Double Hölder Regularity for Incompressible Euler Equations

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March 2023, La Rochelle

The work presented has been done in collaboration with:
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The Incompressible Euler Equations in a domain

The setting is the following:

- $\Omega \subset \mathbb{R}^d$ a **bounded domain** with unit normal n. $\Omega \in C^{2^+}$, so $n \in C^{1^+}$
- Incompressible Euler Equations in Ω,

$$\partial_t u(t) + u(t) \cdot \nabla u(t) = -\nabla p(t)\,,$$

  where $u(t) : \Omega \to \mathbb{R}^d$ and $p(t) : \Omega \to \mathbb{R}$.
- Divergence free condition: $\operatorname{div} u(t) = 0$.
- Boundary conditions: $u(0) = u_0$ and $u(t) \cdot n|_{\partial\Omega} = 0$.

Equation of the pressure: interior equation

$$
\begin{aligned}
-\Delta p(t) &= \operatorname{div}(-\nabla p(t)) = \operatorname{div}(u(t)\cdot\nabla u(t)) \\
&= \partial_i(u^j(t)\partial_j u^i(t)) = \partial^2_{ij}(u^i(t)u^j(t)) \\
&= \operatorname{div}\operatorname{div}(u(t)\otimes u(t))
\end{aligned}
$$

In this talk: $u(t,x) = u(x)$ such that $\operatorname{div} u = 0$ (weakly) and $u\cdot n|_{\partial\Omega} = 0$. p is a solution of

$$-\Delta p = \operatorname{div}\operatorname{div}(u\otimes u)\,,$$

which makes sense whenever u is a (weakly) divergence-free function.

Equation of the pressure: boundary terms

Using that $\partial\Omega$ is a level set of $u \cdot n$,

$$\begin{aligned}\partial_n p &= \nabla p \cdot n = -(u \cdot \nabla u) \cdot n = -u^i \partial_i (u^j) n^j = -u_i \partial_i (u^j n^j) + u^i u^j \partial_i n^j \\ &= -u \cdot \nabla (u \cdot n) + u \otimes u : \nabla n = u \otimes u : \nabla n.\end{aligned}$$

Finally we are considering (weak) solutions of:

$$\left\{ \begin{array}{rcl} -\Delta p & = & \operatorname{div}\operatorname{div}(u \otimes u) \\ \partial_n p & = & u \otimes u : \nabla n\,. \end{array} \right.$$

Question: if $u \in C^\theta(\Omega)$, what is the regularity of p?

Why is that a relevant question?

Regularity of the Lagrangian trajectories: for $u(t)$ a solution to the Euler equation, consider

$$\begin{cases} \dot{X}(t,x) = u(t, X(t,x)) \\ X(0,x) = x \,. \end{cases}$$

Question: Trajectories regularity $u \in L^\infty_t C^\theta_x \Rightarrow X(\cdot, x) \in C^\infty_t$?

Answer : Chemin '92 for $u \in L^\infty_t C^{1,\varepsilon}_x(\mathbb{R}^d)$ (uniqueness class).
Isett '13 : $u \in L^\infty_t C^{1^-}_x(\mathbb{R}^d)$ (non-uniqueness class).

$$\frac{d^2}{dt^2} X(t,x) = \overbrace{(\partial_t + u \cdot \nabla)}^{\frac{D}{Dt} u} u(t, X(t,x)) = -\nabla p(t, X(t,x)).$$

$$p \in L^\infty_t C^\alpha_x \Rightarrow \frac{D}{Dt} u \in L^\infty_t C^{\alpha-1}_x \Rightarrow \frac{d^2}{dt^2} X(\cdot, x) \in C^{\alpha-1}_t \,.$$

Question : Obtain such results on bounded domains Ω.

Main result

Theorem (De Rosa, L., Stefani '23)

Let $\theta \in (0, \frac{1}{2})$ *and* $\Omega \in C^{2^+}$. *Let* u *be a (weakly) divergence-free vector field. Then:*

$$u \in C^{\theta}(\Omega) \Longrightarrow p \in C^{2\theta}(\Omega)\,.$$

1. Endpoint: $u \in C^{\frac{1}{2}}(\Omega) \Longrightarrow p \in C^1_*(\Omega)$.
2. Higher regularity $1/2 < \theta < 1$, $u \in C^{\theta}(\Omega) \Longrightarrow p \in C^{1,2\theta-1}(\Omega)$ (De Rosa, L., Stefani '22, much simpler).
3. Even $u \in C^{\theta}(\Omega) \Longrightarrow p \in C^{\theta}(\Omega)$ does **not** follow from standard elliptic regularity. [Bardos-Titi] used this result (and later proved it) to obtain the positive part of Onsager's conjecture in bounded domains.

Very high regularities (Used by Chemin for example)

The key observation is the computation:

$$\begin{aligned}\operatorname{div}\operatorname{div}(u\otimes u) &= \partial_{ij}(u^i u^j) = \partial_i(u^j\partial_j u^i + u^i\underbrace{\partial_j u^j}_{=0})\\ &= \partial_i(u^j\partial_j u^i)\\ &= \partial_i u^j\partial_j u^i\\ &= \nabla u : \nabla u\,.\end{aligned}$$

Makes sense if $u \in C^1$ for example.

So if $u \in C^{1+\varepsilon}$, $p = (-\Delta)^{-1}\operatorname{div}\operatorname{div}(u\otimes u) = (-\Delta)^{-1}(\underbrace{\nabla u : \nabla u}_{\in C^\varepsilon})$,

"Double" regularity: $\quad u \in C^{1+\varepsilon}(\mathbb{R}^d) \Longrightarrow p \in C^{2+\varepsilon}$

Argument on $\mathbb{R}^2$: [Silvestre '11] and [Isett '13]

We can use Fourier on the euclidean space!

Since

$$p = (-\Delta)^{-1} \operatorname{div} \operatorname{div} u \otimes u = (-\Delta)^{-1} \partial^2_{ij}(u^i u^j),$$

one has $\hat{p}(\xi) = -\frac{\xi^i \xi^j}{|\xi|^2} \widehat{u^i u^j}(\xi)$. With $a(\xi) = -\frac{\xi^i \xi^j}{|\xi|^2}$, $D = -i\nabla$,

$$p = a_0(D)(u \otimes u)$$

where $a(D)$ is an operator of order 0.

Lemma

If $u = \sum_{N \in 2^{\mathbb{N}}} u_N$, $\hat{u}_N(\xi) = \chi(\xi N^{-1})\hat{u}(\xi)$ is a Littlewood-Paley decomposition:

$$\|u\|_{C^\theta} \sim \|u\|_{C^\theta_*} = \|u\|_{B^\theta_{\infty,\infty}} = \sup_N N^\theta \|u_N\|_{L^\infty}.$$

Argument on $\mathbb{R}^2$: [Silvestre '11] and [Isett '13] continued

Write:

$$\begin{aligned} p_N = \mathbf{P}_N p = \mathbf{P}_N a_0(D) \sum_{M,K} u^i_M u^j_K \\ = \sum_{\max\{M,K\}\gtrsim N} \mathbf{P}_N a_0(D)(u^i_M u^j_K) \\ = \sum_{N\lesssim K\sim M} \mathbf{P}_N a_0(D)(u^i_M u^j_K) + \sum_{K\ll M\sim N} \mathbf{P}_N a_0(D)(u^i_M u^j_K) = p^{(1)}_N + p^{(2)}_N \end{aligned}$$

Lemma (Shur test)

$\mathbf{P}_N a_0(D) : L^\infty \to L^\infty$ is continuous (uniformly in N).

$$\|p^{(1)}_N\|_{L^\infty} \lesssim \sum_{K\sim M\gtrsim N} \|u_K\|_{L^\infty}\|u_M\|_{L^\infty} \lesssim \sum_{M\gtrsim N} M^{-2\theta} \lesssim N^{-2\theta}\,.$$

Argument on $\mathbb{R}^2$: [Silvestre '11] and [Isett '13] end

To prove $\|p_N^{(2)}\|_{L^\infty} \lesssim N^{-2\theta}$ we use the **oserved fact**

$$\operatorname{div}\operatorname{div}(u_M \otimes u_K) = \nabla u_M : \nabla u_K \,,$$

and $\|\mathbf{P}_N(-\Delta)^{-1}\|_{L^\infty \to L^\infty} \lesssim N^{-2}$ so that

$$\begin{aligned} \|p_N^{(2)}\|_{L^\infty} &\lesssim \sum_{K \ll M \sim N} \|\mathbf{P}_N(-\Delta)^{-1}(\nabla u_M : \nabla u_K)\|_{L^\infty} \\ &\lesssim N^{-2} \sum_{K \ll M \sim N} K^{1-\theta} M^{1-\theta} \lesssim N^{-2\theta} \,. \end{aligned}$$

Question: Generalization to Ω?

1. Physical proof in $\mathbb{R}^d$ exist (Silvestre '11, Colombo & De Rosa '18): extend to Ω but do not give optimal result (De Rosa, L., Stefani '22)
2. Result is only **local** near the boundary and true for half-space.

First step: extending using boundary conditions wisely (1)

$$p = p_{int} + p_{bounday}$$

$$\partial_n p = u \otimes u : \nabla n \in C^{0^+} \implies p_{boundary} \in C^{1^+} \subset C^{2\theta}$$

The boundary term is not the main issue! p_{int} equation:

$$\begin{cases} -\Delta p & = & \operatorname{div}\operatorname{div}(u \otimes u) + C \text{ in } B_R(x_0) \cap \Omega \\ \partial_n p & = & 0 \text{ on } B_R(x_0) \cap \partial\Omega \end{cases}$$

Straightening of the boundary:

$$\begin{cases} -\partial_i (g^{ij} \sqrt{\det g} \partial_j p) & = & \partial^2_{ij} (\sqrt{\det g} u^i u^j) + C \text{ for } r > 0 \cap \Omega \\ \partial_n p & = & 0 \text{ on } B_R(x_0) \cap \partial\Omega \text{ for } r = 0. \end{cases}$$

Attention: this change of variable is adapted to the geometric situation!

Geometric change of variable: picture

First step: extending using boundary conditions wisely (2)

$u_r(0,\theta) = 0$ and $\partial_r p(0,\theta) = 0$

Extensions: u_θ and p *evenly*, u_r *oddly*.

$\tilde{u}_r, \partial_r \tilde{p}$ are continuous extensions.

Proposition

The equation satisfied is (here $v = \sqrt{\det g}\, u$):

$$-\partial_i(g^{ij}\sqrt{\det g}\,\partial_j q) = \partial^2_{ij}(a v^i v^j) \text{ on } \mathbb{R}^d\,, \tag{1}$$

$$\partial_i v^i = \partial_i(\sqrt{\det g}\, u^i) = \operatorname{div} u = 0\,,$$

$$a = a(r,\theta) \in \mathrm{Lip}\,.$$

Theorem

(1) *satisfies* $v \in C^\theta(\mathbb{R}^d) \Longrightarrow q \in C^{2\theta}(\mathbb{R}^d)$.

Second step: Non-smooth pseudodifferential operators

$$-\partial_i(g^{ij}\sqrt{\det g}\partial_j q) = \partial^2_{ij}(av^i v^j) \Longleftrightarrow E_2 q = \partial^2_{ij}(av^i v^j)$$

where $E_2 = \partial_i \circ E_{1,i} = i\partial_i \circ \mathrm{Op}(g^{ij(x)}\sqrt{g(x)}\xi^j)$ is an **elliptic operator.**

Lemma

$\partial^2_{ij}(a_L v^i_M v^j_K) = a_L \partial_j u^i_M \partial_i v^j_K + \partial_j a_L v^i_M \partial_i v^j_K + \partial_j(a_L v^i_M \partial_i v^j_K).$

If we could write

$$q = \left(\sum_{\substack{M \sim K \\ L \ll M,K}} + \sum_{L \gtrsim M,K} \right) E_2^{-1} \circ \partial^2_{ij}(a_L v^i_M v^j_K)$$
$$\sum_{\substack{K \gg M \\ L \ll M}} E_2^{-1} \circ \left(a_L \partial_j u^i_M \partial_i v^j_K + \partial_j a_L v^i_M \partial_i v^j_K + \partial_j(a_L v^i_M \partial_i v^j_K) \right)$$

we could just do a variant of the LP proof in the euclidean case.

Second step: Problems

1. $E_2 \in \mathrm{Op}(\mathrm{Lip}_x\, S^2)$, *i.e.* order 2 but limited spatial regularity.
2. One cannot compose / invert / have good continuity bounds $C^s_* \to C^t_*$ for such operators.

This is because we work at low spatial regulaity. The extension is responsible for this Lip regularity. Same happens on C^∞ domains.

1. Define $E_{1,i} = E^\sharp_{1,i} + E^\flat_{1-\delta,i}$ where

$$E^\sharp_{1,i} \in \mathrm{Op}(S^1_{1,\delta}) \text{ smooth, and } E^\sharp_{1-\delta,i} \text{ of lower order.}$$

$$A = \mathrm{Op}(a(x,\xi) \Longrightarrow a^\sharp(x,\xi) = \sum_{M \leqslant N^\delta} a_M(x,\xi)\mathbf{P}_N(\xi)$$

2. $\partial_i \circ E^\sharp_{1,i} = E^\sharp_2 \in \mathrm{Op}(S^2_{1,\delta})$ elliptic.
3. Write
$E^\sharp_2 q = -E_{2-\delta} q + [\cdots] \Longrightarrow q = -(E^\sharp_2)^{-1} \circ E_{2-\delta} q + (E^\sharp_2)^{-1}[\cdots]$

Thank you!

Questions?