**Quasi-geostrophic equations, nonlinear Bernstein inequalities and $\alpha$-stable processes.**

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**Abstract :** We prove some functional inequalities for the fractional differentiation operator $(-\Delta)^\alpha$ through the formalism of semi-groups. This gives us an estimate of the regularity of Marchand's weak solutions for the dissipative quasi–geostrophic equation.

**Keywords :** Besov spaces, semi-groups, Bernstein inequalities, quasi-geostrophic equation, stratified Lie groups, $\alpha$-stable processes

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## Introduction.

In this paper, we are interested in the regularity of the weak solutions of the dissipative quasi-geostrophic equation $(QG_\alpha)$, a generalization of the quasi-geostrophic equation $(QG)$ which is related to fluid mechanics [PED] and whose mathematical study was initiated by Constantin, Majda and Tabak [CON] in 1994. The quasi-geostrophic equation $(QG)$ describes the evolution of a function $\theta(t,x)$, $t>0$, $x\in \mathbb{R}^2$ as

$$(1)\qquad \begin{cases} \partial_t\theta + \vec{u}.\vec{\nabla}\theta = 0 \\ \vec{u} = (-R_2\theta, R_1\theta) \\ \theta(0,.) = \theta_0 \end{cases}$$

where $R_i$ is the Riesz transform $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ (so that the vector field $\vec{u}$ is divergence-free : div $\vec{u}=0$).

Throughout the paper, we will denote $\sqrt{-\Delta}$ by $\Lambda$ (this is Calderón's operator). For $0<\alpha\le 1$, the dissipative quasi-geostrophic equation $(QG_\alpha)$ is the equation $(QG)$ penalized by a dissipative term $-\Lambda^{2\alpha}\theta$ :

$$(2)\qquad \begin{cases} \partial_t\theta + \vec{u}.\vec{\nabla}\theta = -\Lambda^{2\alpha}\theta \\ \vec{u} = (-R_2\theta, R_1\theta) \\ \theta(0,.) = \theta_0 \end{cases}$$

In order to deal with irregular solutions, we rewrite the advection term $\vec{u}.\vec{\nabla}\theta$ as $\mathrm{div}(\theta\ \vec{u})$ :

$$(3)\qquad \begin{cases} \partial_t\theta + \mathrm{div}(\theta\ \vec{u}) = -\Lambda^{2\alpha}\theta \\ \vec{u} = (-R_2\theta, R_1\theta) \\ \theta(0,.) = \theta_0 \end{cases}$$

In 1995, Resnick [RES] proved the existence of weak solutions of the equation (3) for $\theta_0\in L^2(\mathbb{R}^2)$ ; those solutions satisfy the inequality

$$(4)\qquad \text{for } t>0,\quad \|\theta(t,.)\|_2^2 + 2\int_0^t\int |\Lambda^\alpha\theta|^2\ dx\ ds \le \|\theta_0\|_2^2$$

so that $\theta\in L^\infty_t L^2\cap L^2_t\dot{H}^\alpha$ where $\dot{H}^\alpha$ is an homogeneous Sobolev space.

In 2008, Marchand [MAR] studied the case of an initial value $\theta_0\in L^p$; he proved the existence of weak solutions to equation (3) when $p\ge 4/3$; moreover, when $p\ge 2$, Marchand's solutions satisfy the inequality

$$(5)\qquad \text{for } t>0,\quad \|\theta(t,.)\|_p^p + p\int_0^t\int \theta|\theta|^{p-2}\Lambda^{2\alpha}\theta\ dx\ ds \le \|\theta_0\|_p^p$$

where the double integral gives a nonnegative contribution, as shown by Córdoba's inequality [COR] [JU]

$$(6)\qquad 2\int |\Lambda^\alpha(|\theta|^{p/2})|^2\ dx \le p\int \theta|\theta|^{p-2}\Lambda^{2\alpha}\theta\ dx.$$

However, the regularity of Marchand's solutions remained unclear.

In this paper, we will establish the regularity of Marchand's solutions in terms of a norm in a Besov space. More precisely, we shall establish a variant of Córdoba's inequality and get that (for $2\le p<\infty$ and $0<\alpha<1$)

$$(7)\qquad \|\theta\|^p_{\dot{B}_p^{2\alpha/p,p}} \le C_p\int \theta|\theta|^{p-2}\Lambda^{2\alpha}\theta\ dx$$

and (for $2 \leq p < \infty$)

$$\|\theta\|^p_{\dot B_p^{2/p,\infty}} \leq C_p \int \theta|\theta|^{p-2}(-\Delta)\theta \; dx \tag{8}$$

where $\dot B_p^{2\alpha/p,p}$ and $\dot B_p^{2/p,\infty}$ are homogeneous Besov spaces. Our method will gives us a new proof of a nonlinear Bernstein inequality given by Danchin [DAN a] : for $\theta \in L^p(\mathbb{R}^n)$ such that its Fourier transform $\hat\theta(\xi)$ is supported in the annulus $1/2 \leq |\xi| \leq 2$,we have, for $1 < p < \infty$

$$A\|\theta\|_p^p \leq \|\vec\nabla(|\theta|^{p/2})\|_2^2 \leq B\|\theta\|_p^p \tag{9}$$

where the constants $A$ and $B$ are positive and depend only on $p$ and on the dimension $n$.

Our main tool will be a precise study of the semi-group $e^{-t\Lambda^{2\alpha}}$. This is a symmetric diffusion semi-group (in the sense given by Stein [STE]) and we will use a representation of the semi-group as a barycentric mean of heat kernels through a formula derived from the theory of $\alpha$-stable processes [ZOL]. For instance, when $\alpha = 1$, we have $e^{-t\Lambda^2} = e^{t\Delta}$ (the heat kernel); for $\alpha = 1/2$, we have $e^{-t\Lambda} = P_t$ the Poisson semi-group. In dimension 1, $e^{-|\xi|}$ is the Fourier transform of $\frac{1}{\pi}\frac{1}{1+x^2}$; we write

$$\frac{1}{\pi}\frac{1}{1+x^2} = \frac{1}{\pi}\int_0^\infty e^{-\sigma}e^{-\sigma x^2}\; d\sigma = \frac{1}{2\pi}\int_0^\infty e^{-\frac{1}{2\sigma}}e^{-\frac{x^2}{2\sigma}}\;\frac{d\sigma}{\sigma^2} \tag{10}$$

and we get

$$e^{-|\xi|} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2\sigma}}e^{-\sigma\frac{\xi^2}{2}}\;\frac{d\sigma}{\sigma^{3/2}} \tag{11}$$

and finally

$$e^{-t\Lambda} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2\sigma}}e^{\sigma\frac{t^2}{2}\Delta}\;\frac{d\sigma}{\sigma^{3/2}}. \tag{12}$$

We shall use a generalization of (12) to the case of $e^{-t\Lambda^{2\alpha}}$.

## 1. One-dimensional stable distributions

The aim of this section is to establish the useful following representation :

**Proposition 1 :**
*For $0 < \alpha < 1$, there exists a probability measure $d\mu_\alpha$ concentrated on $[0, +\infty)$ such that for all $x \in \mathbb{R}$ we have*

$$e^{-|x|^{2\alpha}} = \int_0^{+\infty} e^{-\sigma x^2} d\mu_\alpha(\sigma) \tag{13}$$

**Corollary 1 :**
*Let $\Lambda = \sqrt{-\Delta}$ be the Calderón operator on $\mathbb{R}^n$ and $e^{t\Delta}$ be the heat kernel on $\mathbb{R}^n$. Then the operator $e^{-t\Lambda^{2\alpha}}$ ($t \geq 0$, $0 < \alpha < 1$) may be represented as*

$$e^{-t\Lambda^{2\alpha}} = \int_0^{+\infty} e^{\sigma t^{1/\alpha}\Delta} d\mu_\alpha(\sigma) \tag{14}$$

**Proof :** We need only to prove the Proposition, the Corollary being obvious. We start from the theory of one-dimensional stable processes. The probability density function $d\mu$ of a random variable $X$ is called $\alpha$-stable [ZOL] if its characteristic function $\chi(\xi) = E(e^{iX\xi}) = \int e^{ix\xi}\; d\mu(x)$ is of the form

$$\chi(\xi) = \begin{cases} e^{im\xi - \sigma^\alpha|\xi|^\alpha - i\beta\sigma^\alpha\xi|\xi|^{\alpha-1}\tan(\pi\alpha/2)} & \text{if } \alpha \neq 1 \\ e^{im\xi - \sigma|\xi| + i\beta\sigma\xi\ln|\xi|} & \text{if } \alpha = 1 \end{cases} \tag{15}$$

The admissible values for the parameters are $0 < \alpha \leq 2$ for the stability index $\alpha$, $m \in \mathbb{R}$ for the position parameter $m$, $\sigma \geq 0$ for the scale parameter $\sigma$, and $-1 \leq \beta \leq 1$ for the bias parameter $\beta$. We will write $X \sim S_\alpha(m, \sigma, \beta)$.

In the case of $X \sim S_\alpha(0, \sigma, 1)$ with $0 < \alpha < 1$, we have $\chi(\xi) = e^{-\sigma^\alpha |\xi|^\alpha (1 + i \text{ sgn}(\xi) \ \tan(\pi\alpha/2))}$. If $z^\alpha$ is the holomorphic function defined on $\mathbb{C} \setminus \mathbb{R}^-$ (so that $z^\alpha = |z|^\alpha e^{i\alpha \text{Arg}(z)}$ where the argument of $z$ is taken in $(-\pi, \pi)$), we find that $(i\xi)^\alpha = |\xi|^\alpha e^{i\alpha \text{sgn}(\xi)\pi/2} = \cos(\alpha\pi/2)|\xi|^\alpha (1 + i\text{sgn}(\xi) \tan(\alpha\pi/2))$. Thus, when $X \sim S_\alpha((\cos(\alpha\pi/2))^{-1/\alpha}, 0, 1)$, we have $\chi(\xi) = e^{-(i\xi)^\alpha}$. For $z = \eta + i\xi$ with $\eta \geq 0$, we have $|e^{-z^\alpha}| = e^{-|z|^\alpha \cos(\alpha \text{Arg}(z))} \leq 1$; the Paley–Wiener–Schwartz theorem ensures us that the probability density function $d\mu_\alpha$ of $X$ is supported on $\mathbb{R}^+$ and that, for $z = \xi + i\eta$ with $\eta \geq 0$, we have $e^{(iz)^\alpha} = \int_0^{+\infty} e^{i\sigma z} \, d\mu_\alpha(\sigma)$. When $z = ix^2$, we obtain $e^{-|x|^{2\alpha}} = \int_0^{+\infty} e^{-\sigma x^2} \, d\mu_\alpha(\sigma)$. Thus, Proposition 1 is proved. $\diamond$

**Remark :** formula (13) is well known. See for instance Proposition 1.2.12 in [SAM]. Due to a celebrated theorem of Bernstein [BER], it amounts to say that the function $x > 0 \mapsto e^{-|x|^\alpha}$ is completely monotone, which is easily checked.

## 2. Diffusion semi-groups.

In this section, we consider a symmetric diffusion semi-group as considered by Stein in [STE] :

**Definition 1 :**
*A symmetric diffusion semi-group with infinitesimal generator $L$ is a family of operators $(e^{tL})_{t \geq 0}$ such that :*
*i) $e^{tL}$ is self-adjoint for $t \geq 0$*
*ii) $e^{tL}$ is the convolution operator with a probability density function $p_t(x)$ ($p_t(x) \geq 0$ and $\int p_t(x) \, dx = 1$)*
*iii) $e^{tL} e^{sL} = e^{(t+s)L}$ and, for $f \in L^2$, $\lim_{t \to 0^+} \|e^{tL} f - f\|_2 = 0$*
*We then have*
*iv) $Lf = \lim_{t \to 0} \frac{1}{t}(e^{tL} f - f)$ on a dense subspace of $L^2$*
*v) $\partial_t e^{tL} f = L(e^{tL} f)$*

For classical results on such semi-groups, we refer to the survey of Bakry [BAK]. A crucial result is that, for a convex function $\phi$, we have the Jensen inequality

$$\phi(e^{tL} f) \leq e^{tL} \phi(f) \tag{16}$$

and, by looking at the derivatives of both terms at $t = 0$,

$$\phi'(f) Lf \leq L(\phi f). \tag{17}$$

When $\phi(t) = t^2$, we get $2fL(f) \leq L(f^2)$ : this is the positivity of the square field operator

$$\Gamma(f, g) = \frac{1}{2}(L(f, g) - fL(g) - gL(f)) \tag{18}$$

For $\phi(t) = |t|^\gamma$ with $\gamma > 1$, we find $\gamma f |f|^{\gamma - 2} L(f) \leq L(|f|^\gamma)$. For $\gamma = p/2$ with $2 < p < +\infty$, we multiply the inequality by $|f|^{p/2}$ and we integrate ; we thus get

$$p \int f|f|^{p-2} \ Lf \ dx \leq 2 \int |f|^{p/2} L(|f|^{p/2}) \ dx = -2 \int |\sqrt{-L}(|f|^{p/2})|^2 \ dx. \tag{19}$$

We are now going to generalize (19) by taking into account the sign of $f$ in the RHS of the inequality :

**Theorem 1 :**
*Let $(e^{tL})_{t \geq 0}$ be a symmetric diffusion semi-group. Then :*
*i) For $2 \leq p < +\infty$, we have the inequality*

$$p \int f|f|^{p-2} L(f) \ dx \leq \int f|f|^{\frac{p}{2}-1} L(f|f|^{\frac{p}{2}-1}) \ dx = -\int |\sqrt{-L}(f|f|^{\frac{p}{2}-1})|^2 \ dx. \tag{20}$$

*ii) For $1 \leq p \leq 2$, we have the inequality*

$$4 \int f|f|^{\frac{p}{2}-1} L(f|f|^{\frac{p}{2}-1}) \ dx = -4 \int |\sqrt{-L}(f|f|^{\frac{p}{2}-1})|^2 \ dx \leq p \int f|f|^{p-2} L(f) \ dx \leq 0. \tag{21}$$

**Proof :** We use the convex function $\phi(t) = |t|$,and we find $\text{sgn}(f)\ L(f) \leq L(|f|)$, hence $fL(f) \leq |f|L(|f|)$. We decompose $f$ into $f = f^+ - f^-$ with $f^+ = \frac{f+|f|}{2}$, and we get

$$f^+ L(f^-) + f^- L(f^+) \geq 0 \tag{22}$$

Integrating (21) and using the self-adjointness of $L$ gives $\int f^+ L(f^-)\ dx \geq 0$. The case of $f^+$ and $f^-$ approximating two Dirac masses at separate points gives then that the distribution kernel $K$ of $L$ satisfies $K(x,y) \geq 0$ outside from the diagonal set $x = y$, and we get finally that

$$f^+ L(f^-) = \int_{x \neq y} K(x,y) f^+(x) f^-(y)\ dy \geq 0 \tag{23}$$

and similarly $f^- L(f^+) \geq 0$. In particular, we get that, for $1 \leq p < +\infty$, we have

$$\int (f^+)^{p-1} L(f^-) + (f^-)^{p-1} L(f^+)\ dx \geq 0 \tag{24}$$

On the other hand, we have that $t \mapsto \|e^{tL}\|_p^p$ is nonincreasing, so that (by looking at the derivative at $t = 0$) we have $p \int f|f|^{p-2} Lf\ dx \leq 0$. This inequality together with (24) gives

$$2 \int (f^+)^{p-1} L(f^+) + (f^-)^{p-1} L(f^-)\ dx \leq \int f|f|^{p-2} L(f)\ dx \leq \int (f^+)^{p-1} L(f^+) + (f^-)^{p-1} L(f^-)\ dx. \tag{25}$$

and, similarly, we have for $g = f|f|^{\frac{p}{2}-1}$, $g^+ = (f^+)^{p/2}$ and $g^- = (f^-)^{p/2}$,

$$2 \int g^+ L(g^+) + g^- L(g^-)\ dx \leq \int g\ L(g)\ dx \leq \int g^+ L(g^+) + g^- L(g^-)\ dx. \tag{26}$$

When $p \geq 2$, we write $\|e^{tL} f^+\|_p^p \leq \|e^{tL}(g^+)\|_2^2$ and we get (by looking at the derivative at $t = 0$) that $p \int (f^+)^{p-1} L(f^+)\ dx \leq 2 \int g^+ L(g^+)\ dx$; we have the same inequality for $f^-$ and $g^-$. Thus, (25) and (26) give (20).

When $p \leq 2$, we write $\|e^{tL} g^+\|_2^2 \leq \|e^{tL}(f^+)\|_p^p$ and we get that $2 \int g^+ L(g^+)\ dx \leq p \int (f^+)^{p-1} L(f^+)\ dx$; we have the same inequality for $f^-$ and $g^-$. Thus, (25) and (26) give (21). ◇

## 3. A. et D. Córdoba's inequality and Besov norms

The semi-group $(e^{-t\Lambda^{2\alpha}})_{t \geq 0}$ is a symmetric diffusion semi-group on $\mathbb{R}^n$. The positivity of its kernel is a consequence of the positivity of the heat kernel $e^{t\Delta}$ and of the representation formula given by Corollary 1. Thus, Córdoba's inequality (6) is just a special case of inequality (19). In this section, we shall apply Theorem 1 (generalization of (19)) to the semi-group $(e^{-t\Lambda^{2\alpha}})_{t \geq 0}$. Our application will be based on the following easy lemma :

**Lemma 1 :**

*Let $0 < \gamma \leq 1$. Then for all $a$ and $b$ in $\mathbb{R}$ we have*

$$\left| a|a|^{\gamma-1} - b|b|^{\gamma-1} \right| \leq 2|a-b|^\gamma \tag{27}$$

**Proof :** This is obvious if $ab < 0$ : if $uv < 0$ then $\max(|u|,|v|) \leq |u-v| \leq 2\max(|u|,|v|)$. If $ab \geq 0$, we use the fact that $d_\gamma(x,y) = |x-y|^\gamma$ is a distance on $\mathbb{R}$ and we write $|d_\gamma(a,0) - d_\gamma(b,0)| \leq d_\gamma(a,b)|$. ◇

We may now prove the following extension of Córdoba's inequality, using norms in homogeneous Sobolev and Besov spaces :

**Theorem 2 :**

*(A) Let $0 < \alpha < 1$ and $2 \leq p < +\infty$. Then there is a positive constant $c_{\alpha,p,n} > 0$ such that :*

$$c_{\alpha,p,n} \|f\|^p_{\dot{B}_p^{2\alpha/p,p}} \leq \|f|f|^{\frac{p}{2}-1}\|^2_{\dot{H}^\alpha} = \int |\Lambda^\alpha (f|f|^{\frac{p}{2}-1})|^2\ dx \leq p \int f|f|^{p-2} \Lambda^{2\alpha}(f)\ dx \tag{28}$$

*(B) Let $2 \leq p < +\infty$. Then there is a positive constant $c_{p,n} > 0$ such that :*

$$c_{p,n} \|f\|^p_{\dot{B}_p^{2\alpha/p,\infty}} \leq \|f|f|^{\frac{p}{2}-1}\|^2_{\dot{H}^1} = \int |\vec{\nabla}(f|f|^{\frac{p}{2}-1})|^2 \, dx \leq p \int f|f|^{p-2}(-\Delta f) \, dx \tag{29}$$

*(C) Let $0 < \alpha < 1$ and $\max(1, 2\alpha) < p < 2$. Then there is a positive constant $C_{\alpha,p,n} > 0$ such that :*

$$0 \leq p \int f|f|^{p-2}\Lambda^{2\alpha}(f) \, dx \leq 4\|f|f|^{\frac{p}{2}-1}\|^2_{\dot{H}^\alpha} = 4 \int |\Lambda^\alpha(f|f|^{\frac{p}{2}-1})|^2 \, dx \leq C_{\alpha,p,n}\|f\|^p_{\dot{B}_p^{2\alpha/p,p}} \tag{30}$$

**Proof :** First, we apply Theorem 1 to the symmetric diffusion semi-group $(e^{-t\Lambda^{2\alpha}})_{t\geq 0}$ : (20) gives the RHS inequalities in (28) and (29), while (21) gives the LHS inequality in (30). Thus, the proof of Theorem 2 is reduced to a comparison between a Besov norm and a Sobolev norm.

Besov norms may be defined in various (more or less) equivalent ways. We shall use the characterization of Besov spaces through moduli of continuity. For $\beta \in (0,1)$ and $1 \leq p < \infty$, the norms of $\dot{B}_p^{\beta,p}$ may be defined as

$$\|f\|_{\dot{B}_p^{\beta,p}} = \Big(\int\int \frac{|f(x)-f(y)|^p}{|x-y|^{n+p\beta}} \, dx \, dy\Big)^{\frac{1}{p}} \text{ and } \|f\|_{\dot{B}_p^{\beta,\infty}} = \sup_{h\in\mathbb{R}^n, \ h\neq 0} \frac{\|f(x)-f(x+h)\|_p}{|h|^\beta} \tag{31}$$

Moreover, we have $\dot{H}^\alpha = \dot{B}_2^{\alpha,2}$. Thus, the Sobolev norm $\|f\|_{\dot{H}^\alpha}$ is equivalent, for $\alpha \in (0,1)$, to $\|f\|_{\dot{B}_2^{\alpha,2}} = \sqrt{\int\int \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}} \, dx \, dy}$. For $\alpha = 1$, the Sobolev norm $\|f\|_{\dot{H}^1}$ is equivalent to $\sup_{h\in\mathbb{R}^n, \ h\neq 0} \frac{\|f(x)-f(x+h)\|_2}{|h|}$.

To finish the proof, we use Lemma 1. For $p \geq 2$, we take $\gamma = 2/p$, $a = f(x)|f(x)|^{\frac{p}{2}-1}$, $b = f(y)|f(y)|^{\frac{p}{2}-1}$ and we get

$$|f(x)-f(y)|^p \leq 2^p|f(x)|f(x)|^{\frac{p}{2}-1} - f(y)|f(y)|^{\frac{p}{2}-1}|^2 \tag{32}$$

Using (32) and (31), we then get the LHS inequalities of (28) and (29).

For $p < 2$, we take $\gamma = p/2$, $a = f(x)$, $b = f(y)$ and we get

$$|f(x)|f(x)|^{\frac{p}{2}-1} - f(y)|f(y)|^{\frac{p}{2}-1}|^2 \leq 4|f(x)-f(y)|^p \tag{33}$$

Using (33) and (31), for $2\alpha/p < 1$, we then get the RHS inequality of (30). $\diamond$

## 4. Frequency gaps.

Let $1 < p < +\infty$ and $f \in L^p(\mathbb{R}^n)$ such that the Fourier transform $\hat{f}$ has no low frequency : $\hat{f}(\xi) = 0$ for $|\xi| \leq A$. Then it is well known that the norm of $e^{t\Delta}f$ decays exponentially :

$$\|e^{t\Delta}f\|_p \leq \frac{1}{c_p}e^{-c_p tA^2}\|f\|_p \tag{34}$$

(see for instance Chemin [CHE]). But (34) contains no information for small $t$'s : if $t \leq A^{-2}\frac{1}{c_p}\ln\frac{1}{c_p}$ we have $\|e^{t\Delta}f\|_p \leq \|f\|_p$ and $1 \leq \frac{1}{c_p}e^{-c_p tA^2}$. In this section, we want to prove a more precise estimate :

$$\|e^{t\Delta}f\|_p \leq e^{-c_p tA^2}\|f\|_p \tag{35}$$

We begin with two classical lemmas :

**Lemma 2 :**
*(A) Let $1 \leq p \leq +\infty$ and $g \in L^p(\mathbb{R}^n)$ such that the Fourier transform $\hat{g}$ has no low frequency : $\hat{g}(\xi) = 0$ for $|\xi| \leq A$. Then, for $1 \leq j \leq n$, $\|\frac{\partial_j}{\Delta}g\|_p \leq cA^{-1}\|g\|_p$.*
*(B) Let $1 \leq p \leq +\infty$ and $f \in L^p(\mathbb{R}^n)$ such that the Fourier transform $\hat{f}$ has no low frequency : $\hat{f}(\xi) = 0$ for $|\xi| \leq A$. Then $\|f\|_p \leq cA^{-1}\|\vec{\nabla}f\|_p$.*
*(C) Let $1 \leq p \leq +\infty$ and $f \in L^p(\mathbb{R}^n)$ such that the Fourier transform $\hat{f}$ has no low frequency : $\hat{f}(\xi) = 0$ for $|\xi| \leq A$. Then there exists $F_j \in L^p$ such that $f = \sum_{j=1}^n \partial_j F_j$ with $\|F_j\|_p \leq cA^{-1}\|f\|_p$.*

**Proof :** (A) is obvious : if $\omega \in \mathcal{D}(\mathbb{R}^n)$ is equal to 1 on the ball $B(0,1/4)$ and to 0 outside from the ball $B(0,1/2)$, then the function $k_j$ whose Fourier transform $\hat{k}_j$ is equal to $\hat{k}_j(\xi) = -\frac{i\xi_j}{\|\xi\|^2}(1-\omega(\xi))$ satisfies $k_j \in L^1$. We have $\frac{\partial_j}{\Delta} g = A^{n-1} k_j(Ax) * g$, so that $\|\frac{\partial_j}{\Delta} g\|_p \leq \|k_j\|_1 \|g\|_p$.

For (B) and (C), we just write $f = -\sum_{j=1}^n \frac{\partial_j}{\Delta}\partial_j f = -\sum_{j=1}^n \partial_j \frac{\partial_j}{\Delta} f$. ◇

The following lemma can be found in [KAT] :

**Lemma 3 :**

*Let $1 < p < +\infty$ and $f$ be a $\mathcal{C}^1$ function. If $f \in W^{2,p}(\mathbb{R}^n)$, then we have*

$$-\int f|f|^{p-2}\Delta f \, dx = (p-1)\int_{f(x)\neq 0} |\vec{\nabla} f|^2 \, |f|^{p-2} \, dx \tag{36}$$

**Proof :** For $p \geq 2$, this is obvious. $f|f|^{p-2}$ is $\mathcal{C}^1$ and $\partial_j(f|f|^{p-2}) = (p-1)|f|^{p-2}\partial_j f$. Thus, (36) is a direct consequence of integration by parts.

For $1 < p < 2$, we approximate $f|f|^{p-2}$ by $g_\epsilon = f|f^2+\epsilon^2|^{\frac{p-2}{2}}$ with $\epsilon > 0$. By dominated convergence, we have $-\int f|f|^{p-2}\Delta f \, dx = \lim_{\epsilon\to 0}\int g_\epsilon(-\Delta f)\, dx$. We have $\partial_j(g_\epsilon) = \partial_j f|f^2+\epsilon^2|^{\frac{p-2}{2}}(1+(p-2)\frac{f^2}{f^2+\epsilon^2})$. We consider $\omega \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \leq \omega \leq 1$ and $\omega = 1$ on $B(0,1)$. Then we have

$$-\int \partial_j^2 f g_\epsilon = \lim_{R\to+\infty}\int \partial_j f \, (\omega(x/R)\partial_j g_\epsilon + \frac{1}{R}\partial_j\omega(x/R) g_\epsilon)\, dx = \int |\partial_j f|^2 |f^2+\epsilon^2|^{\frac{p-2}{2}}(1+(p-2)\frac{f^2}{f^2+\epsilon^2})\, dx \tag{37}$$

since $|\int |\partial_j f \frac{1}{R}\partial_j\omega(x/R) g_\epsilon \, dx| \leq R^{-1}\|\partial_j\omega\|_\infty \|f\|^p_{W^{2,p}}$ (and thus goes to 0 as $R$ goes to $+\infty$) and since $\partial_j f \partial_j g_\epsilon \geq 0$ (remark that $p-1 \leq 1+(p-2)\frac{f^2}{x^2+\epsilon^2} \leq 1$), so that we may apply monotonous convergence to $\int \partial_j f \partial_j g_\epsilon \omega(x/R)\, dx$. We may restrict the domain of the integral in the RHS of (37) to the set of $x$ such that $f(x) \neq 0$, since the set of $x$ such that $f(x) = 0$ and $\partial_j f(x) \neq 0$ has Lebesgue measure 0. Thus, we have

$$-\int f|f|^{p-2}\Delta f \, dx = \lim_{\epsilon\to 0^+}\int_{f(x)\neq 0} |\vec{\nabla} f|^2 \, |f^2+\epsilon^2|^{\frac{p-2}{2}} \, (1+(p-2)\frac{f^2}{f^2+\epsilon^2})\, dx \tag{38}$$

Moreover $\epsilon \mapsto |r^2+\epsilon^2|^{\frac{p-2}{2}}$ is nonincreasing function of $\epsilon \in [0,+\infty)$ and we may apply again monotonous convergence to see that

$$\lim_{\epsilon\to 0}\int_{f(x)\neq 0} |\partial_j f|^2 |f^2+\epsilon^2|^{\frac{p-2}{2}} \, dx = \int_{f(x)\neq 0} |\partial_j f|^2 |f|^{p-2}\, dx \tag{39}$$

The inequality $|\vec{\nabla} f|^2 \, |f^2+\epsilon^2|^{\frac{p-2}{2}} \, (1+(p-2)\frac{f^2}{f^2+\epsilon^2}) \geq (p-1)|\vec{\nabla} f|^2 \, |f^2+\epsilon^2|^{\frac{p-2}{2}}$ , together with (38) and (39), gives us that the limit in (39) is finite. The inequality $|\vec{\nabla} f|^2 \, |f^2+\epsilon^2|^{\frac{p-2}{2}} \, (1+(p-2)\frac{f^2}{f^2+\epsilon^2}) \leq |\vec{\nabla} f|^2 \, |f^2+\epsilon^2|^{\frac{p-2}{2}}$, together with (38), gives us by dominated convergence the equality (36). ◇

We may now prove our theorem on frequency gaps :

**Theorem 3 :**

*Let $1 < p < +\infty$ and $f \in L^p(\mathbb{R}^n)$ such that the Fourier transform $\hat{f}$ has no low frequency : $\hat{f}(\xi) = 0$ for $|\xi| \leq A$. Then :*
*(A) If $f \in W^{2,p}$, we have the inequality*

$$c_p\|f\|_p^p \leq A^{-2} p \int f|f|^{p-2}(-\Delta f)\, dx \tag{40}$$

*where the constant $c_p > 0$ depends only on $n$ and $p$.*
*(B) We have the inequality, for all $t \geq 0$,*

$$\|e^{t\Delta} f\|_p \leq e^{-c_p A^2 t}\|f\|_p \tag{41}$$

*where the constant $c_p > 0$ depends only on $n$ and $p$.*
*(C) For $0 < \alpha < 1$ and $t \geq 0$, we have the inequality*

$$\|e^{-t\Lambda^{2\alpha}} f\|_p \leq e^{-c_{\alpha,p}A^{2\alpha}t}\|f\|_p \tag{42}$$

*where the constant $c_{\alpha,p} > 0$ depends only on $n$, $\alpha$ and $p$.*

**Proof :** We may assume (by a density argument) that $f$ is smooth. In order to prove (A), we shall consider the cases $p \geq 2$ and $p < 2$ :
**Case $p \geq 2$ :** We use Lemma 2 and write $f = \sum_{j=1}^n \partial_j F_j$. Then we have

$$\|f\|_p^p = \sum_{j=1}^n \int \partial_j F_j \; f|f|^{p-2} \, dx = -(p-1)\sum_{j=1}^n \int \partial_j f F_j |f|^{p-2} \, dx \tag{43}$$

and by Cauchy–Schwarz

$$\|f\|_p^p \leq (p-1)\sqrt{\int |\vec{\nabla} f|^2 |f|^{p-2} \, dx}\sqrt{\int \sum_j |F_j|^2 |f|^{p-2} \, dx} \tag{44}$$

We conclude with Lemma 2 (C) and Lemma 3.
**Case $p < 2$ :** We use Lemma 2 (B) and write $\|f\|_p \leq cA^{-1}\|\vec{\nabla} f\|_p$. Moreover, when computing the integral $\int |\vec{\nabla} f|^p \, dx$, we may restrict the domain of integration to the set of $x$ such that $f(x) \neq 0$. Then we use Hölder inequality to get

$$\int |\vec{\nabla} f|^p \, dx \leq \Big(\int_{f(x)\neq 0} |\vec{\nabla} f|^2 |f|^{p-2} \, dx\Big)^{p/2}\Big(\int_{f(x)\neq 0} |f|^p \, dx\Big)^{1-\frac{p}{2}} \tag{45}$$

and we conclude with Lemma 2 (B) and Lemma 3.

Thus, (A) is proved. (B) is a direct consequence of (A) : the derivative of $H(t) = \|e^{t\Delta} f\|_p^p$ is equal to $p\int e^{t\Delta} f |e^{t\Delta} f|^{p-2} \Delta(e^{t\Delta} f) \, dx$ and the the derivative of $K(t) = e^{-c_p A^2 t}\|f\|_p^p$ is $-c_p A^2 e^{-c_p A^2 t}\|f\|_p^p$. (A) gives that $H'(t) \leq -c_p A^2 H(t)$; thus, we get, for $J(t) = H(t) - K(t)$, $J'(t) \leq -c_p A^2 J(t)$ and $J(t) \leq J(0)e^{-c_p A^2 t} = 0$. Thus, $H(t) \leq K(t)$ and (B) is proved.

(C) is a consequence of (B) and of the representation formulae (13) and (14) :

$$\|e^{-t\Lambda^{2\alpha}} f\|_p \leq \int_0^\infty \|e^{\sigma t^{1/\alpha}\Delta} f\|_p^p \, d\mu_\alpha(\sigma) \leq \int_0^\infty e^{-c_p A^2 \sigma t^{1/\alpha}}\|f\|_p \, d\mu_\alpha(\sigma) = e^{-(c_p A^2 t^{1/\alpha})^\alpha}\|f\|_p = e^{-c_p^\alpha A^{2\alpha} t}\|f\|_p \tag{46}$$

Thus, (C) is proved. ◇

## 5. Band limited functions.

In this section, we shall estimate the decay of $\|e^{-t\Lambda^{2\alpha}} f\|_p$ by below :

**Theorem 4 :**
*Let $1 < p < +\infty$ and $f \in L^p(\mathbb{R}^n)$ such that the Fourier transform $\hat{f}$ has no high frequency : $\hat{f}(\xi) = 0$ for $|\xi| \geq A$. Then :*
*(A) For $0 < \alpha \leq 1$, we have the inequality*

$$A^{-2\alpha} p \int f|f|^{p-2}\Lambda^{2\alpha} \, dx \leq c_{\alpha,p}\|f\|_p^p \tag{47}$$

*where the constant $c_{\alpha,p} > 0$ depends only on $n$ and $p$.*
*(B) For $0 < \alpha < 1$ and $t \geq 0$, we have the inequality*

$$\|e^{-t\Lambda^{2\alpha}} f\|_p \geq e^{-c_{\alpha,p}A^{2\alpha}t}\|f\|_p \tag{48}$$

*where the constant $c_{\alpha,p} > 0$ depends only on $n$, $\alpha$ and $p$.*

**Proof :** The case $p \geq 2$ is easy. The Bernstein inequalities give us that $\|\Lambda^{2\alpha}(\theta)\|_p \leq cA^{2\alpha}\|\theta\|_p$ and thus (47) is obvious.

When $p < 2$, we use Theorem 1 (21) (or the LHS of Theorem 2 (30) which is valid for $1 < p < 2$) and get that

$$(49) \qquad p\int f|f|^{p-2}\Lambda^{2\alpha}(f)\ dx \leq 4\|f|f|^{\frac{p}{2}-1}\|^2_{\dot{H}^\alpha} \leq 4\|f\|_p^{(1-\alpha)p/2}\|\vec{\nabla}(f|f|^{\frac{p}{2}-1})\|_2^{2\alpha}$$

We approximate $f|f|^{\frac{p-2}{2}}$ by $g_\epsilon = f|f^2+\epsilon^2|^{\frac{p-2}{4}}$ with $\epsilon > 0$. We have $\partial_j g_\epsilon = \partial_j f|f^2+\epsilon^2|^{\frac{p-2}{4}}(1+\frac{p-2}{2}\frac{f^2}{f^2+\epsilon^2})$. We have that

$$(50) \qquad \|\vec{\nabla}g_\epsilon\|_2^2 = \int_{f(x)\neq 0}|\vec{\nabla}f|^2|f^2+\epsilon^2|^{\frac{p-2}{2}}(1+\frac{p-2}{2}\frac{f^2}{f^2+\epsilon^2})^2\ dx \to_{\epsilon>0} (p-1)^2\int_{f(x)\neq 0}|\vec{\nabla}f|^2\ |f|^{p-2}\ dx$$

We use Lemma 3 to get that the limit in (50) is finite; this proves that $\vec{\nabla}(f|f|^{\frac{p}{2}-1}) \in L^2$ and that (using Bernstein inequality)

$$(51) \qquad \|\vec{\nabla}(f|f|^{\frac{p}{2}-1})\|_2^2 = -(p-1)\int f|f|^{p-2}\Delta f\ dx \leq cA^2\|f\|_p^p$$

Thus (A) is proved. (B) is a direct consequence of (A) : the derivative of $H(t) = \|e^{-t\Lambda^{2\alpha}}f\|_p^p$ is equal to $-p\int e^{-t\Lambda^{2\alpha}}f|e^{-t\Lambda^{2\alpha}}f|^{p-2}\Lambda^{2\alpha}(e^{-t\Lambda^{2\alpha}}f)\ dx$ and the the derivative of $K(t) = e^{-c_{\alpha,p}A^2t}\|f\|_p^p$ is $-c_{\alpha,p}A^2e^{-c_{\alpha,p}A^2t}\|f\|_p^p$. (A) gives that $H'(t) \geq -c_{\alpha,p}A^2H(t)$; thus, we get, for $J(t) = H(t) - K(t)$,the inequalities $J'(t) \geq -c_{\alpha,p}A^2J(t)$ and $J(t) \geq J(0)e^{-c_pA^2t} = 0$. Thus, $H(t) \geq K(t)$ and (B) is proved. ◇

## 6. Danchin's inequality.

In this section, we shall discuss the nonlinear Bernstein inequality given by Danchin [DAN a] [DAN b] : for $\theta \in L^p(\mathbb{R}^n)$ such that its Fourier transform $\hat{\theta}(\xi)$ is supported in the annulus $1/2 \leq |\xi| \leq 2$,we have, for $1 < p < \infty$

$$(52) \qquad A\|\theta\|_p^p \leq \|\vec{\nabla}(|\theta|^{p/2})\|_2^2 \leq B\|\theta\|_p^p$$

where the constants $A$ and $B$ are positive and depend only on $p$ and on the dimension $n$. Danchin [DAN a] proved it for $p \in 2\mathbb{N}^*$, then Planchon [PLA] proved it for $p \geq 2$ and finally Danchin gave a proof for $p > 1$ [DAN b]. We shall use our previous results to prove it and generalize it :

**Theorem 5 :**

*Let $1 < p < +\infty$. Let $\theta \in L^p(\mathbb{R}^n)$ such that its Fourier transform $\hat{\theta}(\xi)$ is supported in the annulus $1/2 \leq |\xi| \leq 2$. Then, for $0 < \alpha \leq 1$, we have*

$$(53) \qquad A\|\theta\|_p^p \leq \|\Lambda^\alpha(\theta|\theta|^{p/2-1})\|_2^2 \leq B\|\theta\|_p^p$$

*where the constants $A$ and $B$ are positive and depend only on p, on $\alpha$ and on the dimension $n$.*

**Proof :** Due to the spectral localization of $\theta$, we have

$$(54) \qquad \|\theta\|_p \sim \|\theta\|_{\dot{B}_p^{2\alpha/p,p}} \sim \|\theta\|_{\dot{B}_p^{2\alpha/p,\infty}}$$

The case $p \geq 2$ is easy. (54) and Theorem 2 give us that $A\|\theta\|_p^p \leq \|\Lambda^\alpha(\theta|\theta|^{p/2-1})\|_2^2$. On the other hand, the Bernstein inequalities give us that $\|\Lambda^{2\alpha}(\theta)\|_p \leq Bp^{-1}\|\theta\|_p$ so that, using Theorem 2 again, we have $\|\Lambda^\alpha(\theta|\theta|^{p/2-1})\|_2^2 \leq p\int \theta|\theta|^{p-2}\Lambda^{2\alpha}(\theta)\ dx \leq B\|\theta\|_p^p$.

When $p \leq 2$, we use Theorem 3 : we have $\|e^{-t\Lambda^{2\alpha}}f\|_p^p \leq e^{-c_{\alpha,p}t}\|f\|_p^p$. Looking at the derivatives at $t = 0$ (and using Theorem 2), we get

$$(55) \qquad c_{\alpha,p}\|f\|_p^p \leq p\int f|f|^{p-2}\Lambda^{2\alpha}f\ dx \leq 4\int|\Lambda^\alpha(f|f|^{\frac{p}{2}-1})|^2\ dx$$

On the other hand, (49) and (51) give us the converse inequality. ◇

**Remark :** Theorem 5 has been proved for $p \geq 2$ by Wu [WU] and Chen, Miao and Zhang [CHN].

## 7. Stratified Lie groups.

Since our method is mainly based on the use of symmetric diffusion semigroups, our results may be adapted to various settings. In this section, we pay a few words to the case of the sublaplacian on a stratified Lie group.

We consider a Lie group $G$ and its Lie algebra $\mathcal{G}$ such that $\mathcal{G} = \oplus_{i=1}^r \mathcal{G}_i$ with $[\mathcal{G}_i, \mathcal{G}_j] = \mathcal{G}_{i+j}$ if $i+j \leq r$ and $= \{0\}$ if $i+j > r$. Then $X \in \mathcal{G} \mapsto \exp X$ is a bijection from $\mathcal{G}$ onto $G$, so that we may identify $G$ and $\mathcal{G}$. The Lebesgue measure on $\mathcal{G}$ is then a Haar measure on $G$. We have a modulus on $G$ defined by $|\sum_{i=1}^r X_i|_G = (\sum_{i=1}^r |X_i|^{2\ r!/i})^{\frac{1}{2\ r!}}$ and a dilation operator $\delta_\lambda(\sum_{i=1}^r X_i) = \sum_{i=1}^r \lambda^i X_i$ for $\lambda > 0$. We have $|\delta_\lambda x|_G = \lambda |x|_G$ and $\delta_\lambda(x.y) = \delta_\lambda(x).\delta_\lambda(y)$. We have $d(\delta_\lambda(x)) = \lambda^{\sum_{i=1}^r i \dim \mathcal{G}_i}\ dx = \lambda^Q\ dx$ where $Q = \sum_{i=1}^r i \dim \mathcal{G}_i$ is the homogeneous dimension of $G$.

We fix a basis $(Y_1, \ldots, Y_k)$ of $\mathcal{G}_1$, considered as left-invariant vector fields on $G$. Then the sublaplacian on $G$ is the operator $\mathcal{J} = -\sum_{i=1}^k Y_i^2$. We define the convolution on $G$ by $f * h(x) = \int_G f(xy^{-1})h(y)\ dy = \int_G f(y)h(y^{-1}x)\ dy$. Then $(e^{-t\mathcal{J}})_{t\geq 0}$ is a semi-group of positive self-adjoint convolution operators on $G$, so that the theory of symmetric diffusion semigroups can be applied.

Moreover, we have Sobolev and Besov spaces on $G$, studied by Folland [FOL] and Saka [SAK]. For $0 < s < 1$ and $1 \leq p < +\infty$, the norm of the Besov space $\dot{B}_p^{s,p}$ is equivalent to $\|f\|_{\dot{B}_p^{s,p}} = \big( \int\int \frac{|f(x.y)-f(y)|^p}{|y|_G^{Q+sp}}\ dx\ dy)^{1/p}$. When $p = 2$, the Besov space $\dot{B}_2^{s,2}$ coincides with the Sobolev space $\dot{H}^s = \mathcal{D}(\mathcal{J}^{s/2})$ (normed by $\|f\|_{\dot{H}^s} = \|\mathcal{J}^{s/2} f\|_2$).

Now, a direct adaptation of Theorem 2 gives :

**Theorem 6 :**

*Let $0 < \alpha < 1$ and $2 \leq p < +\infty$. Then there is a positive constant $c_{\alpha,p,G} > 0$ such that :*

$$c_{\alpha,p,G} \|f\|^p_{\dot{B}_p^{2\alpha/p,p}} \leq \|f|f|^{\frac{p}{2}-1}\|^2_{\dot{H}^\alpha} = \int |\mathcal{J}^{\alpha/2}(f|f|^{\frac{p}{2}-1})|^2\ dx \leq p \int f|f|^{p-2} \mathcal{J}^\alpha(f)\ dx \tag{56}$$

## 8. Lie groups of polynomial growth.

In this section, we consider a connected Lie group $G$ and its Lie algebra $\mathcal{G}$, generated from a set of left-invariant vector fields $(X_i)_{1\leq i\leq N}$ (in the sense of Hörmander : $\mathcal{G}$ is generated by the fields $X_i$ and their successive Lie brackets). We condider $dx$ a left-invariant Haar measure on $G$.

We have a Carnot-Carathéodory metric $\rho(x,y) = |y^{-1}.x|_G$ on $G$ associated to the vector fields $X_i$ [COU]. We note $B(x,r)$ for the ball centered at $x \in G$ and with radius $r > 0$, and $V(r)$ for the volume of the ball $V(r) = \int_{|y|_G<r} dy$. The volume obeys to two dimensional orders : for $r < 1$, we have $ar^d \leq V(r) \leq br^d$ for some local dimension $d > 0$ and positive constants $a$, $b$; for $r \geq 1$, either $V$ has a finite dimensional behaviour $ar^D \leq V(r) \leq br^D$ for some $D > 0$ (the dimension at infinity) or $V$ grows exponentially $e^{ar} \leq V(r) \leq e^{br}$. In the first case, $G$ is called a group with polynomial growth (versus exponential growth in the second case).

The sublaplacian on $G$ is the operator $\mathcal{J} = -\sum_{i=1}^N X_i^2$. We define the convolution on $G$ by $f * h(x) = \int_G f(xy^{-1})h(y)\ dy = \int_G f(y)h(y^{-1}x)\ dy$. Then $(e^{-t\mathcal{J}})_{t\geq 0}$ is a semi-group of positive self-adjoint convolution operators on $G$, so that the theory of symmetric diffusion semigroups can be applied.

We can define Sobolev and Besov spaces on $G$ [TRI]. When $p = 2$, the Besov space $\dot{B}_2^{s,2}$ coincides with the Sobolev space $\dot{H}^s = \mathcal{D}(\mathcal{J}^{s/2})$ (normed by $\|f\|_{\dot{H}^s} = \|\mathcal{J}^{s/2} f\|_2$). It is easy to check that Saka's characterization of Besov spaces [SAK] on stratified Lie groups can be extended to the setting of Lie groups with polynomial growth. More precisely, L. Saloff-Coste [SAL] proved the following result :

**Proposition 2 :**

*Let $G$ be a connected Lie group with polynomial growth. For $0 < s < 1$ and $1 \leq p < +\infty$, the norm of the Besov space $\dot{B}_p^{s,p}$ is equivalent to*

$$\|f\|_{\dot{B}_p^{s,p}} = \Big( \int\int \frac{|f(x.y) - f(y)|^p}{|y|_G^{sp}\ V(|y|_G)}\ dx\ dy)^{1/p} \tag{57}$$

Now, a direct adaptation of Theorem 2 gives :

**Theorem 7 :**

*Let $\mathcal{J}$ be the sublaplacian operator on a connected Lie group $G$ with polynomial growth. Let $0 < \alpha < 1$ and $2 \leq p < +\infty$. Then there is a positive constant $c_{\alpha,p,G} > 0$ such that :*

$$c_{\alpha,p,G} \|f\|^p_{\dot{B}_p^{2\alpha/p,p}} \leq \|f|f|^{\frac{p}{2}-1}\|^2_{\dot{H}^\alpha} = \int |\mathcal{J}^{\alpha/2}(f|f|^{\frac{p}{2}-1})|^2 \, dx \leq p \int f|f|^{p-2} \mathcal{J}^\alpha(f) \, dx \tag{58}$$

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