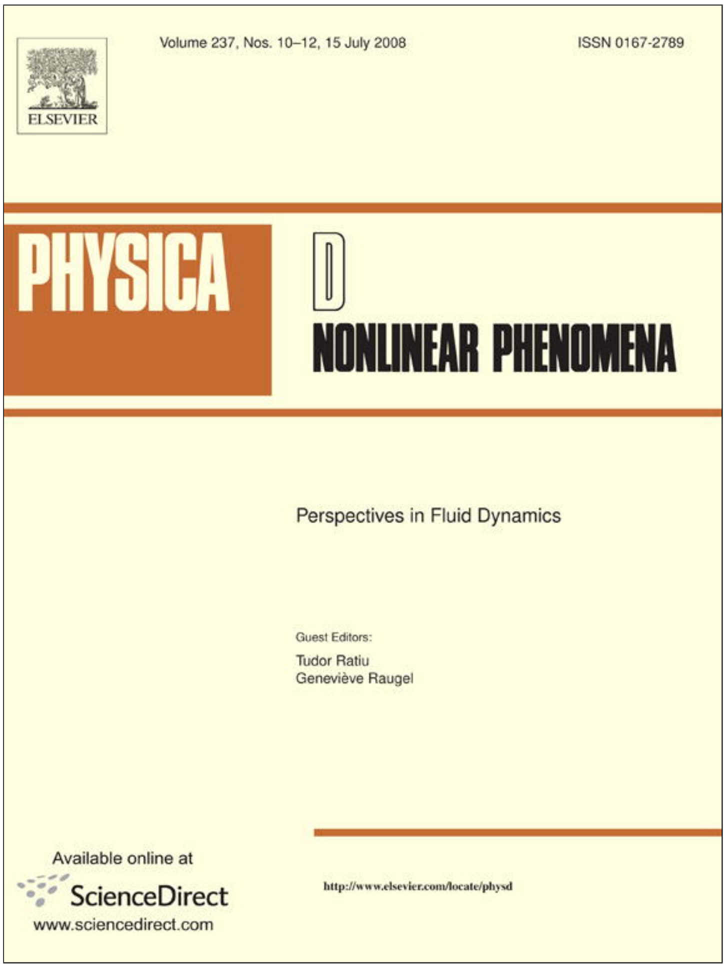
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The Picard iterates for the Navier–Stokes equations in $L^3(\mathbb{R}^3)$

Pierre Gilles Lemarié-Rieusset*

Département de Mathématiques, Université d'Évry, France

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Abstract

We prove that the interval on which the Picard iterates converge to a solution of the Cauchy problem for the 3D Navier–Stokes equations does not depend on the norm in which the convergence is estimated.

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Keywords: Navier–Stokes equations; Mild solutions; Besov spaces

1. Kato's mild solutions

In this paper, we consider the following Cauchy problem for the 3D Navier–Stokes equations: given $\vec{u}_0 \in (L^3(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$, find a solution $\vec{u} \in \mathcal{C}([0, T], (L^3(\mathbb{R}^3))^3)$ of the equations

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - (\vec{u}.\vec{\nabla})\vec{u} - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0, .) = \vec{u}_0 \end{cases} \tag{1}$$

where p is the (unknown) *pressure*, whose action is to maintain the divergence of $\vec{u}$ to be 0 (this *divergence free* condition expresses the incompressibility of the fluid).

In order to solve Eq. (1), we follow Kato [1] and use the Leray–Hopf operator $\mathbb{P}$ which is the orthogonal projection operator on divergence-free vector fields. We thus consider the following Navier–Stokes equations on $\vec{u}(t, x)$:

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla} \cdot (\vec{u} \otimes \vec{u}) \\ \vec{u}(0, .) = \vec{u}_0. \end{cases} \tag{2}$$

Solving the Cauchy problem associated to the initial value $\vec{u}_0$ then amounts to solve the integral equation

$$\vec{u} = \mathrm{e}^{t\Delta}\vec{u}_0 - \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u})\, \mathrm{d}s. \tag{3}$$

In order to solve (3), we define the bilinear operator, B, by

$$B(\vec{f}, \vec{g})(t) = \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\, \mathrm{d}s. \tag{4}$$

* Corresponding address: Département de Mathématiques, Université d'Évry, Bd F. Mitterrand, 91025 Evry, France.
E-mail address: Pierre-Gilles.Lemarie@univ-evry.fr.

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We then define the sequence $\vec{u}^{(n)}$ by

$$\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \quad \text{and} \quad \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)}) \tag{5}$$

and the sequence $\vec{w}^{(n)}$ by

$$\vec{w}^{(n)} = \vec{u}^{(n+1)} - \vec{u}^{(n)}. \tag{6}$$

For every $n \in \mathbb{N}$, we have $\vec{u}^{(n)} \in \mathcal{C}([0, +\infty), (L^3(\mathbb{R}^3))^3)$ and it is well-known that for some positive T we have

$$\sum_{n\in\mathbb{N}} \sup_{0<t<T} \|\vec{w}^{(n)}(t, .)\|_3 < \infty \tag{7}$$

so that the sum

$$\vec{u} = e^{t\Delta}\vec{u}_0 + \sum_{n=0}^{\infty} \vec{w}^{(n)} \tag{8}$$

belongs to $\mathcal{C}([0, T], (L^3(\mathbb{R}^3))^3)$ and is a solution to the Cauchy problem (1). The solution $\vec{u}$ is then known to be smooth.

In this paper, we shall discuss the convergence of the series (8). In order to ensure the convergence of the series, one usually works with weaker norms than the L^3 norm (L^3 is embedded into Besov spaces [2,3] or in the space BMO^{-1} considered by Koch and Tataru [4]). In order to get regularity estimates, one tries to get a contractive estimate in a new norm and this is usually done by taking a smaller value of T. Thus, the series $\sum w^{(n)}$ may be convergent on an interval $[0, T]$ which should depend on the norm in which the terms $w^{(n)}$ are estimated. We shall prove that the interval of convergence does not depend on most norms that are usually used to describe Kato's solutions. Some of those results were previously obtained in [5–7], where they were described as persistency results.

2. Size of the solutions

In 1984, Kato [1] proved the existence of mild solutions in L^p, $p \geq 3$. His construction of mild solutions relies on the fact that the operator $e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}$ is a matrix of convolutions operators (in the x variable) whose kernels $K_{i,j}(t-s, x)$ are controlled by

$$|K_{i,j}(t-s, x)| \leq C \frac{1}{(\sqrt{t-s} + \|x\|)^4}. \tag{9}$$

For $p > 3$, he used the estimate

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\|_p \leq C_p (t-s)^{-\frac{1}{2}-\frac{3}{2p}} \|\vec{f}\|_p \|\vec{g}\|_p \tag{10}$$

to prove the boundedness of B on $L^\infty([0, T], (L^p)^3)$:

$$\|B(\vec{f}, \vec{g})(t, .)\|_p \leq C_p t^{\frac{1}{2}-\frac{3}{2p}} \sup_{0<s<t} \|\vec{f}(s, .)\|_p \sup_{0<s<t} \|\vec{g}(s, .)\|_p. \tag{11}$$

For the critical case $p = 3$, inequality (10) becomes

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\|_3 \leq C \frac{1}{(t-s)} \|\vec{f}\|_3 \|\vec{g}\|_3. \tag{12}$$

This is a very inconvenient estimate for dealing with $\vec{f}$ and $\vec{g}$ in $L^\infty([0, T], (L^3)^3)$, since $\int_0^t \frac{ds}{t-s}$ diverges at the endpoint $s = t$. Kato then used an idea of Weissler [8], namely to use the smoothing properties of the heat kernel (when applied to $\vec{u}_0 \in (L^3)^3$) to search for the existence of a solution in a smaller space of mild solutions ; indeed, whereas the bilinear operator B is unbounded on $\mathcal{C}([0, T], (L^3(\mathbb{R}^3))^3)$ [9], it becomes bounded on the smaller space $\{\vec{f} \in \mathcal{C}([0, T], (L^3(\mathbb{R}^3))^3) \ / \ \sup_{0<t<T} \sqrt{t}\|\vec{f}(t, .)\|_\infty < \infty\}$. Thus, we replace the estimate (12) (which leads to a divergent integral) by the estimates

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\|_3 \leq C \frac{1}{\sqrt{t-s}\sqrt{s}} \|\vec{f}\|_3 \sqrt{s}\|\vec{g}\|_\infty \tag{13}$$

and

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\|_\infty \leq C \frac{1}{\sqrt{t-s}} \min\left(\frac{1}{(t-s)} \|\vec{f}\|_3 \|\vec{g}\|_3, \frac{1}{s}\sqrt{s}\|\vec{f}\|_\infty \sqrt{s}\|\vec{g}\|_\infty\right) \tag{14}$$

which lead to two convergent integrals.

We begin by checking that the introduction of this second norm does not bear any restriction on the interval of convergence for the series (8):

Theorem 1. *Let* $\vec{u}_0 \in (L^3(\mathbb{R}^3))^3$ *with* $\vec{\nabla}.\vec{u}_0 = 0$. *Let the sequence* $\vec{u}^{(n)}$ *be defined by*

$$\vec{u}^{(0)} = \mathrm{e}^{t\Delta}\vec{u}_0 \quad \textit{and} \quad \vec{u}^{(n+1)} = \mathrm{e}^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)}) \tag{15}$$

and the sequence $\vec{w}^{(n)}$ *by*

$$\vec{w}^{(n)} = \vec{u}^{(n+1)} - \vec{u}^{(n)}. \tag{16}$$

Let $T \in (0, +\infty]$. *Then the following assertions are equivalent:*

$$\sum_{n\in\mathbb{N}} \sup_{0<t<T} \|\vec{w}^{(n)}(t,.)\|_3 < \infty, \tag{17}$$

$$\sum_{n\in\mathbb{N}} \sup_{0<t<T} \sqrt{t}\|\vec{w}^{(n)}(t,.)\|_\infty < \infty. \tag{18}$$

Proof. In inequality (13), we have stressed on the role of the L^3 norm of $\vec{f}$ and the L^∞ norm of $\vec{g}$, but we could as well have used the L^∞ norm of $\vec{f}$ and the L^3 norm of $\vec{g}$. We change (13) in a more symmetrical inequality in $\vec{f}$ and $\vec{g}$:

$$\|\mathrm{e}^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\|_3 \le C\frac{1}{\sqrt{t-s}\sqrt{s}}\sqrt{\|\vec{f}\|_3\|\vec{g}\|_3\sqrt{s}\|\vec{f}\|_\infty\sqrt{s}\|\vec{g}\|_\infty}. \tag{19}$$

This gives

$$\sup_{0<t<T}\|B(\vec{f},\vec{g})\|_3 \le C\sqrt{\sup_{0<t<T}\|\vec{f}\|_3 \sup_{0<t<T}\|\vec{g}\|_3 \sup_{0<t<T}\sqrt{t}\|\vec{f}\|_\infty \sup_{0<t<T}\sqrt{t}\|\vec{g}\|_\infty}. \tag{20}$$

In order to estimate $\sup_{0<t<T}\sqrt{t}\|B(\vec{f},\vec{g})\|_\infty$, we write

$$\int_0^{t/2}\|\mathrm{e}^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\|_\infty\,\mathrm{d}s \le C\int_0^{t/2}\frac{1}{(t-s)^{3/2}}\|\vec{f}\|_3\|\vec{g}\|_3\,\mathrm{d}s, \tag{21}$$

so that

$$\sqrt{t}\int_0^{t/2}\|\mathrm{e}^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\|_\infty\,\mathrm{d}s \le C\sup_{0<t<T}\|\vec{f}\|_3\sup_{0<t<T}\|\vec{g}\|_3. \tag{22}$$

On the other hand, we have

$$\int_{t/2}^t\|\mathrm{e}^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\|_\infty\,\mathrm{d}s \le C\int_{t/2}^t\frac{\sqrt{t}}{\sqrt{t-s}\sqrt{s}}\min\left(\frac{\|\vec{f}\|_3\|\vec{g}\|_3}{(t-s)}, \frac{\sqrt{s}\|\vec{f}\|_\infty\sqrt{s}\|\vec{g}\|_\infty}{s}\right)\mathrm{d}s, \tag{23}$$

so that, writing

$$\lambda = \sqrt{\frac{\sup_{0<t<T}\|\vec{f}\|_3\sup_{0<t<T}\|\vec{g}\|_3}{\sup_{0<t<T}\sqrt{t}\|\vec{f}\|_\infty\sup_{0<t<T}\sqrt{t}\|\vec{g}\|_\infty}} \tag{24}$$

and

$$\Phi(\lambda) = \int_0^1\frac{1}{\sqrt{s(1-s)}}\min\left(\frac{\lambda}{1-s}, \frac{1}{\lambda s}\right)\mathrm{d}s, \tag{25}$$

we get

$$\frac{\sqrt{t}\int_{t/2}^t\|\mathrm{e}^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{f}\otimes\vec{g})\|_\infty\,\mathrm{d}s}{\sqrt{\sup_{0<t<T}\|\vec{f}\|_3\sup_{0<t<T}\|\vec{g}\|_3\sup_{0<t<T}\sqrt{t}\|\vec{f}\|_\infty\sup_{0<t<T}\sqrt{t}\|\vec{g}\|_\infty}} \le C\Phi(\lambda). \tag{26}$$

We can easily check that

$$\sup_{\lambda>0}\Phi(\lambda) < \infty. \tag{27}$$

Now, we write

$$\vec{w}^{(n+1)} = B(\vec{u}^{(n)}, \vec{u}^{(n)}) - B(\vec{u}^{(n+1)}, \vec{u}^{(n+1)}) = -B(\vec{u}^{(n)}, \vec{w}^{(n)}) - B(\vec{w}^{(n)}, \vec{u}^{(n+1)}). \tag{28}$$

We define:

$$\alpha_n = \sup_{0<t<T} \|\vec{w}^{(n)}\|_3, \qquad A_n = \sup_{0<t<T} \|\vec{u}^{(n)}\|_3 \tag{29}$$

and

$$\beta_n = \sup_{0<t<T} \sqrt{t}\|\vec{w}^{(n)}\|_\infty, \qquad B_n = \sup_{0<t<T} \sqrt{t}\|\vec{u}^{(n)}\|_\infty. \tag{30}$$

From (20), we get

$$\alpha_{n+1} \leq C\sqrt{\alpha}_n\sqrt{\beta}_n(\sqrt{A_n B_n} + \sqrt{A_{n+1}B_{n+1}}) \tag{31}$$

and thus

$$\alpha_{n+1} \leq C\sqrt{\alpha}_n\sqrt{\beta}_n\sqrt{A_0 + \sum_{p=0}^{n}\alpha_p}\sqrt{B_0 + \sum_{p=0}^{n}\beta_p}. \tag{32}$$

Similarly, from (22) and (26), we get

$$\beta_{n+1} \leq C\alpha_n(A_n + A_{n+1}) + C\sqrt{\alpha_n\beta_n(A_n B_n + A_{n+1}B_{n+1})} \tag{33}$$

and thus

$$\beta_{n+1} \leq C\alpha_n\left(A_0 + \sum_{p=0}^{n}\alpha_p\right) + C\sqrt{\alpha}_n\sqrt{\beta}_n\sqrt{A_0 + \sum_{p=0}^{n}\alpha_p}\sqrt{B_0 + \sum_{p=0}^{n}\beta_p}. \tag{34}$$

We may finish the proof, by using the following lemma: □

Lemma 1. *Let (γ_n), (δ_n) and (ϵ_n) be three sequences of nonnegative real numbers such that:*

$$\sum_{n=0}^{\infty}\gamma_n < \infty \quad \textit{and} \quad \sum_{n=0}^{\infty}\delta_n < \infty \tag{35}$$

and

$$\forall n \in \mathbb{N} \quad \epsilon_{n+1} \leq \gamma_n + \sqrt{\epsilon_n\delta_n\sum_{p=0}^{n}\epsilon_p}. \tag{36}$$

Then, we have

$$\sum_{n=0}^{\infty}\epsilon_n < \infty. \tag{37}$$

Proof. We write

$$\epsilon_{n+1} \leq \gamma_n + \frac{1}{2}\epsilon_n + \frac{1}{2}\delta_n\sum_{p=0}^{n}\epsilon_p. \tag{38}$$

This gives, for $n_0 \in \mathbb{N}$ and $n \geq n_0$

$$\sum_{p=0}^{n+1}\epsilon_p \leq \epsilon_0 + \sum_{p=0}^{n}\gamma_p + \frac{1}{2}\sum_{p=0}^{n}\epsilon_p + \frac{1}{2}\sum_{p=0}^{n_0}\delta_p\sum_{p=0}^{n_0}\epsilon_p + \frac{1}{2}\sum_{p=n_0+1}^{n}\delta_p\sum_{p=0}^{n}\epsilon_p. \tag{39}$$

Choosing n_0 such that

$$\sum_{n_0+1}^{\infty}\delta_p < 1/2, \tag{40}$$

we get

$$\sum_{p=0}^{\infty} \epsilon_p \leq 4 \left(\epsilon_0 + \sum_{p=0}^{\infty} \gamma_p + \frac{1}{2} \sum_{p=0}^{n_0} \delta_p \sum_{p=0}^{n_0} \epsilon_p \right). \tag{41}$$

Thus, Lemma 1 and Theorem 1 are proved. □

Remark. If we work with the Lorentz space $L^{3,\infty}$ instead of the Lebesgue space L^3, then we don't need Weissler's trick of using the smoothing properties of the heat kernel to get mild solutions, since the bilinear operator B is bounded on $\mathcal{C}_*([0,T],(L^{3,\infty}(\mathbb{R}^3))^3)$ [10] (where $\mathcal{C}_*([0,T],(L^{3,\infty}(\mathbb{R}^3))^3)$ is the space of bounded maps from $[0,T]$ to $(L^3)^3$ which are strongly continuous on $(0,T]$ and are *-weakly continuous at $t = 0$). However, we may do the same computations as in the proof of Theorem 1 and see that the solution in $\mathcal{C}_*([0,T],(L^{3,\infty}(\mathbb{R}^3))^3)$ provided by the Picard–Duhamel iterates $\vec{u}^{(n)}$ inherits the good behaviour of the L^∞ norm.

3. Convergence in weaker norms

In the study of mild solutions for the Navier–Stokes equations, weaker norms than the L^3 or the L^∞ norms have been introduced to prove either existence or stability of mild solutions. The weakest norm to be controlled in order to provide existence of mild solutions is the bmo^{-1} norm of the initial value $\vec{u}_0$ (where bmo^{-1} is the space introduced by Koch and Tataru [4]), while stability is described through the control of the norm of the solution $\vec{u}(t,.)$ in the Besov space $B_\infty^{-1,\infty}$ [11] (following ideas of Kozono and co-workers [12,13]). We recall basic definitions and facts about Besov spaces in the appendix.

Recall that $f \in bmo^{-1}$ if and only if, for all positive T, we have

$$\sup_{0<t<T} \sup_{x_0 \in \mathbb{R}^3} t^{-3/2} \int_0^t \int_{B(x_0,\sqrt{t})} |\mathrm{e}^{s\Delta} f|^2 \,\mathrm{d}x \,\mathrm{d}s < \infty. \tag{42}$$

Once again, the introduction of those new norms does not bear any restriction on the interval of convergence for the series (8):

Theorem 2. *Let $\vec{u}_0 \in (L^3(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Let the sequence $\vec{u}^{(n)}$ be defined by*

$$\vec{u}^{(0)} = \mathrm{e}^{t\Delta}\vec{u}_0 \quad \text{and} \quad \vec{u}^{(n+1)} = \mathrm{e}^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)}) \tag{43}$$

and the sequence $\vec{w}^{(n)}$ by

$$\vec{w}^{(n)} = \vec{u}^{(n+1)} - \vec{u}^{(n)}. \tag{44}$$

Let $T \in (0,+\infty)$. Then the following assertions are equivalent:

$$\sum_{n\in\mathbb{N}} \sup_{0<t<T} \|\vec{w}^{(n)}(t,.)\|_3 < \infty, \tag{45}$$

$$\sum_{n\in\mathbb{N}} \sup_{0<t<T} \left(\sqrt{t}\|\vec{w}^{(n)}(t,.)\|_\infty + \sup_{x_0\in\mathbb{R}^3} t^{-3/4} \sqrt{\int_0^t \int_{B(x_0,\sqrt{t})} |\vec{w}^{(n)}(s,x)|^2 \,\mathrm{d}x\,\mathrm{d}s} \right) < \infty, \tag{46}$$

$$\sum_{n\in\mathbb{N}} \sup_{0<t<T} \|\vec{w}^{(n)}(t,.)\|_{B_\infty^{-1,\infty}} < \infty. \tag{47}$$

Remark. Assertions (45) and (47) are equivalent as well to the convergence of the series $\sum_{n\in\mathbb{N}} \sup_{0<t<T} \|\vec{w}^{(n)}(t,.)\|_{bmo^{-1}}$, since we have the continuous imbeddings $L^3 \subset bmo^{-1} \subset B_\infty^{-1,\infty}$.

Proof. In the same way as for proving Theorem 1, we write

$$\vec{w}^{(n+1)} = B(\vec{u}^{(n)}, \vec{u}^{(n)}) - B(\vec{u}^{(n+1)}, \vec{u}^{(n+1)}) = -B(\vec{u}^{(n)}, \vec{w}^{(n)}) - B(\vec{w}^{(n)}, \vec{u}^{(n+1)}). \tag{48}$$

We define:

$$\alpha_n = \sup_{0<t<T} \|\vec{w}^{(n)}\|_3, \qquad A_n = \sup_{0<t<T} \|\vec{u}^{(n)}\|_3, \tag{49}$$

$$\beta_n = \sup_{0<t<T} \sqrt{t}\|\vec{w}^{(n)}\|_\infty, \qquad B_n = \sup_{0<t<T} \sqrt{t}\|\vec{u}^{(n)}\|_\infty, \tag{50}$$

$$\gamma_n = \sup_{0<t<T} \sup_{x_0\in\mathbb{R}^3} t^{-3/4} \sqrt{\int_0^t \int_{B(x_0,\sqrt{t})} |\vec{w}^{(n)}(s,x)|^2 \,\mathrm{d}x\,\mathrm{d}s}, \tag{51}$$

$$C_n = \sup_{0<t<T} \sup_{x_0\in\mathbb{R}^3} t^{-3/4} \sqrt{\int_0^t \int_{B(x_0,\sqrt{t})} |\vec{u}^{(n)}(s,x)|^2 \,\mathrm{d}x\,\mathrm{d}s}, \tag{52}$$

and

$$\delta_n = \sup_{0<t<T} \|\vec{w}^{(n)}\|_{B_\infty^{-1,\infty}}, \qquad D_n = \sup_{0<t<T} \|\vec{u}^{(n)}\|_{B_\infty^{-1,\infty}}. \tag{53}$$

The fact that (45) $\Rightarrow$ (46) is obvious, since $\gamma_n \le C\alpha_n$. The fact that (46) $\Rightarrow$ (47) is easily checked: the operator $\mathbb{P}\vec{\nabla}.$ is bounded from $(L^\infty)^{3\times 3}$ to $(B_\infty^{-1,\infty})^3$, so we shall deal with L^∞ norms. We have

$$\left\| \int_0^{t/2} \mathrm{e}^{(t-s)\Delta} f(s,.)\,\mathrm{d}s \right\|_\infty \le C t^{-3/2} \sup_{x_0\in\mathbb{R}^3} \int_0^t \int_{B(x_0,\sqrt{t})} |f(s,x)|\,\mathrm{d}x\,\mathrm{d}s, \tag{54}$$

and

$$\left\| \int_{t/2}^{t} \mathrm{e}^{(t-s)\Delta} f(s,.)\,\mathrm{d}s \right\|_\infty \le C \sup_{0<s<t} s\|f(s,.)\|\infty. \tag{55}$$

From (48), (54) and (55), we get

$$\delta_{n+1} \le C\gamma_n(C_n + C_{n+1}) + C\beta_n(B_n + B_{n+1}) \le 2C\gamma_n\left(C_0 + \sum_{p=0}^{n}\gamma_p\right) + 2C\beta_n\left(B_0 + \sum_{p=0}^{n}\beta_p\right). \tag{56}$$

The proof that (47) $\Rightarrow$ (45) is not so easy. We use the fact that, for $f \in B_\infty^{-1,\infty}$, we have

$$\sqrt{t}\|\mathrm{e}^{t\Delta} f\|_\infty \le C(1+\sqrt{t})\|f\|_{B_\infty^{-1,\infty}}. \tag{57}$$

We write

$$\vec{w}^{(n+1)}(t,.) = \mathrm{e}^{\frac{t}{2}\Delta}\vec{w}^{(n+1)}(t/2,.) + \vec{v}^{(n)}(t,.), \tag{58}$$

where

$$\vec{v}^{(n)}(t,.) = -\int_0^{\frac{t}{2}} e^{(\frac{t}{2}-s)\Delta}\mathbb{P}\vec{\nabla}.V^{(n)}(t/2+s,.)\,\mathrm{d}s, \tag{59}$$

and

$$V^{(n)} = \vec{u}^{(n)} \otimes \vec{w}^{(n)} + \vec{w}^{(n)} \otimes \vec{u}^{(n+1)}. \tag{60}$$

We then write

$$\|\vec{w}^{(n+1)}(t,.)\|_\infty \le \|\mathrm{e}^{\frac{t}{2}\Delta}\vec{w}^{(n+1)}(t/2,.)\|_\infty + \|\vec{v}^{(n)}(t,.)\|_\infty, \tag{61}$$

hence

$$\|\vec{w}^{(n+1)}(t,.)\|_\infty \le C\left(\frac{1+\sqrt{t}}{\sqrt{t}}\|\vec{w}^{(n+1)}\left(\frac{t}{2},.\right)\|_{B_\infty^{-1,\infty}} + \sqrt{\|\vec{v}^{(n)}(t,.)\|_{B_\infty^{1,\infty}}\|\vec{v}^{(n)}(t,.)\|_{B_\infty^{-1,\infty}}}\right). \tag{62}$$

Moreover, we easily check that

$$\|\vec{v}^{(n)}(t,.)\|_{B_\infty^{-1,\infty}} = \|\vec{w}^{(n+1)}(t,.) - \mathrm{e}^{\frac{t}{2}\Delta}\vec{w}^{(n+1)}(t/2,.)\|_{B_\infty^{-1,\infty}}, \tag{63}$$

hence

$$\|\vec{v}^{(n)}(t,.)\|_{B_\infty^{-1,\infty}} \le \|\vec{w}^{(n+1)}(t,.)\|_{B_\infty^{-1,\infty}} + \|\vec{w}^{(n+1)}(t/2,.)\|_{B_\infty^{-1,\infty}}, \tag{64}$$

while, on the other hand, we have

$$\|\vec{v}^{(n)}(t,.)\|_{B_\infty^{1,\infty}} \le C(1+\sqrt{t}) \sup_{t/2<s<t} \|V^{(n)}(t,.)\|_\infty, \tag{65}$$

(more precisely, it is easy to check that the high frequency term $\|(Id - S_0)\vec{v}^{(n)}(t,.)\|_{B_\infty^{1,\infty}}$ is controled by $\sup_{t/2<s<t}\|V^{(n)}(t,.)\|_\infty$ uniformly in t, while the low frequency term $\|S_0\vec{v}^{(n)}(t,.)\|_{B_\infty^{1,\infty}}$ is controled by $\sqrt{t}\ \sup_{t/2<s<t}\|V^{(n)}(t,.)\|_\infty$ uniformly in t), hence

$$\|\vec{v}^{(n)}(t,.)\|_{B_\infty^{1,\infty}} \le C\frac{1+\sqrt{t}}{t} \sup_{0<s<t}\sqrt{s}\|\vec{w}^{(n)}\|_\infty\Big(\sup_{0<s<t}\sqrt{s}\|\vec{u}^{(n)}\|_\infty + \sup_{0<s<t}\sqrt{s}\|\vec{u}^{(n+1)}\|_\infty\Big). \tag{66}$$

Finally, we get that

$$\beta_{n+1} \le C(1+\sqrt{T})(\delta_{n+1} + \sqrt{\delta_{n+1}\beta_n(B_n + B_{n+1})}), \tag{67}$$

and thus

$$\beta_{n+1} \le C(1+\sqrt{T})\left(\delta_{n+1} + \sqrt{\delta_{n+1}\beta_n\left(B_0 + \sum_{p=0}^{n}\beta_p\right)}\right). \tag{68}$$

We then conclude the proof by using Lemma 1. □

4. Regularity of the solution

It is well-known that the solutions of the Navier–Stokes equations which belong to $\mathcal{C}([0,T]), (L^3(\mathbb{R}^3))^3$ are indeed smooth on $(0,T]$. This regularity is first established in the space variable, then extended to the time variable by differentiating the equations. In the case of the Picard–Duhamel iterates, this can be seen very easily:

Theorem 3. *Let $\vec{u}_0 \in (L^3(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Let the sequence $\vec{u}^{(n)}$ be defined by*

$$\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \quad \text{and} \quad \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)}) \tag{69}$$

and the sequence $\vec{w}^{(n)}$ by

$$\vec{w}^{(n)} = \vec{u}^{(n+1)} - \vec{u}^{(n)}. \tag{70}$$

Let $T \in (0,+\infty)$ and $\sigma > 0$. Then the following assertions are equivalent:

$$\sum_{n\in\mathbb{N}} \sup_{0<t<T} \|\vec{w}^{(n)}(t,.)\|_3 < \infty, \tag{71}$$

$$\sum_{n\in\mathbb{N}} \sup_{0<t<T} t^{\frac{1+\sigma}{2}} \|\vec{w}^{(n)}(t,.)\|_{B_\infty^{\sigma,\infty}} < \infty. \tag{72}$$

Proof. We shall use the well-known inequality

$$\|fg\|_{B_\infty^{\tau,\infty}} \le C_\tau \|f\|_{B_\infty^{\tau,\infty}}\|g\|_\infty + \|f\|_\infty\|g\|_{B_\infty^{\tau,\infty}} \tag{73}$$

for $\tau > 0$ (which is easily checked by using the decomposition of the products into paraproducts). If $\max(0, \sigma-1) < \tau < \sigma$, we get

$$\left\| \int_{t/2}^{t} e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{g}\,ds \right\|_{B_\infty^{\sigma,\infty}} \le C(1+t)t^{\frac{\tau+1-\sigma}{2}} \sup_{t/2<s<t} (\|f\|_{B_\infty^{\tau,\infty}}\|g\|_\infty + \|f\|_\infty\|g\|_{B_\infty^{\tau,\infty}}), \tag{74}$$

(more precisely, the high frequency term $\|(Id - S_0)\int_{t/2}^{t} e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{g}\,ds\|_{B_\infty^{\sigma,\infty}}\|_{B_\infty^{1,\infty}}\|$ is controled by $t^{\frac{\tau+1-\sigma}{2}}\sup_{t/2<s<t}$ $\|(\|f\|_{B_\infty^{\tau,\infty}}\|g\|_\infty + \|f\|_\infty\|g\|_{B_\infty^{\tau,\infty}})\|_\infty$ uniformly in t, while, on the other hand, the low frequency term $\|S_0\int_{t/2}^{t} e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{g}\,ds\|_{B_\infty^{\sigma,\infty}}\|_{B_\infty^{1,\infty}}\|$ is controled by $\sqrt{t}\ \sup_{t/2<s<t}\|(\|f\|_{B_\infty^{\tau,\infty}}\|g\|_\infty + \|f\|_\infty\|g\|_{B_\infty^{\tau,\infty}})\|_\infty$ uniformly in t; of course, we have $\sqrt{t} + t^{\frac{\tau+1-\sigma}{2}} \le C(1+t)t^{\frac{\tau+1-\sigma}{2}}$ for all $t > 0$). We have the interpolation inequality

$$\|f\|_{B_\infty^{\tau,\infty}} \le \|f\|_\infty^{1-\frac{\tau}{\sigma}}\|f\|_{B_\infty^{\sigma,\infty}}^{\frac{\tau}{\sigma}}. \tag{75}$$

We then define:

$$\alpha_n = \sup_{0<t<T}\sqrt{t}\|\vec{w}^{(n)}\|_\infty, \qquad A_n = \sup_{0<t<T}\sqrt{t}\|\vec{u}^{(n)}\|_\infty, \tag{76}$$

and

$$\beta_n = \sup_{0<t<T} t^{\frac{1+\sigma}{2}}\|\vec{w}^{(n)}\|_{B_\infty^{\sigma,\infty}}, \qquad B_n = \sup_{0<t<T} t^{\frac{1+\sigma}{2}}\|\vec{u}^{(n)}\|_{B_\infty^{\sigma,\infty}}. \tag{77}$$

We write again

$$\vec{w}^{(n+1)}(t,.) = e^{\frac{t}{2}\Delta}\vec{w}^{(n+1)}(t/2,.) + \vec{v}^{(n)}(t,.) \tag{78}$$

where

$$\vec{v}^{(n)}(t,.) = -\int_0^{\frac{t}{2}} e^{(\frac{t}{2}-s)\Delta}\mathbb{P}\vec{\nabla}.V^{(n)}(t/2+s,.)\,ds, \tag{79}$$

and

$$V^{(n)} = \vec{u}^{(n)} \otimes \vec{w}^{(n)} + \vec{w}^{(n)} \otimes \vec{u}^{(n+1)}. \tag{80}$$

We then find (using (74) and (75))

$$\beta_{n+1} \leq C\alpha_{n+1} + C(1+T)\left((A_n + A_{n+1})\alpha_n^{1-\frac{\tau}{\sigma}}\beta_n^{\frac{\tau}{\sigma}} + \alpha_n(A_n^{1-\frac{\tau}{\sigma}}B_n^{\frac{\tau}{\sigma}} + A_{n+1}^{1-\frac{\tau}{\sigma}}B_{n+1}^{\frac{\tau}{\sigma}})\right). \tag{81}$$

If $A = A_0 + \sum_{p=0}^{\infty}\alpha_p$, we find

$$\beta_{n+1} \leq C\alpha_{n+1} + C(1+T)\left(A\alpha_n^{1-\frac{\tau}{\sigma}}\beta_n^{\frac{\tau}{\sigma}} + \alpha_n A^{1-\frac{\tau}{\sigma}}\left(B_0 + \sum_{p=0}^{n}\beta_p\right)^{\frac{\tau}{\sigma}}\right), \tag{82}$$

and we conclude (through the Young inequality):

$$\beta_{n+1} \leq C\alpha_{n+1} + \frac{1}{2}\beta_n + C(A(1+T))^{\frac{\sigma}{\sigma-\tau}}\alpha_n + \frac{1}{2}\alpha_n\left(B_0 + \sum_{p=0}^{n}\beta_p\right) + CA\alpha_n \tag{83}$$

which is enough to grant (as in Lemma 1) that the convergence of $\sum_n \alpha_n$ implies the convergence of $\sum_n \beta_n$. Thus, we have proved that (71) $\Rightarrow$ (72).

To prove the converse, we use Kato's theorem to get that, for some small $\delta > 0$, we have

$$\sum_{n=0}^{\infty} \sup_{0<t<\delta} \sqrt{t}\|\vec{w}^{(n)}(t,.)\|_\infty < \infty, \tag{84}$$

and we use the embedding $B_\infty^{\sigma,\infty} \subset L^\infty$ (for $\sigma > 0$) to get

$$\alpha_n \leq \sup_{0<t<\delta} \sqrt{t}\|\vec{w}^{(n)}(t,.)\|_\infty + C\delta^{-\sigma/2}\beta_n \tag{85}$$

which is enough to conclude that the convergence of $\sum_n \beta_n$ implies the convergence of $\sum_n \alpha_n$. □

5. Serrin's exponents

Serrin's theorems on uniqueness or regularity of weak solutions deals with a solution $\vec{u}$ which is $L_t^p L_x^q$ with $2/p+3/q = 1$ [14]. When $\vec{u}$ is a mild solution on $[0,T]$ associated to $\vec{u}_0 \in (L^3)^3$, then $\vec{u} \in (L^p([0,T]), L^q)^3$ for $2/p+3/q = 1$ and $p \geq 3$; the fluctuation $\vec{w} = \vec{u} - e^{t\Delta}\vec{u}_0$ belongs to $(L^p([0,T]), L^q)^3$ for $2/p+3/q = 1$ (and $p \geq 2$)(for a discussion of the regularity of the fluctuation, see [15]).

This can be checked directly on the Picard–Duhamel iterates:

Theorem 4. *Let $\vec{u}_0 \in (L^3(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Let the sequence $\vec{u}^{(n)}$ be defined by*

$$\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \quad \textit{and} \quad \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)}) \tag{86}$$

and the sequence $\vec{w}^{(n)}$ by

$$\vec{w}^{(n)} = \vec{u}^{(n+1)} - \vec{u}^{(n)}. \tag{87}$$

Let $T \in (0,+\infty]$ and p, q such that $p \geq 2$ and $2/p+3/q = 1$. Then the following assertions are equivalent:

$$\sum_{n\in\mathbb{N}} \sup_{0<t<T} \|\vec{w}^{(n)}(t,.)\|_3 < \infty, \tag{88}$$

$$\sum_{n\in\mathbb{N}} \|\vec{w}^{(n)}\|_{L^p((0,T),L^q)} < \infty. \tag{89}$$

Proof. From the Bernstein inequalities, we get the following embeddings for $2/p + 3/q = 1$ and $p \geq 3$:

$$L^3 \subset \dot{B}_3^{0,3} \subset \dot{B}_q^{3/q-1,3} \subset \dot{B}_q^{-2/p,p} \tag{90}$$

and thus

$$\|e^{t\Delta} f\|_{L^p((0,+\infty),L^q)} \leq C\|f\|_3. \tag{91}$$

If $3 \leq p < \infty$, $2/p + 3/q = 1$, $1/q = 1/r - 1/3$, we use the $L^p L^q$ maximal regularity of the heat kernel to get

$$\left\| \int_0^t e^{(t-s)\Delta} \sqrt{-\Delta} F(s,.)\, ds \right\|_{L^p((0,T),L^q)} \leq C\|F\|_{L^p((0,T),L^r)} \tag{92}$$

and thus

$$\|B(\vec{f},\vec{g})\|_{L^p((0,T),L^q)} \leq C\sqrt{\sup_{0<t<T} \|\vec{f}\|_3 \, \|\vec{f}\|_{L^p((0,T),L^q)} \sup_{0<t<T} \|\vec{g}\|_3 \, \|\vec{g}\|_{L^p((0,T),L^q)}}. \tag{93}$$

In the same way as for proving Theorem 1, we then write

$$\vec{w}^{(n+1)} = B(\vec{u}^{(n)},\vec{u}^{(n)}) - B(\vec{u}^{(n+1)},\vec{u}^{(n+1)}) = -B(\vec{u}^{(n)},\vec{w}^{(n)}) - B(\vec{w}^{(n)},\vec{u}^{(n+1)}). \tag{94}$$

We define:

$$\alpha_n = \sup_{0<t<T} \|\vec{w}^{(n)}\|_3, \qquad A_n = \sup_{0<t<T} \|\vec{u}^{(n)}\|_3, \tag{95}$$

and

$$\beta_n = \|\vec{w}^{(n)}\|_{L^p((0,T),L^q)}, \qquad B_n = \|\vec{u}^{(n)}\|_{L^p((0,T),L^q)}. \tag{96}$$

From (93), we get

$$\beta_{n+1} \leq C\sqrt{\alpha_n}\sqrt{\beta_n}(\sqrt{A_n B_n} + \sqrt{A_{n+1}B_{n+1}}) \tag{97}$$

and thus

$$\beta_{n+1} \leq C\sqrt{\alpha_n}\sqrt{\beta_n}\sqrt{A_0 + \sum_{p=0}^{n} \alpha_p}\sqrt{B_0 + \sum_{p=0}^{n} \beta_p}. \tag{98}$$

Due to Lemma 1, we may conclude that the convergence (88) implies the convergence (89) when $p \geq 3$.

Now, we prove the convergence (89) for $2 \leq p < 3$. It is enough to prove it for $p = 2$, since we have, for $2 < p < \infty$ and $2/p + 3/q = 1$,

$$\|f\|_{L^p((0,T),L^q)} \leq \left(\sup_{0<t<T} \|f\|_3\right)^{3/q} \|f\|^{1-3/q}_{L^2((0,T),L^\infty)} \leq \frac{3}{q} \sup_{0<t<T} \|f\|_3 + \frac{q-3}{q}\|f\|_{L^2L^\infty}. \tag{99}$$

We use the $L^r L^s$ maximal regularity for $r = 3/2$ and $s = 9/2$ and for $r = 3$ and $s = 9/4$ and we find

$$\|\sqrt{-\Delta} B(\vec{f},\vec{g})\|_{L^{3/2}((0,T),L^{9/2})} \leq C\|\vec{f}\|_{L^3((0,T),L^9)}\|\vec{g}\|_{L^3((0,T),L^9)}, \tag{100}$$

and

$$\|\sqrt{-\Delta} B(\vec{f},\vec{g})\|_{L^3((0,T),L^{9/4})} \leq C\|\vec{f}\|_{L^3((0,T),L^9)} \sup_{0<t<T} \|\vec{g}\|_3. \tag{101}$$

We use the inequality

$$\|f\|_\infty \leq C\|f\|_{\dot{B}^{2/3,1}_{9/2}} \leq C'\|f\|^{1/2}_{\dot{B}^{1,\infty}_{9/2}}\|f\|^{1/2}_{\dot{B}^{1/3,\infty}_{9/2}} \leq C''\|\sqrt{-\Delta} f\|^{1/2}_{9/2}\|\sqrt{-\Delta} f\|^{1/2}_{9/4} \tag{102}$$

and thus

$$\|B(\vec{f},\vec{g})\|_{L^2((0,T),L^\infty)} \leq C\|\vec{f}\|_{L^3((0,T),L^9)}\sqrt{\|\vec{g}\|_{L^3((0,T),L^9)} \sup_{0<t<T} \|\vec{g}\|_3}. \tag{103}$$

Thus, if we define:

$$\alpha_n = \sup_{0<t<T} \|\vec{w}^{(n)}\|_3, \qquad A_n = \sup_{0<t<T} \|\vec{u}^{(n)}\|_3, \tag{104}$$

$$\beta_n = \|\vec{w}^{(n)}\|_{L^3((0,T),L^9)}, \qquad B_n = \|\vec{u}^{(n)}\|_{L^3((0,T),L^9)} \tag{105}$$

and

$$\gamma_n = \|\vec{w}^{(n)}\|_{L^2((0,T),L^\infty)}, \tag{106}$$

we find from (94) and (103)

$$\gamma_0 \leq C B_0^{3/2} A_0^{1/2} \quad \text{and} \quad \gamma_{n+1} \leq C(B_n\sqrt{\alpha_n\beta_n} + \beta_n\sqrt{A_{n+1}B_{n+1}}), \tag{107}$$

so that the convergence of $\sum_n \alpha_n$ (and, hence, of $\sum_n \beta_n$) implies the convergence of $\sum_n \gamma_n$.

We now prove the converse. We first notice that, for $1/r + 3/(2\sigma) = 1$ and $f \in L^r((0,T), L^\sigma)$, we have

$$\sup_{0<t<T} \|\int_0^t \mathrm{e}^{(t-s)\Delta}\sqrt{-\Delta} f(s,.)\,\mathrm{d}s\|_{\dot{B}_\infty^{-1,\infty}} \leq C\|f\|_{L^r((0,T),L^\sigma)}. \tag{108}$$

This is checked by using the Littlewood–Paley decomposition: we write

$$\left\|\Delta_j \int_0^t \mathrm{e}^{(t-s)\Delta}\sqrt{-\Delta} f(s,.)\,\mathrm{d}s\right\|_\infty \leq C\int_0^t \min\left(2^{j(1+\frac{3}{\sigma})}, \frac{1}{\sqrt{t-s}^{1+\frac{3}{\sigma}}}\right)\|f(s,.)\|_\sigma\,\mathrm{d}s \tag{109}$$

and we conclude by checking (using the equality $1 - 1/r = 3/(2\sigma)$ that

$$\left(\int_0^t \min\left(2^{j\left(1+\frac{3}{\sigma}\right)}, \frac{1}{\sqrt{t-s}^{1+\frac{3}{\sigma}}}\right)^{\frac{r}{r-1}}\mathrm{d}s\right)^{\frac{r-1}{r}} \leq C2^j. \tag{110}$$

From (108), we get for $2/p + 3/q = 1$

$$\sup_{0<t<T}\left\|\int_0^t \mathrm{e}^{(t-s)\Delta}\sqrt{-\Delta}(fg)\,\mathrm{d}s\right\|_{\dot{B}_\infty^{-1,\infty}} \leq C\|f\|_{L^p((0,T),L^q)}\|g\|_{L^p((0,T),L^q)}, \tag{111}$$

and

$$\sup_{0<t<T}\left\|\int_0^t \mathrm{e}^{(t-s)\Delta}\sqrt{-\Delta}(fg)\,\mathrm{d}s\right\|_{\dot{B}_\infty^{-1,\infty}} \leq C\|f\|_{L^p((0,T),L^q)} \sup_{0<t<T}\|g(t,.)\|_3. \tag{112}$$

Now, we define:

$$\alpha_n = \|\vec{w}^{(n)}\|_{L^p((0,T),L^q)}, \qquad A_n = \|\vec{u}^{(n)} - \mathrm{e}^{t\Delta}\vec{u}_0\|_{L^p((0,T),L^q)}, \tag{113}$$

and

$$\beta_n = \sup_{0<t<T}\|\vec{w}^{(n)}\|_{\dot{B}_\infty^{-1,\infty}}. \tag{114}$$

From (94), (111) and (112), we get

$$\beta_{n+1} \leq C\alpha_n(\|\vec{u}_0\|_3 + A_n) \leq C\alpha_n\left(\|\vec{u}_0\|_3 + \sum_{p=0}^n \alpha_p\right). \tag{115}$$

This proves the convergence of (88) (due to Theorem 2). □

Remark. We used the norm of the homogeneous space $\dot{B}_\infty^{-1,\infty}$ and not the norm of the inhomogeneous space $B_\infty^{-1,\infty}$ as in Theorem 2, because we wanted to include the value $T = +\infty$ in the theorem. If we dealt with the nonhomogeneous Besov space, we would find different exponents for t for the low frequencies and the high frequencies (see formulas (65) and (74), for example), and we could not have results valid uniformly on $(0, +\infty)$.

Appendix. Besov spaces

In this appendix, we recall some basic facts on Besov spaces we used throughout the paper. Proofs and further references to Besov spaces can be found in the book [6] (or in the books [16–18]). First, we introduce the well-known Littlewood–Paley decomposition of distributions into dyadic blocks of frequencies:

Definition 1. Let $\varphi_0 \in \mathcal{D}(\mathbb{R}^3)$ be a non-negative radial function such that $|\xi| \leq \frac{1}{2} \Rightarrow \varphi_0(\xi) = 1$ and $|\xi| \geq 1 \Rightarrow \varphi_0(\xi) = 0$. Let ψ_0 be defined as $\psi_0(\xi) = \varphi_0(\xi/2) - \varphi_0(\xi)$. Let S_j and Δ_j be defined as the Fourier multipliers $\mathcal{F}(S_j f) = \varphi_0(\xi/2^j)\mathcal{F}f$ and $\mathcal{F}(\Delta_j f) = \psi_0(\xi/2^j)\mathcal{F}f$. The distribution $\Delta_j f$ is called the j-th dyadic block of the Littlewood–Paley decomposition of f.

For all $N \in \mathbb{Z}$ and all $f \in \mathcal{S}'(\mathbb{R}^3)$ we have

$$f = S_N f + \sum_{j \geq N} \Delta_j f \quad \text{in } \mathcal{S}'(\mathbb{R}^3). \tag{116}$$

This equality is called the Littlewood–Paley decomposition of the distribution f. If moreover $\lim_{N\to-\infty} S_N f = 0$ in $\mathcal{S}'$, then the equality

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f \tag{117}$$

is called the homogeneous Littlewood–Paley decomposition of f.

Then we define the Besov spaces $B_q^{s,p}$:

Definition 2. Let $p, q \in [1, +\infty]$ and $s \in \mathbb{R}$.

(a) The Besov space $B_q^{s,p}(\mathbb{R}^3)$ is the Banach space of distributions $f \in \mathcal{S}'(\mathbb{R}^3)$ such that, for all $j \in \mathbb{N} S_j f \in L^p$ and such that $(2^{js} \parallel \Delta_j f)_{j\in\mathbb{N}} \in l^q$, normed with

$$\|f\|_{B_q^{s,p}} = \|S_0 f\|_p + \left(\sum_{j=0}^{+\infty} 2^{jsq} \|\Delta_j f\|_p^q\right)^{1/q}. \tag{118}$$

(b) For $s < 3/p$, the homogeneous Besov space $\dot{B}_q^{s,p}(\mathbb{R}^3)$ is the Banach space of distributions $f \in \mathcal{S}'(\mathbb{R}^3)$ such that $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ in $\mathcal{S}'(\mathbb{R}^3)$ and such that, for all $j \in \mathbb{Z}$, $\Delta_j f \in L^p$ with $(2^{js} \parallel \Delta_j f)_{j\in\mathbb{Z}} \in l^q$, normed with

$$\|f\|_{\dot{B}_q^{s,p}} = \left(\sum_{j=-\infty}^{+\infty} 2^{jsq} \|\Delta_j f\|_p^q\right)^{1/q}. \tag{119}$$

We have the obvious embeddings

$$B_q^{s,p} \subset \dot{B}_q^{s,p} \quad \text{for } 0 < s < 3/p \quad \text{and} \quad \dot{B}_q^{s,p} \subset B_q^{s,p} \quad \text{for } s < 0 \tag{120}$$

and

$$\text{for } 1 \leq q_1 \leq q_2 \leq +\infty, \quad B_{q_1}^{s,p} \subset B_{q_2}^{s,p} \quad \text{and} \quad \dot{B}_{q_1}^{s,p} \subset \dot{B}_{q_2}^{s,p}. \tag{121}$$

An important result of harmonic analysis states that

$$\text{for } 1 < p < +\infty, \dot{B}^{0,p}_{\min(p,2)} \subset L^p \subset \dot{B}^{0,p}_{\max(p,2)}. \tag{122}$$

The Bernstein inequalities on L^p norms state that there exists constants C_{p_1,p_2} for $1 \leq p_1 \leq p_2 \leq +\infty$ such that

$$\text{for } j \in \mathbb{Z} \text{ and for } f \in \mathcal{S}'(\mathbb{R}^3), \quad \|S_j f\|_{p_2} \leq C_{p_1,p_2} 2^{3j(\frac{1}{p_1} - \frac{1}{p_2})} \|S_j f\|_{p_1} \tag{123}$$

which implies that, for $1 \leq q \leq +\infty$ and $s \in \mathbb{R}$,

$$\text{for } 1 \leq p_1 \leq p_2 \leq +\infty, \quad B_q^{s,p_1} \subset B_q^{s-3(\frac{1}{p_1}-\frac{1}{p_2}),p_2} \quad \text{and} \quad \dot{B}_q^{s,p_1} \subset \dot{B}_q^{s-3(\frac{1}{p_1}-\frac{1}{p_2}),p_2}. \tag{124}$$

The Bernstein inequalities on derivatives state that there exists constants C_α for $\alpha \in \mathbb{N}^3$ such that

$$\text{for } j \in \mathbb{Z}, 1 \leq p \leq \infty \text{ and for } f \in \mathcal{S}'(\mathbb{R}^3), \quad \left\|\frac{\partial^\alpha}{\partial x^\alpha} S_j f\right\|_p \leq C_\alpha 2^{j|\alpha|} \|S_j f\|_p \tag{125}$$

which implies that $\frac{\partial^\alpha}{\partial x^\alpha}$ is a bounded map from $B_q^{s,p}$ to $B_q^{s-|\alpha|,p}$ and from $\dot{B}_q^{s,p}$ to $\dot{B}_q^{s-|\alpha|,p}$. Similarly, we find that $\frac{\partial^\alpha}{\partial x^\alpha}$ is a bounded map from L^∞ to $\dot{B}_\infty^{-|\alpha|,\infty}$.

The Riesz transforms operate boundedly on the dyadic blocks: there exists a constant C_0 and constants C_p for $1 < p < +\infty$ such that, for all $j \in \mathbb{Z}$ and all $f \in \mathcal{S}'(\mathbb{R}^3)$, for $k = 1, \ldots, 3$

$$\text{for } 1 < p < +\infty, \quad \|R_k S_j f\|_p \leq C_p \|S_j f\|_p \tag{126}$$

and

$$\text{for } 1 \leq p \leq +\infty, \quad \|R_k \Delta_j f\|_p \leq C_0 \|\Delta_j f\|_p. \tag{127}$$

In particular, we see easily that the operator $\mathbb{P}\vec{\nabla}.$ is bounded from $(L^\infty)^{3\times3}$ to $(B_\infty^{-1,\infty})^3$.

A useful criterion to check whether a distribution f belongs to a Besov space is the following one: if $s > 0$, $1 \leq p \leq +\infty$ and $1 \leq q \leq +\infty$, $f = \sum_{j \in \mathbb{N}} f_j$ where the Fourier transforms $\hat{f}_j$ are supported in balls $B(0, C2^j)$ (where C doesn't depend on j) and if $(2^{js}\|f_j\|_p)_{j\in\mathbb{N}} \in l^q$, then f belongs to $B_q^{s,p}$. A similar criterion holds for all $s \in \mathbb{R}$, if we request that, for $j > 0$, the Fourier transforms $\hat{f}_j$ are supported in coronas $\{\xi \in \mathbb{R}^3 \ / \ \gamma 2^j \leq \|\xi\| \leq C2^j\}$ (where $\gamma > 0$ doesn't depend on j). Due to this criterion, one is lead to split a product $fg = (S_0 f + \sum_{j\in\mathbb{Z}} \Delta_j f)(S_0 g + \sum_{j\in\mathbb{Z}} \Delta_j g)$ into pieces well localized in frequency

$$fg = \pi(f, g) + \pi(g, f) + R(f, g) \tag{128}$$

where the paraproduct $\pi(f, g)$ contains the terms whose frequency is determined mainly by g

$$\pi(f, g) = \sum_{j=2}^{+\infty} S_{j-2} f \Delta_j g, \tag{129}$$

the paraproduct $\pi(g, f)$ similarly contains the terms whose frequency is determined mainly by f and $R(f, g)$ is the remainder

$$R(f, g) = S_0 f S_2 g + \Delta_0 f S_3 g + \Delta_1 f S_4 g + \sum_{j=2}^{+\infty} \sum_{l=-2}^{+2} \Delta_j f \Delta_{j+l} g. \tag{130}$$

This decomposition and the criterion allows one to check very easily the well-known inequality

$$\|fg\|_{B_q^{s,p}} \leq C_{s,p,q}(\|f\|_{B_q^{s,p}} \|g\|_\infty + \|f\|_\infty \|g\|_{B_q^{s,p}}), \tag{131}$$

for $s > 0$, $1 \leq p \leq +\infty$ and $1 \leq q \leq +\infty$.

Besov spaces may be characterized through the heat kernel:

Lemma 2. *Let $1 \leq p \leq +\infty$, $1 \leq q \leq +\infty$ and $s < 0$.*

(a) *Let $T > 0$. $f \in \mathcal{S}'(\mathbb{R}^3)$ belongs to $B_q^{s,p}(\mathbb{R}^3)$ if and only if $\mathrm{e}^{t\Delta} f \in L^p$ for all $t > 0$ and $t^{|s|/2}\|\mathrm{e}^{t\Delta} f\|_p \in L^q((0, T), \frac{\mathrm{d}t}{t})$. Moreover, the norm of $B_q^{s,p}$ is equivalent to the norm $\|\mathrm{e}^{T\Delta} f\|_p + \left(\int_0^T t^{q|s|}\|\mathrm{e}^{t\Delta}\|_p^q \frac{\mathrm{d}t}{t}\right)^{1/q}$.*

(b) *$f \in \mathcal{S}'(\mathbb{R}^3)$ belongs to $\dot{B}_q^{s,p}(\mathbb{R}^3)$ if and only if $\mathrm{e}^{t\Delta} f \in L^p$ for all $t > 0$ and $t^{|s|/2}\|\mathrm{e}^{t\Delta} f\|_p \in L^q((0, \infty), \frac{\mathrm{d}t}{t})$. The norm of $\dot{B}_q^{s,p}$ is equivalent to $\left(\int_0^{+\infty} t^{q|s|}\|\mathrm{e}^{t\Delta}\|_p^q \frac{\mathrm{d}t}{t}\right)^{1/q}$.*

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