Almost-Sure Scattering for NLS at Mass Regularity in dimensions $d \geqslant 2$

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The NLS equation

$u(t,x) : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{C}$ satisfying

$$\begin{cases} i\partial_t u + \Delta u &= |u|^{p-1} u \\ u(t=0) &= u_0 \in H^s(\mathbb{R}^d)\,. \end{cases} \tag{NLS}$$

for $d \geqslant 2$ and $p \in [1, 1 + \frac{4}{d}]$.

Two formal conserved quantities:

$$M(t) = \|u(t)\|_{L^2}^2 \text{ and } E(t) = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{p+1}\|u(t)\|_{L^{p+1}}^{p+1}\,.$$

Regularities: L^2 and H^1.

Scattering for NLS

Duhamel formula for (NLS) reads:

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-t')\Delta}(|u(t')|^{p-1}u(t'))\,\mathrm{d}t'\,.$$

We say that $u(t)$ scatters forward in time in H^s if

$$\begin{aligned} e^{-it\Delta}u(t) - u_0 &= -i\int_0^t e^{-it'\Delta}(|u(t')|^{p-1}u(t'))\,\mathrm{d}t' \\ &\xrightarrow[t\to\infty]{} u_+\,, \end{aligned}$$

in H^s, for some $u_+ \in H^s$.

Scattering at H^1 regularity

$$i\partial_t u + \Delta u = |u|^{p-1}u \tag{NLS}$$

[Colliander, Keel, Staffilani, Takaoa, Tao] proved that for $(d,p) = (3,5)$, global well-posedness and scattering holds in $H^1(\mathbb{R}^3)$.

Probabilistic improvement on this result are of the following flavour: [Killip, Murphy, Visan + Dodson, Lührmann, Mendelson] etc. prove that for some $s < 1$ almost every initial data in H^s give rise to global solutions $u(t) = e^{it\Delta}u_0 + v(t)$ where $v \in \mathcal{C}^0(\mathbb{R}, H^1)$ scatters in H^1 using *deterministic* methods on v and *probabilistic* estimates on $e^{it\Delta}u_0$.

In this work we investigate what happens at regularity $\simeq L^2$ using completely different methods.

Deterministic results

Theorem (Deterministic theory)

Let $p \geqslant 1$ and $d \geqslant 2$. Then:

1. *For $p \in \left[1, 1+\frac{4}{d}\right]$ the Cauchy problem for (NLS) is globally well-posed in $L^2(\mathbb{R}^d)$ and ill-posed in L^2 if $p > 1+\frac{4}{d}$.*
2. *For $p \leqslant 1+\frac{2}{d}$ and for every $u_0 \in L^2$, the solutions do not scatter in L^2, neither forward nor backward in time.*
3. *For $d \geqslant 2$, $p \in \left(1+\frac{2}{d}, 1+\frac{4}{d-2}\right)$ and initial data in H^1 scattering in L^2 holds.*
4. *For $d \geqslant 2$, $p = 1+\frac{4}{d}$ and radial initial data in L^2, scattering holds in L^2.*

Works of Christ, Colliander and Tao; Dodson; Barab; Tsutsumi, Yajima.

The main result

Set $X^0_{\mathrm{rad}} = \bigcap_{s>0} H^{-s}_{\mathrm{rad}}$, that almost radial L^2 functions.

Theorem (Chapter 4, Theorem 4.3)

Let $d \in \{2, \dots, 24\}$ and $p \in]1 + \frac{2}{d}, 1 + \frac{4}{d}]$. There exists a measure μ on X^0_{rad} and $\sigma > 0$ such that for μ-almost every $u_0 \in X^0_{\mathrm{rad}}$ there exists a unique solution to (NLS) in the space

$$u_L + \mathcal{C}^0(\mathbb{R}, H^\sigma)\,,$$

where $u_L(t) = e^{it\Delta} u_0$. Furthermore, there is a $\sigma' > 0$ such that the solution u scatters in $H^{\sigma'}$ as $t \to \pm\infty$, that is there exists $u_\pm \in H^{\sigma'}$ such that

$$\| e^{-it\Delta} u(t) - u_0 - u_\pm \|_{H^{\sigma'}} \underset{t \to \pm\infty}{\longrightarrow} 0\,.$$

1. **Slow growth of L^{p+1} and Sobolev norms $\Longrightarrow$ scattering**
2. A general globalisation argument designed for this
3. Fitting (NLS) into this framework using the *lens transform*
4. Constructing the quasi-invariant measures needed
5. Constructing a probabilistic local and global theory
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The scattering $e^{-it\Delta}u(t) - u_0 \to u_+$ in $H^{\sigma'}$ results from:

- A global bound $\sup_{t>0} \|e^{-it\Delta}u(t) - (u_0 + u_+)\|_{H^{\sigma'+\varepsilon}} \leqslant C$
- Precised scattering in $H^{-\sigma_0}$:

$$\|e^{-it\Delta}u(t) - (u_0 + u_+)\|_{H^{-\sigma_0}} \lesssim t^{-C}\,.$$

We only explain the second part. Duhamel formula:

$$e^{-it\Delta}u(t) - u_0 = -i\int_0^t e^{-it'\Delta}(|u|^{p-1}u)\,\mathrm{d}t'$$

Expect $u_+ = -i\int_0^\infty e^{-it'\Delta}(|u|^{p-1}u)\,\mathrm{d}t'$

Crude bound:

$$
\begin{aligned}
\|e^{-it\Delta}u(t) - (u_0 + u_+)\|_{H^{-\sigma_0}} &\lesssim \left\| \int_t^\infty e^{-it'\Delta}(|u|^{p-1}u)\,\mathrm{d}t' \right\|_{H^{-\sigma_0}} \\
&\lesssim \int_t^\infty \||u|^{p-1}u\|_{H^{-\sigma_0}}\,\mathrm{d}t' \\
&\lesssim \int_t^\infty \|v(t')\|_{L^{p+1}}^p\,\mathrm{d}t',
\end{aligned}
$$

we used Sobolev embedding $L^{\frac{p+1}{p}} \hookrightarrow H^{-\sigma_0}$.

This is motivation for obtaining slowly growing estimates for $u \in L^{p+1}$ or $u \in H^\sigma$.

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Invariant measure and global solutions

Flow ϕ_t of a PDE: $u(t) = \phi_t u_0$. μ is invariant on H^s if for any measurable sets $A \subset H^s$, $\mu(\phi_{-t}A) = \mu(A)$ that is $\mathbb{P}(u(t) \in A) = \mathbb{P}(u_0 \in A)$.
Assumptions.

1. μ is invariant for under ϕ_t on H^s;
2. $\mu(\|u\|_{H^s} > \lambda) \lesssim e^{-c\lambda^2}$;
3. Good LWP theory: if $\|u_0\|_{H^s} \leqslant \lambda$ then a solution exists on $[0, \tau]$, $\tau \sim \frac{1}{\lambda^K}$ and $\|u(t)\|_{H^s} \leqslant 2\lambda$ for $t \leqslant \tau$.

Proposition

Then for μ-almost every $u_0 \in H^s$ there exists a global solution u such that:

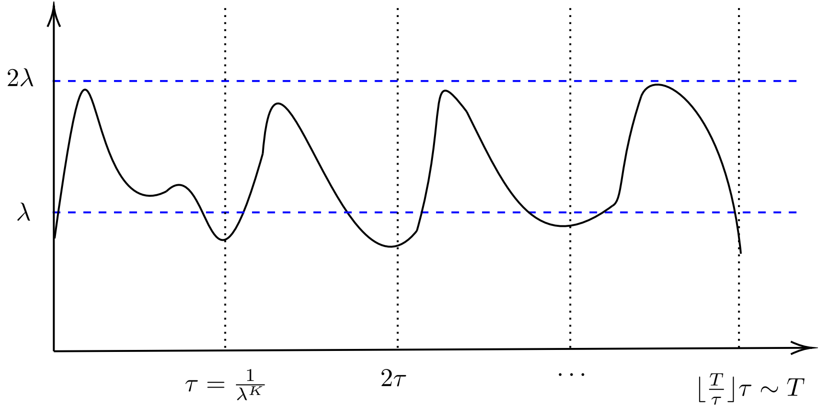
$$\|u(t)\|_{H^s} \lesssim \log^{\frac{1}{2}}(t)\,.$$

The picture

Goal: Set of measure $\geqslant 1-\varepsilon$ made of solutions with logarithmic growth until time T.

Local well-posedness time $\tau = \frac{1}{\lambda^K}$.

$$G_{\lambda,T} := \{u_0, \text{ for all } n \leqslant \lfloor \tfrac{T}{\tau} \rfloor, \|u(n\tau)\|_{H^s} \leqslant \lambda\}.$$



The proof

Set $B_{\lambda,T} := H^s \setminus G_{\lambda,T} \subset \bigcup_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \{u_0, \|u(n\tau)\|_{H^s} > \lambda\}$ the set of potentially *bad* initial data.

$$\begin{aligned} \mu(B_{\lambda,T}) &\leqslant \sum_{n=0}^{\lfloor \frac{T}{\tau} \rfloor} \mathbb{P}(\|u(n\tau)\|_{H^s} > \lambda) \\ &\leqslant \frac{T}{\tau} \mathbb{P}(\|u_0\|_{H^s} > \lambda) \\ &\leqslant T\lambda^K e^{-c\lambda^2}, \end{aligned}$$

thus $\mu(B_{\lambda,T}) \leqslant \varepsilon$ for $\lambda \sim \log^{1/2}\left(\frac{T}{\varepsilon}\right)$.

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Looking for invariant measures

For which systems is it easy to construct invariant measures?

In finite dimension. A good class of equations: Hamiltonian equations. Let:

$$\partial_t p = -\partial_q H(p,q) \quad \partial_t q = \partial_p H(p,q)\,,$$

Then $d\mu = \underbrace{Z^{-1}}_{Renorm.Cst.} \; e^{-H(p,q)}\mathrm{d}p\mathrm{d}q$ is invariant, because:

1. H is conserved by the flow.
2. $\mathrm{d}p\mathrm{d}q$ is conserved (Liouville).

Problems:

- (NLS) is infinite dimensional.
- $-\Delta$ has uncountable spectrum when acting on $\mathbb{R}^d$.

Compactification of (NLS)

We solve the second problem by applying a transformation to (NLS), the *lens* transform, mapping u solution to (NLS) to a v solution to (HNLS):

$$i\partial_t v - Hv = \cos(2t)^{-\alpha(p,d)}|v|^{p-1}v \text{ on } (-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}^d \,, \qquad \text{(HNLS)}$$

where $\alpha(p,d) = 2 - \frac{d}{2}(p-1) \in [0,1)$. and (Appendix B for explanation)

$$v(t,x) := \mathcal{L}u(t,x) = \frac{1}{\cos(2t)^{\frac{d}{2}}} u\left(\frac{\tan(2t)}{2}, \frac{x}{\cos(2t)}\right) \exp\left(-\frac{i|x|^2\tan(2t)}{2}\right) \,,$$

$H = -\Delta + |x|^2$ the harmonic oscillator with **discrete spectrum**:

$$He_n = (4n+d)e_n =: \lambda_n^2 e_n \,, n \geqslant 0 \,.$$

Denote $\|u\|_{\mathcal{W}^{s,p}} = \|H^{\frac{s}{2}}u\|_{L^p}$, and $\mathcal{H}^s = \mathcal{W}^{s,2}$.

Scattering for (HNLS)

Scattering for (NLS) $\iff$ scattering for (HNLS), this time it is simpler since

$$e^{itH} v(t) - u_0 = \int_0^t e^{-i(t-t')H} \left(\cos(2t)^{-\alpha(p,d)} |v|^{p-1} v \right) \mathrm{d}t' \,,$$

so as $t \to \infty$ expect

$$v_+ = \int_0^{\frac{\pi}{4}} e^{-i(t-t')H} \left(\cos(2t)^{-\alpha(p,d)} |v|^{p-1} v \right) \mathrm{d}t' \,.$$

$$\| e^{itH} v(t) - (u_0 + v_+) \|_{\mathcal{H}^{-\sigma_0}} \lesssim \int_t^{\frac{\pi}{4}} \cos(2t')^{-\alpha(p,d)} \| v(t') \|_{L^{p+1}}^p \, \mathrm{d}t' \,,$$

Since $\cos(2t') \sim (\pi/4 - t')$ as $t \to \pi/4$, and $\alpha(p,d) \in (0,1)$ only needs:

$$\| v(t) \|_{L^{p+1}} \lesssim \log^{\alpha}\left(\frac{\pi}{4} - t\right) .$$

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Hamiltonian setting (Section 4)

Writing $v = \sum_{n\geqslant 0} v_n e_n$ and $v_n = q_n + ip_n$ one has an infinite dimensional hamiltonian system for

$$H = H_t(\{p_n, q_n\}_{n\geqslant 0}) = \frac{1}{2}\|v\|^2_{\mathcal{H}^1} + \frac{\cos(2t)^{-\alpha(p,d)}}{p+1}\|v\|^{p+1}_{L^{p+1}},$$

Dependence of the Hamiltonian on $t \Longrightarrow$ do not expect invariance of the formal measure

$$\begin{aligned}\nu_t(A) &= \int_A \exp(-H_t(v))\,\mathrm{d}v \\ &= \int_A \exp\left(\frac{\cos(2t)^{-\alpha(p,d)}}{p+1}\|v\|^{p+1}_{L^{p+1}}\right) d\mu(v),\end{aligned}$$

with $\mathrm{d}\mu := e^{-\|v\|^2_{\mathcal{H}^1}}\mathrm{d}v = \bigotimes_{n\geqslant 0} e^{-\lambda_n^2 v_n^2}\mathrm{d}v_n$ (Gaussian densities)

Construction of the Gaussian measure

Definition

μ is the image measure by the *randomisation map*:

$$u_0^\omega(x) := \sum_{n\geqslant 0} \frac{g_n^\omega}{\lambda_n} e_n(x)\,,$$

where $(g_n)_{n\geqslant 0}$ are i.i.d. Gaussian random variables.

Lemma

1. *μ is supported by $\mathcal{X}^0_{\mathrm{rad}} = \bigcap_{s>0} \mathcal{H}^{-s}_{\mathrm{rad}}$.*
2. *For μ-almost every u there holds that $u \in \mathcal{X}^0_{\mathrm{rad}} \setminus \bigcup_{\varepsilon\geqslant 0} \mathcal{H}^{\varepsilon}_{\mathrm{rad}}$.*
3. *For any $v \in \mathcal{X}^0_{\mathrm{rad}}$ and any $\varepsilon > 0$ there holds $\mu\left(B(v,\varepsilon)\right) > 0$.*

μ is a Gaussian measure, with *good* non-smoothing properties.

The quasi-invariant measure analysis

Define $\mathrm{d}\nu_t(v) := C_t e^{-\cos(2t)^{-\alpha(p,d)} \frac{\|v\|^{p+1}}{p+1}} \mathrm{d}\mu(v)$.

Let ϕ_t be the flow of:

$$i\partial_t v - Hv = \cos(2t)^{-\alpha(p,d)} |v|^{p-1} v\,.$$

Proposition (Quasi-invariant bound (Proposition 4.4, Section 4))

1. *For all $t \geqslant 0$, $\mu \ll \nu_t \ll \mu$.*
2. *For all t and Borelian set A,*

$$\nu_0(A) \leqslant \nu_t(\phi_t A)^{\cos(2t)^{\alpha(p,d)}}$$

Remark. This bound is *bad* for $t \to \pi/4$, it is a bound by 1.

Assumptions. Generalise the assumption of the globalisation theorem.

1. $\nu_0(\phi_{-t}A) \leqslant \nu_t(A)^{\cos(2t)^{\alpha(p,d)}}$ is the "quasi-invariance".
2. For any $\sigma > 0$, $\nu_t(\|u\|_{\mathcal{H}^{-\sigma}} > \lambda) \leqslant \mu(\|u\|_{\mathcal{H}^{-\sigma}} > \lambda) \leqslant e^{-c\lambda^{-2}}$.
3. Good LWP theory in $\mathcal{X}^0_{\mathrm{rad}}$?

Proposition (Lemma 4.23 (Section 5))

For μ-almost every $u_0 \in \mathcal{X}^0_{\mathrm{rad}}$ there exists a global solution $v = v_L + w \in v_L + \mathcal{C}^0(\mathbb{R}, \mathcal{H}^\sigma)$ to (HNLS), obeying

$$\|w(t)\|_{H^s}, \|w\|_{L^q_{[0,t]}\mathcal{W}^{s,r}} \lesssim \left(\frac{\pi}{4} - t\right)^{\frac{-\alpha(p,d)}{2}} \log^{\frac{1}{2}}\left(\frac{\pi}{4} - t\right).$$

when $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $s \leqslant \sigma$.

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Local theory in $\mathcal{X}^0_{\text{rad}}$

Affine ansatz: $v(t) = e^{-itH}u_0^\omega + v(t) = v_L(t) + w(t)$, where

$$i\partial_t w - Hw(t) = \cos(2t)^{-\alpha(p,d)}|v_L(t) + w(t)|^{p-1}(v_L(t) + w(t)).$$

Proposition (Proposition 4.3 (Section 3))

There exist $\sigma > 0$ and $T \in [0, \pi/4)$ such that local well-posedness for w holds in some space $Y^s \hookrightarrow \mathcal{C}^0([0, T], \mathcal{H}^\sigma)$.

The proof is a fixed-point argument on

$$\Phi(w)(t) = -i \int_0^t e^{-i(t-t')H} \cos(2t)^{-\alpha(p,d)} \Big(|v_L + w|^{p-1}(v_L + w)\Big) \, \mathrm{d}t'$$

in an *adapted* space $Y^\sigma = \bigcap_{(q,r):\frac{2}{q}+\frac{d}{r}=\frac{d}{2}} L^q_T \mathcal{W}^{s,\sigma}$

Difficulty of the local theory

Crude Bound in $L^\infty_T\mathcal{H}^\sigma$, and use $\cos(2t)^{-\alpha(p,d)} \lesssim (\pi/4 - T)^{-\alpha(p,d)}$.

$$\begin{aligned}\|\Phi(v)\|_{L^\infty_T\mathcal{H}^\sigma} &\lesssim_T \int_0^T \||v_L+w|^{p-1}(u_L+v)\|_{\mathcal{H}^s}\,\mathrm{d}t \\ &\lesssim_T \||v_L|^{p-1}v_L\|_{L^1_T\mathcal{H}^s} + \||w|^{p-1}w\|_{L^1_T\mathcal{H}^\sigma} + \underbrace{\{\text{ other terms.}\}}_{\text{Handled similarly}}\end{aligned}$$

- The term $\||w|^{p-1}w\|_{L^1_T\mathcal{H}^\sigma}$ may be controlled (for some p) by deterministic methods, since $w \in \mathcal{H}^\sigma$.
- $v_L \notin \mathcal{H}^\sigma$ so we have to control $|v_L|^{p-1}v_L$.

$\longrightarrow$ We use randomness in $v_L = \displaystyle\sum_{n\geqslant 0} e^{-it\lambda_n^2}\frac{g_n^\omega}{\lambda_n}e_n(x)$

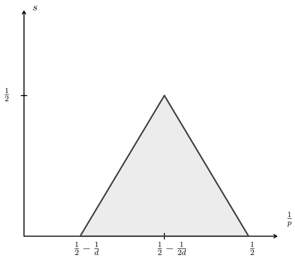
Random linear gain

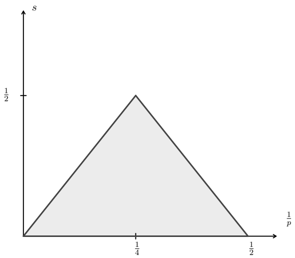
Lemma (Lemma 4.12 (Section 2))

Outside a set of probability at most $Ce^{-c\lambda^2}$ there holds

$$\left\| v_L \right\|_{L^q_T \mathcal{W}^{s_p^-,p}} \leqslant T^{\frac{1}{q}} \lambda \,,$$

where $s_p = \begin{cases} d\left(\frac{1}{2} - \frac{1}{p}\right) & \text{for } 2 \leqslant p < \frac{2d}{d-1} \\ 1 - d\left(\frac{1}{2} - \frac{1}{p}\right) & \text{for } \frac{2d}{d-1} \leqslant p \leqslant \frac{2d}{d-2} \,. \end{cases}$





Random linear gain: preparation of the proof

With the lemma, write $H^{\frac{\sigma}{2}}(|v_L|^{p-1}v_L) \simeq |v_L|^{p-1}H^{\frac{\sigma}{2}}v_L$ so that Hölder yields

$$\big\||v_L|^{p-1}v_L\big\|_{L^1_T\mathcal{H}^\sigma} \lesssim \|v_L\|^{p-1}_{L^{2(p-1)}{}_T L^{4(p-1)}} \|v_L\|_{L^2_T\mathcal{W}^{\sigma,4}} ,$$

when $(d,\sigma) = (2,2)$ this works, e.g.

For the proof we use:

Lemma (Kolmogorov, Paley, Zygmund)

For complex numbers $(a_n)_{n\geqslant 0}$,

$$\left\|\sum_{n\geqslant 2} a_n g_n^\omega\right\|_{L^r_\Omega} \lesssim \sqrt{r}\left(\sum_{n\geqslant 0}|a_n|^2\right)^{\frac{1}{2}} .$$

Random linear gain: proof (random decoupling)

For $s < s_p$, write $H^{\frac{s}{2}} v_L(t,x) = \sum_{n \geqslant 0} \lambda_n^s e^{-it\lambda_n^2} \frac{g_n^\omega}{\lambda_n} e_n(x)$. The lemma with $a_n = \lambda_n^{s-1} e^{-it\lambda_n^2}$ implies

$$\left\| \sum_{n \geqslant 0} \lambda_n^s e^{-it\lambda_n^2} \frac{g_n^\omega}{\lambda_n} e_n(x) \right\|_{L^a_\Omega} \lesssim \sqrt{a} \left(\sum_{n \geqslant 0} \lambda_n^{2(s-1)} |e_n(x)|^2 \right)^{\frac{1}{2}}$$

Combinend with Minkowski for $a \geqslant q, r$ we get

$$\begin{aligned} \left\| \sum_{n \geqslant 0} \lambda_n^s e^{-it\lambda_n^2} \frac{g_n^\omega}{\lambda_n} e_n(x) \right\|_{L^a_\Omega L^q_T L^r_x} &\lesssim \sqrt{a} \left\| \left(\sum_{n \geqslant 0} \lambda_n^{2(s-1)} |e_n(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^q_T L^r_x} \\ &\lesssim T^{\frac{1}{q}} \left(\sum_{n \geqslant 0} \lambda_n^{2(s-1)} \|e_n\|^2_{L^r} \right)^{\frac{1}{2}} \end{aligned}$$

Random linear gain: proof (deterministic estimate)

Finally we have obtained

$$\|u_L\|_{L^a_\Omega L^q_T \mathcal{W}^{s_r^-,r}} \lesssim T^{\frac{1}{q}} \left(\sum_{n \geqslant 0} \lambda_n^{2(s-1)} \|e_n\|^2_{L^r} \right)^{\frac{1}{2}} .$$

We conclude with deterministic estimates:

Lemma (Estimates of Hermite functions)

$$\|e_n\|_{L^r} \lesssim \begin{cases} \lambda_n^{-d(1/2-1/r)} & \text{for } 2 \leqslant r < \frac{2d}{d-1} \\ \lambda_n^{d(1/2-1/r)-1} & \text{for } \frac{2d}{d-1} < r \leqslant \frac{2d}{d-2} . \end{cases}$$

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A pointwise L^{p+1} bound

Lemma ((4.28) in Corollary 4.21)

Set $A_{t,\lambda} := \{u_0 \in X^0,\ \|\phi_t u_0\|_{L^{p+1}} > \lambda\}$ *then*

$$\nu_0(A_{t,\lambda}) \leqslant e^{-\frac{\lambda^{p+1}}{p+1}}.$$

Proof. Quasi-invariance brings $\nu_0(A_{t,\lambda}) \leqslant \nu_t(\phi_t A_{t,\lambda})^{\cos(2t)^{\alpha(p,d)}}$. For $u \in \phi_t A_{t,\lambda}$ one has $\|u\|_{L^{p+1}} > \lambda$.

$$\nu_0(A_{t,\lambda}) \leqslant \left(e^{-\frac{\lambda^{p+1}}{p+1}\cos(2t)^{-\alpha(p,d)}}\mu(\phi_t A_{t,\lambda})\right)^{\cos(2t)^{\alpha(p,d)}} \leqslant e^{-\frac{\lambda^{p+1}}{p+1}}.$$

We used that $\mathrm{d}\nu_t(u) = e^{-\frac{\|u\|_{L^{p+1}}^{p+1}}{p+1}\cos(2t)^{-\alpha(p,d)}}\mathrm{d}\mu(u)$.

Framework for L^{p+1} bounds

Proposition (Lemma 4.25, Section 5)

For almost-every $u_0 \in \mathcal{X}^0_{\mathrm{rad}}$, there holds

$$\|v(t)\|_{L^{p+1}} \lesssim \log^{\frac{1}{p+1}}\left(\frac{\pi}{4} - t\right) \text{ for all } t \in \left[0, \frac{\pi}{4}\right) .$$

Application of the globalisation argument on intervals $[0, t]$!

- Chop $[0, t]$ in intervals $[t_n, t_{n+1}]$, $n = 0, \dots, N$ with $t_n = n\tau$.
- For each time t_n we have $\|v(t_n)\|_{L^{p+1}} \leqslant \lambda$ for $u_0 \in U_{t_n}$ such that $\mu(\mathcal{X}^0_{\mathrm{rad}} \setminus U_{t_n}) \leqslant e^{-\frac{\lambda^{p+1}}{p+1}}$.
- ⚠ Not possible to take $\bigcup_{t\in[0,\frac{\pi}{4})} U_t$, not a countable union ⚠
- Check that with high probability for $t \in [t_n, t_{n+1}]$ $\|v(t) - t(t_n)\|_{L^{p+1}} \lesssim |t - t_n|^\gamma$.

Variation of L^{p+1} norms

Lemma

With "high probability" there holds $\|v(t_1) - v(t_2)\|_{L^{p+1}} \lesssim |t_1 - t_2|^\beta$.

Case (1): low values of p implies control by Sobolev norms and $L^\infty_t L^{p+1}_x$.

Case (2): higher values need an extra argument. We write

$$v(t_2) - v(t_1) = \underbrace{(e^{-i(t-t_1)H} - 1)v(t_1)}_{=I} \\ - i \underbrace{\int_{t_1}^{t} e^{-i(t-t')H}(\cos(2t')^{\alpha(p,d)}|v(t')|^{p-1}v(t'))\mathrm{d}t'}_{=II}$$

Treatment of II

Recall the dispersion estimate for $q \geqslant 2$,

$$\|e^{itH}v\|_{L^q} \lesssim t^{-d\left(\frac{1}{2}-\frac{1}{q}\right)}\|v\|_{L^{q'}} .$$

Applied with $q = p + 1$ we get By dispersion we have:

$$\begin{aligned}\|II\|_{L^{p+1}} &\lesssim \int_{t_1}^{t} \frac{\cos(2t')^{-\alpha(p,d)}}{|t-s|^{d\left(\frac{1}{2}-\frac{1}{p}\right)}}\|v(t')\|_{L^{p+1}}^p dt' \\ &\lesssim \left(\frac{\pi}{4}-t\right)^{-\alpha(p,d)} |t_2-t_1|^{\beta}\|v\|^p_{L^r_{[t_1,t_2]}L^{p+1}_x} ,\end{aligned}$$

Which is controlable if $\|v\|^p_{L^r_{[t_1,t_2]}L^{p+1}_x}$ is. Requires: $s > d\left(\frac{1}{2}-\frac{1}{p+1}\right)-\frac{2}{r}$.

Control of $\|v\|_{L^\infty L^{p+1}_x}$ by Y^s if $s > d\left(\frac{1}{2}-\frac{1}{p+1}\right)$

Treatment of I

Sobolev embedding in time:

$$\begin{aligned}\|(e^{-i(t-t_1)H}-1)v(t_1)\|_{L^\infty L^{p+1}} &\lesssim |t-t_1|^\beta \|e^{-i(t-t_1)}v(t_1)\|_{\mathcal{C}^{0,\beta}_t L^{p+1}}\\ &\lesssim |t_2-t_1|^\beta \|\|e^{-i(t-t_1)}v(t_1)\|_{W^{\varepsilon,q}_t L^{p+1}}\,.\end{aligned}$$

Then use that $|\partial_t|^\varepsilon e^{itH}v = H^\varepsilon e^{itH}v$ hence:

$$\|I\|_{L^\infty L^{p+1}} \lesssim \|e^{-i(t-t_1)H}v(t_1)\|_{L^q_t \mathcal{W}^{2\varepsilon,p+1}}\,,$$

and *random estimates* to finish.

Where does the Lens transform comes from?

Scaling invariance: $u(s,y) \mapsto \lambda^{-\frac{d}{2}} u\left(\frac{s}{\lambda^2}, \frac{y}{\lambda}\right)$.

When $\lambda = \lambda(s)$, we define new variables (t,x) such that $dx = \frac{dy}{\lambda(s)}$ and $dt = \frac{ds}{\lambda^2(s)}$, i.e., $t = \int_0^t \frac{ds}{\lambda^2(s)}$.

- Good way to compactify time is $t = \frac{1}{2}\arctan(2s)$ thus $\lambda = \cos(2s)$.
- New equation is $i\partial_t v + \Delta_x v - i\frac{\lambda'(t)}{\lambda(t)} x \cdot \nabla_x v + \cdots$. We gauge this last term by absoring it in an exponential.