High-Regularity Invariant Measures for Euler Equations and Growth of the Solutions

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Euler Equations, $d \in \{2,3\}$

$u(t) : \mathbb{T}^d \longrightarrow \mathbb{R}^d$, $p(t) : \mathbb{T}^d \longrightarrow \mathbb{R}$ satisfying

$$\begin{cases} \partial_t u + u \cdot \nabla u & = & -\nabla p \\ \nabla \cdot u & = & 0 \\ u(0) & = & u_0 \in H^s(\mathbb{T}^d)\,. \end{cases}$$

Leray projector: $\mathbf{P} : L^2 \to L^2_{div}$,

$$\begin{cases} \partial_t u + B(u,u) & = & 0 \\ u(0) & = & u_0 \in H^s_{\mathrm{div}}(\mathbb{T}^d)\,. \end{cases}$$

with $B(u,v) := \mathbf{P}\,(u \cdot \nabla v)$.

$$-\Delta p = \nabla \cdot (u \cdot \nabla u)\,.$$

Deterministic results ($d = 2$)

Global well-posedness in H^s holds for large s.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{H^s}^2 = -(B(u,u),u)_{H^s}\,.$$

Modulo lower order error terms:

$$\begin{aligned}(B(u,u),u)_{H^s} &\sim (\nabla^s(u\cdot\nabla u),\nabla^s u)_{L^2}\\ &\sim (u\cdot\nabla\nabla^s u,\nabla^s u)_{L^2} + (\nabla u\cdot\nabla^s u,\nabla^s u)_{L^2}\,.\end{aligned}$$

Property of B: $(B(u,v),v)_{L^2} = 0$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{H^s}^2 \lesssim \underbrace{\|\nabla u\|_{L^\infty}}_{?}\|u(t)\|_{H^s}^2\,.$$

With interpolation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{H^s}^2 \lesssim \|\nabla u(t)\|_{L^p}\|u(t)\|_{H^s}^{2+\frac{2}{p}}\,.$$

Deterministic results ($d = 2$)

Remark: $\Omega(t) = \nabla \wedge u(t)$ satisfies a transport equation

$$\partial_t \Omega + u \cdot \nabla \Omega = 0\,.$$

Calderón-Zygmund: $\|\nabla u(t)\|_{L^p} \lesssim p\|\Omega(t)\|_{L^p} \lesssim p\|\Omega_0\|_{L^p}$.

With $y(t) := \|u(t)\|_{H^s}$ we get:

$$y'(t) \leqslant Cpy(t)^{1+\frac{2}{p}} \leqslant Cy(t)\log y(t)\,,$$

in the end we obtain $\|u(t)\|_{H^s} \leqslant Ce^{Ce^{Ct}}$.

Theorem (Matching lower bounds)

(Kiselev-Šverak) In a disc, double exponential growth can occur.
(Zlatoš) Best result on the torus: exponential growth on arbitrary long time.

Conjecture. Almost-surely double exponential growth does not occur.

Main result

Flow ϕ_t of a PDE: $u(t) = \phi_t u_0$. μ is invariant on H^s if for any measurable sets $A \subset H^s$, $\mu(\phi_{-t}A) = \mu(A)$ that is:

$$\mathbb{P}(u(t) \in A) = \mathbb{P}(u_0 \in A)\,.$$

Theorem

For $s > 2$ $(d = 2)$ and $s > 7/2$ $(d = 3)$ there exists measure μ_s on H^s, invariant under the flow of the Euler equation and such that:

1. *Almost sure global well-posedness holds on the support of the measure. (Thm 5.2/5.5)*
2. *Growth estimates of the form $\|u(t)\|_{H^s} \lesssim t^{\alpha(s)}$. (Thm. 5.9/5.11)*

Similar results in other settings : Sy (on NLS), Sy-Flödes (on SQG).

Invariant measure as a tool of globalisation

1. **Invariant measure + LWP $\Longrightarrow$ slowly growing global solutions.**
2. Construction of invariant measures.
3. Properties of such measures.

Growth of Sobolev Norms with Invariant Measures

Assumptions.

1. μ is invariant on H^s;
2. $\mu(\|u\|_{H^s} > \lambda) \lesssim e^{-c\lambda^2}$;
3. Good LWP theory: if $\|u_0\|_{H^s} \leqslant \lambda$ then a solution exists on $[0, \tau]$, $\tau \sim \frac{1}{\lambda^K}$ and $\|u(t)\|_{H^s} \leqslant 2\lambda$ for $t \leqslant \tau$.

Proposition

Then for μ-almost every $u_0 \in H^s$ there exists a global solution u such that:

$$\|u(t)\|_{H^s} \lesssim \log^{\frac{1}{2}}(t)\,.$$

Growth of Sobolev Norms with Invariant Measures

Assumptions. Really flexible, except **(3)**.

1. $\mu(\phi_{-t}A) \leqslant \mu(A)$ "quasi-invariance".
2. $\mathbb{E}_\mu[\|u\|_{H^s}^2] < \infty$, hence $\mu(\|u\|_{H^s} > \lambda) \lesssim \lambda^{-2}$.
3. Good LWP theory: if $\|u_0\|_{H^s} \leqslant \lambda$ then a solution exists on $[0, \tau]$, $\tau \sim \frac{1}{\lambda^K}$ and $\|u(t)\|_{H^s} \leqslant 2\lambda$ for $t \leqslant \tau$.

Proposition (Section 5 (if $d = 2$ a global flow is already known))

Then for μ-almost every $u_0 \in H^s$ there exists a global solution u such that:

$$\|u(t)\|_{H^s} \lesssim F(t)\,.$$

Conservation Laws and Invariant Measures

For which systems is it easy to construct invariant measures?

In finite dimension. A good class of equations: Hamiltonian equations. Let:

$$\partial_t p = -\partial_q H(p,q) \quad \partial_t q = \partial_p H(p,q)\,,$$

Then $d\mu = \underbrace{Z^{-1}}_{Renorm.Cst.} e^{-H(p,q)}\mathrm{d}p\mathrm{d}q$ is invariant, because:

1. H is conserved by the flow.
2. $\mathrm{d}p\mathrm{d}q$ is conserved (Liouville).

Conservation Laws and Invariant Measures

Two-dimensional case. Let $\Omega(x) = \sum_n \Omega_n e^{in\cdot x}$:

1. Conservation law for Euler in $2d$: $E = \|\Omega\|_{L^2}^2 = \sum_n |\Omega_n|^2$.
2. Guessing invariant measure:

$$d\mu(u) = Ce^{-\sum_n \Omega_n^2} \prod_n d\Omega_n$$

3. μ is the law of the random variable:

$$\Omega^\omega(x) = \sum_n g_n(\omega) e^{in\cdot x} .$$

4. Issue: $\mathrm{supp}(\mu) \sim \{\Omega \in H^{-1-}\} \sim \{u \in H^{0-}\}$. (See Flandoli).

Hyper-dissipative method (1)

Hyper-dissipative stochastic regularization (HVE_ν^2) and (HVE_ν^3)

$$\begin{cases} \partial_t u_\nu + B(u_\nu, u_\nu) + \nu L u_\nu & = \sqrt{\nu}\eta \\ u_\nu(0) & = u_0 \in H^s_{\mathrm{div}}(\mathbb{T}^d)\,, \end{cases}$$

with $L = (-\Delta)^{1+\delta}$ and $\eta = \partial_t \xi$ with:

$$\xi(t) = \sum_n \phi_n \beta_n(t) e^{in\cdot x}\,,$$

white in time ($\beta_n \sim$ brownian), smooth in space.

$$\sum |n|^{2k} |\phi_n|^2 =: \mathcal{B}_k\,.$$

Stochastic Navier-Stokes $\longrightarrow \delta = 0$.

Scaling balance between νL and $\sqrt{\nu}\eta$

Hyper-dissipative method (2)

Follwing Kuksin:

1. Construct invariant measures μ_ν to the approximate equations

$$\partial_t u_\nu + B(u_\nu, u_\nu) + \nu L u_\nu = \sqrt{\nu}\eta\,,$$

   using a Krylov-Bogolyubov argument. (Section

2. Prove uniform estimates for the μ_ν.

3. Pass to the limit $\nu \to 0$.

Hyper-viscous method (2)

Follwing Kuksin:

1. Construct invariant measures μ_ν to the approximate equations

$$\partial_t u_\nu + B(u_\nu, u_\nu) + \nu L u_\nu = \sqrt{\nu}\eta\,,$$

   using a Krylov-Bogolyubov argument.

   Classical μ_ν invariant in H^1: Section 3 for solutions, Section 4.1/4.2

2. Prove uniform estimates for the μ_ν.

   Crucial step: Proposition 5.3/5.4

3. Pass to the limit $\nu \to 0$.

   Easy once one has uniform estimate: Section 4.3

Proving uniform estimates: Itô formula

$$u_\nu(t) = u_0 + \int_0^t (-\nu L u_\nu - B(u_\nu, u_\nu))\,\mathrm{d}t' + \sqrt{\nu}\sum_n \phi_n \beta_n(t) e_n\,,$$

then for any functional F:

$$\begin{aligned}\mathbb{E}[F(u_\nu(t)] - \mathbb{E}[F(u_0)] &= -\nu \int_0^t \mathbb{E}[F'(u_\nu; Lu_\nu)\,\mathrm{d}t' \\ &\quad - \int_0^t \mathbb{E}[F'(u_\nu; B(u_\nu, u_\nu)]\mathrm{d}t' \\ &\quad + \frac{\nu}{2}\int_0^t \sum_n |\phi_n|^2 \mathbb{E}[F''(u; e_n, e_n)]\,\mathrm{d}t'\,.\end{aligned}$$

Dimension 2

Itô formua to $F(u) = \|u\|_{H^1}^2$ gives

$$\mathbb{E}\left[\|u_\nu(t)\|_{H^1}^2\right] - \mathbb{E}\left[\|u_0\|_{H^1}^2\right] = \int_0^t ((I) + (II))\, \mathrm{d}t',$$

where (I) is the deterministic contribution:

$$(I) := -\mathbb{E}\left[(B(u_\nu, u_\nu), u_\nu)_{H^1} + \nu(Lu_\nu, u_\nu)_{H^1}\right],$$

and (II) is the Itô correction: $(II) := \frac{\nu}{2}\sum_n |n|^2|\phi_n|^2 =: \frac{\nu}{2}\mathcal{B}_1$.

1. Algebraic cancellation: $(B(u,u),u)_{H^1} = 0$. (Dimension 2 used here!)
2. $\nu(Lu, u)_{H^1} = \nu\|u\|_{H^{2+\delta/2}}^2$.
3. Measure invariance $\mathbb{E}\left[\|u_\nu(t)\|_{H^1}^2\right] = \mathbb{E}\left[\|u_0\|_{H^1}^2\right]$.

Dimension 2 (Cont.)

From this:

$$\int_0^t \left(\nu \mathbb{E}[\|u_\nu(t')\|^2_{H^{2+\delta}}] - \nu \frac{\mathcal{B}_1}{2}\right) \mathrm{d}t' = \mathbb{E}\int_0^t (B(u_\nu, u_\nu), u_\nu)_{H^1}\,\mathrm{d}t' = 0\,,$$

thus

$$\mathbb{E}\left[\|u_\nu(t)\|^2_{H^{2+\delta}}\right] = \frac{\mathcal{B}_1}{2} \text{ independently of } \nu\,.$$

Obtained:

- Measures μ_ν, invariant for $(HVE)_\nu$.
- Uniform estimate $\mathbb{E}_{\mu_\nu}[\|u\|^2_{H^{2+\delta}}] = \frac{\mathcal{B}_1}{2}$.
- $2+\delta > 2$: local theory of the form $\tau_{LWP} \sim \lambda^{-K}$ is available.

Possible to pass to the limit.

Summary (dimension 2)

Obtained:

- Measure μ, invariant for the Euler equations.
- $\mathbb{E}_{\mu_\nu}[\|u\|^2_{H^{2+\delta}}] \leqslant \frac{\mathcal{B}_1}{2}$, in particular $\mu(H^{2+\delta}) = 1$.
- Almost-surely $\|u(t)\|_{H^s} \lesssim t^{\alpha(s)}$ (C.f. Bourgain's globalisation argument).

Dimension 3 case

- Key algebraic cancellation: only in L^2, $(B(u,v),v)_{L^2}=0$: Itô to $F(u)=\|u\|_{L^2}^2$:

$$\mathbb{E}_{\mu_\nu}[\|u\|^2_{H^{1+\delta}}]=\frac{\mathcal{B}_0}{2}\,.$$

- Leads to invariant measure μ, $\mu(H^{1+\delta})=1$ and $\mathbb{E}_\mu[\|u\|^2_{H^{1+\delta}}]\leqslant\frac{\mathcal{B}_0}{2}$. $1+\delta>\frac{3}{2}+1$
- Further approximation problem: $\mu=\mu_N$ for approximate equation (truncation in low frequencies)

$$\partial_t u_N+B_N(u_N,u_N)=0\,,$$

$u_N(0)=\mathbf{P}_{\leqslant N}u_0$ and $B_N(u,v)=\mathbf{P}_{\leqslant N}B(u,v)$.

- Again uniform estimate $\mathbb{E}_{\mu_N}[\|u\|_{H^{1+\delta}}]\leqslant\frac{\mathcal{B}_0}{2}$ and $N\to\infty$.

Is it finished?

1. Invariant measure + LWP $\Longrightarrow$ slowly growing global solutions. ✔
2. Construction of invariant measures in H^s, $s > \frac{d}{2} + 1$. ✔
   - $\mu(H^{2+\delta}) = 1$ ✔
   - $\mathbb{E}_\mu[\|u\|^2_{H^{2+\delta}}] < \infty$ ✔
   - $2 + \delta > \frac{d}{2} + 1$ ✔
3. **Properties of such measures. Question: what does μ look like?**

Properties of the invariant measures: *partial* answer

Proposition ($d=2$ Thm 5.4, see Section 6)

The measure μ constructed satisfies:

$$\mu(u\,, \|u\|_{L^2} \in \Gamma) \lesssim |\Gamma|\,,$$

in particular μ does not possess any atom.

Proposition ($d=3$ Thm. 5.6, see Section 6)

The measure μ constructed satisfies the following alternative:

1. *μ does not have any atom, and for* $\mathrm{dist}(\Gamma, 0) > \varepsilon$*,*

$$\mu(u\,, \|u\|_{L^2} \in \Gamma) \lesssim_\varepsilon |\Gamma|\,.$$

2. $\mu = \delta_0$

What precludes the $\mu = \delta_0$ case in dimension 2?

- Algebraic miracle $(B(u,u),u)_{L^2} = 0 \Longrightarrow \mathbb{E}_{\mu_\nu}[\|u\|^2_{H^{1+\delta}}] = \frac{\mathcal{B}_0}{2}$.
- Algebraic miracle $(B(u,u),u)_{H^1} = 0 \Longrightarrow \mathbb{E}_{\mu_\nu}[\|u\|^2_{H^{2+\delta}}] = \frac{\mathcal{B}_1}{2}$.
- Weak convergence $\mu_\nu \to \mu$ in $H^{2+\delta-\varepsilon}$.

Expect that $\mathbb{E}_{\mu_\nu}[\|u\|^2_{H^{1+\delta}}] \to \mathbb{E}_{\mu}[\|u\|^2_{H^{1+\delta}}] = \frac{\mathcal{B}_0}{2}$, precluding δ_0.

Only have $\mathbb{E}_{\mu_\nu}[\|u\|^2_{H^{1+\delta}} \mathbf{1}_{\|u\|_{H^{1+\delta}} \leqslant R}] \xrightarrow[\nu \to 0]{} \mathbb{E}_{\mu}[\|u\|^2_{H^{1+\delta}} \mathbf{1}_{\|u\|_{H^{1+\delta}} \leqslant R}]$.

Proposition (Prop 5.6 (iv))

The correct limiting statement is:

$$\mathbb{E}_{\mu}[\|u\|^2_{H^1}] \geqslant C(\mathcal{B}_0, \mathcal{B}_1, \delta) > 0\,.$$

What precludes the $\mu = \delta_0$ case in dimension 2?

Proposition (Prop 5.6 (iv))

$$\mathbb{E}_\mu[\|u\|_{H^1}^2] \geqslant C(\mathcal{B}_0, \mathcal{B}_1, \delta) > 0\,.$$

Proof.

- By Itô + H^1 algebraic miracle: $\mathbb{E}_{\mu_\nu}\left[e^{c\|u\|_{H^1}^2}\right] \leqslant C < \infty$.
- Implies $\mathbb{E}_{\mu_\nu}[\|u\|_{H^1}^2 \mathbf{1}_{\|u\|_{H^1} > R}] \leqslant R^{-2}\mathbb{E}_{\mu_\nu}[\|u\|_{H^1}^4] \leqslant CR^{-2}$.

This gives

$$\begin{aligned}\lim_{\nu\to 0} \mathbb{E}_{\mu_\nu}[\|u\|_{H^1}^2] &= \lim_{R\to\infty}\lim_{\nu\to 0} \mathbb{E}_{\mu_\nu}[\|u\|_{H^1}^2 \mathbf{1}_{\|u\|_{H^1} > R}] \\ &= \lim_{R\to\infty} \mathbb{E}_{\mu}[\|u\|_{H^1}^2 \mathbf{1}_{\|u\|_{H^1} > R}] = \mathbb{E}_\mu[\|u\|_{H^1}^2]\,.\end{aligned}$$

Last step: $\frac{\mathcal{B}_0}{2} = \mathbb{E}_{\mu_\nu}[\|u\|_{H^{1+\delta}}^2] \leqslant \mathbb{E}_{\mu_\nu}[\|u\|_{H^1}^2]^{1-\theta}\,\underbrace{\mathbb{E}_{\mu_\nu}[\|u\|_{H^{2+\delta}}^2]^\theta}_{=\mathcal{B}_1/2}$.

Proof of no other atom than 0

Let Γ be such that $\mathrm{dist}(\Gamma, 0) > r$. Reduces to:

Proposition (Prop. 5.9)

There holds: $\mu_\nu(u, \|u\|_{L^2} \in \Gamma) \leqslant C(r)|\Gamma|$.

Probability detour. For processes $y(t) = y_0 + \int_0^t x(s)\,\mathrm{d}s + \sum_n \phi_n \beta_n(t)$, there exists a "local time" field $\Lambda_t^\omega(a)$ such that:

$$\frac{\mathcal{B}_0}{2} \int_0^t \mathbf{1}_\Gamma(y(t'))\,\mathrm{d}t' = \int_\Gamma \Lambda_t^\omega(a)\,\mathrm{d}a\,.$$

- $\int_\Gamma \mathbb{E}[\Lambda_t(a)]\,\mathrm{d}a = \frac{t\mathcal{B}_0}{2}\mathbb{E}[\mathbf{1}_\Gamma(y_0)]$.
- $\mathbb{E}[\Lambda_t(a)] = -t\mathbb{E}[\mathbf{1}_{[a,\infty)}(y_0)x_0]$ (From Itô formula)

Proof of no other atom than 0 (end)

Uniform control on $\mathbb{E}_{\mu_\nu}[\|u\|^2_{H^{1+\delta}}]$ imply:

$$\mathbb{E}_{\mu_\nu}\left[\mathbf{1}_\Gamma(\|u\|_{L^2})\sum_n |\phi_n|^2|u_n|^2\right] \lesssim |\Gamma| \text{ where } u_n = (u, e_n)_{L^2}\,.$$

Goal: lower bound on $\sum_n |\phi_n|^2|u_n|^2$.

Remark: let $\Omega_\varepsilon = \{v, \|v\|_{L^2} \leqslant \varepsilon \text{ or } \|v\|_{H^{1+\delta}} \geqslant \varepsilon^{-1/2}\}$, then for $u \in \Omega_\varepsilon^c$:

$$\sum_n |\phi_n|^2|u_n|^2 \geqslant \kappa(\varepsilon)\,.$$

Let $\varepsilon < r$ so that $\{\|u\|_{L^2} \in \Gamma\} \cap \Omega_\varepsilon \subset \{\|u\|_{H^{1+\delta}} > \varepsilon^{-1/2}\}$,

$$\mu_\nu(\{\|u\|_{L^2} \in \Gamma\} \cap \Omega_\varepsilon) \lesssim \varepsilon\,.$$

Finally

$$\begin{aligned}\mu_\nu(\|u\|_{L^2} \in \Gamma) &= \mu_\nu(\{\|u\|_{L^2} \in \Gamma\} \cap \Omega_\varepsilon) + \mu_\nu(\{\|u\|_{L^2} \in \Gamma\} \cap \Omega_\varepsilon^c)\\ &\lesssim \varepsilon + \kappa(\varepsilon)^{-1}|\Gamma|\,.\end{aligned}$$

μ_ν construction: Krylov-Bogolyubov method (1/2)

Given $\lambda_t = \mathcal{L}(u_\nu(t))$ where $\mathcal{L}(u_0) = \delta_0$, existence of the invariant measure comes if:

1. The measure $\bar{\lambda}_t := \frac{1}{t}\int_0^t \lambda_{t'}\,\mathrm{d}t'$ are uniformly tight in H^1.
2. Some continuity of the trajectories (this is OK).

Local and global theory Use $e^{-tL} : H^s \to H^{s+2(1+\delta)}$.

$$u_\nu(t) = (\text{Linear evol. of } \xi(t)) + (\text{Smoother Deterministic Term})$$

- Subcritical space for $s > \frac{d}{2} + 1 - 2(1+\delta) = -2\delta$: we have LWP in H^1.
- Global theory: apply deterministic arguments, works since Navier-Stokes is globally well-posed in H^1 in dimension 2.

Krylov-Bogolyubov method (2/2)

(1) By compactness of $H^1 \to H^{2+\delta}$, reduces to:

$$\sup_{t>0} \bar{\lambda}_t(u, \|u\|_{H^{2+\delta}} > R) \underset{R\to\infty}{\longrightarrow} 0\,.$$

$$\begin{aligned}\bar{\lambda}_t(u, \|u\|_{H^{2+\delta}} > R) &\leqslant \frac{1}{t}\int_0^t \mathbb{P}(\|u_\nu(t')\|_{H^{2+\delta}} > R)\,\mathrm{d}t' \\ &\leqslant \frac{C}{tR^2}\int_0^t \mathbb{E}\left[\|u_\nu(t')\|^2_{H^{2+\delta}}\right]\,\mathrm{d}t'\end{aligned}$$

Itô formula:

$$\mathbb{E}[\|u_\nu(t)\|^2_{H^1}] - \underbrace{\mathbb{E}[\|u_0\|^2_{H^1}]}_{=0} + \nu\int_0^t \mathbb{E}[\|u_\nu(t')\|^2_{H^1}]\,\mathrm{d}t' = \nu\mathcal{B}_1 t\,,$$

this gives

$$\bar{\lambda}_t(\|u\|_{H^{2+\delta}} > R) \leqslant \frac{C}{R^2}\,.$$

No concentration around zero (1/4)

Remark. (1) results from the fact that the measures μ_ν do not possess any atom and the uniform estimate

$$\mu_\nu(0 < \|u\|_{L^2} \leqslant \varepsilon) \lesssim \varepsilon\,.$$

What *should* be used in the proof:

1. Invariance of the measure μ;
2. Some properties of the bilinear form $B(u,u) = \mathbb{P}u \cdot \nabla u$;
3. The high regularity support of the measure.

Better to estimate $\mathbb{P}(a \leqslant \|u\|_{L^2} \leqslant \varepsilon) \lesssim \varepsilon$, by writing:

$$\mathbb{P}(0 < \|u\|_{L^2} \leqslant \varepsilon) = \lim_{\beta \to 0} \frac{1}{\beta} \int_0^\beta \mathbb{P}(a < \|u\|_{L^2} \leqslant \varepsilon)\,\mathrm{d}a\,,$$

hence we need:

$$\frac{1}{\varepsilon\beta}\mathbb{E}\int_0^\beta \mathbf{1}_{(a,\varepsilon)}(\|u\|_{L^2})\,\mathrm{d}a \lesssim 1\,.$$

No concentration around zero (2/4)

Starting point:

$$\frac{1}{\varepsilon\beta}\mathbb{E}\int_0^\beta \mathbf{1}_{(a,\varepsilon)}(\|u\|_{L^2})\,\mathrm{d}a \lesssim \frac{1}{\beta}\int_0^\beta \frac{\mathbf{1}_{(a,\varepsilon)}(\|u\|_{L^2})}{\|u\|_{L^2}}\,\mathrm{d}a\,.$$

Tool. Local time for martingales and Itô formulas:

$$\mathbb{E}_{\mu_\nu}\left[\int_0^\beta \mathbf{1}_{(a,\infty)}(\|u\|_{L^2}^2)A(u)\,\mathrm{d}a\right] + \sum_n |\phi_n|^2\mathbb{E}_{\mu_\nu}\left[\mathbf{1}_{(0,\beta)}(g(\|u\|_{L^2}^2))(g'(\|u\|_{L^2}^2)|u_n|)^2\right] = 0$$

$A(u) := g'(\|u\|_{L^2}^2)\left(\frac{\mathcal{B}_0}{2} - \|\nabla u\|_{L^2}^2\right) + g''(\|u\|_{L^2}^2)\sum_n |\phi_n|^2|u_n|^2$ Properties $B(u,u)$ and invariance is used.

No concentration around zero (3/4)

Take $g(x) = \sqrt{x}$ and get:

$$\mathbb{E}_{\mu_\nu}\left[\int_0^\beta \mathbf{1}_{(a,\infty)}(\|u\|_{L^2}^2) A(u)\,\mathrm{d}a\right] \leqslant 0\,,$$

that is:

$$\mathbb{E}_{\mu_\nu}\left[\int_0^\beta \frac{\mathbf{1}_{(a,\infty)(\|u\|_{L^2})}}{\|u\|_{L^2}^3}\left(\mathcal{B}_0\|u\|_{L^2}^2 - \sum_n |\phi_n|^2|u_n|^2\right)\mathrm{d}a\right] \leqslant \int_0^\beta \mathbb{E}_{\mu_\nu}\left[\frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2}\right]\mathrm{d}a\,.$$

- $\mathcal{B}_0\|u\|_{L^2}^2 - \sum_n |\phi_n|^2|u_n|^2 \gtrsim \|u\|_{L^2}^2$;
- High regularity: $\|\nabla u\|_{L^2}^2 \lesssim \|u\|_{L^2}\|\nabla^2 u\|_{L^2}$ and

$$\mathbb{E}_{\mu_\nu}[\|u\|_{H^2}] \lesssim 1\,.$$

No concentration around zero (4/4)

We summarize:

$$
\begin{aligned}
\frac{1}{\varepsilon}\mathbb{E}\int_0^\beta \mathbf{1}_{(a,\varepsilon)}(\|u\|_{L^2})\,\mathrm{d}a &\lesssim \mathbb{E}\int_0^\beta \frac{\mathbf{1}_{(a,\varepsilon)}(\|u\|_{L^2})}{\|u\|_{L^2}^2}\,\mathrm{d}a \\
&\lesssim \mathbb{E}_{\mu_\nu}\left[\int_0^\beta \frac{\mathbf{1}_{(a,\infty)(\|u\|_{L^2})}}{\|u\|_{L^2}^3}\left(\mathcal{B}_0\|u\|_{L^2}^2 - \sum_n |\phi_n|^2|u_n|^2\right)\mathrm{d}a\right] \\
&\lesssim \int_0^\beta \mathbb{E}_{\mu_\nu}\left[\frac{\|\nabla u\|^2}{\|u\|_{L^2}}\right]\mathrm{d}a \\
&\lesssim \beta\,.
\end{aligned}
$$