu_{v_0}\left(w \in L^2_G, \|(\mathrm{Id}-\Pi_K)(w-v_0)\|_{L^2_G} \leqslant \frac{\varepsilon}{2}\right)\,.$$

To handle the second term in the right-hand side, since $\Pi_K v_0$ tends to v_0 in L^2_G as $K \to \infty$, one has that for large enough $K > 0$,

$$\mu_{v_0}\left(w \in L^2_G, \|(\mathrm{Id}-\Pi_K)(w-v_0)\|_{L^2_G} \leqslant \frac{\varepsilon}{2}\right) \geqslant \mu\left(w \in L^2_G, \|(\mathrm{Id}-\Pi_K)w\|_{L^2_G} \leqslant \frac{\varepsilon}{4}\right)\,.$$

But we have

$$\mu_{v_0}\left(w \in L^2_G, \|(\mathrm{Id}-\Pi_K)w\|_{L^2_G} > \frac{\varepsilon}{4}\right) \underset{K\to\infty}{\longrightarrow} 0\,,$$

since from the Markov inequality,

$$\mu_{v_0}\left(w \in L^2_G, \|(\mathrm{Id}-\Pi_K)w\|_{L^2_G} > \frac{\varepsilon}{4}\right) = \mathbb{P}\left(\|(\mathrm{Id}-\Pi_K)v^\omega\|_{L^2_G} > \frac{\varepsilon}{4}\right) \lesssim \varepsilon^{-2}\mathbb{E}\left[\|(\mathrm{Id}-\Pi_K)v^\omega\|^2_{L^2_G}\right] \lesssim \varepsilon^{-2} \sum_{|(I,m)|>K} \|v_{I,m}\|^2_{L^2_G}\,,$$

which goes to zero as K goes to infinity. We therefore fix $K > 0$ large enough so that

$$\mu_{v_0}\left(w \in L^2_G, \|(\mathrm{Id}-\Pi_K)(w-v_0)\|_{L^2_G} \leqslant \frac{\varepsilon}{2}\right) > 0\,.$$

It only remains to handle the first term in the right-hand side and prove that

$$\mu_{v_0}\left(w \in L^2_G, \|\Pi_K(w-v_0)\|_{L^2_G} \leqslant \frac{\varepsilon}{2}\right) > 0\,. \tag{6.18}$$

Let $c > 0$ small enough so that the following inclusion holds:

$$\bigcap_{|(I,m)|<K} \left\{w \in L^2_G \mid \exists \widetilde{X}_{I,m} \in \mathbb{R}, w_{I,m} = \widetilde{X}_{I,m} v_{I,m} \text{ and } \|(\widetilde{X}_{I,m}-1)v_{I,m}\|_{L^2_G} \leqslant c\varepsilon\right\} \subset \left\{w \in L^2_G \mid \|\Pi_K(w-v_0)\|_{L^2_G} \leqslant \frac{\varepsilon}{2}\right\}\,.$$

This implies by independence of the random variables $X_{I,m}$ that

$$\mu_{v_0}\left(w \in L^2_G, \|\Pi_K(w-v_0)\|_{L^2_G} \leqslant \frac{\varepsilon}{2}\right) \geqslant \prod_{|(I,m)|\leqslant K} \mathbb{P}\left(\|(X_{I,m}(\omega)-1)v_{I,m}\|_{L^2_G} \leqslant c\varepsilon\right),$$

which is positive thanks to the assumption $\mathbb{P}(|X_{I,m}-1|<r)>0$ for all $r>0$, thus (6.18) holds. □

4. Action of the Laplace operator

In the remaining of this article, our general strategy is to group the decompositions (6.1) of u and v in dyadic packets (6.9)

$$u = \sum_{A\in 2^{\mathbb{N}}} u_A \quad \text{and} \quad v = \sum_{B\in 2^{\mathbb{N}}} v_B\,,$$

and write the estimates for the H^k norm of the product uv in terms of the L^2 norms of products $u_A v_B$ for dyadic A and B.

In this section, we consider two elements $u, v \in H^k_G$ and provide useful estimates for the H^k norm of the products $u_A v_B$ and $u_A v$ in terms of the L^2 norms of the products $u_A v_B$ for dyadic A and B.

4.1. The action of $-\Delta_G$ on a product of functions. In this part, we prove that for $u, v \in H^k_G$, the term $-\Delta_G(uv)$ can be expressed thanks to shifted versions of u and v, defined as follows.

Definition 6.6 (δ-shifted functions)**.** Let and $u \in H^k_G$, decomposed as (6.1) $u = \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} u_{I,m}$, where $u_{I,m}$ is defined in (6.8) as

$$\mathcal{F}_{y\to\eta}(u_{I,m})(x,\eta) = f_{I,m}(\eta) h_m(|\eta|^{\frac12}x)\,,$$

and $f_{I,m}(\eta) = f_m(\eta)\mathbf{1}_{|\eta|\in[I,2I]}$.

For $\delta \in D_1 := \{-1,0,1\}\times\{+,-\}$ we write $\delta = (\delta_0, \pm)$ and for $m\in\mathbb{N}$, we write $m+\delta$ as a shortcut for $m+\delta := m+\delta_0$. We introduce the shifted function u^δ from its decomposition (6.1) $u^\delta = \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} u^\delta_{I,m}$ as follows: for all $(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}$,

$$\mathcal{F}_{y\to\eta}(u^\delta_{I,m})(x,\eta) = F^\delta_{I,m}(\eta) f_{I,m}(\eta) h_{m+\delta}(|\eta|^{\frac12}x)\,,$$

and if $A\in 2^{\mathbb{N}}$ is the dyadic integer such that $(m+1)I \sim A$,

$$F^\delta_{I,m}(\eta) := \begin{cases} \left(\frac{(2m+1)|\eta|}{4A}\right)^{\frac12} & \text{if } \delta \in \{(-1,+),(1,+)\} \\ \left(\frac{(2m+1)}{4A}\right)^{\frac12} \frac{\eta}{\sqrt{|\eta|}} & \text{if } \delta \in \{(-1,-),(1,-)\} \\ \frac{(2m+1)|\eta|}{4A} & \text{if } \delta = (0,+) \\ 1 & \text{if } \delta = (0,-). \end{cases} \tag{6.19}$$

Note that by definition, for all $k \geqslant 0$ and $u \in H^k_G$, we have $\|u^\delta\|_{H^k_G} \leqslant \|u\|_{H^k_G}$.

Lemma 6.26 (Action of Δ_G)**.** *Let $A, B \in 2^{\mathbb{N}}$, $I, I \in 2^{\mathbb{Z}}$ and $m, n \in \mathbb{N}$ such that $(m+1)I \sim A$ and $(n+1)J \sim B$. We denote $D_2 := D_1 \times D_1$. Then there holds:*

$$(\mathrm{Id} - \Delta_G)(u_{I,m} v_{J,n}) = \sum_{(\delta_1,\delta_2)\in D_2} C_{A,B}(\delta_1,\delta_2) u^{\delta_1}_{I,m} v^{\delta_2}_{J,n}\,,$$

where for some explicit numerical constants $c(\delta_1, \delta_2)$,

$$C_{A,B}(\delta_1,\delta_2) = \begin{cases} 4A & \textit{if } (\delta_1,\delta_2) = ((0,-),(0,+)) \\ 4B & \textit{if } (\delta_1,\delta_2) = ((0,+),(0,-)) \\ c(\delta_1,\delta_2)\sqrt{AB} & \textit{if } (\delta_1,\delta_2) \in (\{-1,1\}\times\{+,-\})^2 \\ 1 & \textit{if } (\delta_1,\delta_2) = ((0,-),(0,-)) \\ 0 & \textit{if } (\delta_1,\delta_2) = ((0,+),(0,+)),. \end{cases}$$

Proof. Taking the Fourier transform of $u_{I,m}v_{J,n}$ in y, we transform the product into a convolution product

$$\mathcal{F}_{y\to\eta}(u_{I,m}v_{J,n})(x,\eta) = \int f_{I,m}(\eta_1) h_m(|\eta_1|^{\frac12}x) g_{J,n}(\eta-\eta_1) h_n(|\eta-\eta_1|^{\frac12}x)\mathrm{d}\eta_1.$$

In Fourier variable, the Grushin operator acts as $\partial_{xx} - x^2|\eta|^2$, therefore we have

$$\mathcal{F}(\Delta_G(u_{I,m}v_{J,n}))(x,\eta) = \int f_{I,m}(\eta_1) g_{J,n}(\eta-\eta_1)(\partial_{xx}-x^2|\eta|^2)\left(h_m(|\eta_1|^{\frac12}x)h_n(|\eta-\eta_1|^{\frac12}x)\right)\mathrm{d}\eta_1.$$

We now use the fact that the Hermite functions are eigenvectors of the Harmonic oscillator:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} h_m(x) = -(2m+1)h_m + x^2 h_m$$

to deduce the formula

$$\begin{aligned}(\partial_{xx} - x^2|\eta|^2)&(h_m(|\eta_1|^{\frac12}x)h_n(|\eta-\eta_1|^{\frac12}x)) \\ &= -\left((2m+1)|\eta_1| + (2n+1)|\eta-\eta_1|\right)h_m(|\eta_1|^{\frac12}x)h_n(|\eta-\eta_1|^{\frac12}x) \\ &\quad + x^2(|\eta_1|^2 + |\eta-\eta_1|^2 - |\eta|^2)h_m(|\eta_1|^{\frac12}x)h_n(|\eta-\eta_1|^{\frac12}x) \\ &\quad + 2\partial_x(h_m(|\eta_1|^{\frac12}x))\partial_x(h_n(|\eta-\eta_1|^{\frac12}x))\,.\end{aligned}$$

Note that if η_1 and $\eta - \eta_1$ have the same sign, then $|\eta_1|^2 + |\eta-\eta_1|^2 - |\eta|^2 = -2\eta_1(\eta-\eta_1)$, and if η_1 and $\eta-\eta_1$ are of opposite sign, $|\eta_1|^2 + |\eta-\eta_1|^2 - |\eta|^2 = 2|\eta_1||\eta-\eta_1| = -2\eta_1(\eta-\eta_1)$.

We now use the identities

$$xh_m(x) = \sqrt{\frac{m}{2}}h_{m-1} + \sqrt{\frac{m+1}{2}}h_{m+1}\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}x}h_m(x) = \sqrt{\frac{m}{2}}h_{m-1} - \sqrt{\frac{m+1}{2}}h_{m+1}\,,$$

to remove the weight x^2 and simplify the products $\partial_x(h_m(|\eta_1|^{\frac{1}{2}}x))\partial_x(h_n(|\eta - \eta_1|^{\frac{1}{2}}x))$. We deduce that

$$\begin{aligned}
(\partial_{xx}-&x^2|\eta|^2)(h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta-\eta_1|^{\frac{1}{2}}x))\\
&= -\left((2m+1)|\eta_1| + (2n+1)|\eta-\eta_1|\right)h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta-\eta_1|^{\frac{1}{2}}x)\\
&\quad - 2\frac{\eta_1(\eta-\eta_1)}{\sqrt{|\eta_1||\eta-\eta_1|}}\left(\sqrt{\frac{m}{2}}h_{m-1}(|\eta_1|^{\frac{1}{2}}x) + \sqrt{\frac{m+1}{2}}h_{m+1}(|\eta_1|^{\frac{1}{2}}x)\right)\\
&\qquad\qquad \times\left(\sqrt{\frac{n}{2}}h_{n-1}(|\eta-\eta_1|^{\frac{1}{2}}x) + \sqrt{\frac{n+1}{2}}h_{n+1}(|\eta-\eta_1|^{\frac{1}{2}}x)\right)\\
&\quad + 2\sqrt{|\eta_1||\eta-\eta_1|}\left(\sqrt{\frac{m}{2}}h_{m-1}(|\eta_1|^{\frac{1}{2}}x) - \sqrt{\frac{m+1}{2}}h_{m+1}(|\eta_1|^{\frac{1}{2}}x)\right)\\
&\qquad\qquad \times\left(\sqrt{\frac{n}{2}}h_{n-1}(|\eta-\eta_1|^{\frac{1}{2}}x) - \sqrt{\frac{n+1}{2}}h_{n+1}(|\eta-\eta_1|^{\frac{1}{2}}x)\right)\,.
\end{aligned}$$

Now, we observe that by definition of the support of $\mathcal{F}_{y\to\eta}(u_{I,m})(x,\eta)$ and $\mathcal{F}_{y\to\eta}(v_{J,n})(x,\eta)$, we have $1+(2m+1)|\eta_1| \in [A, 4A]$ and $1+(2n+1)|\eta-\eta_1| \in [B, 4B]$. Therefore, for some numerical constants $c(\delta_1,\delta_2)$, using the notations (6.19) for $F^{\delta}_{I,m}$, one can write

$$\begin{aligned}
(1-\partial_{xx}+&x^2|\eta|^2)(h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta-\eta_1|^{\frac{1}{2}}x))\\
&= \left(1+4AF^{(0,+)}_{I,m}(\eta_1)+4BF^{(0,+)}_{J,n}(\eta-\eta_1)\right)h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta-\eta_1|^{\frac{1}{2}}x)\\
+ \sum_{(\delta_1,\delta_2)\in(\{-1,1\}\times\{+,-\})^2} & c(\delta_1,\delta_2)\sqrt{AB}F^{\delta_1}_{I,m}(\eta_1)F^{\delta_2}_{J,n}(\eta-\eta_1)h_{m+\delta_1}(|\eta_1|^{\frac{1}{2}}x)h_{n+\delta_2}(|\eta-\eta_1|^{\frac{1}{2}}x)\,.
\end{aligned}$$

We denote $D_2 = D_1^2$. Then we define $C_{A,B}((0,-),(0,-)) = 1$, $C_{A,B}((0,+),(0,+)) = 0$, $C_{A,B}((0,+),(0,-)) = 2A$, $C_{A,B}((0,-),(0,+)) = 2B$ and $C_{A,B}(\delta_1,\delta_2) = c(\delta_1,\delta_2)\sqrt{AB}$ if $(\delta_1,\delta_2)\in(\{-1,1\}\times\{+,-\})^2$. With the notation from Definition 6.6 (and a similar notation for the $v^{\delta_2}_{J,n}$), we conclude that

$$\begin{aligned}
&\mathcal{F}((\mathrm{Id}-\Delta_G)(u_{I,m}v_{J,n}))(x,\eta)\\
&= \sum_{(\delta_1,\delta_2)\in D_2} C_{A,B}(\delta_1,\delta_2)\int f^{\delta_1}_{I,m}(\eta_1)g^{\delta_2}_{J,n}(\eta-\eta_1)h_{m+\delta_1}(|\eta_1|^{\frac{1}{2}}x)h_{n+\delta_2}(|\eta-\eta_1|^{\frac{1}{2}}x)\,\mathrm{d}\eta_1\,,
\end{aligned}$$

so that

$$(\mathrm{Id}-\Delta_G)(u_{I,m}v_{J,n}) = \sum_{(\delta_1,\delta_2)\in D_2} C_{A,B}(\delta_1,\delta_2)u^{\delta_1}_{I,m}v^{\delta_2}_{J,n}\,. \qquad \square$$

Lemma 6.27 (Action of Δ_G for the product of three terms)**.** *There exists a finite set $D_3 \subset D_1 \times D_1 \times D_1$ such that the following holds. Let $u^{(1)}, u^{(2)}, u^{(3)} \in L^2_G$, $A_1, A_2, A_3 \in 2^{\mathbb{N}}$, $I_1, I_2, I_3 \in 2^{\mathbb{Z}}$ and $m_1, m_2, m_3 \in \mathbb{N}$ such that $(m_i+1)I_i \sim A_i$ for $i = 1, 2, 3$. Then*

$$(\mathrm{Id} - \Delta_G)(u^{(1)}_{I_1,m_1} u^{(2)}_{I_2,m_2} u^{(3)}_{I_3,m_3}) = \sum_{\delta=(\delta_1,\delta_2,\delta_3)\in D_3} C_{A_1,A_2,A_3}(\delta)(u^{(1)}_{I_1,m_1})^{\delta_1}(u^{(2)}_{I_2,m_2})^{\delta_2}(u^{(3)}_{I_3,m_3})^{\delta_3} ,$$

where the shifted functions $(u^{(i)}_{I_i,m_i})^{\delta_i}$ are defined in Definition 6.6 and for all $\delta \in D_3$,

$$|C_{A_1,A_2,A_3}(\delta)| \lesssim \max\{A_1, A_2, A_3\} .$$

Proof. Taking the Fourier transform in y, we transform the product $u^{(1)}_{I_1,m_1} u^{(2)}_{I_2,m_2} u^{(3)}_{I_3,m_3}$ into a convolution product

$$\begin{aligned}\mathcal{F}_{y\to\eta}(u^{(1)}_{I_1,m_1} u^{(2)}_{I_2,m_2} u^{(3)}_{I_3,m_3})(x,\eta) = \int f^{(1)}_{I_1,m_1}(\eta_1) f^{(2)}_{I_2,m_2}(\eta_2) f^{(3)}_{I_3,m_3}(\eta-\eta_1-\eta_2)& \\ h_{m_1}(|\eta_1|^{\frac12}x)h_{m_2}(|\eta_2|^{\frac12}x)h_{m_3}(|\eta-\eta_1-\eta_2|^{\frac12}x)\mathrm{d}\eta_1\mathrm{d}\eta_2 &.\end{aligned}$$

Then, we write $h_{m_1,m_2,m_3}(\eta_1,\eta_2,\eta_3) := h_{m_1}(|\eta_1|^{\frac12}x)h_{m_2}(|\eta_2|^{\frac12}x)h_{m_3}(|\eta_3|^{\frac12}x)$ and expand

$$\begin{aligned}&(\partial_{xx} - x^2|\eta_1+\eta_2+\eta_3|^2)(h_{m_1,m_2,m_3}(\eta_1,\eta_2,\eta_3)) \\ &\quad = -\left((2m_1+1)|\eta_1| + (2m_2+1)|\eta_2| + (2m_3+1)|\eta_3|\right)h_{m_1,m_2,m_3}(\eta_1,\eta_2,\eta_3) \\ &\qquad + x^2(|\eta_1|^2+|\eta_2|^2+|\eta_3|^2 - |\eta_1+\eta_2+\eta_3|^2)h_{m_1,m_2,m_3}(\eta_1,\eta_2,\eta_3) \\ &\qquad + 2(\mathbb{I}_{1,2} + \mathbb{I}_{1,3} + \mathbb{I}_{2,3}) ,\end{aligned}$$

where

$$\begin{aligned}\mathbb{I}_{i,j} &= \partial_x(h_{m_i}(|\eta_i|^{\frac12}x))\partial_x(h_{m_j}(|\eta_j|^{\frac12}x)) \\ &= \sqrt{|\eta_i||\eta_j|}\left(\sqrt{\frac{m_i}{2}}h_{m_i-1}(|\eta_i|^{\frac12}x) - \sqrt{\frac{m_i+1}{2}}h_{m_i+1}(|\eta_i|^{\frac12}x)\right) \\ &\qquad \left(\sqrt{\frac{m_j}{2}}h_{m_j-1}(|\eta_j|^{\frac12}x) - \sqrt{\frac{m_j+1}{2}}h_{m_j+1}(|\eta_j|^{\frac12}x)\right) .\end{aligned}$$

In particular, expanding $\mathbb{I}_{i,j}$, this term writes as a combination of terms

$$C_{A_1,A_2,A_3}(\delta)F^{\delta_1}_{I_1,m_1}(\eta_1)F^{\delta_2}_{I_2,m_2}(\eta_2)F^{\delta_3}_{I_3,m_3}(\eta_3)h_{m_1+\delta_1,m_2+\delta_2,m_3+\delta_3}(\eta_1,\eta_2,\eta_3) , \tag{6.20}$$

where $|C_{A_1,A_2,A_3}(\delta)| \lesssim \max\{A_1, A_2, A_3\}$ for $\delta = (\delta_1, \delta_2, \delta_3)$ in some finite set $D_3 \subset D_1^3$.

By comparing the signs of η_1, η_2 and η_3, one can see that $|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2 - |\eta_1+\eta_2+\eta_3|^2$ is a linear combination of terms $|\eta_i\eta_j|$ and $\eta_i\eta_j$ for $i \neq j$. We now use the identity

$$xh_m(x) = \sqrt{\frac{m}{2}}h_{m-1} + \sqrt{\frac{m+1}{2}}h_{m+1}$$

to remove the weight x^2 and write the term

$$x^2(|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2 - |\eta_1+\eta_2+\eta_3|^2)h_{m_1,m_2,m_3}(\eta_1,\eta_2,\eta_3)$$

as a linear combination of terms as (6.20) above. Finally, we conclude that

$$\begin{aligned}\mathcal{F}(&(\mathrm{Id} - \Delta_G)(u^{(1)}_{I_1,m_1}u^{(2)}_{I_2,m_2}u^{(3)}_{I_3,m_3}))(x,\eta)\\ &= \sum_{\delta\in D_3} C_{A_1,A_2,A_3}(\delta)\int (f^{(1)}_{I_1,m_1})^{\delta_1}(\eta_1)(f^{(2)}_{I_2,m_2})^{\delta_2}(\eta_2)(f^{(3)}_{I_3,m_3})^{\delta_3}(\eta-\eta_1-\eta_2)\\ &\qquad\qquad h_{m_1+\delta_1}(|\eta_1|^{\frac12}x)h_{m_2+\delta_2}(|\eta_2|^{\frac12}x)h_{m_3+\delta_3}(|\eta-\eta_1-\eta_2|^{\frac12}x)\mathrm{d}\eta_1\mathrm{d}\eta_2\end{aligned}$$

for some $|C_{A_1,A_2,A_3}(\delta)| \lesssim \max\{A_1,A_2,A_3\}$, leading to the lemma. □

4.2. A bound on the Sobolev norm of a high-low product. We are now able to estimate the H^ℓ norm of a product u_Av_B by its L^2 norm as follows.

Corollary 6.28. *Let $\ell \in [0,2]$. Then there exists $C > 0$ such that for all $u, v \in H^\ell_G$, for all $A, B \in 2^{\mathbb{N}}$, there holds*

$$\|u_Av_B\|_{H^\ell_G} \leqslant C\max\{A,B\}^{\frac{\ell}{2}} \sum_{(\delta_1,\delta_2)\in D_2} \|(u_A)^{\delta_1}(v_B)^{\delta_2}\|_{L^2_G}\,.$$

For general $\ell \geqslant 0$, the same result holds up to taking bigger finite sets D_2 (depending on ℓ) and applying the shift several times for each function.

Proof. We proceed by interpolation.

In the case $\ell = 0$, there is nothing to do. In the case $\ell = 2$, we use Lemma 6.26 to write that for any $(m+1)I \sim A$ and $(n+1)J \sim B$,

$$(\mathrm{Id} - \Delta_G)(u_{I,m}v_{J,n}) = \sum_{(\delta_1,\delta_2)\in D_2} C_{A,B}(\delta_1,\delta_2)u^{\delta_1}_{I,m}v^{\delta_2}_{J,n}\,,$$

and thus by summation and taking the L^2_G norm, we obtain

$$\|(\mathrm{Id} - \Delta_G)(u_Av_B)\|^2_{L^2_G} \lesssim \Bigg\| \sum_{(\delta_1,\delta_2)\in D_2} C_{A,B}(\delta_1,\delta_2) \sum_{\substack{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}\\(m+1)I\sim A}}\ \sum_{\substack{(J,n)\in 2^{\mathbb{Z}}\times\mathbb{N}\\(n+1)J\sim B}} u^{\delta_1}_{I,m}v^{\delta_2}_{J,n}\Bigg\|^2_{L^2_G}\,.$$

We use the triangle inequality on the sum over (δ_1,δ_2) and the fact that $|C_{A,B}(\delta_1,\delta_2)| \lesssim \max(A,B)$ to deduce that

$$\begin{aligned}\|(\mathrm{Id}-\Delta_G)(u_Av_B)\|_{L^2_G}^2 &\lesssim A^2 \sum_{(\delta_1,\delta_2)\in D_2} \Bigg\| \sum_{\substack{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}\\ (m+1)I\sim A}} \sum_{\substack{(J,n)\in 2^{\mathbb{Z}}\times\mathbb{N}\\ (n+1)J\sim B}} u_{I,m}^{\delta_1} v_{J,n}^{\delta_2} \Bigg\|_{L^2_G}^2 \\ &\lesssim A^2 \sum_{(\delta_1,\delta_2)\in D_2} \|(u_A)^{\delta_1}(v_B)^{\delta_2}\|_{L^2_G}^2 .\end{aligned}$$

To get the result when $\ell\in(0,2)$, it only remains to interpolate thanks to the inequality

$$\|u_Av_B\|_{H^\ell_G} \leqslant \|u_Av_B\|_{L^2_G}^{1-\frac{\ell}{2}} \|u_Av_B\|_{H^2_G}^{\frac{\ell}{2}}$$

when $u_A, v_B \in H^2_G$, and conclude by density.

Similarly, for even integer ℓ, we apply Lemma 6.26 successively $\frac{\ell}{2}$ times to get the estimate, and conclude by interpolation for exponents between ℓ and $\ell+2$. $\square$

Using Lemma 6.27 instead of 6.26, we get the following adaptation of Corollary 6.28 for the product of three terms.

Corollary 6.29. *Let $\ell\in[0,2]$ and $u,v,w\in H^\ell_G$. Then for all $A,B,C\in 2^{\mathbb{N}}$, there holds*

$$\|u_Av_Bw_C\|_{H^\ell_G} \lesssim \max\{A,B,C\}^{\frac{\ell}{2}} \sum_{(\delta_1,\delta_2,\delta_3)\in D_3} \|(u_A)^{\delta_1}(v_B)^{\delta_2}(w_C)^{\delta_3}\|_{L^2_G} .$$

For general $\ell\geqslant 0$, the same result holds up to taking bigger finite sets D_3 (depending on ℓ) and applying the shift several times for each function.

Proof. By interpolation over ℓ, it is sufficient to establish the following inequality for even integer ℓ: for all $B,C\leqslant A$,

$$\left\|(\mathrm{Id}-\Delta_G)^{\ell/2}(u_Av_Bw_C)\right\|_{L^2_G}^2 \lesssim \sum_{(\delta_1,\delta_2,\delta_3)\in D_3} A^\ell \|(u_A)^{\delta_1}(v_B)^{\delta_2}(w_C)^{\delta_3}\|_{L^2_G}^2 ,$$

then apply this result to $u, P_{\leqslant A}v$ and $P_{\leqslant A}w$.

Since in the case $\ell=0$ there is nothing to do, let us assume $\ell=2$. Using Lemma 6.27, we can write

$$(\mathrm{Id}-\Delta_G)(u_{I_1,m_1}v_{I_2,m_2}w_{I_3,m_3}) = \sum_{(\delta_1,\delta_2,\delta_3)\in D_3} C_{A,B,C}(\delta_1,\delta_2,\delta_3) u_{I_1,m_1}^{\delta_1} v_{I_2,m_2}^{\delta_2} w_{I_3,m_3}^{\delta_3} ,$$

and therefore

$$\|(\mathrm{Id}-\Delta_G)(u_A v_B w_C)\|_{L^2_G}^2$$
$$\lesssim \Bigg\| \sum_{(\delta_1,\delta_2,\delta_3)\in D_3} C_{A,B,C}(\delta_1,\delta_2,\delta_3) \sum_{m_1,m_2,m_3\in\mathbb{N}} \sum_{\substack{(m_1+1)I_1\sim A\\ (m_2+1)I_2\sim B\\ (m_3+1)I_3\sim C}} u^{\delta_1}_{I_1,m_1} v^{\delta_2}_{I_2,m_2} w^{\delta_3}_{I_3,m_3} \Bigg\|_{L^2_G}^2 .$$

We use the triangle inequality on the sum over $(\delta_1,\delta_2,\delta_3)$ and the fact that $|C_{A,B,C}(\delta_1,\delta_2,\delta_3)| \lesssim \max(A,B,C) \leqslant A$ to deduce that

$$\|(\mathrm{Id}-\Delta_G)(u_A v_B w_C)\|_{L^2_G}^2 \lesssim A^2 \sum_{(\delta_1,\delta_2,\delta_3)\in D_3} \|(u_A)^{\delta_1}(v_B)^{\delta_2}(w_C)^{\delta_3}\|_{L^2_G}^2 .$$

In the case of a general even integer ℓ, successive applications of Lemma 6.27 lead to a similar result.

$\square$

4.3. A refined Sobolev estimate of a product. Finally, we estimate the H^ℓ norm of a product $u_A v$ by the L^2 norm of products $u_A v_B$ and the H^ℓ norm of v as follows.

In rough terms, we should have the following. Let $\ell \in [0,2]$, $u,v\in L^2_G$ and $A\in 2^{\mathbb{N}}$. Then for all $\varepsilon>0$, we have

$$\|uv\|_{H^\ell_G}^2 \lesssim \sum_{(\delta_1,\delta_2)\in D_2} \sum_{A,B:B\leqslant A} A^{\ell+\varepsilon} \|(u_A)^{\delta_1}(v_B)^{\delta_2}\|_{L^2_G}^2 + \sum_{\delta\in D_1}\sum_{A} A^{\varepsilon}\|(u_A)^{\delta_1}\|_{L^\infty_G}^2 \|v\|_{H^\ell_G}^2 ,$$

where we recall that D_1 is some finite set and $D_2 = D_1\times D_1$.

However, in order to get nice mapping properties in L^p spaces for $p\neq 2$ which are necessary during the course of this proof, see Appendix C, we introduce a cutoff function $\chi\in\mathcal{C}^\infty_c[0,1)$ such that $\chi\equiv 1$ on $[0,\frac12]$. Then, for $A\in 2^{\mathbb{N}}$, we define the projection $P_{\leqslant A}$ as the Fourier multiplier

$$P_{\leqslant A} = \chi\left(\frac{\mathrm{Id}-\Delta_G}{A}\right),$$

which is a smooth counterpart for the projection on the Sobolev modes $1+(2m+1)|\eta|\leqslant A$, acting on the decomposition (6.1) as

$$\mathcal{F}_{y\to\eta}(P_{\leqslant A}u)(x,\eta) = \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} \chi\left(\frac{1+(2m+1)|\eta|}{A}\right)\widehat{u_{I,m}}(x,\eta).$$

Note that since on the support of $\widehat{u_{I,m}}$, we have $|\eta|\in[I,2I]$, the sum can be restricted to the indices $1+(2m+1)I\leqslant A$. The projection $P_{\leqslant A}$ commutes with the block decomposition (6.9) and the Grushin operator: for $B\in 2^{\mathbb{N}}$, we have

$$P_{\leqslant A}(v_B) = (P_{\leqslant A}v)_B$$

and for $k \in \mathbb{R}$, we have

$$P_{\leqslant A}((-\Delta_G)^{k/2}v) = (-\Delta_G)^{k/2}(P_{\leqslant A}v).$$

Moreover, for all $u \in L^2_G$, we have $\|P_{\leqslant A}u\|_{L^2_G} \leqslant \|u\|_{L^2_G}$. We also denote $P_{>A} = \mathrm{Id} - P_{\leqslant A}$.

Lemma 6.30 (Sobolev norm for the product of two terms)**.** *Let $\ell \in [0,2]$, $u, v \in L^2_G$ and $A \in 2^{\mathbb{N}}$. Then for all $\varepsilon > 0$, we have*

$$\|u_A v\|^2_{H^\ell_G} \lesssim \sum_{(\delta_1,\delta_2)\in D_2} \sum_{B: B\leqslant A} A^\ell B^\varepsilon \|(u_A)^{\delta_1}(P_{\leqslant A}v_B)^{\delta_2}\|^2_{L^2_G} + \sum_{\delta\in D_1} \|(u_A)^{\delta_1}\|^2_{L^\infty_G}\|v\|^2_{H^\ell_G},$$

where we recall that $D_1 = \{-1, 0, 1, \varnothing\}$ and $D_2 \subset D_1 \times D_1$.

As a consequence, we have

$$\|uv\|^2_{H^\ell_G} \lesssim \sum_{(\delta_1,\delta_2)\in D_2} \sum_{A,B: B\leqslant A} A^{\ell+\varepsilon} \|(u_A)^{\delta_1}(P_{\leqslant A}v_B)^{\delta_2}\|^2_{L^2_G} + \sum_{\delta\in D_1}\sum_{A} A^\varepsilon \|(u_A)^{\delta_1}\|^2_{L^\infty_G}\|v\|^2_{H^\ell_G}.$$

For general $\ell \geqslant 0$, the same result holds up to taking bigger finite sets D_1, D_2 (depending on ℓ) and applying the shift several times for each function.

Proof. The consequence is a simple application of the Cauchy-Schwarz' inequality.

We decompose $v = P_{\leqslant A}v + P_{>A}v$ and $v = \sum_{B\in 2^{\mathbb{N}}} v_B$. Since $(P_{\leqslant A}v_B) = v_B$ for $4B \leqslant A$ and $(P_{\leqslant A}v_B) = 0$ for $B > A$, we have

$$\|u_A v\|^2_{H^\ell_G} \lesssim \left\| \sum_{B\leqslant A} u_A(P_{\leqslant A}v_B)\right\|^2_{H^\ell_G} + \left\| \sum_{4B > A} u_A(P_{>A}v_B)\right\|^2_{H^\ell_G}.$$

We treat the two parts separately, and in each case we proceed by interpolation between successive even integers ℓ.

Step 1: upper bound for $\left\| \sum_{B\leqslant A} u_A(P_{\leqslant A}v_B)\right\|_{H^\ell_G}$. Concerning the first term, thanks to the Cauchy-Schwarz inequality, we have

$$\left\| \sum_{B\leqslant A} (\mathrm{Id} - \Delta_G)^{\ell/2}(u_A(P_{\leqslant A}v_B))\right\|^2_{L^2_G} \lesssim \left(\sum_{B\leqslant A} B^{-\varepsilon}\right)\left(\sum_{B\leqslant A} B^{\varepsilon}\left\|(\mathrm{Id} - \Delta_G)^{\ell/2}(u_A(P_{\leqslant A}v_B))\right\|^2_{L^2_G}\right),$$

and as $\sum_{B\in 2^{\mathbb{N}}} B^{-\varepsilon} \lesssim 1$, we infer that

$$\left\| \sum_{B\leqslant A} (\mathrm{Id} - \Delta_G)^{\ell/2}(u_A(P_{\leqslant A}v_B))\right\|^2_{L^2_G} \lesssim \sum_{B\leqslant A} B^{\varepsilon}\|(\mathrm{Id} - \Delta_G)^{\ell/2}(u_A(P_{\leqslant A}v_B))\|^2_{L^2_G}.$$

But using Corollary 6.28 to $(P_{\leqslant A}v)$ instead of v, we have

$$\left\|(\mathrm{Id} - \Delta_G)^{\ell/2}(u_A v_B)\right\|^2_{L^2_G} \lesssim A^\ell \sum_{(\delta_1,\delta_2)\in D_2} \|(u_A)^{\delta_1}(P_{\leqslant A}v_B)^{\delta_2}\|^2_{L^2_G}.$$

Step 2: upper bound for $\| \sum_{4B>A} u_A(P_{>A}v_B)\|_{H^\ell_G}$. For the second term, we establish the following estimate by interpolation:

$$\Big\| \sum_{4B>A} (\mathrm{Id} - \Delta_G)^{\ell/2}(u_A(P_{>A}v_B))\Big\|^2_{L^2_G} \lesssim \sum_{\delta \in D_1} \|(u_A)^{\delta_1}\|^2_{L^\infty_G} \|v\|^2_{H^\ell_G} . \tag{6.21}$$

In the case $\ell = 0$, we only need to note that from the orthogonality of the different modes v_B, we have

$$\Big\| u_A \sum_{4B>A} (P_{>A}v_B)\Big\|^2_{L^2_G} \lesssim \|u_A\|^2_{L^\infty_G} \Big\| \sum_{4B>A} (P_{>A}v_B)\Big\|^2_{L^2_G} \lesssim \|u_A\|^2_{L^\infty_G} \|v\|^2_{L^2_G} .$$

In the case $\ell = 2$, we use Lemma 6.26 to get that for any $m, n \in \mathbb{N}$ and $I, J \in 2^{\mathbb{Z}}$ such that $(m+1)I \sim A$ and $(n+1)J \sim B$,

$$(\mathrm{Id} - \Delta_G)(u_{I,m}(P_{>A}v)_{J,n}) = \sum_{(\delta_1,\delta_2)\in D_2} C_{A,B}(\delta_1, \delta_2) u^{\delta_1}_{I,m}(P_{>A}v)^{\delta_2}_{J,n} ,$$

and therefore obtain

$$\Big\| \sum_{4B>A} (\mathrm{Id} - \Delta_G) u_A(P_{>A}v_B)\Big\|^2_{L^2_G} \lesssim \Big\| \sum_{(\delta_1,\delta_2)\in D_2} \sum_{4B>A} C_{A,B}(\delta_1, \delta_2) \sum_{m,n\in\mathbb{N}} \sum_{\substack{(m+1)I\sim A\\(n+1)J\sim B}} u^{\delta_1}_{I,m}(P_{>A}v)^{\delta_2}_{J,n}\Big\|^2_{L^2_G} .$$

We use the triangle inequality on the sum over (δ_1, δ_2) to get that for fixed A,

$$\begin{aligned}
\Big\| \sum_{4B>A} (\mathrm{Id} - \Delta_G) u_A(P_{>A}v_B)\Big\|^2_{L^2_G} &\lesssim \sum_{(\delta_1,\delta_2)\in D_2} \Big\| \sum_{4B>A} C_{A,B}(\delta_1, \delta_2) \sum_{m,n\in\mathbb{N}} \sum_{\substack{(m+1)I\sim A\\(n+1)J\sim B}} u^{\delta_1}_{I,m}(P_{>A}v)^{\delta_2}_{J,n}\Big\|^2_{L^2_G} \\
&= \sum_{(\delta_1,\delta_2)\in D_2} \Big\| \sum_{4B>A} C_{A,B}(\delta_1, \delta_2) u^{\delta_1}_{A}(P_{>A}v_B)^{\delta_2}\Big\|^2_{L^2_G} \\
&\lesssim \sum_{(\delta_1,\delta_2)\in D_2} \|u^{\delta_1}_A\|^2_{L^\infty_G} \Big\| \sum_{4B>A} C_{A,B}(\delta_1, \delta_2) (P_{>A}v_B)^{\delta_2}\Big\|^2_{L^2_G} .
\end{aligned}$$

Now, by orthogonality and the fact that $|C_{A,B}(\delta_1, \delta_2)| \lesssim B$ for $4B > A$, we have

$$\begin{aligned}
\Big\| \sum_{4B>A} C_{A,B}(\delta_1, \delta_2)(P_{>A}v_B)^{\delta_2}\Big\|^2_{L^2_G} &= \sum_{4B>A} |C_{A,B}(\delta_1, \delta_2)|^2 \|(P_{>A}v_B)^{\delta_2}\|^2_{L^2_G} \\
&\lesssim \sum_{4B>A} B^2 \|(P_{>A}v_B)\|^2_{L^2_G} \lesssim \|v\|^2_{H^2_G} ,
\end{aligned}$$

which concludes the proof in the case $\ell = 2$.

Estimate (6.21) is now a consequence of an interpolation result, stated and proven in Lemma 6.39, based on the Stein's interpolation theorem. Instead of using this lemma, one could invoke the interpolation result between Sobolev spaces $[H^0, H^\ell]_\theta = H^{\theta\ell}$ for $\theta \in [0,1]$, that holds for Sobolev spaces on $\mathbb{R}^d$ (see for instance [BL76]) and could be extended to the setting of the Grushin operator. For general ℓ, successive uses of Lemma 6.26 lead to the result. $\square$

In what follows, we will actually use the trilinear version of Lemma 6.30.

Corollary 6.31 (Sobolev norm for the product of three terms)**.** *Let $\ell \in [0,2]$ and $u, v, w \in L^2_G$. Then for all $\varepsilon > 0$ there holds*

$$\|uvw\|^2_{H^\ell_G} \lesssim \sum_{\substack{(\delta_1,\delta_2,\delta_3)\in D_3\\ A,B,C: B,C\leqslant A}} A^{\ell+\varepsilon}\|(u_A)^{\delta_1}(P_{\leqslant A}v_B)^{\delta_2}(P_{\leqslant A}w_C)^{\delta_3}\|^2_{L^2_G} + \sum_{\delta_1\in D_1}\sum_A A^\varepsilon\|(u_A)^{\delta_1}\|^2_{L^\infty_G}\|v\|^2_{H^\ell_G}\|w\|^2_{H^\ell_G}.$$

** For $\ell > 2$, a similar result holds up to taking bigger finite sets D_3 and taking successive shifted functions.*

Proof. We mimick the proof of Lemma 6.30. It is enough to establish that for every A, we have

$$\|u_A vw\|^2_{H^\ell_G} \lesssim \sum_{\substack{(\delta_1,\delta_2,\delta_3)\in D_3\\ B,C: B,C\leqslant A}} A^{\ell}(BC)^\varepsilon\|(u_A)^{\delta_1}(P_{\leqslant A}v_B)^{\delta_2}(P_{\leqslant A}w_C)^{\delta_3}\|^2_{L^2_G} + \sum_{\delta_1\in D_1}\|(u_A)^{\delta_1}\|^2_{L^\infty_G}\|v\|^2_{H^\ell_G}\|w\|^2_{H^\ell_G}.$$

We start by writing that

$$\|u_A vw\|^2_{H^\ell_G} \lesssim \left\|\sum_{B,C\leqslant A} u_A(P_{\leqslant A}v_B)(P_{\leqslant A}w_C)\right\|^2_{H^\ell_G} + \left\|\sum_{B,C} u_A\left(v_Bw_C - (P_{\leqslant A}v_B)(P_{\leqslant A}w_C)\right)\right\|^2_{H^\ell_G}.$$

Step 1: upper bound for $\|\sum_{B,C\leqslant A} u_A(P_{\leqslant A}v_B)(P_{\leqslant A}w_C)\|_{H^\ell_G}$. By the Cauchy-Schwarz inequality, we have

$$\left\|\sum_{B,C\leqslant A}(\mathrm{Id}-\Delta_G)^{\ell/2}(u_A(P_{\leqslant A}v_B)(P_{\leqslant A}w_C))\right\|^2_{L^2_G}$$
$$\lesssim \left(\sum_{B,C\leqslant A}(BC)^{-\varepsilon}\right)\left(\sum_{B,C\leqslant A}(BC)^\varepsilon\left\|(\mathrm{Id}-\Delta_G)^{\ell/2}(u_A(P_{\leqslant A}v_B)(P_{\leqslant A}w_C))\right\|^2_{L^2_G}\right)$$

$$\lesssim \sum_{B,C\leqslant A}(BC)^{\varepsilon}\left\|(\mathrm{Id}-\Delta_G)^{\ell/2}(u_A(P_{\leqslant A}v_B)(P_{\leqslant A}w_C))\right\|_{L^2_G}^2 .$$

We then conclude thanks to Corollary 6.29 that

$$\left\|\sum_{B,C\leqslant A}(\mathrm{Id}-\Delta_G)^{\ell/2}(u_A(P_{\leqslant A}v_B)(P_{\leqslant A}w_C))\right\|_{L^2_G}^2 \lesssim \sum_{\substack{(\delta_1,\delta_2,\delta_3)\in D_3\\ B,C:B,C\leqslant A}} A^{\ell}(BC)^{\varepsilon}\|(u_A)^{\delta_1}(P_{\leqslant A}v_B)^{\delta_2}(P_{\leqslant A}w_C)^{\delta_3}\|_{L^2_G}^2 .$$

Step 2: upper bound for $\left\|\sum_{B,C}u_A\left(v_Bw_C-(P_{\leqslant A}v_B)(P_{\leqslant A}w_C)\right)\right\|_{H^\ell_G}$. We split

$$v_Bw_C-(P_{\leqslant A}v_B)(P_{\leqslant A}w_C)=(P_{>A}v_B)w_C+(P_{\leqslant A}v_B)(P_{>A}w_C).$$

For the first term, using the algebra property of H^ℓ_G for $\ell>\frac{3}{2}$ we write

$$\left\|\sum_{B,C}u_A(P_{>A}v_B)w_C\right\|_{H^\ell_G}^2 \lesssim \left\|\sum_{B}u_A(P_{>A}v_B)\right\|_{H^\ell_G}^2\left\|\sum_{C:C\leqslant A}w_C\right\|_{H^\ell_G}^2$$
$$\lesssim \left\|\sum_{B:4B>A}(\mathrm{Id}-\Delta_G)^{\ell/2}(u_A(P_{>A}v_B))\right\|_{L^2_G}^2\|w\|_{H^\ell_G}^2 ,$$

hence inequality (6.21) in the Step 2 of Lemma 6.30 applies and gives the expected bound. A similar treatment can be applied to the second term. ☐

5. Deterministic bilinear estimates

In this section, we first establish deterministic bilinear and trilinear estimates for the product of two Hermite functions with rescaling, of the form $h_m(\alpha_1\cdot)h_n(\alpha_2\cdot)$ and $h_{m_1}(\alpha_1\cdot)h_{m_2}(\alpha_2\cdot)h_{m_3}(\alpha_3^{\cdot})$. We then deduce deterministic bilinear estimates for the products $u_{I,m}v_{J,n}$.

5.1. Bilinear estimates for rescaled Hermite functions. We start by establishing bilinear estimates for the rescaled Hermite functions, which will be at the core of our future bilinear and trilinear smoothing estimates. Let us recall that $\lambda_m=\sqrt{2m+1}$.

Lemma 6.32. *Let $\alpha>0$, $m,n\in\mathbb{N}$, and assume that $\lambda_n\leqslant\frac{\alpha}{4}\lambda_m$. Then there holds*

$$\|h_mh_n(\alpha\cdot)\|_{L^2}^2\lesssim\frac{1}{\alpha\lambda_m}.$$

Proof. We cut the Hermite functions between the space regions delimited by the pointwise estimates of Corollary 6.15.

(1) In the region $R_1 := \left\{|x| \leqslant \frac{1}{2}\lambda_m\right\}$, we use the bound $|h_m(x)| \lesssim \lambda_m^{-\frac{1}{2}}$. Therefore there holds

$$\|h_m h_n(\alpha\cdot)\|^2_{L^2(R_1)} \lesssim \lambda_m^{-1}\|h_n(\alpha\cdot)\|^2_{L^2(R_1)} \lesssim \lambda_m^{-1}\alpha^{-1}\,.$$

(2) In the region $R_2 := \left\{\frac{1}{2}\lambda_m \leqslant |x| \leqslant 2\lambda_m\right\}$, we know by assumption that $2\lambda_n \leqslant \frac{\alpha}{2}\lambda_m \leqslant \alpha|x|$. We therefore use the bounds $|h_m(x)| \lesssim \left(\lambda_m^{\frac{2}{3}} + |x^2 - \lambda_m^2|\right)^{-\frac{1}{4}}$ and $|h_n(\alpha x)| \lesssim e^{-c(\alpha x)^2}$ with $c = \frac{1}{8}$, and get

$$\|h_m h_n(\alpha\cdot)\|^2_{L^2(R_2)} \lesssim \int_{\frac{1}{2}\lambda_m}^{2\lambda_m} \frac{e^{-2c\alpha^2x^2}}{\sqrt{\lambda_m^{\frac{2}{3}} + |x^2 - \lambda_m^2|}}\,\mathrm{d}x\,.$$

We fix the limits of the integral by setting $y = \frac{x}{\lambda_m}$, which leads to the bound:

$$\begin{aligned}\int_{\frac{1}{2}\lambda_m}^{2\lambda_m} \frac{e^{-2c\alpha^2x^2}}{\sqrt{\lambda_m^{\frac{2}{3}} + |x^2 - \lambda_m^2|}}\,\mathrm{d}x &= \int_{\frac{1}{2}}^{2} \frac{e^{-2c(\alpha\lambda_m)^2y^2}}{\sqrt{\lambda_m^{-\frac{4}{3}} + |y^2 - 1|}}\,\mathrm{d}y \\ &\lesssim \int_{\frac{1}{2}}^{2} \frac{e^{-2c(\alpha\lambda_m)^2y^2}}{\sqrt{|y^2 - 1|}}\,\mathrm{d}y \\ &\lesssim e^{-\frac{c}{2}(\alpha\lambda_m)^2}\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}y}{\sqrt{|y^2 - 1|}}\,.\end{aligned}$$

Now we remark that $e^{-\frac{c}{2}(\alpha\lambda_m)^2} \lesssim \frac{1}{\alpha\lambda_m}$, giving the required estimate.

(3) In the last region $R_3 := \{|x| \geqslant 2\lambda_m\}$ we use the exponential bounds for the two terms $|h_m(x)| \leqslant e^{-cx^2}$ and $|h_n(\alpha x)| \leqslant e^{-c(\alpha x)^2}$ with $c = \frac{1}{8}$. This leads to

$$\|h_m h_n(\alpha\cdot)\|^2_{L^2(R_3)} \lesssim \int_{|x|\geqslant 2\lambda_m} e^{-2cx^2(1+\alpha^2)}\mathrm{d}x\,.$$

Then using that for $C = 2c(1 + \alpha^2)$ and $X = 2\lambda_m$, we have

$$\int_X^\infty e^{-Cx^2}\mathrm{d}x \leqslant \frac{1}{2CX}\int_X^\infty 2Cxe^{-Cx^2}\mathrm{d}x \leqslant \frac{1}{2CX}\,,$$

this gives the result.

All the previous bounds put together imply the lemma. $\square$

In what follows, we will use the bilinear estimate from Lemma 6.32 under the following form.

Corollary 6.33 (Bilinear estimates for rescaled Hermite functions)**.** *Let* $m, n \in \mathbb{N}$ *and* $\alpha_1, \alpha_2 > 0$. *Then*

$$\|h_m(\alpha_1\cdot)\, h_n(\alpha_2\cdot)\|_{L^2}^2 \lesssim \min\left\{\frac{1}{\alpha_1^2(2n+1)}, \frac{1}{\alpha_2^2(2m+1)}\right\}^{\frac{1}{2}}.$$

Proof. In the first scenario, assume that $\alpha_1\sqrt{2n+1} \leqslant \frac{1}{4}\alpha_2\sqrt{2m+1}$. We start with a change of variable:

$$\|h_m(\alpha_1\cdot)\, h_n(\alpha_2\cdot)\|_{L^2}^2 = \frac{1}{\alpha_1}\left\|h_m h_n\left(\frac{\alpha_2}{\alpha_1}\cdot\right)\right\|_{L^2}^2.$$

Denote $\alpha := \frac{\alpha_2}{\alpha_1}$. Since $\alpha_1\sqrt{2n+1} \leqslant \frac{1}{4}\alpha_2\sqrt{2m+1}$, we have $\lambda_n \leqslant \frac{\alpha}{4}\lambda_m$. Then Lemma 6.32 implies

$$\left\|h_m h_n\left(\frac{\alpha_2}{\alpha_1}\cdot\right)\right\|_{L^2}^2 \lesssim \frac{1}{\alpha\sqrt{2m+1}} = \sqrt{\frac{\alpha_1^2}{\alpha_2^2(2m+1)}}.$$

Consequently,

$$\|h_m(\alpha_1\cdot)h_n(\alpha_2\cdot)\|_{L^2}^2 \lesssim \sqrt{\frac{1}{\alpha_2^2(2m+1)}}.$$

This is enough to conclude since by assumption, we have $\sqrt{\frac{1}{\alpha_2^2(2m+1)}} \leqslant \sqrt{\frac{1}{\alpha_1^2(2n+1)}}$.

In the second scenario, assume that $\alpha_2\sqrt{2m+1} \leqslant \frac{1}{4}\alpha_1\sqrt{2n+1}$. We exchange the roles of m and n and the roles of α_1 and α_2 to get

$$\|h_m(\alpha_1\cdot)h_n(\alpha_2\cdot)\|_{L^2}^2 \lesssim \sqrt{\frac{1}{\alpha_1^2(2n+1)}}.$$

In the last scenario, there exists $C_0 > 0$ such that $\frac{1}{4}\alpha_2\sqrt{2m+1} \leqslant \alpha_1\sqrt{2n+1} \leqslant 4\alpha_2\sqrt{2m+1}$. Then from Hölder's inequality, one obtains the bound

$$\|h_m(\alpha_1\cdot)h_n(\alpha_2\cdot)\|_{L^2}^2 \lesssim \|h_m(\alpha_1\cdot)\|_{L^4}^2\|h_n(\alpha_2\cdot)\|_{L^4}^2.$$

From the L^4 norm estimate $\|h_m\|_{L^4} \leqslant \frac{C}{(2m+1)^{\frac{1}{8}}}$, $m \in \mathbb{N}$ (see Lemma 6.14), we deduce

$$\|h_m(\alpha_1\cdot)h_n(\alpha_2\cdot)\|_{L^2}^2 \lesssim \frac{1}{(\alpha_1^2(2m+1))^{\frac{1}{4}}(\alpha_2^2(2n+1))^{\frac{1}{4}}}.$$

Since $\alpha_1\sqrt{2n+1}$ and $\alpha_2\sqrt{2m+1}$ differ from a factor at most 4, in this situation, one can bound the right-hand side by either $\sqrt{\frac{1}{\alpha_2^2(2m+1)}}$ or $\sqrt{\frac{1}{\alpha_1^2(2n+1)}}$. ☐

Corollary 6.34 (Trilinear estimates for rescaled Hermite functions)**.** *Let* $m_1, m_2, m_3 \in \mathbb{N}$ *and* $\alpha_1, \alpha_2, \alpha_3 > 0$. *Then*

$$\|h_{m_1}(\alpha_1\cdot)h_{m_2}(\alpha_2\cdot)h_{m_3}(\alpha_3\cdot)\|_{L^2}^2 \lesssim \frac{1}{\alpha_1\alpha_2\alpha_3}\min\left\{\frac{\alpha_i\alpha_j}{(2m_i+1)^{\frac{1}{6}}(2m_j+1)^{\frac{1}{2}}}\right\}_{i\neq j\in\{1,2,3\}}.$$

As a consequence, if $A \in 2^{\mathbb{N}}$, $I_1, I_2, I_3 \in 2^{\mathbb{Z}}$ *are such that* $(m_1+1)I_1 \sim A$, *for* $\alpha_1^2 \in [I_1, 2I_1]$, $\alpha_2^2 \in [I_2, 2I_2]$ *and* $\alpha_3^2 \in [I_3, 2I_3]$, *we have*

$$\alpha_1\alpha_2\alpha_3 \|h_{m_1}(\alpha_1\cdot)h_{m_2}(\alpha_2\cdot)h_{m_3}(\alpha_3\cdot)\|_{L^2}^2 \lesssim C_{\{I_i,m_i\}}^2 \,,$$

where one can choose

$$C_{\{I_i,m_i\}}^2 = \frac{\langle I_1\rangle (I_2I_3)^{\frac{1}{4}}}{A^{\frac{1}{2}}(2m_2+1)^{\frac{1}{12}}(2m_3+1)^{\frac{1}{12}}} \,.$$

Note that the gain with power $\frac{1}{12}$ induced by the L^∞ norm on the Hermite functions is not necessary in the sequel, but we keep this exponent for the sake of security.

Proof. We start by breaking the symmetry of roles of the three terms

$$\|h_{m_1}(\alpha_1\cdot)h_{m_2}(\alpha_2\cdot)h_{m_3}(\alpha_3\cdot)\|_{L^2}^2 \leqslant \|h_{m_1}(\alpha_1\cdot)\|_{L^\infty}^2 \|h_{m_2}(\alpha_2\cdot)h_{m_3}(\alpha_3\cdot)\|_{L^2}^2 \,.$$

Then we use Lemma 6.14 and Corollary 6.33 to bound

$$\|h_{m_1}(\alpha_1\cdot)h_{m_2}(\alpha_2\cdot)h_{m_3}(\alpha_3\cdot)\|_{L^2}^2 \lesssim \frac{1}{(2m_1+1)^{\frac{1}{6}}} \min\left\{\frac{1}{\alpha_2(2m_3+1)^{\frac{1}{2}}}, \frac{1}{\alpha_3(2m_2+1)^{\frac{1}{2}}}\right\} \,.$$

By symmetry between the roles of m_1, m_2, m_3, this implies

$$\|h_{m_1}(\alpha_1\cdot)h_{m_2}(\alpha_2\cdot)h_{m_3}(\alpha_3\cdot)\|_{L^2}^2 \lesssim \frac{1}{\alpha_1\alpha_2\alpha_3} \min\left\{\frac{\alpha_i\alpha_j}{(2m_i+1)^{\frac{1}{6}}(2m_j+1)^{\frac{1}{2}}}\right\}_{i\neq j\in\{1,2,3\}} \,.$$

In order to establish the second bound, for every $j = 1, 2, 3$, we have $\alpha_j \leqslant (2I_j)^{\frac{1}{2}}$, therefore

$$\min\left\{\frac{\alpha_i\alpha_j}{(2m_i+1)^{\frac{1}{6}}(2m_j+1)^{\frac{1}{2}}}\right\}_{i\neq j\in\{1,2,3\}} \lesssim \min\left\{\frac{I_i^{\frac{1}{2}}I_j^{\frac{1}{2}}}{(2m_i+1)^{\frac{1}{6}}(2m_j+1)^{\frac{1}{2}}}\right\}_{i\neq j\in\{1,2,3\}} \,.$$

Now, let us remark that

$$\min\left\{\frac{\alpha_i\alpha_j}{(2m_i+1)^{\frac{1}{6}}(2m_j+1)^{\frac{1}{2}}}\right\}_{i\neq j\in\{1,2,3\}} \leqslant \frac{I_2^{\frac{1}{2}}I_1^{\frac{1}{2}}}{(2m_2+1)^{\frac{1}{6}}(2m_1+1)^{\frac{1}{2}}} \,,$$

and since the same bound holds when exchanging the roles of (I_2, m_2) and (I_3, m_3), we obtain by interpolation

$$\min\left\{\frac{\alpha_i\alpha_j}{(2m_i+1)^{\frac{1}{6}}(2m_j+1)^{\frac{1}{2}}}\right\}_{i\neq j\in\{1,2,3\}} \leqslant \frac{(I_2I_3)^{\frac{1}{4}}\langle I_1\rangle}{((2m_2+1)(2m_3+1))^{\frac{1}{12}}A^{\frac{1}{2}}} \,. \qquad \square$$

5.2. Bilinear and trilinear estimates for the product of unimodal blocks. In this part, we provide a bilinear estimate for the "building blocks" $u_{I,m}v_{J,n}$, where $u_{I,m}$ and $v_{J,n}$ are frequency localized and unimodal.

Proposition 6.7 (Bilinear block estimate)**.** *Let $m,n\geqslant 0$ and $I,J\in 2^{\mathbb{Z}}$. Let $u_{I,m}$ (resp. $v_{J,n}$) in L^2_G defined by (6.8)*

$$\mathcal{F}_{y\to\eta}(u_{I,m})(x,\eta)=f_{I,m}(\eta)h_m(|\eta|^{\frac12}x)$$

resp.

$$\mathcal{F}_{y\to\eta}(v_{J,n})(x,\eta)=g_{J,n}(\eta)h_n(|\eta|^{\frac12}x)\,,$$

the function $f_{I,m}$ (resp. $g_{J,n}$) being supported on the set $\{|\eta|\in[I,2I]\}$ (resp. on the set $\{|\eta|\in[J,2J]\}$). Then, one has

$$\|u_{I,m}v_{J,n}\|^2_{L^2_G}\lesssim \min\{I,J\}\min\left\{\frac{I}{2m+1},\frac{J}{2n+1}\right\}^{\frac12}\|u_{I,m}\|^2_{L^2_G}\|v_{J,n}\|^2_{L^2_G}\,.$$

As a consequence,

$$\|u_{I,m}v_{J,n}\|^2_{L^2_G}\lesssim \min\left\{\frac{J\langle I\rangle}{(1+(2m+1)I)^{\frac12}},\frac{I\langle J\rangle}{(1+(2n+1)J)^{\frac12}}\right\}\|u_{I,m}\|^2_{L^2_G}\|v_{J,n}\|^2_{L^2_G}\,.$$

Proof. From the Parseval formula, we have $\|u_{I,m}v_{J,n}\|^2_{L^2_G}=\|\hat u_{I,m}*\hat v_{J,n}\|^2_{L^2_{x,\eta}}$, where the convolution is the classical convolution product in the variable η. We expand the norm of this convolution product and obtain

$$\|u_{I,m}v_{J,n}\|^2_{L^2_G}=\int_{\mathbb{R}^3}f_{I,m}(\eta_1)f_{I,m}(\eta_2)g_{J,n}(\eta-\eta_1)g_{J,n}(\eta-\eta_2)\mathbb{I}_{m,n}(\eta_1,\eta_2,\eta-\eta_1,\eta-\eta_2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta\,,$$

with

$$\mathbb{I}_{m,n}(\eta_1,\eta_2,\eta_1',\eta_2'):=\int_{\mathbb{R}}h_m(|\eta_1|^{\frac12}x)h_m(|\eta_2|^{\frac12}x)h_n(|\eta_1'|^{\frac12}x)h_n(|\eta_2'|^{\frac12}x)\,\mathrm{d}x\,.$$

By Cauchy-Schwarz' inequality, one has

$$|\mathbb{I}_{m,n}(\eta_1,\eta_2,\eta_1',\eta_2')|\leqslant\|h_m(|\eta_1|^{\frac12}\cdot)h_n(|\eta_1'|^{\frac12}\cdot)\|_{L^2}\|h_m(|\eta_2|^{\frac12}\cdot)h_n(|\eta_2'|^{\frac12}\cdot)\|_{L^2}\,.$$

Using Corollary 6.33, we deduce the estimate

$$|\mathbb{I}_{m,n}(\eta_1,\eta_2,\eta_1',\eta_2')|\lesssim\min\left\{\frac{1}{|\eta_1|(2n+1)},\frac{1}{|\eta_1'|(2m+1)}\right\}^{\frac14}\min\left\{\frac{1}{|\eta_2|(2n+1)},\frac{1}{|\eta_2'|(2m+1)}\right\}^{\frac14}\,.$$

Going back to the blocks $u_{I,m}$ and $v_{J,n}$, this estimate implies

$$\|u_{I,m}v_{J,n}\|^2_{L^2_G}\lesssim\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\min\left\{\frac{1}{|\eta_1|(2n+1)},\frac{1}{|\eta-\eta_1|(2m+1)}\right\}^{\frac14}|f_{I,m}(\eta_1)g_{J,n}(\eta-\eta_1)|\,\mathrm{d}\eta_1\right)^2\mathrm{d}\eta\,.$$

By extracting the minimum, we conclude

$$\|u_{I,m}v_{J,n}\|^2_{L^2_G}\lesssim\min\left\{\frac{1}{\sqrt{2n+1}}\|\frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}}*|g_{J,n}|\|^2_{L^2},\frac{1}{\sqrt{2m+1}}\||f_{I,m}|*\frac{|g_{J,n}(\cdot)|}{|\cdot|^{1/4}}\|^2_{L^2}\right\}\,.$$

Using the symmetry of the roles of u and v, we estimate for instance $\|\frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} * |g_{J,n}|\|_{L^2}^2$. We apply Young's inequality

$$\|\frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} * |g_{J,n}|\|_{L^2}^2 \leqslant \|\frac{f_{I,m}(\cdot)}{|\cdot|^{1/4}}\|_{L^1}^2 \|g_{J,n}\|_{L^2}^2 .$$

Thanks to Cauchy-Schwarz' inequality, since the interval $[I, 2I]$ has length I, we get

$$\|\frac{f_{I,m}(\cdot)}{|\cdot|^{1/4}}\|_{L^1}^2 \lesssim I\|\frac{f_{I,m}(\cdot)}{|\cdot|^{1/4}}\|_{L^2}^2 .$$

Moreover, we know that $\|g_{J,n}\|_{L^2}^2 \lesssim \sqrt{J}\|v_{J,n}\|_{L^2}^2$, therefore

$$\|\frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} * |g_{J,n}|\|_{L^2}^2 \leqslant I\sqrt{J}\|u_{I,m}\|_{L^2_G}^2 \|v_{J,n}\|_{L^2_G}^2 .$$

But we also have from Young's inequality

$$\|\frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} * |g_{J,n}|\|_{L^2}^2 \leqslant \|\frac{f_{I,m}(\cdot)}{|\cdot|^{1/4}}\|_{L^2}^2 \|g_{J,n}\|_{L^1}^2 ,$$

where from Cauchy-Schwarz' inequality, $\|g_{J,n}\|_{L^1}^2 \leqslant J\|g_{J,n}\|_{L^2}^2 \lesssim J^{\frac{3}{2}}\|v_{J,n}\|_{L^2_G}^2$, so that actually

$$\|\frac{|f_{I,m}(\cdot)|}{|\cdot|^{1/4}} * |g_{J,n}|\|_{L^2}^2 \leqslant \min\{I, J\}\sqrt{J}\|u_{I,m}\|_{L^2_G}^2 \|v_{J,n}\|_{L^2_G}^2 .$$

To conclude, using the symmetry of the roles of u and v, we have proven that

$$\|u_{I,m}v_{J,n}\|_{L^2_G}^2 \lesssim \min\{I, J\} \min\left\{\frac{I}{2m+1}, \frac{J}{2n+1}\right\}^{\frac{1}{2}} \|u_{I,m}\|_{L^2_G}^2 \|v_{J,n}\|_{L^2_G}^2 . \qquad \square$$

6. Probabilistic bilinear and trilinear estimates

In this section we provide the proof of Theorem 2 (i). The main idea of proof is to use the deterministic bilinear estimate for the products $u_{I,m}v_{J,n}$ given by Proposition 6.7 in order to prove a smoothing effect on quadratic and cubic expressions of a random function.

6.1. Bilinear estimate for random interactions. To deal with purely random interactions of the form $(u_0^\omega)^2$ and $|u_0^\omega|^2$, we heavily rely on the frequency decoupling offered by the randomization. This is akin to the bilinear estimates obtained in [BTT13].

Let us recall some notation. We fix a sequence of independent identically distributed subgaussian random variables $(X_{I,m})_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}}$ and $u_0 \in H^k_G$. We use decomposition (6.2)

$$u_0^\omega = \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} X_{I,m}(\omega)u_{I,m} .$$

For all $t \in \mathbb{R}$, we consider the time evolution under the linear flow (LS-G) with initial data $u_{I,m}$ denoted $z_{I,m}(t) = e^{it\Delta_G} u_{I,m}$, $(I, m) \in 2^{\mathbb{Z}} \times \mathbb{N}$. Then, we define the random counterpart z^ω of u as in (6.11)

$$z^\omega(t) = e^{it\Delta_G} u_0^\omega = \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} X_{I,m}(\omega) z_{I,m}(t)\,.$$

Proof of (6.4) *in Theorem 2 (i).* We establish estimate (6.4), that is, we bound the norms of the products $\|(z^\omega)^2\|_{L^q_T H^\ell_G}$ and $\||z^\omega|^2\|_{L^q_T H^\ell_G}$ for $\ell = k + \frac{3}{2}$.

Step 1: reduction of inequality (6.4) *to deterministic estimates.* We start with $\|(z^\omega)^2\|_{L^q_T H^\ell_G}$. Fix $t \in [0, T]$ and $(x, y) \in \mathbb{R}^2$. Applying Corollary 6.20 (i) with the norm $L^q_T L^2_{x,y}$ and $\Psi_{(I,m),(J,n)} = (\mathrm{Id} - \Delta_G)^{\ell/2}(z_{I,m} z_{J,n})$, we obtain that outside a set of probability at most e^{-cR^2} there holds

$$\|(z^\omega)^2\|^2_{L^q_T H^\ell_G} \leqslant R^4 \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} \sum_{(J,n)\in 2^{\mathbb{Z}}\times\mathbb{N}} \|z_{I,m} z_{J,n}\|^2_{L^q_T H^\ell_G}\,.$$

It remains to prove that for every $u_0, v_0 \in \mathcal{X}^k_1$, denoting $z_{I,m}(t) = e^{it\Delta_G} u_{I,m}$ and $\tilde{z}_{J,n}(t) = e^{it\Delta_G} v_{J,n}$ for $t \in \mathbb{R}$, we have

$$\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} \sum_{(J,n)\in 2^{\mathbb{Z}}\times\mathbb{N}} \|z_{I,m} \tilde{z}_{J,n}\|^2_{L^q_T H^\ell_G} \lesssim T^{\frac{2}{q}} \|u_0\|^2_{\mathcal{X}^k_1} \|v_0\|^2_{\mathcal{X}^k_1}\,. \tag{6.22}$$

Indeed we will conclude by taking $u_0 = v_0$.

Similarly for $\||z^\omega|^2\|_{L^q_T H^\ell_G}$, applying Corollary 6.20 (i) with the norm $L^q_T L^2_{x,y}$ and $\Psi_{(I,m),(J,n)} = (\mathrm{Id} - \Delta_G)^{\ell/2}(z_{I,m} \overline{z_{J,n}})$, we obtain that outside a set of probability at most e^{-cR^2} there holds

$$\||z^\omega|^2\|^2_{L^q_T H^\ell_G} \leqslant R^4 \sum_{(I,m),(J,n)\in 2^{\mathbb{Z}}\times\mathbb{N}} \|z_{I,m} \overline{z_{J,n}}\|^2_{L^q_T H^\ell_G} + R^4 \left(\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} \||z_{I,m}|^2\|_{L^q_T H^\ell_G} \right)^2 .$$

The upper bound is handled using inequality (6.22) and establishing

$$\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} \|z_{I,m} \tilde{z}_{I,m}\|_{L^q_T H^\ell_G} \lesssim T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}^k_1} \|v_0\|_{\mathcal{X}^k_1}\,, \tag{6.23}$$

which we apply to u and $v = \overline{u}$.

Step 2: Proof of (6.22). We claim that (6.22) is a consequence of the time independent inequality

$$\sum_{(I,m),(J,n)\in 2^{\mathbb{Z}}\times\mathbb{N}} \|u_{I,m} v_{J,n}\|^2_{H^\ell_G} \lesssim \|u_0\|^2_{\mathcal{X}^k_1} \|v_0\|^2_{\mathcal{X}^k_1}\,. \tag{6.24}$$

Indeed we apply (6.24) to $z_{I,m}(t) = e^{it\Delta_G} u_{I,m}$ and $\tilde{z}_{J,n}(t) = e^{it\Delta_G} v_{J,n}$ for $t \in \mathbb{R}$ instead of $u_{I,m}$ and $v_{J,n}$, then we use $\mathcal{X}^k_1$ isometry property of the linear flow, where we recall that we have defined the norm (6.3)

$$\|u_0\|_{\mathcal{X}_1^k}^2 = \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} (1+(2m+1)I)^k \langle I\rangle \|u_{I,m}\|_{L^2}^2 .$$

Finally we integrate in time using the Hölder inequality, yielding to (6.22).

In order to prove (6.24), fix $I, J \in 2^{\mathbb{Z}}$ and $m, n \in \mathbb{N}$. Let $A, B \in 2^{\mathbb{N}}$ such that $(m+1)I \sim A$ and $(n+1)J \sim B$. We apply Corollary 6.28 and get

$$\|u_{I,m}v_{J,n}\|_{H_G^\ell}^2 \lesssim \max\{A,B\}^\ell \|u_{I,m}v_{J,n}\|_{L_G^2}^2 .$$

Then, from the unit block bound of Proposition 6.7, it follows that

$$\|u_{I,m}v_{J,n}\|_{H_G^\ell}^2 \lesssim \max\{A,B\}^\ell \min\left\{\frac{J\langle I\rangle}{A^{\frac12}}, \frac{I\langle J\rangle}{B^{\frac12}}\right\} \|u_{I,m}\|_{L_G^2}^2 \|v_{J,n}\|_{L_G^2}^2 . \tag{6.25}$$

Separating the cases $A \leqslant B$ and $A > B$, we obtain the bounds

$$\begin{aligned}\sum_{(I,m),(J,n)\in 2^{\mathbb{Z}}\times\mathbb{N}} \|u_{I,m}v_{J,n}\|_{H_G^\ell}^2 \lesssim & \sum_{\substack{(I,m),(J,n)\in 2^{\mathbb{Z}}\times\mathbb{N}\\ A\leqslant B}} B^{\ell-\frac12} I\langle J\rangle \|u_{I,m}\|_{L_G^2}^2 \|v_{J,n}\|_{L_G^2}^2 \\ & + \sum_{\substack{(I,m),(J,n)\in 2^{\mathbb{Z}}\times\mathbb{N}\\ A> B}} A^{\ell-\frac12} J\langle I\rangle \|u_{I,m}\|_{L_G^2}^2 \|v_{J,n}\|_{L_G^2}^2 ,\end{aligned}$$

implying (6.24) since $\ell - \frac12 = k$.

Step 3: Proof of (6.23). We claim that (6.23) is a consequence of the time independent inequality

$$\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} \|u_{I,m}v_{I,m}\|_{H_G^\ell} \lesssim \|u_0\|_{\mathcal{X}_1^k} \|v_0\|_{\mathcal{X}_1^k} . \tag{6.26}$$

Indeed we apply (6.26) to $z_{I,m}(t) = e^{it\Delta_G}u_{I,m}$ and $\widetilde{z}_{I,m}(t) = e^{it\Delta_G}v_{I,m}$ for $t \in \mathbb{R}$ instead of $u_{I,m}$, then we use $\mathcal{X}_1^k$ isometry property of the linear flow and integrate in time using the Hölder inequality. This yields (6.23).

In order to prove (6.26), for $I \in 2^{\mathbb{Z}}$ and $m \in \mathbb{N}$, we denote $A \in 2^{\mathbb{N}}$ such that $(m+1)I \sim A$. From (6.25) applied to $I = J$, $m = n$ and $A = B$, we therefore write

$$\sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} \|u_{I,m}v_{I,m}\|_{H_G^\ell} \lesssim \sum_{(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}} A^{\frac{\ell}{2}-\frac14} \langle I\rangle \|u_{I,m}\|_{L_G^2} \|v_{I,m}\|_{L_G^2} .$$

An application of Cauchy-Schwarz' inequality implies (6.26) since $\ell - \frac12 = k$. ☐

6.2. Trilinear estimate for random interactions.

Proof of (6.5) *in Theorem 2 (i).* We now estimate $\||z^\omega|^2 z^\omega\|_{L_T^q H_G^\ell}$.

Step 1: reduction of inequality (6.5) *to deterministic estimates.* Using Corollary 6.20 (*ii*), we know that outside a set of probability at most e^{-cR^2}, there holds

$$\| |z^\omega|^2 z^\omega \|^2_{L^q_T H^\ell_G} \leqslant R^6 \sum_{(I_1,m_1),(I_2,m_2),(I_3,m_3)\in 2^{\mathbb{Z}}\times\mathbb{N}} \| z_{I_1,m_1} z_{I_2,m_2} \overline{z_{I_3,m_3}} \|^2_{L^q_T H^\ell_G} + R^6 \sum_{(I_2,m_2)\in 2^{\mathbb{Z}}\times\mathbb{N}} \left(\sum_{(I_1,m_1)\in 2^{\mathbb{Z}}\times\mathbb{N}} \| |z_{I_1,m_1}|^2 z_{I_2,m_2} \|_{L^q_T H^\ell_G} \right)^2 . \tag{6.27}$$

As in the two subsections above, using Hölder's inequality in time, inequality (6.5) is now a consequence of the time independent inequalities

$$\sum_{(I_1,m_1),(I_2,m_2),(I_3,m_3)\in 2^{\mathbb{Z}}\times\mathbb{N}} \| u_{I_1,m_1} u_{I_2,m_2} \overline{u_{I_3,m_3}} \|^2_{H^\ell_G} \lesssim \|u_0\|^6_{\mathcal{X}^k_1} , \tag{6.28}$$

$$\sum_{(I_2,m_2)\in 2^{\mathbb{Z}}\times\mathbb{N}} \left(\sum_{(I_1,m_1)\in 2^{\mathbb{Z}}\times\mathbb{N}} \| |u_{I_1,m_1}|^2 u_{I_2,m_2} \|_{H^\ell_G} \right)^2 \lesssim \|u_0\|^6_{\mathcal{X}^k_1} . \tag{6.29}$$

Step 2: Proof of (6.28)*.* We rather prove inequality

$$\sum_{(I_1,m_1),(I_2,m_2),(I_3,m_3)\in 2^{\mathbb{Z}}\times\mathbb{N}} \| u^{(1)}_{I_1,m_1} u^{(2)}_{I_2,m_2} u^{(3)}_{I_3,m_3} \|^2_{H^\ell_G} \lesssim \|u^{(1)}\|^2_{\mathcal{X}^k_1} \|u^{(2)}\|^2_{\mathcal{X}^k_1} \|u^{(2)}\|^2_{\mathcal{X}^k_1} , \tag{6.30}$$

for fixed $u^{(1)}, u^{(2)}, u^{(3)} \in L^2_G$ decomposed as in (6.1), as this implies (6.28) with $u^{(1)} = u^{(2)} = u_0$ and $u^{(3)} = \overline{u_0}$. Let A_1, A_2, A_3 be the dyadic integers such that $(m_1+1)I_1 \sim A_1$, $(m_2+1)I_2 \sim A_2$ and $(m_3+1)I_3 \sim A_3$.

We apply Corollary 6.29 and get

$$\| u^{(1)}_{I_1,m_1} u^{(2)}_{I_2,m_2} u^{(3)}_{I_3,m_3} \|^2_{H^\ell_G} \lesssim \max\{A_1,A_2,A_3\}^\ell \| u^{(1)}_{I_1,m_1} u^{(2)}_{I_2,m_2} u^{(3)}_{I_3,m_3} \|^2_{L^2_G} .$$

Assuming up to permutation that $\max\{A_1,A_2,A_3\} = A_1$, we deduce

$$\| u^{(1)}_{I_1,m_1} u^{(2)}_{I_2,m_2} u^{(3)}_{I_3,m_3} \|^2_{H^\ell_G} \lesssim A_1^\ell \| u^{(1)}_{I_1,m_1} u^{(2)}_{I_2,m_2} \|^2_{L^2_G} \| u^{(3)}_{I_3,m_3} \|^2_{L^\infty_G} .$$

Then, from the unit block bound of Proposition 6.7, it follows that if one has $\max\{A_1,A_2,A_3\} = A_1$, then

$$\| u^{(1)}_{I_1,m_1} u^{(2)}_{I_2,m_2} u^{(3)}_{I_3,m_3} \|^2_{H^\ell_G} \lesssim A_1^{\ell-\frac12} \langle I_1\rangle \langle I_2\rangle \| u^{(1)}_{I_1,m_1} \|^2_{L^2_G} \| u^{(2)}_{I_2,m_2} \|^2_{L^2_G} \| u^{(3)}_{I_3,m_3} \|^2_{L^\infty_G} . \tag{6.31}$$

This implies that

$$\sum_{\substack{(I_1,m_1),(I_2,m_2),(I_3,m_3)\in 2^{\mathbb{Z}}\times\mathbb{N}\\ \max\{A_1,A_2,A_3\}=A_1}} \|u^{(1)}_{I_1,m_1}u^{(2)}_{I_2,m_2}u^{(3)}_{I_3,m_3}\|^2_{H^\ell_G}$$
$$\lesssim \sum_{\substack{(I_1,m_1),(I_2,m_2),(I_3,m_3)\in 2^{\mathbb{Z}}\times\mathbb{N}\\ \max\{A_1,A_2,A_3\}=A_1}} A_1^\ell \frac{\langle I_1\rangle\langle I_2\rangle}{A_1^{\frac12}}\|u^{(1)}_{I_1,m_1}\|^2_{L^2_G}\|u^{(2)}_{I_2,m_2}\|^2_{L^2_G}\|u^{(3)}_{I_3,m_3}\|^2_{L^\infty_G}.$$

By definition of $\mathcal{X}^k_\rho$, we get that for $k=\ell-\frac12$,

$$\sum_{\substack{(I_1,m_1),(I_2,m_2),(I_3,m_3)\in 2^{\mathbb{Z}}\times\mathbb{N}\\ \max\{A_1,A_2,A_3\}=A_1}} \|u^{(1)}_{I_1,m_1}u^{(2)}_{I_2,m_2}u^{(3)}_{I_3,m_3}\|^2_{H^\ell_G}$$
$$\lesssim \|u^{(1)}\|^2_{\mathcal{X}^k_1}\|u^{(2)}\|^2_{\mathcal{X}^0_1}\sum_{(I_3,m_3)\in\mathbb{N}^*\times\mathbb{N}}\|u^{(3)}_{I_3,m_3}\|^2_{L^\infty_G}.$$

It only remains to use estimate (6.15) from Lemma 6.22 and the embedding $W^{\varepsilon,p}_G\hookrightarrow L^\infty_G$ for arbitrary small $\varepsilon>0$ and $p>\frac{3}{\varepsilon}$ to deduce that one has

$$\sum_{(I_3,m_3)\in\mathbb{N}^*\times\mathbb{N}}\|u^{(3)}_{I_3,m_3}\|^2_{L^\infty_G}\lesssim\|u^{(3)}\|^2_{\mathcal{X}^{-\zeta(p)+\varepsilon}_{\zeta(p)+\frac32-\frac3p}}.$$

Remark that for large p we have $\mathcal{X}^k_1\hookrightarrow\mathcal{X}^{-\zeta(p)+\varepsilon}_{\zeta(p)+\frac32-\frac3p}$. This is indeed the case because $\varepsilon+\frac12-\frac3p\leqslant k$ for small ε since $k>\frac12$. We conclude that

$$\sum_{\substack{(I_1,m_1),(I_2,m_2),(I_3,m_3)\in 2^{\mathbb{Z}}\times\mathbb{N}\\ \max\{A_1,A_2,A_3\}=A_1}} \|u^{(1)}_{I_1,m_1}u^{(2)}_{I_2,m_2}u^{(3)}_{I_3,m_3}\|^2_{H^\ell_G}\lesssim\|u^{(1)}\|^2_{\mathcal{X}^k_1}\|u^{(2)}\|^2_{\mathcal{X}^k_1}\|u^{(3)}\|^2_{\mathcal{X}^k_1}.$$

By symmetry, this inequality is also valid when $\max\{A_1,A_2,A_3\}=A_2$ or A_3. This implies (6.30) and therefore (6.28).

Step 3: Proof of (6.29). In order to prove (6.29), fix $I_1,I_2\in 2^{\mathbb{Z}}$ and $m_1,m_2\in\mathbb{N}$.

If $\max\{A_1,A_2\}=A_1$, we use inequality (6.31) with $u^{(1)}=u^{(2)}=u_0$ and $u^{(3)}=\overline{u_0}$:

$$\|u_{I_1,m_1}u_{I_2,m_2}\overline{u}_{I_1,m_1}\|^2_{H^\ell_G}\lesssim A_1^\ell\frac{\langle I_1\rangle\langle I_2\rangle}{A_1^{\frac12}}\|u_{I_1,m_1}\|^2_{L^2_G}\|u_{I_2,m_2}\|^2_{L^2_G}\|u_{I_1,m_1}\|^2_{L^\infty_G}.$$

Otherwise, we have $\max\{A_1,A_2\}=A_2$. In this case, we still use inequality (6.31) with $u^{(1)}=u^{(2)}=u_0$ and $u^{(3)}=\overline{u_0}$:

$$\|u_{I_2,m_2}u_{I_1,m_1}\overline{u}_{I_1,m_1}\|^2_{H^\ell_G}\lesssim A_2^\ell\frac{\langle I_1\rangle\langle I_2\rangle}{A_2^{\frac12}}\|u_{I_2,m_2}\|^2_{L^2_G}\|u_{I_1,m_1}\|^2_{L^2_G}\|u_{I_1,m_1}\|^2_{L^\infty_G}.$$

We deduce by summation that

$$
\begin{aligned}
&\sum_{(I_2,m_2)\in 2^{\mathbb{Z}}\times\mathbb{N}}\Bigg(\sum_{(I_1,m_1)\in 2^{\mathbb{Z}}\times\mathbb{N}}\||u_{I_1,m_1}|^2u_{I_2,m_2}\|_{H^\ell_G}\Bigg)^2\\
&\lesssim \sum_{(I_2,m_2)\in 2^{\mathbb{Z}}\times\mathbb{N}}\Bigg(\sum_{(I_1,m_1)\in 2^{\mathbb{Z}}\times\mathbb{N}} A_1^{\frac{\ell}{2}-\frac14}(\langle I_1\rangle\langle I_2\rangle)^{\frac12}\|u_{I_1,m_1}\|_{L^2_G}\|u_{I_2,m_2}\|_{L^2_G}\|u_{I_1,m_1}\|_{L^\infty_G}\Bigg)^2\\
&+\sum_{(I_2,m_2)\in 2^{\mathbb{Z}}\times\mathbb{N}}\Bigg(\sum_{(I_1,m_1)\in 2^{\mathbb{Z}}\times\mathbb{N}}\|u_{I_1,m_1}\|_{L^2_G}A_2^{\frac{\ell}{2}-\frac14}(\langle I_1\rangle\langle I_2\rangle)^{\frac12}\|u_{I_2,m_2}\|_{L^2_G}\|u_{I_1,m_1}\|_{L^\infty_G}\Bigg)^2.
\end{aligned}
$$

Now we apply Cauchy-Schwarz' inequality and get

$$
\begin{aligned}
&\sum_{(I_2,m_2)\in 2^{\mathbb{Z}}\times\mathbb{N}}\Bigg(\sum_{(I_1,m_1)\in 2^{\mathbb{Z}}\times\mathbb{N}}\||u_{I_1,m_1}|^2u_{I_2,m_2}\|_{H^\ell_G}\Bigg)^2\\
&\lesssim \sum_{(I_2,m_2)\in 2^{\mathbb{Z}}\times\mathbb{N}}\Bigg(\sum_{(I_1,m_1)\in 2^{\mathbb{Z}}\times\mathbb{N}} A_1^{\ell-\frac12}\langle I_1\rangle\|u_{I_1,m_1}\|^2_{L^2_G}\Bigg)\langle I_2\rangle\|u_{I_2,m_2}\|^2_{L^2_G}\Bigg(\sum_{(I_1,m_1)\in 2^{\mathbb{Z}}\times\mathbb{N}}\|u_{I_1,m_1}\|^2_{L^\infty_G}\Bigg)\\
&+\sum_{(I_2,m_2)\in 2^{\mathbb{Z}}\times\mathbb{N}}\Bigg(\sum_{(I_1,m_1)\in 2^{\mathbb{Z}}\times\mathbb{N}}\langle I_1\rangle\|u_{I_1,m_1}\|^2_{L^2_G}\Bigg)A_2^{\ell-\frac12}\langle I_2\rangle\|u_{I_2,m_2}\|^2_{L^2_G}\Bigg(\sum_{(I_1,m_1)\in 2^{\mathbb{Z}}\times\mathbb{N}}\|u_{I_1,m_1}\|^2_{L^\infty_G}\Bigg).
\end{aligned}
$$

In only remains to use the definition (6.3) of $\mathcal{X}^k_1$ and the estimate proven in the above step $\sum_{(I_1,m_1)\in 2^{\mathbb{Z}}\times\mathbb{N}}\|u_{I_1,m_1}\|^2_{L^\infty_G}\lesssim\|u_0\|^2_{\mathcal{X}^k_1}$ to get (6.29). □

7. Deterministic-probabilistic trilinear estimate

In this section, we establish Theorem 2 (ii). Let $\varepsilon_0>0$ such that $u_0\in\mathcal{X}^k_{1+\varepsilon_0}$ and $v,w\in L^\infty_T H^\ell_G$. We recall that z, v and w have a decomposition

$$
z=\sum_{A\in 2^{\mathbb{N}}} z_A=\sum_{A\in 2^{\mathbb{N}}}\sum_{\substack{(I_1,m_1)\in 2^{\mathbb{Z}}\times\mathbb{N}\\ (m_1+1)I_1\sim A}} z_{I_1,m_1}\,,
$$

$$
v=\sum_{B\in 2^{\mathbb{N}}} v_B=\sum_{B\in 2^{\mathbb{N}}}\sum_{\substack{(I_2,m_2)\in 2^{\mathbb{Z}}\times\mathbb{N}\\ (m_2+1)I_2\sim B}} v_{I_2,m_2}
$$

and

$$
w=\sum_{C\in 2^{\mathbb{N}}} w_C=\sum_{C\in 2^{\mathbb{N}}}\sum_{\substack{(I_3,m_3)\in 2^{\mathbb{Z}}\times\mathbb{N}\\ (m_3+1)I_3\sim B}} w_{I_3,m_3}\,.
$$

Let $\varepsilon > 0$ to be chosen later. Using Corollary 6.31, we get that for all t, there holds

$$\|z^\omega vw(t)\|_{H^\ell_G}^2 \lesssim \sum_{\substack{(\delta_1,\delta_2,\delta_3)\in D_3\\ A,B,C:B,C\leqslant A}} A^{\ell+\varepsilon}\|(z_A^\omega)^{\delta_1}(P_{\leqslant A}v_B)^{\delta_2}(P_{\leqslant A}w_C)^{\delta_3}(t)\|_{L^2_G}^2 + \sum_{\substack{\delta_1\in D_1\\ A\in 2^{\mathbb{N}}}} A^\varepsilon\|(z_A^\omega)^{\delta_1}(t)\|_{L^\infty_G}^2\|v(t)\|_{H^\ell_G}^2\|w(t)\|_{H^\ell_G}^2 . \tag{6.32}$$

Using estimate (6.14) for the second term in the right hand side, we infer that outside of a set of probability e^{-cR^2}, there holds

$$\sum_{A\in 2^{\mathbb{N}}} A^\varepsilon\|z_A^\omega\|_{L^q_TL^\infty_G}^2 \lesssim_\varepsilon R^2T^{\frac{2}{q}}\|u_0\|_{\mathcal{X}_1^k}^2 .$$

Moreover, since applying the shifts $\delta_1\in D_1$ to every mode z_A give equivalent estimates for the L^p norms, this leads similarly to

$$\left\|\left(\sum_{\substack{\delta_1\in D_1\\ A\in 2^{\mathbb{N}}}} A^\varepsilon\|(z_A^\omega)^{\delta_1}\|_{L^\infty_G}^2\right)^{1/2}\|v\|_{H^\ell_G}\|w\|_{H^\ell_G}\right\|_{L^q_T} \lesssim RT^{\frac{1}{q}}\|u_0\|_{\mathcal{X}_1^k}\|v\|_{L^\infty_TH^\ell_G}\|w\|_{L^\infty_TH^\ell_G} .$$

For the sake of simplicity, we note that the shifted function u^δ is nothing but a shift of the indices $(I,m)\in 2^{\mathbb{Z}}\times\mathbb{N}$ of order at most one and a multiplication of every mode (I,m) by a function of modulus at most 1 in Fourier variable. Similarly, the projection $P_{\leqslant A}$ is a multiplication of every mode (I,m) by a function of modulus at most 1 in Fourier variable. Therefore, we assume without loss of generality that in the right-hand side of (6.32) there is no shift ($\delta_1=\delta_2=\delta_3=\varnothing$) and no projection $P_{\leqslant A}$, up to applying the proof to $(P_{\leqslant A}v_B)^{\delta_1}$ instead of v_B and doing a similar transformation for w. In order to estimate $\|z^\omega vw\|_{L^q_TH^\ell_G}$, we therefore write

$$\left\|\left(\sum_{A,B,C:B,C\leqslant A} A^{\ell+\varepsilon}\|z_A^\omega v_Bw_C\|_{L^2_G}^2\right)^{1/2}\right\|_{L^q_T} = \left\|\sum_{A,B,C:B,C\leqslant A} A^{\ell+\varepsilon}\|z_A^\omega v_Bw_C\|_{L^2_G}^2\right\|_{L^{q/2}_T}^{1/2} ,$$

and our aim is to prove that for our choice of q and ε, then with probability greater than $1-e^{-cR^2}$, we have

$$\left\|\sum_{A,B,C:B,C\leqslant A} A^{\ell+\varepsilon}\|z_A^\omega v_Bw_C\|_{L^2_G}^2\right\|_{L^{q/2}_T} \lesssim_\varepsilon R^2T^{\frac{2}{q}}\|u_0\|_{\mathcal{X}_{1+\varepsilon_0}^k}^2\|v\|_{L^\infty_TH^\ell_G}^2\|w\|_{L^\infty_TH^\ell_G}^2 .$$

By homogeneity, it is enough to prove that there exists C_ε such that for every $R>0$, with probability greater than $1-e^{-cR^2}$, for every $v,w\in L^\infty_TH^\ell_G$

satisfying $\|v\|_{L^\infty_T H^\ell_G} \leqslant 1$ and $\|w\|_{L^\infty_T H^\ell_G} \leqslant 1$,

$$\left\| \sum_{A,B,C:B,C\leqslant A} A^{\ell+\varepsilon} \|z^\omega_A v_B w_C(t)\|^2_{L^2_G} \right\|_{L^{q/2}_T} \leqslant C_\varepsilon R^2 T^{\frac{2}{q}} \|u_0\|^2_{\mathcal{X}^k_{1+\varepsilon_0}}. \tag{6.33}$$

7.1. Step 1: Decoupling z^ω from vw in (6.33). We fix $t \in \mathbb{R}$ and $A, B, C \in 2^{\mathbb{N}}$ such that $B, C \leqslant A$. Then we use the Plancherel formula to get

$$\begin{aligned}\|z^\omega_A v_B w_C(t)\|^2_{L^2_G} &= \int \mathrm{d}y \int \mathrm{d}x (z^\omega_A \overline{z^\omega_A})(x,y)(v_B\overline{v_B}w_C\overline{w_C})(x,y)\\ &= \int \mathrm{d}\eta \int \mathrm{d}x (\widehat{z^\omega_A} * \widehat{\overline{z^\omega_A}})(x,\eta)(\widehat{v_B} * \widehat{\overline{v_B}} * \widehat{w_C} * \widehat{\overline{w_C}})(x,\eta).\end{aligned}$$

We use decomposition (6.8). For $(I,m) \in 2^{\mathbb{Z}} \times \mathbb{N}$, we denote

$$\mathcal{F}_{y\to\eta}(z^\omega_{I,m})(t,x,\eta) = f^\omega_{I,m}(t,\eta) h_m(\sqrt{|\eta|}x),$$

where $f^\omega_{I,m}(t,\eta) = X_{I,m}(\omega)e^{-t(2m+1)|\eta|} f_{I,m}(\eta)$ and $f_{I,m}(\eta) = f_m(\eta)\mathbf{1}_{|\eta|\in[I,2I]}$. Similarly for v, we write

$$\mathcal{F}_{y\to\eta}(v_{I,m})(t,x,\eta) = g^\omega_{I,m}(t,\eta) h_m(\sqrt{|\eta|}x),$$

where the dependence of $g^\omega_{I,m}$ along the variables t and ω is not explicit. For w we do the same by making use of functions $\widetilde{g}^\omega_{I,m}$. Then we expand everything:

$$\begin{aligned}\|z^\omega_A v_B w_C(t)\|^2_{L^2_G} = &\int \mathrm{d}\eta \int \mathrm{d}x \int \mathrm{d}\eta_1 \int \mathrm{d}\eta_2 \int \mathrm{d}\eta_3 \int \mathrm{d}\eta_4\\ &\sum_{\substack{m_1,m_1'\in\mathbb{N}\\ (m_1+1)I_1,(m_1'+1)I_1'\sim A}} f^\omega_{I_1,m_1}(t,\eta_1)\overline{f^\omega_{I_1',m_1'}(t,\eta-\eta_1)} h_{m_1}(\sqrt{|\eta_1|}x) h_{m_1'}(\sqrt{|\eta-\eta_1|}x)\\ &\sum_{\substack{m_2,m_2'\in\mathbb{N}\\ (m_2+1)I_2,(m_2'+1)I_2'\sim B}} g^\omega_{I_2,m_2}(t,\eta_2)\overline{g^\omega_{I_2',m_2'}(t,\eta_3)} h_{m_2}(\sqrt{|\eta_2|}x) h_{m_2'}(\sqrt{|\eta_3|}x)\\ &\sum_{\substack{m_3,m_3'\in\mathbb{N}\\ (m_3+1)I_3,(m_3'+1)I_3'\sim C}} \widetilde{g}^\omega_{I_3,m_3}(t,\eta_4)\overline{\widetilde{g}^\omega_{I_3',m_3'}(t,\eta-\eta_2-\eta_3-\eta_4)} h_{m_3}(\sqrt{|\eta_4|}x) h_{m_3'}(\sqrt{|\eta-\eta_2-\eta_3-\eta_4|}x).\end{aligned}$$

Our aim is to apply the probabilistic decoupling to the series involving products $f^\omega_{I_1,m_1}\overline{f^\omega_{I_1',m_1'}}$. However, since v and w may depend on ω, we first isolate the terms involving $g^\omega_{I_2,m_2}$, $g^\omega_{I_2',m_2'}$, $\widetilde{g}^\omega_{I_3,m_3}$ and $\widetilde{g}^\omega_{I_3',m_3'}$.

The expanded formula is rather long, we reorganize it as follows. We define

$$\begin{aligned}\mathbb{J}_{\{m_i,m_i'\}}(\eta,\eta_1,\eta_2,\eta_3,\eta_4) := &\int \mathrm{d}x h_{m_1}(\sqrt{|\eta_1|}x) h_{m_1'}(\sqrt{|\eta-\eta_1|}x) h_{m_2}(\sqrt{|\eta_2|}x) h_{m_2'}(\sqrt{|\eta_3|}x)\\ &h_{m_3}(\sqrt{|\eta_4|}x) h_{m_3'}(\sqrt{|\eta-\eta_2-\eta_3-\eta_4|}x).\end{aligned} \tag{6.34}$$

We define the "random" part

(6.35)
$$\mathbf{J}^\omega_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'}(t,\eta,\eta_2,\eta_3,\eta_4) := \int \mathrm{d}\eta_1 \sum_{\substack{(I_1,m_1),(I_1',m_1')\in 2^{\mathbb{Z}}\times\mathbb{N}\\ (m_1+1)I_1,(m_1'+1)I_1'\sim A}} f^\omega_{I_1,m_1}(t,\eta_1)\overline{f^\omega_{I_1',m_1'}(t,\eta-\eta_1)}$$
$$\mathbb{J}_{\{m_i,m_i'\}}(\eta,\eta_1,\eta_2,\eta_3,\eta_4)|\eta_2\eta_3\eta_4(\eta-\eta_2-\eta_3-\eta_4)|^{1/4}$$
$$\mathbf{1}_{|\eta_2|\in[I_2,2I_2]}\mathbf{1}_{|\eta_3|\in[I_2',2I_2']}\mathbf{1}_{|\eta_4|\in[I_3,2I_3]}\mathbf{1}_{|\eta-\eta_2-\eta_3-\eta_4|\in[I_3',2I_3']}\,,$$
whereas the "deterministic" part is written
$$\mathbf{K}^\omega_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'}(t,\eta,\eta_2,\eta_3,\eta_4) = \frac{g^\omega_{I_2,m_2}(t,\eta_2)}{|\eta_2|^{1/4}}\frac{\overline{g^\omega_{I_2',m_2'}(t,\eta_3)}}{|\eta_3|^{1/4}}$$
$$\times\frac{\widetilde{g}^\omega_{I_3,m_3}(t,\eta_4)}{|\eta_4|^{1/4}}\frac{\overline{\widetilde{g}^\omega_{I_3',m_3'}(t,\eta-\eta_2-\eta_3-\eta_4)}}{|\eta-\eta_2-\eta_3-\eta_4|^{1/4}}\,.$$
Then the expanded formula becomes
$$\|z_A^\omega v_B w_C(t)\|^2_{L^2_G} = \int\mathrm{d}\eta\int\mathrm{d}\eta_2\int\mathrm{d}\eta_3\int\mathrm{d}\eta_4 \sum_{\substack{(I_2,m_2),(I_2',m_2')\in 2^{\mathbb{Z}}\times\mathbb{N}\\ (m_2+1)I_2,(m_2'+1)I_2'\sim B}}\ \sum_{\substack{(I_3,m_3),(I_3',m_3')\in 2^{\mathbb{Z}}\times\mathbb{N}\\ (m_3+1)I_3,(m_3'+1)I_3'\sim C}}$$
$$\mathbf{J}^\omega_{A,I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'}(t,\eta,\eta_2,\eta_3,\eta_4)$$
$$\mathbf{K}^\omega_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'}(t,\eta,\eta_2,\eta_3,\eta_4)\,.$$
Taking the absolute value, this leads by summation to a condensed expression
$$\left\|\sum_{A,B,C:B,C\leqslant A} A^{\ell+\varepsilon}\|z_A^\omega v_B w_C(t)\|^2_{L^2_G}\right\|_{L_T^{q/2}} \leqslant \|A^{\ell+\varepsilon}\mathbf{J}^\omega\mathbf{K}^\omega\|_{L^p_\psi}\,,$$
where, ψ stands for a huge tuple
$$\psi = (T,A,B,C,I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3',\eta,\eta_2,\eta_3,\eta_4)\,.$$
Moreover, we take $L^p_\psi = L^{p_1}_{\psi_1}\dots L^{p_d}_{\psi_d}$ norm with the exponents
$$L^p_\psi = L_T^{q/2}L^1_A L^1_{B,C,I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3',\eta,\eta_2,\eta_3,\eta_4}\,,$$
where the L^{p_j} norm implicitely denotes the ℓ^{p_j} norm when we consider sequence spaces. It is now possible to take successive Hölder inequalities, as one easily checks that for $\frac{1}{q_1}+\frac{1}{q_1'}=\frac{1}{p_1}$ and $\frac{1}{q_2}+\frac{1}{q_2'}=\frac{1}{p_2}$, we have
$$\|fg\|_{L_x^{p_1}L_y^{p_2}} \leqslant \left\|\|f\|_{L_y^{q_2}}\|g\|_{L_y^{q_2'}}\right\|_{L_x^{p_1}} \leqslant \|f\|_{L_x^{q_1}L_y^{q_2}}\|g\|_{L_x^{q_1'}L_y^{q_2'}}\,.$$
Our purpose is to isolate the $L_T^\infty H_G^\ell$ norm of vw by using Hölder inequalities. For this, we note that it better to split between $(BC)^\ell\mathbf{K}^\omega$ and $A^{\ell+\varepsilon}\mathbf{J}^\omega(BC)^{-\ell}$. As we will see below, our actual splitting also involves some extra other terms which aim at balancing the two terms and make every series converge.

We fix p_1 and p_2 to be very large exponents in $[2, \infty)$ to be chosen later, with respective conjugate exponents denoted p_1' and p_2': $\frac{1}{p_1} + \frac{1}{p_1'} = 1$ and $\frac{1}{p_2} + \frac{1}{p_2'} = 1$. For the term involving $\mathbf{J}^\omega$, we choose the exponents

$$L_\psi^{p(J)} = L_T^{q/2} L_A^1 L_{B,C}^{p_2} L^2_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'} L^{p_1}_{\eta,\eta_2,\eta_3,\eta_4}$$

and for the term involving $\mathbf{K}^\omega$, we choose the exponents

$$L_\psi^{p(K)} = L_T^\infty L_A^\infty L_{B,C}^{p_2'} L^2_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'} L^{p_1'}_{\eta,\eta_2,\eta_3,\eta_4} .$$

The reason why we introduce the exponents p_1 and p_2 instead of taking them directly equal to ∞ is that we need finite exponents in order to apply the probabilistic decoupling to $\mathbf{J}^\omega$.

Our choice of exponents is compatible with the assumptions for applying successive Hölder inequalities, so that

$$\Big\| \sum_{A,B,C: B,C\leqslant A} A^{\ell+\varepsilon} \| z_A^\omega v_B w_C\|^2_{L^2_G} \Big\|_{L_T^{q/2}} \tag{6.36}$$
$$\leqslant \|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L_\psi^{p(J)}} \|(BC)^{\ell-\varepsilon_2}(I_2I_2'I_3I_3')^{-1/2+\varepsilon_1}\mathbf{K}^\omega\|_{L_\psi^{p(K)}} .$$

The exponents $\varepsilon_1, \varepsilon_2 > 0$ will be chosen later depending on the following step.

7.2. Step 2: Evaluating the deterministic part $\mathbf{K}^\omega$ in (6.36).

We first estimate the term

$$\|(BC)^{\ell-\varepsilon_2}(I_2I_2'I_3I_3')^{-1/2+\varepsilon_1}\mathbf{K}^\omega\|_{L_\psi^{p(K)}}$$

with

$$L_\psi^{p(K)} = L_A^\infty L_T^\infty L_{B,C}^{p_2'} L^2_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'} L^{p_1'}_{\eta,\eta_2,\eta_3,\eta_4} .$$

For the intuition, we should keep in mind that $p_1', p_2' \approx 1$. By definition, of $\mathbf{K}^\omega$, we have

$$|\mathbf{K}^\omega_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'}(t,\eta,\eta_2,\eta_3,\eta_4)| = \frac{|g^\omega_{I_2,m_2}(t,\eta_2)|}{|\eta_2|^{1/4}} \frac{|g^\omega_{I_2',m_2'}(t,\eta_3)|}{|\eta_3|^{1/4}} \frac{|\widetilde{g}^\omega_{I_3,m_3}(t,\eta_4)|}{|\eta_4|^{1/4}} \frac{|\widetilde{g}^\omega_{I_3',m_3'}(t,\eta-\eta_2-\eta_3-\eta_4)|}{|\eta-\eta_2-\eta_3-\eta_4|^{1/4}} ,$$

therefore

$$\|\mathbf{K}^\omega\|_{L^{p_1'}_{\eta,\eta_2,\eta_3,\eta_4}} = \left\| \left(\frac{|g^\omega_{I_2,m_2}|}{|\cdot|^{1/4}}\right)^{p_1'} * \left(\frac{|g^\omega_{I_2',m_2'}|}{|\cdot|^{1/4}}\right)^{p_1'} * \left(\frac{|\widetilde{g}^\omega_{I_3,m_3}|}{|\cdot|^{1/4}}\right)^{p_1'} * \left(\frac{|\widetilde{g}^\omega_{I_3',m_3'}|}{|\cdot|^{1/4}}\right)^{p_1'} (t,\eta) \right\|_{L^1_\eta}^{1/p_1'} .$$

We apply Young's inequality

$$\|\mathbf{K}^\omega\|_{L^{p_1'}_{\eta,\eta_2,\eta_3,\eta_4}} \leqslant \left\|\left(\frac{|g^\omega_{I_2,m_2}|}{|\cdot|^{1/4}}\right)^{p_1'} * \left(\frac{|g^\omega_{I_2',m_2'}|}{|\cdot|^{1/4}}\right)^{p_1'}(t,\eta)\right\|^{1/p_1'}_{L^1_\eta} \left\|\left(\frac{|\widetilde{g}^\omega_{I_3,m_3}|}{|\cdot|^{1/4}}\right)^{p_1'} * \left(\frac{|\widetilde{g}^\omega_{I_3',m_3'}|}{|\cdot|^{1/4}}\right)^{p_1'}(t,\eta)\right\|^{1/p_1'}_{L^1_\eta}.$$

Applying Young's inequality again, we get

$$\left\|\left(\frac{|g^\omega_{I_2,m_2}|}{|\cdot|^{1/4}}\right)^{p_1'} * \left(\frac{|g^\omega_{I_2',m_2'}|}{|\cdot|^{1/4}}\right)^{p_1'}(t,\eta)\right\|^{1/p_1'}_{L^1_\eta} \leqslant \left\|\left(\frac{|g^\omega_{I_2,m_2}|}{|\cdot|^{1/4}}\right)^{p_1'}(t,\eta)\right\|^{1/p_1'}_{L^1_\eta} \left\|\left(\frac{|g^\omega_{I_2',m_2'}|}{|\cdot|^{1/4}}\right)^{p_1'}(t,\eta)\right\|^{1/p_1'}_{L^1_\eta}.$$

But since

$$\left\|\left(\frac{|g^\omega_{I_2',m_2'}|}{|\cdot|^{1/4}}\right)^{p_1'}(t,\eta)\right\|^{1/p_1'}_{L^1_\eta} = \left\|\frac{|g^\omega_{I_2',m_2'}|}{|\cdot|^{1/4}}(t,\eta)\right\|_{L^{p_1'}_\eta},$$

the Hölder inequality with the choice of exponents $\frac{1}{p_1'} = \frac{1}{2} + \frac{p_1-2}{2p_1}$ leads to

$$\begin{aligned}\left\|\left(\frac{|g^\omega_{I_2,m_2}|}{|\cdot|^{1/4}}\right)^{p_1'}(t,\eta)\right\|^{1/p_1'}_{L^1_\eta} &\leqslant \left\|\frac{|g^\omega_{I_2,m_2}|}{|\cdot|^{1/4}}(t,\eta)\right\|_{L^2_\eta} \left\|\mathbf{1}_{|\eta|\in[I_2,2I_2]}\right\|_{L^{2p_1/(p_1-2)}_\eta} \\ &\leqslant \|v_{I_2,m_2}(t)\|_{L^2_G}(I_2)^{(p_1-2)/(2p_1)}.\end{aligned}$$

By symmetry of the roles of I_2, m_2 and I_2', m_2', we deduce

$$\left\|\left(\frac{|g^\omega_{I_2,m_2}|}{|\cdot|^{1/4}}\right)^{p_1'} * \left(\frac{|g^\omega_{I_2',m_2'}|}{|\cdot|^{1/4}}\right)^{p_1'}(t,\eta)\right\|^{1/p_1'}_{L^1_\eta} \leqslant \|v_{I_2,m_2}(t)\|_{L^2_G}\|v_{I_2',m_2'}(t)\|_{L^2_G}\left(I_2I_2'\right)^{(p_1-2)/(2p_1)}.$$

By symmetry of the roles of u and v, we conclude

$$\|\mathbf{K}^\omega\|_{L^{p_1'}_{\eta,\eta_2,\eta_3,\eta_4}} \leqslant \|v_{I_2,m_2}(t)\|_{L^2_G}\|v_{I_2',m_2'}(t)\|_{L^2_G}\|w_{I_3,m_3}(t)\|_{L^2_G}\|w_{I_3',m_3'}(t)\|_{L^2_G}\left(I_2I_2'I_3I_3'\right)^{(p_1-2)/(2p_1)}.$$

Note that $\frac{p_1-2}{2p_1} = \frac{1}{2} - \frac{1}{p_1} = \frac{1}{2} - \varepsilon_1$ where $\varepsilon_1 = \frac{1}{p_1}$ is very small. This leads to

$$\begin{aligned}&\|(I_2I_2'I_3I_3')^{-1/2+\varepsilon_1}\mathbf{K}^\omega\|_{L^2_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'}L^{p_1'}_{\eta,\eta_2,\eta_3,\eta_4}} \\ &\leqslant \left(\sum_{\substack{m_2,m_2',m_3,m_3',I_2,I_2',I_3,I_3' \\ (m_2+1)I_2,(m_2'+1)I_2'\sim B \\ (m_3+1)I_3,(m_3'+1)I_3'\sim C}} \|v_{I_2,m_2}(t)\|^2_{L^2_G}\|v_{I_2',m_2'}(t)\|^2_{L^2_G}\|w_{I_3,m_3}(t)\|^2_{L^2_G}\|w_{I_3',m_3'}(t)\|^2_{L^2_G}\right)^{1/2} \\ &= \|v_B(t)\|^2_{L^2_G}\|w_C(t)\|^2_{L^2_G}.\end{aligned}$$

Finally we get

$$\|(BC)^{\ell-\varepsilon_2}(I_2I_2'I_3I_3')^{-1/2+\varepsilon_1}\mathbf{K}^\omega\|_{L^{p_2'}_{B,C}L^2_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'}L^{p_1'}_{\eta,\eta_2,\eta_3,\eta_4}} \leqslant \left(\sum_{B,C}(BC)^{(\ell-\varepsilon_2)p_2'}\|v_B(t)\|^{2p_2'}_{L^2_G}\|w_C(t)\|^{2p_2'}_{L^2_G}\right)^{1/p_2'}.$$

The exponent p_2' is greater than 1, which could be troublesome, but we recall that we have chosen v and w satisfying $\|v\|_{L^\infty_T H^\ell_G} \leqslant 1$ and $\|w\|_{L^\infty_T H^\ell_G} \leqslant 1$ in the assumption of the desired estimate (6.33). Therefore, for every t, B, C, we have $\|v_B(t)\|_{L^2_G} \leqslant 1$ and $\|w_C(t)\|_{L^2_G} \leqslant 1$, leading to

$$\|(BC)^{\ell-\varepsilon_2}(I_2I_2'I_3I_3')^{-1/2+\varepsilon_1}\mathbf{K}^\omega\|_{L^{p_2'}_{B,C}L^2_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'}L^{p_1'}_{\eta,\eta_2,\eta_3,\eta_4}} \leqslant \left(\sum_{B,C}(BC)^{(\ell-\varepsilon_2)p_2'}\|v_B(t)\|^{2}_{L^2_G}\|w_C(t)\|^{2}_{L^2_G}\right)^{1/p_2'}.$$

We choose ε_2 such that $(\ell-\varepsilon_2)p_2' = \ell$, i.e. $\varepsilon_2 = \frac{\ell}{p_2}$. Then the upper bound is bounded by

$$\begin{aligned}\|(BC)^{\ell-\varepsilon_2}(I_2I_2'I_3I_3')^{-1/2+\varepsilon_1}\mathbf{K}^\omega\|_{L^{p_2'}_{B,C}L^2_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'}L^{p_1'}_{\eta,\eta_2,\eta_3,\eta_4}} &\leqslant \left(\sum_{B,C}\|v_B(t)\|^{2}_{H^\ell_G}\|w_B(t)\|^{2}_{H^\ell_G}\right)^{1/p_2'}\\ &\leqslant (\|v(t)\|_{H^\ell_G}\|w(t)\|_{H^\ell_G})^{2/p_2'}\\ &\leqslant 1\,.\end{aligned}$$

Taking the $L^\infty_T L^\infty_A$ norm, we conclude that

$$\|(BC)^{\ell-\varepsilon_2}(I_2I_2'I_3I_3')^{-1/2+\varepsilon_1}\mathbf{K}^\omega\|_{L^{p(K)}_\psi} \leqslant 1\,.$$

7.3. Step 3: Evaluating the random part $\mathbf{J}^\omega$ in (6.36). We now estimate the term

$$\|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L^{p(J)}_\psi}$$

with $\varepsilon_1 = \frac{1}{p_1}$, $\varepsilon_2 = \frac{\ell}{p_2}$ and

$$L^{p(J)}_\psi = L^{q/2}_T L^1_A L^{p_2}_{B,C} L^2_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'} L^{p_1}_{\eta,\eta_2,\eta_3,\eta_4}\,.$$

Step 3.1: probabilistic decoupling. We first apply the probabilistic decoupling: let us check that the assumptions in order to apply Corollary 6.20 (*iii*) are met.

It is now more convenient to write

$$A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega = \sum_{\substack{m_1,m_1',I_1,I_1'\\(m_1+1)I_1,(m_1'+1)I_1'\sim A}} X_{I_1,m_1}(\omega)\overline{X_{I_1',m_1'}(\omega)}\Psi_{(I_1,m_1),(I_1',m_1')}(\psi)\,,$$

where from the definition (6.35) of $\mathbf{J}^\omega$, we have

$$\begin{aligned}\Psi_{(I_1,m_1),(I_1',m_1')}(\psi) = A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\int \mathrm{d}\eta_1 f_{I_1,m_1}(t,\eta_1)\overline{f_{I_1',m_1'}}(t,\eta-\eta_1)\\ \mathbb{J}_{\{m_i,m_i'\}}(\eta,\eta_1,\eta_2,\eta_3,\eta_4)|\eta_2\eta_3\eta_4(\eta-\eta_2-\eta_3-\eta_4)|^{1/4}\\ \mathbf{1}_{|\eta_2|\in[I_2,2I_2]}\mathbf{1}_{|\eta_3|\in[I_2',2I_2']}\mathbf{1}_{|\eta_4|\in[I_3,2I_3]}\mathbf{1}_{|\eta-\eta_2-\eta_3-\eta_4|\in[I_3',2I_3']}\,,\end{aligned}\tag{6.37}$$

and we recall that $\mathbb{J}$ is defined in (6.34).

Applying Corollary 6.20 *(iii)* with

$$\psi_- = (T,A,B,C,I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'),\quad \psi_+ = (\eta,\eta_2,\eta_3,\eta_4)\,,$$
$$L^{p_-(J)}_{\psi_-} = L^{q/2}_T L^1_A L^{p_2}_{B,C} L^2_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'}\,,\quad L^{p_+(J)}_{\psi_+} = L^{p_1}_{\eta,\eta_2,\eta_3,\eta_4}\,,$$

since

$$\|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L^{p(J)}_\psi} = \Bigg\|\sum_{\substack{m_1,m_1',I_1,I_1'\\(m_1+1)I_1,(m_1'+1)I_1'\sim A}} X_{I_1,m_1}(\omega)\overline{X_{I_1',m_1'}(\omega)}\Psi_{(I_1,m_1),(I_1',m_1')}(\psi)\Bigg\|_{L^{p(J)}_\psi}\,,$$

we get that outside of a set of probability e^{-cR^2}, there holds

$$\begin{aligned}\|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L^{p(J)}_\psi}\\ \leqslant R^2\Bigg\|\Bigg(\sum_{\substack{m_1,m_1',I_1,I_1'\\(m_1+1)I_1,(m_1'+1)I_1'\sim A}}\|\Psi_{(I_1,m_1),(I_1',m_1')}(\psi)\|^2_{L^{p_+(J)}_{\psi_+}}\Bigg)^{1/2}\Bigg\|_{L^{p_-(J)}_{\psi_-}}\\ + R^2\Bigg\|\sum_{\substack{m_1,I_1\\(m_1+1)I_1\sim A}}\|\Psi_{(I_1,m_1),(I_1,m_1)}(\psi)\|_{L^{p_+(J)}_{\psi_+}}\Bigg\|_{L^{p_-(J)}_{\psi_-}}\,.\end{aligned}\tag{6.38}$$

Step 3.2: deterministic trilinear estimates. Now, we apply the deterministic trilinear estimates from Corollary 6.34 in order to gain $\frac{1}{2}$ additional derivatives, *i.e.* $\frac{1}{2}$ powers of A.

Let us fix I_1, I_1', m_1, m_1' and ψ. We estimate $\Psi_{(I_1,m_1),(I_1',m_1')}(\psi)$ defined in (6.37). We start with $\mathbb{J}$ defined in (6.34) as

$$\mathbb{J}_{\{m_i,m_i'\}}(\eta,\eta_1,\eta_2,\eta_3,\eta_4) = \int \mathrm{d}x h_{m_1}(\sqrt{|\eta_1|}x)h_{m_1'}(\sqrt{|\eta-\eta_1|}x$$
$$h_{m_2}(\sqrt{|\eta_2|}x)h_{m_2'}(\sqrt{|\eta_3|}x)h_{m_3}(\sqrt{|\eta_4|}x)h_{m_3'}(\sqrt{|\eta-\eta_2-\eta_3-\eta_4|}x)\,.$$

We apply Cauchy-Schwarz' inequality

$$|\mathbb{J}_{\{m_i,m_i'\}}(\eta,\eta_1,\eta_2,\eta_3,\eta_4)| \leqslant \|h_{m_1}(\sqrt{|\eta_1|}x)h_{m_2}(\sqrt{|\eta_2|}x)h_{m_3}(\sqrt{|\eta_4|}x)\|_{L^2_x}$$
$$\|h_{m_1'}(\sqrt{|\eta-\eta_1|}x)h_{m_2'}(\sqrt{|\eta_3|}x)h_{m_3'}(\sqrt{|\eta-\eta_2-\eta_3-\eta_4|}x)\|_{L^2_x}\,.$$

We apply Corollary 6.34 for $\alpha_1^2 = |\eta_1| \in [I_1, 2I_1]$, $\alpha_2^2 = |\eta_2| \in [I_2, 2I_2]$ and $\alpha_3^2 = |\eta_4| \in [I_3, 2I_3]$:

$$\alpha_1\alpha_2\alpha_3\|h_{m_1}(\alpha_1\cdot)h_{m_2}(\alpha_2\cdot)h_{m_3}(\alpha_3\cdot)\|_{L^2}^2 \lesssim C^2_{\{I_i,m_i\}}\,,$$

where

$$C^2_{\{I_i,m_i\}} = \frac{\langle I_1\rangle(I_2I_3)^{1/4}}{A^{1/2}((2m_2+1)(2m_3+1))^{1/12}}\,.$$

We also apply Corollary 6.34 for $\alpha_1^2 = |\eta-\eta_1| \in [I_1', 2I_1']$, $\alpha_2^2 = |\eta_3| \in [I_2', 2I_2']$ and $\alpha_3^2 = |\eta-\eta_2-\eta_3-\eta_4| \in [I_3', 2I_3']$. We conclude that

$$|\eta_1(\eta-\eta_1)\eta_2\eta_3\eta_4(\eta-\eta_2-\eta_3-\eta_4)|^{1/4}|\mathbb{J}_{\{m_i,m_i'\}}(\eta,\eta_1,\eta_2,\eta_3,\eta_4)| \lesssim C_{\{I_i,m_i\}}C_{\{I_i',m_i'\}}\,.$$

Moreover, since

$$\int \mathrm{d}\eta_1 \left|\frac{f_{I_1,m_1}(t,\eta_1)}{|\eta_1|^{1/4}}\frac{\overline{f_{I_1',m_1'}(t,\eta-\eta_1)}}{|\eta-\eta_1|^{1/4}}\right| = \left(\frac{|f_{I_1,m_1}|}{|\cdot|^{1/4}}\right) * \left(\frac{|f_{I_1',m_1'}|}{|\cdot|^{1/4}}\right)(t,\eta)\,,$$

we get the estimate

$$|\Psi_{(I_1,m_1),(I_1',m_1')}(\psi)| \leqslant A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}$$
$$C_{\{I_i,m_i\}}C_{\{I_i',m_i'\}}\left(\frac{|f_{I_1,m_1}|}{|\cdot|^{1/4}}\right) * \left(\frac{|f_{I_1',m_1'}|}{|\cdot|^{1/4}}\right)(t,\eta)$$
$$\mathbf{1}_{|\eta_2|\in[I_2,2I_2]}\mathbf{1}_{|\eta_3|\in[I_2',2I_2']}\mathbf{1}_{|\eta_4|\in[I_3,2I_3]}\mathbf{1}_{|\eta-\eta_2-\eta_3-\eta_4|\in[I_3',2I_3']}\,.$$

Step 3.3: Integration bounds. It is now time to evaluate $\|\Psi_{(I_1,m_1),(I_1',m_1')}(\psi)\|_{L^{p_+(J)}_{\psi_+}}$ with

$$L^{p_+(J)}_{\psi_+} = L^{p_1}_{\eta,\eta_2,\eta_3,\eta_4}\,.$$

First, as indicators functions are bounded by 1, we have

$$\left\|\mathbf{1}_{|\eta_2|\in[I_2,2I_2]}\mathbf{1}_{|\eta_3|\in[I_2',2I_2']}\mathbf{1}_{|\eta_4|\in[I_3,2I_3]}\mathbf{1}_{|\eta-\eta_2-\eta_3-\eta_4|\in[I_3',2I_3']}\right\|_{L^{p_1}_{\eta_2,\eta_3,\eta_4}}$$
$$\leqslant \left\|\mathbf{1}_{|\eta_2|\in[I_2,2I_2]}\mathbf{1}_{|\eta_3|\in[I_2',2I_2']}\mathbf{1}_{|\eta_4|\in[I_3,2I_3]}\right\|^{1/p_1}_{L^1_{\eta_2,\eta_3,\eta_4}} \leqslant (I_2I_2'I_3)^{1/p_1}\,.$$

By symmetry of roles, recalling that $\varepsilon_1 = \frac{1}{p_1}$, we write that

$$\left\|\mathbf{1}_{|\eta_2|\in[I_2,2I_2]}\mathbf{1}_{|\eta_3|\in[I_2',2I_2']}\mathbf{1}_{|\eta_4|\in[I_3,2I_3]}\mathbf{1}_{|\eta-\eta_2-\eta_3-\eta_4|\in[I_3',2I_3']}\right\|_{L^{p_1}_{\eta_2,\eta_3,\eta_4}} \leqslant (I_2I_2'I_3I_3')^{3\varepsilon_1/4} \,.$$

Hence we have discarded the integration over η_1,η_2,η_3 and the term

$$\mathbf{1}_{|\eta_2|\in[I_2,2I_2]}\mathbf{1}_{|\eta_3|\in[I_2',2I_2']}\mathbf{1}_{|\eta_4|\in[I_3,2I_3]}\mathbf{1}_{|\eta-\eta_2-\eta_3-\eta_4|\in[I_3',2I_3']}\,,$$

$$\|\Psi_{(I_1,m_1),(I_1',m_1')}(\psi)\|_{L^{p_1}_{\eta_2,\eta_3,\eta_4}} \leqslant A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1/4}$$
$$C_{\{I_i,m_i\}}C_{\{I_i',m_i'\}}\left(\frac{|f_{I_1,m_1}|}{|\cdot|^{1/4}}\right)*\left(\frac{|f_{I_1',m_1'}|}{|\cdot|^{1/4}}\right)(\eta)\,.$$

Then, we integrate in η. Fixing I_1,I_1',m_1,m_1', we have thanks to Young's inequality with $1+\frac{1}{p_1} = 2\frac{p_1+1}{2p_1}$,

$$\left\|\left(\frac{|f_{I_1,m_1}|}{|\cdot|^{1/4}}\right)*\left(\frac{|f_{I_1',m_1'}|}{|\cdot|^{1/4}}\right)(\eta)\right\|_{L^{p_1}_\eta} \leqslant \left\|\left(\frac{|f_{I_1,m_1}|}{|\cdot|^{1/4}}\right)\right\|_{L^{2p_1/(p_1+1)}_\eta}\left\|\left(\frac{|f_{I_1',m_1'}|}{|\cdot|^{1/4}}\right)\right\|_{L^{2p_1/(p_1+1)}_\eta}$$

and from Hölder's inequality with $\frac{p_1+1}{2p_1} = \frac{1}{2}+\frac{1}{2p_1}$,

$$\begin{aligned}\left\|\left(\frac{|f_{I_1,m_1}|}{|\cdot|^{1/4}}\right)*\left(\frac{|f_{I_1',m_1'}|}{|\cdot|^{1/4}}\right)(t,\eta)\right\|_{L^{p_1}_\eta} &\leqslant (I_1I_1')^{1/(2p_1)}\left\|\left(\frac{|f_{I_1,m_1}|}{|\cdot|^{1/4}}\right)(t,\eta)\right\|_{L^2_\eta}\left\|\left(\frac{|f_{I_1',m_1'}|}{|\cdot|^{1/4}}\right)(t,\eta)\right\|_{L^2_\eta}\\ &= (I_1I_1')^{1/(2p_1)}\|z_{I_1,m_1}(t)\|_{L^2_G}\|z_{I_1',m_1'}(t)\|_{L^2_G}\\ &= (I_1I_1')^{1/(2p_1)}\|u_{I_1,m_1}\|_{L^2_G}\|u_{I_1',m_1'}\|_{L^2_G}\,,\end{aligned}$$

where in the latter equality we have used the L^2 isometry property of the linear flow. We have proven that

$$\|\Psi_{(I_1,m_1),(I_1',m_1')}(\psi)\|_{L^{p_+(J)}_{\psi_+}} \leqslant A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1/4}$$
$$C_{\{I_i,m_i\}}C_{\{I_i',m_i'\}}(I_1I_1')^{1/(2p_1)}\|u_{I_1,m_1}\|_{L^2_G}\|u_{I_1',m_1'}\|_{L^2_G}\,.$$

Recalling that $\varepsilon_1 = \frac{1}{p_1}$ and replacing $C^2_{\{I_i,m_i\}} = \frac{\langle I_1\rangle(I_2I_3)^{1/4}}{A^{1/2}((2m_2+1)(2m_3+1))^{1/12}}$ by its value, this leads to

$$\|\Psi_{(I_1,m_1),(I_1',m_1')}(\psi)\|_{L^{p_+(J)}_{\psi_+}} \leqslant A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1/4}$$
$$\frac{(\langle I_1\rangle\langle I_1'\rangle)^{1/2}(I_2I_2'I_3I_3')^{1/8}}{A^{1/2}((2m_2+1)(2m_2'+1)(2m_3+1)(2m_3'+1))^{1/24}}(I_1I_1')^{\varepsilon_1/2}\|u_{I_1,m_1}\|_{L^2_G}\|u_{I_1',m_1'}\|_{L^2_G}\,.$$

Step 3.4: decoupled summation. We now consider the sums over I_1,I_1',m_1,m_1' in (6.38).

For the second term in the right-hand side of (6.38), we take $I_1' = I_1$, $m_1' = m_1$ and incorporate the term $\langle I_1\rangle^{1+\varepsilon_1}$, which is the only term still dependent of I_1, m_1 in the upper bound. By summation, this leads to

$$\sum_{\substack{m_1, I_1\\(m_1+1)I_1\sim A}} \langle I_1\rangle^{1+\varepsilon_1}\|u_{I_1,m_1}\|^2_{L^2_G} = \|u_A\|^2_{\mathcal{X}^0_{1+\varepsilon_1}}\,.$$

For the first term in the right-hand side of (6.38), we also have

$$\left(\sum_{\substack{m_1,m_1',I_1,I_1'\\(m_1+1)I_1,(m_1'+1)I_1'\sim A}} \langle I_1\rangle^{1+\varepsilon_1}\langle I_1'\rangle^{1+\varepsilon_1}\|u_{I_1,m_1}\|^2_{L^2_G}\|u_{I_1',m_1'}\|^2_{L^2_G}\right)^{1/2} = \|u_A\|^2_{\mathcal{X}^0_{1+\varepsilon_1}}\,.$$

Therefore (6.38) becomes

$$\begin{aligned}
&\|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L^{p(J)}_\psi}\\
&\lesssim R^2\left\|A^{\ell-\frac12+\varepsilon}(BC)^{-\ell+\varepsilon_2}\frac{(I_2I_2'I_3I_3')^{1/2+1/8-\varepsilon_1/4}}{((2m_2+1)(2m_2'+1)(2m_3+1)(2m_3'+1))^{1/24}}\|u_A\|^2_{\mathcal{X}^0_{1+\varepsilon_1}}\right\|_{L^{p_-(J)}_{\psi_-}}\,,
\end{aligned}$$

where

$$L^{p_-(J)}_{\psi_-} = L^{q/2}_T L^1_A L^{p_2}_{B,C} L^2_{I_2,I_2',I_3,I_3',m_2,m_2',m_3,m_3'}\,.$$

Step 3.5: Hölder bounds. Let us now compute the norm $L^2_{I_2,m_2}$. Thanks to the condition $1+(2m_2+1)I_2\in[B,2B]$ implying $m_2\leqslant\frac{B}{I_2}$ and $I_2\leqslant B$, we have

$$\begin{aligned}
\sum_{\substack{m_2,I_2\\(m_2+1)I_2\sim B}}\frac{I_2^{1+1/4-\varepsilon_1/2}}{(m_2+1)^{1/12}} &\leqslant \sum_{I_2\in2^{\mathbb{Z}}:I_2\leqslant B} I_2^{1+1/4-\varepsilon_1/2}\sum_{m_2\leqslant\frac{B}{I_2}}\frac{1}{(m_2+1)^{1/12}}\\
&\leqslant \sum_{I_2\in2^{\mathbb{Z}}:I_2\leqslant B} I_2^{1+1/4-\varepsilon_1/2}\left(\frac{B}{I_2}\right)^{1-1/12}\\
&\lesssim B^{1-1/12}\sum_{I_2\in2^{\mathbb{Z}}:I_2\leqslant B} I_2^{1/4+1/12-\varepsilon_1/2}\,.
\end{aligned}$$

The series over $I_2\leqslant1$ is convergent since for $\varepsilon_1>0$ sufficiently small, the exponent $1/4+1/12-\varepsilon_1/2$ is positive. Moreover, one can see that the gain of $1/12$ is actually useless here in the argument. For the series over $I_2\geqslant1$, we get a bound $B^{1/4+1/12-\varepsilon_1/2}$, so that

$$\sum_{\substack{m_2,I_2\\(m_2+1)I_2\sim B}}\frac{I_2^{1+1/4-\varepsilon_1/2}}{(m_2+1)^{1/12}}\lesssim B^{1+1/4-\varepsilon_1/2}\lesssim B^{5/4}\,.$$

We do the same for all the other indices $I_2', m_2', I_3, m_3, I_3', m_3'$ and get

$$\|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L^{p(J)}_\psi} \lesssim R^2\left\|A^{\ell-1/2+\varepsilon}(BC)^{-\ell+\varepsilon_2+5/4}\|u_A\|^2_{\mathcal{X}^0_{1+\varepsilon_1}}\right\|_{L^{q/2}_T L^1_A L^{p_2}_{B,C}}.$$

We now take the L^{p_2} norm over B, C. Since $\ell > \frac{3}{2}$, the exponent in front of the term (BC) is $-\ell+\varepsilon_2+\frac{5}{4} < -\frac{3}{2}+\varepsilon_2+\frac{5}{4}$. When $\varepsilon_2 < \frac{1}{4}$ is small, we see that this exponent is negative, so that the L^{p_2} norm over B and C is convergent and bounded by some constant C_0. This leads to

$$\|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L^{p(J)}_\psi} \lesssim R^2\left\|A^{\ell-\frac{1}{2}+\varepsilon}\|u_A\|^2_{\mathcal{X}^0_{1+\varepsilon_1}}\right\|_{L^{q/2}_T L^1_A}.$$

We finally conclude that

$$\|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L^{p(J)}_\psi} \lesssim R^2\|\|u_0\|^2_{\mathcal{X}^{\ell-\frac{1}{2}+\varepsilon}_{1+\varepsilon_1}}\|_{L^{q/2}_T}.$$

It only remains to take the $L^{q/2}_T$ norm, but one can see that our upper bound does not depend on T anymore, so this only adds a $T^{2/q}$ factor:

$$\|A^{\ell+\varepsilon}(BC)^{-\ell+\varepsilon_2}(I_2I_2'I_3I_3')^{1/2-\varepsilon_1}\mathbf{J}^\omega\|_{L^{p(J)}_\psi} \lesssim R^2T^{2/q}\|u_0\|^2_{\mathcal{X}^{\ell-\frac{1}{2}+\varepsilon}_{1+\varepsilon_1}}.$$

When $\ell+\varepsilon < k+\frac{1}{2}$ (this is equivalent to taking ε small enough), we conclude that

$$\|z^\omega vw\|^2_{L^q_T H^\ell_G} \lesssim_\varepsilon R^2T^{\frac{2}{q}}\|u_0\|^2_{\mathcal{X}^k_{1+\varepsilon_1}},$$

where $\varepsilon_1 = \frac{1}{p_1}$ can be chosen arbitrarily small and in particular no greater than ε_0. Using the homogeneity, we remove the assumption $\|v\|_{L^\infty_T H^\ell_G} \leqslant 1$, $\|w\|_{L^\infty_T H^\ell_G} \leqslant 1$ and deduce that up to removing an extra set of probability not larger than e^{-cR^2}, inequality (6.6) holds. This concludes the proof of Theorem 2 (ii).

8. Local well-posedness

This section is devoted to the proof of Theorem 1 .

We fix $k \in (1, \frac{3}{2})$ and $u_0 \in H^k_G$. We assume that there exists $\varepsilon_0 > 0$ such that $u_0 \in \mathcal{X}^k_{1+\varepsilon}$. We denote by u_0^ω its associate randomization, defined by (6.2), and recall that we write $z^\omega(t) := e^{it\Delta_G}u_0^\omega$ for the solution to the linear flow (LS-G) associated to the initial data u_0^ω. We seek for a solution u to (NLS-G) of the form

$$u(t) = z^\omega(t) + v(t),$$

where $v(0) = 0$ and $v(t) \in H^\ell_G$ with $\ell \in (\frac{3}{2}, k+\frac{1}{2})$. We will prove local well-posedness for $v \in \mathcal{C}^0\left([0,T], H^\ell_G\right)$ solving

$$\begin{cases} i\partial_t v - \Delta_G v = |z^\omega + v|^2(z^\omega + v) \\ v(0) = 0\,, \end{cases} \tag{6.39}$$

thanks to a fixed point argument. We consider the map $\Phi : v \in \mathcal{C}^0\left([0,T], H^\ell_G\right) \mapsto \Phi(v) \in \mathcal{C}^0\left([0,T], H^\ell_G\right)$ defined by

$$\Phi(v) : t \in [0,T] \mapsto -i\int_0^t e^{i(t-t')\Delta_G}\left(|z^\omega + v|^2(z^\omega + v)\right)\,\mathrm{d}t'\,. \tag{6.40}$$

Observe that by the Duhamel formula, v solves (6.39) if and only if $\Phi(v) = v$. Note that v may depend on ω.

We introduce the following set of initial data:

$$E_{R,T} = \{\omega \in \Omega \mid \ (6.41), (6.42), (6.43) \text{ and } \ (6.44) \text{ hold}\}\,,$$

where

$$\|(z^\omega)^2\|_{L^1_T H^\ell_G} + \||z^\omega|^2\|_{L^1_T H^\ell_G} \leqslant TR^2 \|u_0\|^2_{\mathcal{X}^k_1}\,, \tag{6.41}$$

$$\||z^\omega|^2 z^\omega\|_{L^1_T H^\ell_G} \leqslant TR^3 \|u_0\|^3_{\mathcal{X}^k_1}\,, \tag{6.42}$$

$$\|z^\omega v w\|_{L^1_T H^\ell_G} \leqslant TR\|u_0\|_{\mathcal{X}^k_1}\|v\|_{L^\infty_T H^\ell_G}\|w\|_{L^\infty_T H^\ell_G} \text{ for all } v, w \in L^\infty_T H^\ell_G\,, \tag{6.43}$$

$$\|z^\omega\|^2_{L^2_T L^\infty_G} \leqslant TR^2 \|u_0\|^2_{\mathcal{X}^k_1}\,. \tag{6.44}$$

We have the following estimate of $E_{R,T}$.

Lemma 6.35. *Let $u_0 \in H^k_G$. Then there exists a constant $c > 0$ which depends on the basis function u_0 of the randomization, such that for all $R, T > 0$, $\mathbb{P}(\Omega \setminus E_{R,T}) \leqslant e^{-cR^2}$.*

Proof. Outside a set of probability at most e^{-cR^2} the bounds (6.41) and (6.42) follow from Theorem 2 (i). Similarly, and (6.43) follows from Theorem 2 (ii) with $u_0 \in \mathcal{X}^k_1 \subset \mathcal{X}^{k-\varepsilon}_{1+\varepsilon}$ for ε chosen small enough so that $\ell < k - \varepsilon + \frac{1}{2}$. Moreover, (6.44) follows from Proposition 6.3. □

The key estimate in proving Theorem 1 is the following.

Proposition 6.8 (A priori estimate). *Let $u_0^\omega \in E_{R,T}$ and $\ell \in (\frac{3}{2}, k+\frac{1}{2})$. Then for any $v \in \mathcal{C}^0([0,T], H^\ell_G)$ there holds*

$$\|\Phi(v)\|_{L^\infty_T H^\ell_G} \lesssim T\left(\|v\|^3_{L^\infty_T H^\ell_G} + (R\|u_0\|_{\mathcal{X}^k_1})^3\right)\,. \tag{6.45}$$

Similarly, for any $v_1, v_2 \in \mathcal{C}^0([0,T], H^\ell_G)$ there holds

(6.46)
$$\|\Phi(v_2)-\Phi(v_1)\|_{L^\infty_T H^\ell_G} \lesssim T\|v_2-v_1\|_{L^\infty_T H^\ell_G}\left((R\|u_0\|_{\mathcal{X}^k_1})^2 + \|v_1\|^2_{L^\infty_T H^\ell_G} + \|v_2\|^2_{L^\infty_T H^\ell_G}\right).$$

Proof. Let $\omega \in E_{R,T}$. Estimates (6.45) and (6.46) reduce to multilinear estimates, as a consequence of the triangle inequality in the Duhamel formula (6.40) and the fact that $e^{it\Delta_G}$ is an isometry in H^ℓ_G. Indeed, for $t \in [0,T]$, there holds

$$\begin{aligned}\|\Phi(v)(t)\|_{H^\ell_G} &\leqslant \int_0^t \|e^{i(t-t')\Delta_G}|z^\omega+v|^2(z^\omega+v)(t')\|_{H^\ell_G}\,\mathrm{d}t' \\ &\lesssim \int_0^t \||z^\omega+v|^2(z^\omega+v)(t')\|_{H^\ell_G}\,\mathrm{d}t' \\ &\lesssim \||z^\omega+v|^2(z^\omega+v)\|_{L^1_T H^\ell_G}.\end{aligned}$$

We expand the cubic term and bound

$$\begin{aligned}\|\Phi(v)\|_{L^\infty_T H^\ell_G} \lesssim{}& \||z^\omega|^2 z^\omega\|_{L^1_T H^\ell_G} + \|(z^\omega)^2\bar{v}\|_{L^1_T H^\ell_G} + \||z^\omega|^2 v\|_{L^1_T H^\ell_G} \\ &+ \|z^\omega|v|^2\|_{L^1_T H^\ell_G} + \|z^\omega\bar{v}^2\|_{L^1_T H^\ell_G} + T\||v|^2 v\|_{L^\infty_T H^\ell_G}.\end{aligned} \tag{6.47}$$

Similarly, let $v_1, v_2 \in H^\ell_G$, then we have

$$\|\Phi(v_2)-\Phi(v_1)\|_{L^\infty_T H^\ell_G} \lesssim \||z^\omega+v_2|^2(z^\omega+v_2) - |z^\omega+v_1|^2(z^\omega+v_1)\|_{L^1_T H^\ell_G},$$

so that

$$\begin{aligned}\|\Phi(v_2)-\Phi(v_1)\|_{L^\infty_T H^\ell_G} \lesssim{}& \|(z^\omega)^2(\overline{v_2-v_1})\|_{L^1_T H^\ell_G} + \||z^\omega|^2(v_2-v_1)\|_{L^1_T H^\ell_G} \\ &+ \|z^\omega(|v_2|^2-|v_1|^2)\|_{L^1_T H^\ell_G} + \|z^\omega(\bar{v}_2^2-\bar{v}_1^2)\|_{L^1_T H^\ell_G} + T\||v_2|^2v_2-|v_1|^2v_1\|_{L^\infty_T H^\ell_G}.\end{aligned} \tag{6.48}$$

We now provide upper bounds for all the terms in (6.47) and (6.48).

- We begin with $(z^\omega)^2\bar{v}$, $(z^\omega)^2(\overline{v_2-v_1})$, $|z^\omega|^2 v$ and $|z^\omega|^2(v_2-v_1)$. Let $w = v$ or $w = v_2 - v_1$. By the product law in H^k_G of Proposition 6.1, we have

  $$\|(z^\omega)^2\bar{w}(t)\|_{H^\ell_G} \lesssim \left(\|(z^\omega)^2(t)\|_{H^\ell_G} + \|z^\omega(t)\|^2_{L^\infty_G}\right)\|w(t)\|_{H^\ell_G}.$$

  Using (6.41) and (6.44) from the assumption $\omega \in E_{R,T}$, this gives:

  $$\begin{aligned}\|(z^\omega)^2\bar{w}\|_{L^1_T H^\ell_G} &\lesssim \left(\|(z^\omega)^2\|_{L^1_T H^\ell_G} + \|z^\omega\|^2_{L^2_T L^\infty_G}\right)\|w\|_{L^\infty_T H^\ell_G} \\ &\lesssim TR^2\|u_0\|^2_{\mathcal{X}^k_1}\|w\|_{L^\infty_T H^\ell_G}.\end{aligned}$$

  Similarly one has

  $$\||z^\omega|^2 w\|_{L^1_T H^\ell_G} \lesssim TR^2\|u_0\|^2_{\mathcal{X}^k_1}\|w\|_{L^\infty_T H^\ell_G}.$$

  We have both proven

  $$\|(z^\omega)^2\bar{v}\|_{L^1_T H^\ell_G} + \||z^\omega|^2 v\|_{L^1_T H^\ell_G} \lesssim TR^2\|u_0\|^2_{\mathcal{X}^k_1}\|v\|_{L^\infty_T H^\ell_G}$$

and

$$\|(z^\omega)^2\overline{(v_2-v_1)}\|_{L^1_T H^\ell_G} + \||z^\omega|^2(v_2-v_1)\|_{L^1_T H^\ell_G} \lesssim TR^2\|u_0\|^2_{\mathcal{X}^k_1}\|v_2-v_1\|_{L^\infty_T H^\ell_G}\,.$$

- Let us estimate $z^\omega|v|^2$, $z^\omega(|v_2|^2-|v_1|^2)$, $z^\omega\bar{v}^2$ and $z^\omega(\bar{v}_2^2-\bar{v}_1^2)$. Using (6.43) from the assumption that $\omega \in E_{R,T}$, we infer:

$$\|z^\omega|v|^2\|_{L^1_T H^\ell_G} + \|z^\omega\bar{v}^2\|_{L^1_T H^\ell_G} \lesssim TR\|u_0\|_{\mathcal{X}^k_1}\|v\|^2_{L^\infty_T H^\ell_G}$$

and

$$\begin{aligned}\|z^\omega(|v_2|^2-|v_1|^2)\|_{L^1_T H^\ell_G} &+ \|z^\omega(\bar{v}_2^2-\bar{v}_1^2)\|_{L^1_T H^\ell_G}\\ &\lesssim TR\|u_0\|_{\mathcal{X}^k_1}(\|v_2+v_1\|^2_{L^\infty_T H^\ell_G}+\|v_2-v_1\|^2_{L^\infty_T H^\ell_G})\|v_2-v_1\|^2_{L^\infty_T H^\ell_G}\\ &\lesssim TR\|u_0\|_{\mathcal{X}^k_1}(\|v_2\|^2_{L^\infty_T H^\ell_G}+\|v_1\|^2_{L^\infty_T H^\ell_G})\|v_2-v_1\|^2_{L^\infty_T H^\ell_G}\,.\end{aligned}$$

- Observe that thanks to the algebra property of H^ℓ_G (since $\ell > \frac{3}{2}$) of Lemma 6.1, we have

$$\||v|^2 v\|_{L^\infty_T H^\ell_G} \lesssim \|v\|^3_{L^\infty_T H^\ell_G}$$

and

$$\||v_2|^2 v_2 - |v_1|^2 v_1\|_{L^\infty_T H^\ell_G} \lesssim (\|v_2\|_{L^\infty_T H^\ell_G}+\|v_1\|_{L^\infty_T H^\ell_G})^2\|v_2-v_1\|_{L^\infty_T H^\ell_G}\,.$$

All these bounds combined together with assumption (6.42) in estimates (6.47) and (6.48) imply (6.45) and (6.46). ☐

Proof of Theorem 1. Let $\omega \in E_{R,T}$. Thanks Proposition 6.8, we know that there exists $C>0$ such that the map $\Phi : \mathcal{C}^0([0,T],H^\ell_G) \to \mathcal{C}^0([0,T],H^\ell_G)$ is bounded Lipschitz on finite balls: if $\|v\|_{L^\infty_T H^\ell_G} \leqslant R\|u_0\|_{\mathcal{X}^k_1}$, we have

$$\|\Phi(v)\|_{L^\infty_T H^\ell_G} \leqslant CT(R\|u_0\|_{\mathcal{X}^k_1})^3\,,$$

and for $\|v_1\|_{L^\infty_T H^\ell_G} \leqslant R$ and $\|v_2\|_{L^\infty_T H^\ell_G} \leqslant R$, we have

$$\|\Phi(v_2)-\Phi(v_1)\|_{L^\infty_T H^\ell_G} \leqslant CT(R\|u_0\|_{\mathcal{X}^k_1})^2\|v_2-v_1\|_{L^\infty_T H^\ell_G}\,.$$

Thus, taking $T = \frac{1}{2C(R\|u_0\|_{\mathcal{X}^k_1})^2}$, we see that Φ stabilizes the ball $B(0,R\|u_0\|_{\mathcal{X}^k_1})$ in $\mathcal{C}^0([0,T],H^\ell_G)$, moreover, Φ is a contraction on the ball $B(0,R\|u_0\|_{\mathcal{X}^k_1})$. The existence and uniqueness of v solving (6.39) then follows from standard contraction mapping arguments.

We have obtained that for any $\omega \in E_{R,T}$ (and $T = \frac{1}{2C(R\|u_0\|_{\mathcal{X}^k_1})^2}$), there exists a unique solution to (NLS-G) in the space

$$e^{it\Delta_G}u_0^\omega + \mathcal{C}^0([0,T],H^\ell_G) \subset \mathcal{C}^0([0,T],H^k_G)\,.$$

Then the set

$$E := \bigcup_{k\geqslant 1}\bigcap_{n\geqslant k} E_{n,\frac{1}{2n^2}}$$

satisfies the requirements of Theorem 1 (i). Indeed, it remains to see that $\mathbb{P}(\Omega \setminus E) = 0$. Since the sequence of sets $\bigcup_{n\geqslant k} E_{n,\frac{1}{2n^2}}$ is non-increasing, we have

$$\mathbb{P}(\Omega \setminus E) \leqslant \limsup_{k\to\infty} \mathbb{P}\left(\bigcup_{n\geqslant k} E_{n,\frac{1}{2n^2}}\right) \leqslant \limsup_{k\to\infty} \sum_{n\geqslant k} e^{-cn^2},$$

which is 0 since $\sum e^{-cn^2}$ converges. □

Appendix A. Pointwise estimates on Hermite functions

The purpose of this appendix is to explain how to prove the estimates of Corollary 6.15 as a consequence of the pointwise estimates on the Hermite functions from Theorem 6.12.

Proof of Corollary 6.15. We study the bounds on distinct regions of space. Let us fix $m \in \mathbb{N}$.

(1) For $|x| \leqslant \frac{1}{2}\lambda_m$, there holds $|x^2 - \lambda_m^2| \geqslant \frac{3}{4}\lambda_m^2$ thus

$$|h_m(x)| \lesssim |x^2 - \lambda_m^2|^{-\frac{1}{4}} \lesssim \lambda_m^{-\frac{1}{2}}.$$

(2) For $\frac{1}{2}\lambda_m \leqslant |x| \leqslant \lambda_m - \lambda_m^{-\frac{1}{3}}$, we have $x^2 \leqslant \lambda_m^2 - 2\lambda_m^{\frac{2}{3}} + \lambda_m^{-\frac{2}{3}} \leqslant \lambda_m^2 - \lambda_m^{\frac{2}{3}}$ thus $|\lambda_m^2 - x^2| = (\lambda_m^2 - x^2) \geqslant \lambda_m^{\frac{2}{3}}$, which implies that:

$$\lambda_m^{\frac{2}{3}} + |\lambda_m^2 - x^2| \leqslant 2|\lambda_m^2 - x^2|.$$

Finally, we get

$$|h_m(x)| \lesssim |\lambda_m^2 - x^2|^{-\frac{1}{4}} \lesssim \left(\lambda_m^{\frac{2}{3}} + |\lambda_m^2 - x^2|\right)^{-\frac{1}{4}}.$$

(3) For $||x| - \lambda_m| \leqslant \lambda_m^{-\frac{1}{3}}$, we have $|x^2 - \lambda_m^2| \leqslant ||x| - \lambda_m| \cdot ||x| + \lambda_m| \lesssim \lambda_m^{\frac{2}{3}}$, so that

$$|h_m(x)| \lesssim \lambda_m^{-\frac{1}{6}} = (\lambda_m^{\frac{2}{3}})^{-\frac{1}{4}} \lesssim \left(\lambda_m^{\frac{2}{3}} + |\lambda_m^2 - x^2|\right)^{-\frac{1}{4}}.$$

(4) For $\lambda_m + \lambda_m^{-\frac{1}{3}} \leqslant |x| \leqslant 2\lambda_m$ there holds $|x^2 - \lambda_m^2| \gtrsim \lambda_m^{\frac{2}{3}}$, thus the crude bound $e^{-s_m(x)} \leqslant 1$ gives

$$|h_m(x)| \leqslant \frac{e^{-s_m(x)}}{|x^2 - \lambda_m^2|^{\frac{1}{4}}} \lesssim \left(\lambda_m^{\frac{2}{3}} + |x^2 - \lambda_m^2|\right)^{-\frac{1}{4}}.$$

(5) Let $|x| \geqslant 2\lambda_m$. Observe that by change of variable $t = \lambda_m y$ we have

$$s_m(x) = \lambda_m^2 \int_1^{\frac{x}{\lambda_m}} \sqrt{y^2 - 1}\, \mathrm{d}y \geqslant \lambda_m^2 \int_1^{\frac{x}{\lambda_m}} (y - 1)\, \mathrm{d}y = \frac{(x - \lambda_m)^2}{2},$$

where we used that for $y \geqslant 1$, $\sqrt{y^2-1} \geqslant y-1$. Then, observe that $x-\lambda_m \geqslant \frac{x}{2}$ by definition of x. This implies $s_m(x) \geqslant \frac{x^2}{8}$ and finally, since $|x^2-\lambda_m^2| \geqslant \lambda_m^2 \geqslant 1$, we conclude:

$$|h_m(x)| \lesssim \frac{e^{-s_m(x)}}{|x^2-\lambda_m^2|^{\frac{1}{4}}} \lesssim e^{-\frac{1}{8}x^2}\,. \qquad \square$$

Appendix B. Algebra property and local theory

Appendix B.1. Proof of the functional inequalities. In the Grushin case, the proof of Proposition 6.1 is a consequence of the following results. In the context of the Heisenberg sub-Laplacian, the proof of Proposition 6.1 about the algebra property of the Sobolev spaces H^k can be found in [BG01] and relies on representation theoretic formulæ.

Lemma 6.36 (See the proof of Lemma 3.6 in [BFKG16])**.** *Let $H = \partial_{xx} + x^2$ denote the Harmonic oscillator. For all $k \geqslant 0$, there exists $C(k) > 0$ such that for all $m \in \mathbb{N}$,*

$$\frac{1}{C(k)}\|H^{k/2}h_m\|_{L^2_x} \leqslant \|\partial_x^k h_m\|_{L^2_x} + \|x^k h_m\|_{L^2_x} \leqslant C(k)\|H^{k/2}h_m\|_{L^2_x}\,.$$

Corollary 6.37. *For all $k \geqslant 0$, there exists $C(k) > 0$ such that for all $u \in H^k_G$, there holds*

$$\frac{1}{C(k)}\|(\mathrm{Id}-\Delta_G)^{k/2}u\|_{L^2_G} \leqslant \|\langle\partial_x\rangle^k u\|_{L^2_G}+\|\langle x\partial_y\rangle^k u\|_{L^2_G} \leqslant C(k)\|(\mathrm{Id}-\Delta_G)^{k/2}u\|_{L^2_G}\,.$$

Proof. We decompose u as

$$\mathcal{F}_{y\to\eta}u(x,\eta) = \sum_m f_m(\eta)h_m(|\eta|^{\frac{1}{2}}x)\,.$$

Then

$$\|(\mathrm{Id}-\Delta_G)^{k/2}u\|_{L^2_x}^2 = \sum_m \int |f_m(\eta)|^2 \mathrm{d}\eta \int (1+(2m+1)|\eta|)^k h_m(|\eta|^{\frac{1}{2}}x)^2 \mathrm{d}x.$$

Hence we see that $\|(\mathrm{Id}-\Delta_G)^{k/2}u\|_{L^2_G} \sim_k \|u\|_{L^2_x} + \|(-\Delta_G)^{k/2}u\|_{L^2_G}$, and

$$\begin{aligned}\|(-\Delta_G)^{k/2}u\|_{L^2_G}^2 &\sim_k \sum_m \int |f_m(\eta)|^2 \mathrm{d}\eta \int ((2m+1)|\eta|)^k h_m(|\eta|^{\frac{1}{2}}x)^2 \mathrm{d}x \\ &= \sum_m \int |f_m(\eta)|^2 \mathrm{d}\eta \int |\eta|^k (H^{k/2}h_m)(|\eta|^{\frac{1}{2}}x)^2 \mathrm{d}x.\end{aligned}$$

Now we use a change of variables to get

$$\|(-\Delta_G)^{k/2}u\|_{L^2_G}^2 \sim_k \sum_m \int |f_m(\eta)|^2 \mathrm{d}\eta |\eta|^{k-1/2} \int (H^k h_m)(x)^2 \mathrm{d}x.$$

Then we use that $\|H^{k/2}h_m\|_{L^2_x} \sim_k \|\partial_x^k h_m\|_{L^2_x} + \|x^k h_m\|_{L^2_x}$ and get

$$\begin{aligned}\|(-\Delta_G)^{k/2}u\|^2_{L^2_G} \sim_k \sum_m \int |f_m(\eta)|^2 \mathrm{d}\eta |\eta|^{k-1/2} \int ((\partial_x^k + x^k)h_m)(x)^2 \mathrm{d}x \\ = \sum_m \int |f_m(\eta)|^2 \mathrm{d}\eta \int |\eta|^k ((\partial_x^k + x^k)h_m)(|\eta|^{\frac12}x)^2 \mathrm{d}x \\ = \sum_m \int |f_m(\eta)|^2 \mathrm{d}\eta \int (\partial_x^k + (|\eta| x)^k) h_m (|\eta|^{\frac12}x)^2 \mathrm{d}x \\ = \|\partial_x^k u\|^2_{L^2_G} + \|(x\partial_y)^k u\|^2_{L^2_G}.\end{aligned}$$

□

We are now ready to give a proof of the product laws.

Proof of Proposition 6.1. (i) We start by using the above Corollary 6.37 and get

$$\|uv\|_{H^k_G} \lesssim \|\langle\partial_x\rangle^k(uv)\|_{L^2_{x,y}} + \|\langle x\partial_y\rangle^k(uv)\|_{L^2_{x,y}}.$$

Now, the classical product rule in Sobolev spaces applied in x (resp. in y) implies

$$\|\langle\partial_x\rangle^k(uv)\|_{L^2_x} \lesssim \|\langle\partial_x\rangle^k u\|_{L^2_x}\|v\|_{L^\infty_x} + \|u\|_{L^\infty_x}\|\langle\partial_x\rangle^k v\|_{L^2_x},$$

resp.

$$\|\langle x\partial_y\rangle^k(uv)\|_{L^2_y} \lesssim \|\langle x\partial_y\rangle^k u\|_{L^2_y}\|v\|_{L^\infty_y} + \|u\|_{L^\infty_y}\|\langle x\partial_y\rangle^k v\|_{L^2_y},$$

which combined with Hölder estimates in y (resp. x) yields

$$\begin{aligned}\|uv\|_{H^k_G} \lesssim & \|\langle\partial_x\rangle^k u\|_{L^2_{x,y}}\|v\|_{L^\infty_{x,y}} + \|u\|_{L^\infty_{x,y}}\|\langle\partial_x\rangle^k v\|_{L^2_{x,y}} \\ & + \|\langle x\partial_y\rangle^k u\|_{L^2_{x,y}}\|v\|_{L^\infty_{x,y}} + \|u\|_{L^\infty_{x,y}}\|\langle x\partial_y\rangle^k v\|_{L^2_{x,y}}.\end{aligned}$$

Proposition 6.1 (i) follows by an other application of Corollary 6.37.

(ii) is a direct consequence of (i) and the Sobolev embedding $H^k_G \hookrightarrow L^\infty_G$ when $k > \frac32$.

(iii) When p is an integer, the result follows from (ii) by iteration. □

Lemma 6.38 (Limit Sobolev embedding)**.** *The following statements hold.*

(i) There exists $C > 0$ such that for every $p > 2$ and $u \in H^{\frac32}_G$, there holds

$$\|u\|_{L^p_G} \leqslant C\sqrt{p}\|u\|_{H^{\frac32}_G}. \tag{6.49}$$

(ii) (Brezis-Gallouët) For any $k > \frac32$, there exists $C_k > 0$ such that there holds

$$\|u\|_{L^\infty_G} \leqslant C_k \|u\|_{H^{\frac32}_G} \log^{\frac12}\left(1 + \frac{\|u\|_{H^k_G}}{\|u\|_{H^{\frac32}_G}}\right). \tag{6.50}$$

Proof. (i) Let $p > 2$ and $u \in L^p_G$. By the triangle inequality we have

$$\|u\|_{L^p_G} \leqslant \sum_{A \in 2^{\mathbb{N}}} \|u_A\|_{L^p_G} .$$

Then, the Sobolev embedding yields

$$\|u\|_{L^p_G} \lesssim \sum_{A \in 2^{\mathbb{N}}} \|u_A\|_{H_G^{3(\frac{1}{2}-\frac{1}{p})}} \lesssim \sum_{A \in 2^{\mathbb{N}}} A^{-\frac{3}{2p}} \|u_A\|_{H_G^{\frac{3}{2}}} .$$

An application of the Cauchy-Schwarz inequality and orthogonality provide us with

$$\|u\|_{L^p_G} \lesssim \left(\sum_{A \in 2^{\mathbb{N}}} A^{-\frac{3}{p}} \right)^{\frac{1}{2}} \|u\|_{H_G^{\frac{3}{2}}} ,$$

which gives the conclusion since $\sum_{A \in 2^{\mathbb{N}}} A^{-\frac{3}{p}} \sim Cp$ as p goes to infinity.

(ii) Let $A_0 \geqslant 1$ be a dyadic integer. Start by writing $u = \sum_{A \leqslant A_0} u_A + \sum_{A > A_0} u_A$, and by Cauchy-Schwarz

$$\|u\|_{L^\infty} \leqslant \log^{\frac{1}{2}}(A_0) \left(\sum_{A \leqslant A_0} \|u_A\|^2_{L^\infty} \right)^{\frac{1}{2}} + \sum_{A > A_0} \|u_A\|^2_{L^\infty} .$$

Observe that for $p \geqslant 2$, $\|u_A\|_{L^p_G} \lesssim A^{\frac{3}{2}(\frac{1}{2}-\frac{1}{p})} \|u_A\|_{L^2_G}$, which gives when letting $p \to \infty$

$$\|u_A\|_{L^\infty_G} \lesssim \|u_A\|_{H_G^{\frac{3}{2}}} , \tag{6.51}$$

and this latter inequality also implies

$$\|u_A\|_{L^\infty_G} \lesssim A^{-\frac{1}{2}(k-\frac{3}{2})} \|u_A\|_{H^k_G} . \tag{6.52}$$

The bound (6.51) when $A \leqslant A_0$ and (6.52) when $A > A_0$ imply

$$\|u\|_{L^\infty_G} \lesssim \|u\|_{H_G^{\frac{3}{2}}} \log^{\frac{1}{2}} A_0 + A_0^{-\frac{1}{2}(k-\frac{3}{2})} \|u\|_{H^k_G} ,$$

which gives the result after optimisation in A_0. □

Appendix B.2. Deterministic local Cauchy theory. We finish this appendix with a summary of well-posedness results for (NLS-$\mathbb{H}^1$) (resp. (NLS-G)) for $k > 2$ (resp. $k > \frac{3}{2}$). To the best of our knowledge, the best well-posedness result for (NLS-$\mathbb{H}^1$) is the following.

Proposition 6.9 (Well-posedness for (NLS-$\mathbb{H}^1$), see [BG01])**.** *For $k > 2$, the Cauchy problem for (NLS-$\mathbb{H}^1$) is locally well-posed in $\mathcal{C}^0([0,T^*), H^k(\mathbb{H}^1))$ and $T^* = T^*(\|u_0\|_{H^k(\mathbb{H}^1)}) \gtrsim \|u_0\|^{-2}_{H^k(\mathbb{H}^1)}$.*

Similarly, we recall the best local theory for (NLS-G).

Proposition 6.10 (Well-posedness theory for (NLS-G))**.** *The following well-posedness statements hold:*

(i) For $k > \frac{3}{2}$, the Cauchy problem for (NLS-G) is locally well-posed in $\mathcal{C}([0,T^*), H^k_G)$. Moreover, for $u_0 \in H^k_G$, the maximal time T^* satisfies $T^* \gtrsim \|u_0\|^{-2}_{L^\infty_G}$.

(ii) The blow-up criterion can be refined:

$$T^* < \infty \Longrightarrow \|u(t)\|_{H^{\frac{3}{2}}_G} \underset{t\to T^*}{\longrightarrow} \infty\,.$$

Sketch of proof for Proposition 6.10. We only give formal arguments, which are easily converted into fully rigorous proofs by standard means.

(i) Let $k > \frac{3}{2}$. Applying the operator $(-\Delta_G)^{\frac{k}{2}}$ to (NLS-G), then multiplying by $(-\Delta_G)^{\frac{k}{2}}u$ and integrating by parts in space, we compute that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{H^k_G} \lesssim \|(-\Delta_G)^{\frac{k}{2}}(|u|^2u)(-\Delta_G)^{\frac{k}{2}}\bar{u}\|_{L^1_G}\,.$$

Applying the Hölder inequality, the algebra property of Lemma 6.1 and the Hölder inequality again, we finally arrive at the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{H^k_G} \lesssim \|u\|^2_{L^\infty_G}\|u\|^2_{H^k_G}\,,$$

which by the Sobolev embedding and the Grönwall inequality gives an *a priori* estimate in H^k and implies the local theory.

(ii) This follows from the energy estimate and inequality (6.50), which give

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{H^k_G} \lesssim \|u\|^2_{L^\infty_G}\|u\|^2_{H^k_G} \lesssim \|u\|^2_{H^{\frac{3}{2}}}\|u\|^2_{H^k_G}\log\left(1+\frac{\|u\|_{H^k_G}}{\|u\|_{H^{\frac{3}{2}}}}\right)\,,$$

which implies the result after an application of Grönwall's inequality. □

Note that a similar argument to that of (ii), which relies on the inequality (6.49) can be used to prove that, if solutions exists in $H^{\frac{3}{2}}_G$, they are unique. This argument goes back to Yudovich [Yud63], then has been used by Vladimirov [Vla84] in the context of Schrödinger equations. Namely, if u_1, u_2 are two $H^{\frac{3}{2}}_G$ solutions in $L^\infty([0,T), H^{\frac{3}{2}}_G)$, introduce $\phi(t) = \|u_1(t)-u_2(t)\|^2_{L^2_G}$ and fix $T_1 < T$, we prove that $\phi(t) = 0$ for all $t \in [0,T_1]$. Denote by $w(t) = u_1(t)-u_2(t)$. Then

$$\phi'(t) = 2\int_{\mathbb{R}^2} w'(t)\bar{w}(t)\,\mathrm{d}x = 2i\int_{\mathbb{R}^2}\Delta w(t)\bar{w}(t)\,\mathrm{d}x - 2i\int_{\mathbb{R}^2}(|u_1|^2u_1-|u_2|^2u_2)\bar{w}(t)\,\mathrm{d}x$$

Since the first terms equals $-2i\|\nabla w\|^2_{L^2} \in i\mathbb{R}$ and ϕ is real-valued, we have

$$\phi'(t) \leqslant 2\left|\int_{\mathbb{R}^2}(|u_1|^2u_1-|u_2|^2u_2)\bar{w}(t)\,\mathrm{d}x\right| \lesssim \int_{\mathbb{R}^2}|w(t)|^2\left(|u_1(t)|^2+|u_2(t)|^2\right)\mathrm{d}x\,.$$

Then for all $t \leqslant T_1$, we have

$$\phi'(t) \lesssim \int_{\mathbb{R}^2} |(u_1 - u_2)(t)|^{2(1-1/p)}(|u_1(t)|^{2(1+1/p)} + |u_2(t)|^{2(1+1/p)})\,\mathrm{d}x$$
$$\lesssim \sqrt{p}\phi(t)^{1-\frac{1}{p}} \left(\|u_1\|^{2(1+1/p)}_{H^{\frac{3}{2}}_G} + \|u_2\|^{2(1+1/p)}_{H^{\frac{3}{2}}_G} \right),$$

where we use (6.49) in the last step. Since $\|u_1(t)\|_{L^\infty([0,T_1],H^{\frac{3}{2}})} = C(T_1)$ and $2(1+\frac{1}{p}) \leqslant 3$, we obtain

$$\phi'(t) \lesssim \sqrt{p}\phi(t)^{1-\frac{1}{p}}$$

The, we integrate on $[0,t]$ to get

$$\phi(t) \leqslant \left(\frac{t}{\sqrt{p}}\right)^p,$$

which goes to 0 as $p \to \infty$, hence $\phi = 0$.

Appendix C. An interpolation lemma

The aim of this appendix is to prove that if inequality (6.21)

$$\Big\| \sum_{2B>A} (\mathrm{Id} - \Delta_G)^{\ell/2}(u_A(P_{>A}v_B)) \Big\|^2_{L^2_G} \lesssim \sum_{\delta \in D_1} \|(u_A)^{\delta_1}\|^2_{L^\infty_G} \|v\|^2_{H^\ell_G}$$

holds for $\ell = 0$ and $\ell = 2$, then this inequality holds for all $\ell \in [0,2]$ by interpolation. Writing $w = (\mathrm{Id} - \Delta_G)^{\ell/2} v$, we have

$$\Big\| \sum_{B>A} (\mathrm{Id} - \Delta_G)^{\ell/2}(u_A(P_{>A}v_B)) \Big\|_{L^2_G} = \Big\| (\mathrm{Id} - \Delta_G)^{\ell/2} \Big(u_A \sum_{2B>A} (\mathrm{Id} - \Delta_G)^{-\ell/2}(P_{>A}w)_B \Big) \Big\|_{L^2_G}.$$

Lemma 6.39 (Interpolation lemma)**.** *Let $\ell \in [0,1]$, then for any $w \in L^2_G$, there holds*

$$\|(\mathrm{Id} - \Delta_G)^\ell \left(u_A \chi_{>A} (\mathrm{Id} - \Delta_G)^{-\ell} w\right)\|_{L^2_G} \lesssim \left(\sum_{\delta \in D_1} \|(u_A)^{\delta_1}\|^2_{L^\infty_G} \right)^{1/2} \|w\|_{L^2_G}.$$

The proof of this lemma is an application of Stein's interpolation theorem in the case when the operators are bounded by the same constant.

Theorem 6.40 (Stein interpolation theorem [Ste56], Theorem 1)**.** *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 0,1$ be two measured spaces, $p_i, q_i \in [1,\infty]$, $S := \{z \in \mathbb{C} \mid 0 < Re(z) < 1\}$ and $(T_z)_{z\in\bar{S}}$ a family of operators from simple functions in $L^1(\mu_1)$ to μ_2-measurable functions. Assume that there exists $c < \pi$ such that the following holds.*

(i) For any fixed simple functions f and g on Ω_0 and Ω_1 respectively, $z \in \bar{S} \mapsto \int_{\Omega_1} T_z(f) g \,\mathrm{d}\mu_1$ is continuous on $\bar{S}$ and holomorphic in S, and satisfies

$$\sup_{z \in S} e^{-c|\mathrm{Im}(z)|} \log \left| \int_{\Omega_1} T_z(f) g \,\mathrm{d}\mu_1 \right| < \infty .$$

(ii) The operator $T_z : L^{p_0}(\Omega_0) \to L^{q_0}(\Omega_1)$ is continuous whenever $Re(z) = 0$: there exists C_0 such that for all $f \in L^{p_0}(\Omega_0)$ and $r \in \mathbb{R}$,

$$\|T_{0+ir} f\|_{L^{q_0}} \leqslant C_0(r) \|f\|_{L^{p_0}},$$

and similarly, whenever $Re(z) = 1$, the operator $T_z : L^{p_1}(\Omega_0) \to L^{q_1}(\Omega_1)$ is continuous: there exists C_1 such that for all $f \in L^{p_1}(\Omega_0)$ and $r \in \mathbb{R}$,

$$\|T_{1+ir} f\|_{L^{q_1}} \leqslant C_1(r) \|f\|_{L^{p_1}} .$$

Moreover, for $i \in \{0, 1\}$,

$$\sup_{r \in \mathbb{R}} e^{-c|r|} \log |C_i(r)| < \infty .$$

Then for $\theta \in [0,1]$, setting $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{p_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, the operator $T_\theta : L^{p_\theta}(\Omega_0) \to L^{q_\theta}(\Omega_1)$ is bounded. More precisely, there exists $C_\theta = C(\theta, C_0, C_1)$ such that

$$\|T_\theta f\|_{L^{q_\theta}} \leqslant C_\theta \|f\|_{L^{p_\theta}}, \quad f \in L^{p_\theta}(\Omega_0), \quad r \in \mathbb{R}.$$

Proof of Lemma 6.39. We apply Theorem 6.40 to the operators

$$\left(\sum_{\delta \in D_1} \|(u_A)^{\delta_1}\|^2_{L^\infty_G} \right)^{-1/2} T_z ,$$

with

$$T_z f := (\mathrm{Id} - \Delta_G)^z \left(u_A P_{>A} (\mathrm{Id} - \Delta_G)^{-z} f \right) ,$$

and $(\mathrm{Id} - \Delta_G)^z = \exp\left(z \log(\mathrm{Id} - \Delta_G)\right)$. We make the choice $\Omega_0 = \Omega_1 = \mathbb{R}^2$, $\mu_0 = \mu_1$ is the Lebesgue measure λ, and $p_0 = p_1 = p_\theta = q_\theta = q_0 = q_1 = 2$.

Observe that for any real r, the operator $(\mathrm{Id} - \Delta_G)^{ir}$ acts by rotation of the Fourier coefficients in L^2_G thus it is bounded in L^2 with norm 1. Indeed, the decomposition $\mathcal{F}(w)(x, \eta) = \sum_{m \in \mathbb{N}} f_m(\eta) h_m(|\eta|^{\frac{1}{2}} x)$ leads to

$$\mathcal{F}((\mathrm{Id} - \Delta_G)^{ir} w)(x, \eta) = \sum_{m \in \mathbb{N}} (1 + (2m+1)|\eta|)^{ir} f_m(\eta) h_m(|\eta|^{\frac{1}{2}} x) ,$$

so that we can conclude by orthogonality of the decomposition. This implies that for all $w \in L^2(\mathbb{R}^2)$ and $z \in \bar{S}$, we have

$$\|T_z f\|_{L^2(\mathbb{R}^2)} = \|T_{\mathrm{Re}(z)}((\mathrm{Id} - \Delta_G)^{-ir} f)\|_{L^2(\mathbb{R}^2)} ,$$

with $\|(\mathrm{Id} - \Delta_G)^{-ir} w\|_{L^2(\mathbb{R}^2)} = \|w\|_{L^2(\mathbb{R}^2)}$. Thanks to the fact that inequality (6.21) holds in the case $\ell = 0$ and $\ell = 2$, assumption (ii) from Theorem 6.40 then holds with some constant functions $C_0 = C_1$ independent of $r = \mathrm{Im}(z)$.

We now establish assumption (i). Fix two simple functions f, g on $\mathbb{R}^2$. Then the map $z \in \bar{S} \mapsto \int_{\mathbb{R}^2} T_z(f) g \, \mathrm{d}\lambda$ is continuous and holomorphic. Moreover, for all $z \in S$, we have

$$\left| \int_{\mathbb{R}^2} T_z(f) g \, \mathrm{d}\lambda \right| \leqslant \|T_z f\|_{L^2(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)} \,.$$

We write $z = \ell + ir$ with $(\ell, r) \in [0,1] \times \mathbb{R}$. When $\ell = 0$, we simply write

$$\|T_z f\|_{L^2_G} \lesssim \|u_A\|_{L^\infty_G} \|P_{>A} f\|_{L^2_G} \,,$$

and observe that $\|u_A\|^2_{L^\infty_G} \lesssim \|u_A\|^2_{H^{\frac{3}{2}+\varepsilon}_G} \lesssim A^{\frac{3}{2}+\varepsilon} \|u_A\|^2_{L^2_G} < +\infty$. Otherwise, if $\ell > 0$, start by an application of the product law in Proposition 6.1, which gives

$$\begin{aligned}
\|T_z f\|_{L^2_G} &= \|u_A P_{>A} (\mathrm{Id} - \Delta_G)^{-z} f\|_{H^\ell_G} \\
&\lesssim \|u_A\|_{L^\infty_G} \|P_{>A} (\mathrm{Id} - \Delta_G)^{-z} f\|_{H^\ell_G} + \|u_A\|_{H^h_G} \|P_{>A} (\mathrm{Id} - \Delta_G)^{-z} f\|_{L^\infty_G} \\
&\lesssim \|u_A\|_{L^\infty_G} \|f\|_{L^2_G} + \|u_A\|_{H^\ell_G} \|P_{>A} (\mathrm{Id} - \Delta_G)^{-ir} f\|_{W^{-\ell,\infty}_G} \,.
\end{aligned}$$

Then we observe that $\|u_A\|_{H^\ell_G} \lesssim A^{\frac{\ell}{2}} \|u_A\|_{L^2_G} < \infty$, $\|u_A\|_{L^\infty_G} < \infty$, as well as $\|f\|_{L^2_G} < \infty$. It remains to study $\|P_{>A}(\mathrm{Id} - \Delta_G)^{-ir} f\|_{W^{-\ell,\infty}_G}$. We first use the dual Sobolev embedding $L^p_G \hookrightarrow W^{-\ell,\infty}_G$ where $\frac{1}{p} - \frac{\ell}{3} = 0$ (so that $3 \leqslant p < \infty$):

$$\|P_{>A}(\mathrm{Id} - \Delta_G)^{-ir} f\|_{W^{-\ell,\infty}_G} \lesssim \|P_{>A}(\mathrm{Id} - \Delta_G)^{-ir} f\|_{L^p_G} \,.$$

Now, we conclude by using the continuity of $P_{>A}(\mathrm{Id} - \Delta_G)^{-ir}$ on L^p_G.

Indeed, $P_{>A}(\mathrm{Id} - \Delta_G)^{-ir} = F(-\Delta_G)$, where

$$F(\lambda) = \left(1 - \chi\left(\frac{1+\lambda}{A}\right)\right)(1+\lambda)^{ir}.$$

Since $\chi \in \mathcal{C}^\infty_c[0,1)$, one knows that $F \in W^{2,\infty}(\mathbb{R})$. Now, Theorem 1 in [MS12] implies that for all $p \in (1, \infty)$, $F(-\Delta_G)$ is bounded in $\mathcal{L}(L^p(\mathbb{R}^2))$. $\square$

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RÉSUMÉ

Cette thèse étudie l'influence des données initiales aléatoires dans les équations aux dérivées partielles qui peuvent être dispersives ou provenir de la mécanique des fluides. L'aléatoire permet la construction de solutions locales, globales et, dans certains cas, l'étude du comportement asymptotique alors qu'aucune théorie déterministe n'est connue. Les principales méthodes aléatoires pour les équations aux dérivées partielles sont exposées dans les deux premiers chapitres puis, dans le chapitre 3, on obtient l'existence presque-sûre de solutions faibles et globales pour des équations d'ondes non-linéaires, à un niveau de régularité plus faible que celui de l'énergie. Dans le chapitre 4, on construit des solutions globales aux équations de Schrödinger non-linéaires posées sur l'espace euclidien. Ces solutions sont construites à un très faible niveau de régularité et exhibent un comportement de diffusion en temps long. Le chapitre 5 se concentre sur l'étude de la croissance des normes de Sobolev pour les solutions des équations d'Euler à l'aide de mesures invariantes produites à un haut niveau de régularité. Enfin, le dernier chapitre étudie le problème de Cauchy local pour les équations de Schrödinger non-linéaires posées pour le laplacien de Grushin. On construit de façon probabiliste des solutions à un niveau de régularité plus faible que la théorie déterministe.

MOTS CLÉS

Aléatoire ; équation ; différentielle.

ABSTRACT

This dissertation studies the effect of the randomisation of initial data for dispersive and fluid partial differential equations. Randomness makes possible the construction of local and global solutions, and allows in some cases the study of the asymptotic behaviour in situations for which no deterministic theory is known. The results of this dissertation are proven using variations of two important methods which are exposed in Chapter 2. In Chapter 3 we obtain the almost-sure existence of weak and global solutions for nonlinear wave equations at a lower level of regularity than the energy level. In Chapter 4 we construct global solutions to nonlinear Schrödinger equations on the Euclidean space. These solutions are of very low regularity and scatter. Chapter 5 focuses on the study of the growth of Sobolev norms for solutions of the Euler equations using invariant measures produced at high regularity. Finally, the last chapter studies the local Cauchy problem for the nonlinear Schrödinger equations posed for the Grushin Laplacian. Using a probabilistic method, we construct solutions at a lower level of regularity than what the best deterministic theory can produce.

KEYWORDS

Random ; equation ; differential.