The Navier–Stokes equations in the critical Morrey–Campanato space.

Pierre Gilles LEMARIÉ–RIEUSSET
Département de Mathématiques, Université d'Évry
Pierre-Gilles.Lemarie@univ-evry.fr

Abstract :
    We shall discuss various points on solutions of the 3D Navier–Stokes equations from the point of view of Morrey–Campanato spaces (global solutions, strong-weak uniqueness, the role of real interpolation, regularity).

The classical *Navier-Stokes equations* describe the motion of a Newtonian fluid; we consider only the case when the fluid is viscous, homogeneous, incompressible and fills the entire space and is not submitted to external forces; then, the equations describing the evolution of the motion $\vec{u}(t,x)$ of the fluid element at time t and position x are given by:

(1)
$$\begin{cases} \rho \ \partial_t \vec{u} = \mu \ \Delta \vec{u} - \rho \ (\vec{u}.\vec{\nabla}) \ \vec{u} - \vec{\nabla} p \\ \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

ρ is the (constant) *density* of the fluid, and we may assume with no loss of generality that $\rho = 1$. μ is the *viscosity* coefficient, and we may assume as well that $\mu = 1$. p is the (unknown) *pressure*, whose action is to maintain the divergence of $\vec{u}$ to be 0 (this *divergence free* condition expresses the incompressibility of the fluid).

We shall use the scaling property of equations (1). When $(\vec{u}, p)$ is a solution on $(0,T) \times \mathbb{R}^3$ of the Cauchy problem associated to equations (1) and initial value $\vec{u}_0$, then, for every $\lambda > 0$ and every $x_0 \in \mathbb{R}^3$,

$$(\lambda \vec{u}(\lambda^2 t, \lambda(x - x_0)), \lambda^2 p(\lambda^2 t, \lambda(x - x_0)))$$

is a solution on $(0, \lambda^{-2}T) \times \mathbb{R}^3$ of the Cauchy problem with initial value $\lambda \vec{u}_0(\lambda(x - x_0))$. Therefore, we shall consider the Cauchy problem with initial value in a critical shift-invariant Banach space : we shall require that $\vec{u}_0 \in B^3$ where B is a Banach space of distributions such that, for every $\lambda > 0$ and every $x_0 \in \mathbb{R}^3$ and for every $f \in B$, we

have $\|f(x-x_0)\|_B = \|f\|_B$ and $\lambda\|f(\lambda x)\|_B = \|f\|_B$. If we require moreover that B is continuously embedded in the space of locally square integrable functions, then B must be embedded into the homogeneous Morrey–Campanato space $\dot{M}^{2,3}$ which will play a prominent part throughout the paper.

1. The Navier–Stokes equations in the critical Morrey–Campanato space.

In order to solve equations (1), we use the Leray–Hopf operator $\mathbb{P}$ which is the orthogonal projection operator on divergence-free vector fields. We thus consider the following Navier–Stokes equations on $\vec{u}(t,x)$, $t \in (0,\infty)$, $x \in \mathbb{R}^3$:

$$(2) \qquad \begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}\cdot(\vec{u}\otimes\vec{u}) \\[2mm] \vec{\nabla}.\vec{u} = 0 \end{cases}$$

(Every solution of (2) is a solution of (1). Conversely, under the restriction that $\vec{u}$ vanishes at infinity in a weak sense, every solution of (1) is a solution of (2); see [FURLT 00] or [LEM 02]). Solving the Cauchy problem associated to the initial value $\vec{u}_0$ then amounts to solve the integral equation

$$(3) \qquad \vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\ ds.$$

In order to solve (3), we define the bilinear operator B by

$$(4) \qquad B(\vec{u},\vec{v})(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\ ds.$$

We are going to describe the solutions of (3) when $\vec{u}_0$ belongs to the homogeneous Morrey–Campanato space $\dot{M}^{2,3}$.

Definition 1 :
For $1 < p \le q < \infty$, the homogeneous Morrey–Campanato space $\dot{M}^{p,q}(\mathbb{R}^3)$ is defined as the space of locally p-integrable functions f such that

$$(5) \qquad \sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{3(1/q - 1/p)} \Big(\int_{|x-x_0|<R} |f(x)|^p\ dx\Big)^{1/p} < \infty;$$

the predual of $\dot{M}^{p,q}$ is then the space of functions f which may be decomposed as a series $\sum_{n\in\mathbb{N}} \lambda_n f_n$ with f_n supported by a ball $B(x_n, R_n)$ with $R_n > 0$, $f_n \in L^{p/(p-1)}$, $\|f_n\|_{p/(p-1)} \le R_n^{3(1/q-1/p)}$ and $\sum_{n\in\mathbb{N}} |\lambda_n| < \infty$.

2

For $p = 1 \leq q < \infty$, the homogeneous Morrey–Campanato space $\dot{M}^{1,q}(\mathbb{R}^3)$ is defined as the space of locally bounded measures μ such that

$$(6) \qquad \sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{3(\frac{1}{q}-1)} |\mu|(B(x_0, R)) < \infty;$$

the predual of $\dot{M}^{p,q}$ is then the space of functions f which may be decomposed as a series $\sum_{n \in \mathbb{N}} \lambda_n f_n$ with f_n supported by a ball $B(x_n, R_n)$ with $R_n > 0$, f_n continuous, $\|f_n\|_\infty \leq R_n^{3(1/q-1)}$ and $\sum_{n \in \mathbb{N}} |\lambda_n| < \infty$.

When the initial value $\vec{u}_0$ belongs to $(\dot{M}^{2,3})^3$, we may search for a solution in two ways : either we use the formalism of mild solutions introduced by Kato in the study of solutions on Lebesgue spaces [KAT 84], or we use a mollification of the equations and then construct a weak solution through energy estimates and compactnes criteria (a process introduced by Leray in the study of solutions in L^2 [LER 34]).

We then have the following result :

Theorem 1 :
Let $\vec{u}_0 \in (\dot{M}^{2,3}(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Then the fixed-point problem

$$(7) \qquad \vec{u} = e^{t\Delta} \vec{u}_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}.(\vec{u} \otimes \vec{u}) \; ds$$

can be solved in the following three cases :
(A) Local mild solution for a regular initial value :
If $\vec{u}_0$ belongs more precisely to $(\tilde{M}^{2,3})^3$, where $\tilde{M}^{2,3}$ is the closure of $\mathcal{D}(\mathbb{R}^3)$ in $\dot{M}^{2,3}$, then there exists a positive $T = T(\vec{u}_0)$ such that the sequence $\vec{u}^{(n)}$ defined by

$$(8) \qquad \vec{u}^{(0)} = e^{t\Delta} \vec{u}_0 \text{ and } \vec{u}^{(n+1)} = e^{t\Delta} \vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)})$$

remains bounded in the space $(\mathcal{E}_T)^3$ where $\mathcal{E}_T$ is defined as

$$f \in \mathcal{E}_T \Leftrightarrow f \in L^2_{loc}((0,T) \times \mathbb{R}^3), \; \sup_{0 < t < T} t^{1/4} \|f(t,.)\|_{\dot{M}^{4,6}} < \infty \text{ and } \sup_{0 < t < T} t^{1/2} \|f(t,.)\|_\infty < \infty$$

and normed by

$$\|f\|_{\mathcal{E}_T} = \sup_{0 < t < T} t^{1/4} \|f(t,.)\|_{\dot{M}^{4,6}} + \sup_{0 < t < T} t^{1/2} \|f(t,.)\|_\infty.$$

Moreover, the sequence $\vec{u}^{(n)}$ converges in $(\mathcal{E}_T)^3$ to a solution $\vec{u}$ of (7) which belongs to $\mathcal{C}([0,T], (\dot{M}^{2,3})^3)$.
(B) Global mild solution for a small initial value :
There exists a positive constant ϵ_0 such that when $\vec{u}_0$ belongs to $(\dot{M}^{2,3})^3$ with

$$\|\vec{u}_0\|_{\dot{M}^{2,3}} \leq \epsilon_0,$$

3

then the sequence $\vec{u}^{(n)}$ defined by

$$\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \ \text{and} \ \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)})$$

remains bounded in the space $(\mathcal{E}_\infty)^3$ where $\mathcal{E}_\infty$ is defined as

$$f \in \mathcal{E}_\infty \Leftrightarrow f \in L^2_{loc}((0,+\infty) \times \mathbb{R}^3), \ \sup_{0<t} t^{1/4}\|f(t,.)\|_{\dot{M}^{4,6}} < \infty \ \text{and} \ \sup_{0<t} t^{1/2}\|f(t,.)\|_\infty < \infty$$

and normed by

$$\|f\|_{\mathcal{E}_\infty} = \sup_{0<t} t^{1/4}\|f(t,.)\|_{\dot{M}^{4,6}} + \sup_{0<t} t^{1/2}\|f(t,.)\|_\infty.$$

Moreover, the sequence $\vec{u}^{(n)}$ converges in $(\mathcal{E}_\infty)^3$ to a solution $\vec{u}$ of (7) which satisfies $\sup_{0<t}\|\vec{u}(t,.)\|_{\dot{M}^{2,3}} < \infty$.

(C) Global weak solution for a general initial value :

   Let $\omega \in \mathcal{D}(\mathbb{R}^3)$ with $\omega \geq 0$ and $\int_{\mathbb{R}^3} \omega \ dx = 1$; then the mollified equations are given for $\epsilon > 0$ by

(9)
$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.((\vec{u}*\omega_\epsilon) \otimes \vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{cases}$$

where $\omega_\epsilon = \frac{1}{\epsilon^3}\omega(\frac{x}{\epsilon})$. The equations (9) have a unique global solution $\vec{u}_\epsilon$ such that

(10)
$$\sup_{x_0 \in \mathbb{R}^3, \ R>0, \ t>0} \frac{1}{R+\sqrt{t}} \int_{\|x-x_0\|<R} |\vec{u}_\epsilon(t,x)|^2 \ dx < \infty.$$

We have moreover

(11)
$$\sup_{\epsilon>0} \sup_{x_0 \in \mathbb{R}^3, \ R>0, \ t>0} \frac{1}{R+\sqrt{t}} \int_{\|x-x_0\|<R} |\vec{u}_\epsilon(t,x)|^2 \ dx < \infty.$$

and

(12)
$$\sup_{\epsilon>0} \sup_{x_0 \in \mathbb{R}^3, \ t>0} \frac{1}{\sqrt{t}} \int_0^t \int_{\|x-x_0\|<\sqrt{t}} |\vec{\nabla} \otimes \vec{u}_\epsilon(s,x)|^2 \ ds \ dx < \infty.$$

There exists a sequence $(\epsilon_k)_{k \in \mathbb{N}}$ (depending on $\vec{u}_0$) such that ϵ_k decreases to 0 and $\vec{u}_{\epsilon_k}$ converges strongly in $L^2_{loc}((0,\infty) \times \mathbb{R}^3)$ to a solution $\vec{u}$ of (7). Moreover, this solution satisfies the local energy inequality : the pressure p such that

$$\partial_t \vec{u} = \Delta\vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla}p \ \text{for all} \ \phi \in \mathcal{D}(Q)$$

may be chosen in $L^{3/2}_{loc}((0,\infty) \times \mathbb{R}^3))$ and such that for all $\phi \in \mathcal{D}((0,\infty) \times \mathbb{R}^3)$ with $\phi \geq 0$ we have

(13) $\quad 2\iint |\vec{\nabla} \otimes \vec{u}|^2 \phi \ dx \ dt \leq \iint |\vec{u}|^2(\partial_t\phi + \Delta\phi) \ dx \ dt + \iint (|\vec{u}|^2 + 2p)(\vec{u}.\vec{\nabla})\phi \ dx \ dt.$

4

We discuss the proof of points (A) and (B) in Section 2, and the proof of point (C) in Section 3. Whereas points (A) and (B) are classical (they were first proved, in the setting of Morrey–Campanato spaces, by Kato [KAT 92]; see also [TAY 92], [FED 93] and [CAN 95]), point (C) may appear new (it is however a straightforward consequence of the theory of weak solutions in L^2_{uloc} developed in [LEM 02]).

2. Mild solutions for the Navier–Stokes equations.

In the formalism of mild solutions, we try to solve (7) by the fixed-point algorithm : $\vec{u} = \lim_{n\to\infty} \vec{u}^{(n)}$ with $\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0$ and $\vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)})$.

The resolution of this fixed-point problem is based on a general tool for multilinear equations in a Banach space :

Lemma 1 :
   Let E be a Banach space and T a bounded bilinear operator on E

(14) $$\|T(x,y)\|_E \leq C_0 \|x\|_E \|y\|_E.$$

Let $x_0 \in E$ with $\|x_0\|_E < \frac{1}{4C_0}$. Then, the equation $x = x_0 + T(x,x)$ has at least one solution. More precisely, it has one unique solution $x \in E$ such that $\|x\|_E \leq \frac{1}{2C_0}$.

This lemma is straightfoward, since the mapping $x \mapsto x_0 + T(x,x)$ is then a contraction on the closed ball $\bar{B}(0, \|x_0\|_E + \frac{1}{4C_0})$. We may remark that the existence and uniqueness result holds in the closed ball [AUST 99], [LEM 02], [LEM 05] (even though the mapping $x \mapsto x_0 + T(x,x)$ is no longer a contraction) and can be extended to more general multilinear operators [LEM 05] :

Lemma 2 :
   Let $k \geq 2$. Let E be a Banach space and T_k be a bounded k-linear operator on E

$$\|T_k(x_1, \ldots, x_k)\|_E \leq C_0 \|x_1\|_E \ldots \|x_k\|_E.$$

Let

(15) $$r_k = (C_0 k)^{-\frac{1}{k-1}} \frac{k-1}{k} \quad \text{and} \quad R_k = \frac{k}{k-1} r_k.$$

Let B_k be the closed ball $B_k = \bar{B}(0, R_k)$. If $x_0 \in E$ with $\|x_0\|_E \leq r_k$, then the equation $x = x_0 + T_k(x, \ldots, x)$ has at least one solution. More precisely, it has one unique solution x_∞ in B_k.

5

We may moreover precise the speed of convergence of the Picard-Duhamel approximation of x_∞ [LEM 05]:

Lemma 3 :

   Under the assumptions of Lemma 2, let x_n be defined from x_0 by $x_{n+1} = x_0 + T_k(x_n, \ldots, x_n)$. If $\|x_0\|_E \le r_k$, then we have

$$(16) \qquad \limsup_{n \to \infty} \, n^2 \|x_{n+1} - x_n\|_E \le \frac{2R_k}{k-1}.$$

Of course, if $\|x_0\|_E < r_k$, the rate of convergence is much better (exponentially decreasing, due to contractivity : $\|x_{n+1} - x_n\|_E \le (\frac{\|x_0\|_E}{r_k})^{(n+1)(k-1)} \|x_0\|_E$).

We now come back to the Navier–Stokes equations. The construction of mild solutions relies on the fact that the operator $e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}$ is a matrix of convolutions operators (in the x variable) whose kernels $K_{i,j}(t-s,x)$ are controlled by

$$|K_{i,j}(t-s,x)| \le C \frac{1}{(\sqrt{t-s} + \|x\|)^4}.$$

In 1984, Kato [KAT 84] proved the existence of mild solutions in L^p, $p \ge 3$. For $p > 3$, he used the estimate

$$(17) \qquad \|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{v})\|_p \le C_p (t-s)^{-\frac{1}{2} - \frac{3}{2p}} \|\vec{u}\|_p \|\vec{v}\|_p$$

to prove the boundedness of B on $L^\infty([0,T], (L^p)^3)$:

$$(18) \qquad \|B(\vec{u}, \vec{v})(t,.)\|_p \le C_p t^{\frac{1}{2} - \frac{3}{2p}} \sup_{0<s<t} \|\vec{u}(s,.)\|_p \sup_{0<s<t} \|\vec{v}(s,.)\|_p.$$

For the critical case $p = 3$, inequality (17) becomes

$$(19) \qquad \|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{v})\|_3 \le C \frac{1}{(t-s)} \|\vec{u}\|_3 \|\vec{v}\|_3.$$

This is a very unconvenient estimate for dealing with $\vec{u}$ and $\vec{v}$ in $L^\infty([0,T], (L^3)^3)$, since $\int_0^t \frac{ds}{t-s}$ diverges both at the endpoints $s = 0$ and $s = t$. Kato then used an idea of Weissler [WEI 81], namely to use the smoothing properties of the heat kernel (when applied to $\vec{u}_0 \in (L^3)^3$) to search for the existence of a solution in a smaller space of mild solutions ; indeed, whereas the bilinear operator B is unbounded on $\mathcal{C}([0,T], (L^3(\mathbb{R}^3))^3)$ [ORU 98], it becomes bounded on the smaller space $\{\vec{f} \in \mathcal{C}([0,T], (L^3(\mathbb{R}^3))^3) \, / \, \sup_{0<t<T} \sqrt{t}\|\vec{f}(t,.)\|_\infty < \infty\}$. Thus, we replace the estimate (19) (which leads to a divergent integral) by the estimates

$$(20) \qquad \|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{v})\|_3 \le C \frac{1}{\sqrt{t-s}\sqrt{s}} \|\vec{u}\|_3 \sqrt{s}\|\vec{v}\|_\infty$$

and

(21) $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\|_\infty \leq C\dfrac{1}{\sqrt{t-s}}\min(\dfrac{1}{(t-s)}\|\vec{u}\|_3\|\vec{v}\|_3, \dfrac{1}{s}\sqrt{s}\|\vec{u}\|_\infty\sqrt{s}\|\vec{v}\|_\infty)$

which lead to two convergent integrals.

In the same way, in order to construct mild solutions in $\dot{M}^{2,3}$, one uses the smoothing properties of the heat kernel :

(22) $$\sup_{t>0}\sqrt{t}\,\|e^{t\Delta}\vec{u}_0\|_\infty \leq C\|\vec{u}_0\|_{\dot{M}^{2,3}}.$$

From (22), we find as well that

(23) $$\|e^{t\Delta}\vec{u}_0\|_{\dot{M}^{4,6}} \leq \sqrt{\|e^{t\Delta}\vec{u}_0\|_{\dot{M}^{2,3}}\|e^{t\Delta}\vec{u}_0\|_\infty} \leq \sqrt{C}t^{-1/4}\|\vec{u}_0\|_{\dot{M}^{2,3}}.$$

Thus, $\left(e^{t\Delta}\vec{u}_0\right)_{0<t<T}$ belongs to the space $(\mathcal{E}_T)^3$ defined in Theorem 1, point (A) $(T<\infty)$ or point (B) $(T=\infty)$. Then, the proof of points (A) and (B) relies on the estimates

(24) $$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\|_{\dot{M}^{4,6}} \leq C\dfrac{1}{(t-s)^{1/2}\,s^{3/4}}s^{1/4}\|\vec{u}\|_{\dot{M}^{4,6}}\,s^{1/2}\|\vec{v}\|_\infty$$

and

(25) $\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\|_\infty \leq C\dfrac{1}{\sqrt{t-s}}\min(\dfrac{s^{1/4}\|\vec{u}\|_{\dot{M}^{4,6}}s^{1/4}\|\vec{v}\|_{\dot{M}^{4,6}}}{\sqrt{t-s}\sqrt{s}}, \dfrac{\sqrt{s}\|\vec{u}\|_\infty\sqrt{s}\|\vec{v}\|_\infty}{s})$

which prove that the bilinear operator B is bounded on $(\mathcal{E}_T)^3$:

(26) $$\|B(\vec{u},\vec{v})\|_{\mathcal{E}_T} \leq C_0\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T}$$

where C_0 does not depend on $T\in(0,+\infty]$. Since we have

(27) $$\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} \leq C\|\vec{u}_0\|_{\dot{M}^{2,3}}$$

and

(28) $$\lim_{T\to 0}\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} = 0 \text{ when } \vec{u}_0\in(\tilde{M}^{2,3})^3,$$

we get the convergence of $\vec{u}^{(n)}$ in $(\mathcal{E}_T)^3$ when $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T}\leq\frac{1}{4C_0}$, i.e. when $\vec{u}_0$ is small $(T=+\infty)$ or when $\vec{u}_0$ is regular and T is small enough. Moreover, B is obviously bounded from $(\mathcal{E}_T)^3\times(\mathcal{E}_T)^3$ to $(L^\infty((0,T),\dot{M}^{2,3})^3$. Thus, the convergence of $\vec{u}^{(n)}$ to a solution $\vec{u}$ holds as well in the norm of $(L^\infty(\dot{M}^{2,3})^3$ as in the norm of $(\mathcal{E}_T)^3$. Moreover, when $\vec{u}_0$ is regular, it is easy to check by induction on n that $\vec{u}^{(n)}$ belongs to the space

$$\{\vec{f}\in(\mathcal{E}_T)^3 \,/\, \lim_{t\to 0}t^{1/4}\|\vec{f}\|_{\dot{M}^{4,6}} = \lim_{t\to 0}t^{1/2}\|\vec{f}\|_\infty = 0\}\cap\mathcal{C}([0,T],(\tilde{M}^{2,3})^3)$$

7

which is closed in $(\mathcal{E}_T)^3 \cap (L^\infty(\dot{M}^{2,3}))^3$. Thus, one gets the proof of points (A) and (B) of Theorem 1.

3. Maximal solutions.

The Navier–Stokes equations are locally well-posed in L^∞, since the bilinear operator B is bounded on $(\mathcal{F}_T)^3$, where $\mathcal{F}_T$ is defined as

$$f \in \mathcal{F}_T \Leftrightarrow f \in L^2_{loc}((0,T) \times \mathbb{R}^3) \text{ and } \sup_{0<t<T} \|f(t,.)\|_\infty < \infty$$

and normed by

$$\|f\|_{\mathcal{F}_T} = \sup_{0<t<T} \|f(t,.)\|_\infty.$$

We easily can check that

$$(29) \qquad \|B(\vec{u},\vec{v})(t,.)\|_\infty \le C_0 t^{\frac{1}{2}} \sup_{0<s<t} \|\vec{u}(s,.)\|_\infty \sup_{0<s<t} \|\vec{v}(s,.)\|_\infty.$$

so that, by lemma 1 or 2, we may conclude that the Navier–Stokes equations associated to $\vec{u}_0 \in (L^\infty)^3$

$$(30) \qquad \vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) \, ds$$

has a (unique) mild solution in $(\mathcal{F}_T)^3$ with

$$(31) \qquad T = \frac{1}{(4C_0\|\vec{u}_0\|_\infty)^2}.$$

Thus, the solution described in point (A) of Theorem 1 can be continued on a maximal interval $[0,T^*)$, with

$$(32) \qquad T^* = +\infty \text{ or } \liminf_{t\to T^*} \sqrt{T^*-t}\|\vec{u}(t,.)\|_\infty > \frac{1}{4C_0}.$$

This maximal solution remains in $\dot{M}^{2,3}$:

Proposition 1 :
    Let $\vec{u}_0 \in (\tilde{M}^{2,3})^3$ and let $\vec{u}$ be the maximal continuation in $L^\infty_{loc}((0,T^*),(L^\infty)^3)$ of the mild solution associated to $\vec{u}_0$ by Theorem 1. Then, $\vec{u} \in \mathcal{C}([0,T^*),(\tilde{M}^{2,3})^3)$.

This is a direct consequence of Theorem 1. Indeed, if $T < T^*$ is such that $\vec{u} \in \mathcal{C}([0,T),(\tilde{M}^{2,3})^3)$ and if $\delta \in (0,T)$, then $\vec{u}$ is uniformly bounded on $[\delta,T]$; moreover, we have, for $t \in [\delta,T]$,

$$(33) \qquad \vec{u}(t,.) = e^{(t-\delta)\Delta}\vec{u}(\delta,.) - \int_\delta^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) \, ds,$$

8

hence

$$(34) \qquad \|\vec{u}(t,.)\|_{\dot{M}^{2,3}} \leq \|\vec{u}(\delta,.)\|_{\dot{M}^{2,3}} + C \sup_{\delta \leq s \leq T} \|\vec{u}(s,.)\|_{\infty} \int_{\delta}^{t} \frac{1}{(t-s)} \|\vec{u}(s,.)\|_{\dot{M}^{2,3}} \, ds$$

and the Gronwall lemma shows that the norm $\|\vec{u}(t,.)\|_{\dot{M}^{2,3}}$ remains bounded as $t \to T$. If $0 < t_0 < T$, the Kato algorithm provides a mild solution in $\mathcal{C}([t_0, t_0 + t_1], (\tilde{M}^{2,3})^3)$ where t_1 is such that, for a positive constant ϵ_0,

$$\sup_{0 < t < t_1} t^{1/4} \|e^{t\Delta} \vec{u}(t_0,.)\|_{\dot{M}^{4,6}} + t^{1/2} \|e^{t\Delta} \vec{u}(t_0,.)\|_{\infty} \leq \epsilon_0,$$

hence, at least, for t_1 such that

$$t_1^{1/4} \sqrt{\|\vec{u}(t_0,.)\|_{\dot{M}^{2,3}} \|\vec{u}(t_0,.)\|_{\infty}} + t_1^{1/2} \|\vec{u}(t_0,.)\|_{\infty} \leq \epsilon_0,$$

while the same Kato algorithm provides a mild solution in $L^{\infty}([t_0, t_0 + t_2], (L^{\infty})^3)$ where t_2 is such that, for a positive constant ϵ_1,

$$t_2^{1/2} \|\vec{u}(t_0,.)\|_{\infty} \leq \epsilon_1.$$

Moreover, by uniqueness in $L^{\infty}([t_0, t_0 + t_2], (L^{\infty})^3)$, this mild solution coincides withthe maximal solution $\vec{u}$. Thus, we may conclude that $\vec{u}$ remains in $\mathcal{C}([0, t_0 + \inf(t_1, t_2)), (\tilde{M}^{2,3})^3)$ and that we may choose t_0 such that $t_0 + \inf(t_1, t_2) > T$. This proves Proposition 1.

4. Global weak solutions for the Navier–Stokes equations.

The proof of Point (C) in Theorem 1 is based on some local energy estimates for the solution of the mollified equations

$$(35 - \epsilon) \qquad \begin{cases} \partial_t \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \mathbb{P}\vec{\nabla}.((\vec{u}_\epsilon * \omega_\epsilon) \otimes \vec{u}_\epsilon) \\ \vec{\nabla}.\vec{u}_\epsilon = 0 \\ \vec{u}_\epsilon(0,.) = \vec{u}_0 \end{cases}$$

These estimates are described in [LEM 02] in the study of the Navier–Stokes equations in L_{uloc}^2, where L_{uloc}^2 is the space of uniformly locally square integrable functions :

$$f \in L_{uloc}^2 \Leftrightarrow \sup_{x_0 \in \mathbb{R}^3} \int_{\|x - x_0\| < 1} |f(x)|^2 \, dx < \infty$$

normed with

$$\|f\|_{L_{uloc}^2} = \sup_{x_0 \in \mathbb{R}^3} \sqrt{\int_{\|x - x_0\| < 1} |f(x)|^2 \, dx}.$$

We recall the main result proved in [LEM 02] :

9

Proposition 2 :

Let $\vec{u}_0 \in (L^2_{uloc}(\mathbb{R}^3))^3$ be such that $\vec{\nabla}.\vec{u}_0 = 0$. Define $\alpha_0 = \|\vec{u}_0\|_{L^2_{uloc}}$ and $\alpha_1 = \min(1, \alpha_0)$. Then, there exists a positive constant C_0 (which does not depend on $\vec{u}$ nor on ϵ) such that the equations $(35-\epsilon)$ have a solution $\vec{u}_\epsilon$ on $(0, T_0) \times \mathbb{R}^3$ with $T_0 = \min(1, \frac{\alpha_1^2}{\alpha_0^2 C_0^4})$ and such that for all $0 < t < T_0$ we have

$$(36) \qquad \|\vec{u}_\epsilon(t, .)\|_{L^2_{uloc}} \le \sqrt{C_0}\|\vec{u}_0\|_{L^2_{uloc}} \left(1 - \frac{\alpha_0^2 C_0^4}{\alpha_1^2}t\right)^{-1/4}$$

and

$$(37) \qquad \sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{\|x-x_0\|<1} |\vec{\nabla} \otimes \vec{u}_\epsilon(s, x))|^2 \, dx \, ds \le C_0\|\vec{u}_0\|_{L^2_{uloc}}^2 \left(1 - \frac{\alpha_0^2 C_0^4}{\alpha_1^2}t\right)^{-1/2}.$$

We now use the scaling property of the Navier–Stokes equations. When $\vec{v}_\epsilon$ is a solution on $(0, T) \times \mathbb{R}^3$ of the Cauchy problem associated to equations $(35 - \epsilon)$ and initial value $\vec{v}_0$, then, for every $\lambda > 0$, $\lambda\vec{v}_\epsilon(\lambda^2 t, \lambda x)$ is a solution on $(0, \lambda^{-2}T) \times \mathbb{R}^3$ of the Cauchy problem associated to equations $(35 - \frac{\epsilon}{\lambda})$ with initial value $\lambda\vec{v}_0(\lambda x)$. We now use the following points :
- we have uniqueness of the solutions of $(35 - \epsilon)$ in the space $L^\infty((L^2_{uloc})^3)$
- the constant C_0 in Proposition 2 does not depend on ϵ
- when $u \in \dot{M}^{2,3}$, then $\sup_{\lambda>0} \|\lambda u(\lambda x)\|_{L^2_{uloc}} = \|u\|_{\dot{M}^{2,3}}$.

Thus, applying Proposition 2 to $\lambda\vec{u}_0(\lambda x)$ and to equations $(35-\frac{\epsilon}{\lambda})$, we find a solution $\vec{v}_{\frac{\epsilon}{\lambda}}$ defined on $(0, T_0)$ where T_0 depends only on $\|\vec{u}_0\|_{\dot{M}^{2,3}}$ and not on ϵ nor λ, hence the solution $\vec{u}_\epsilon$ of $(35 - \epsilon)$ associated to $\vec{u}_0$ satisfies $\vec{u}_\epsilon = \frac{1}{\lambda}\vec{v}_{\frac{\epsilon}{\lambda}}(\frac{t}{\lambda^2}, \frac{x}{\lambda})$, hence is defined on $(0, \lambda^2 T_0)$ and satisfies for all $0 < t < \lambda^2 T_0$

$$(38) \qquad \sup_{x_0 \in \mathbb{R}^3} \int_{\|x-x_0\|<\lambda} |\vec{u}_\epsilon(t, x)|^2 \, dx \le C_0\lambda\|\lambda u_0(\lambda x)\|_{L^2_{uloc}}^2 \left(1 - \frac{\alpha_0^2 C_0^4}{\alpha_1^2\lambda^2}t\right)^{-1/2}$$

and

$$(39) \qquad \sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{\|x-x_0\|<\lambda} |\vec{\nabla} \otimes \vec{u}_\epsilon(s, x))|^2 \, dx \, ds \le C_0\lambda\|\lambda u_0(\lambda x)\|_{L^2_{uloc}}^2 \left(1 - \frac{\alpha_0^2 C_0^4}{\alpha_1^2\lambda^2}t\right)^{-1/2}.$$

Since $\vec{u}_\epsilon$ is defined on $(0, \lambda^2 T_0)$ for every positive λ, $\vec{u}_\epsilon$ is defined on $(0, +\infty)$. Moreover, we may estimate $\int_{\|x-x_0\|<R} |\vec{u}_\epsilon(t, x)|^2 \, dx$ by using (38) : if $t \le \frac{1}{4}R^2 T_0$, then

$$\int_{\|x-x_0\|<R} |\vec{u}_\epsilon(t, x)|^2 \, dx \le C_0 R\|R u_0(Rx)\|_{L^2_{uloc}}^2 \left(1 - \frac{\alpha_0^2 C_0^4}{\alpha_1^2 R^2}t\right)^{-1/2} \le 2C_0 R\|u_0\|_{\dot{M}^{2,3}}^2,$$

while if $t \ge \frac{1}{4}R^2 T_0$ we have

$$\int_{\|x-x_0\|<R} |\vec{u}_\epsilon(t, x)|^2 \, dx \le \int_{\|x-x_0\|<\sqrt{\frac{4t}{T_0}}} |\vec{u}_\epsilon(t, x)|^2 \, dx \le 2C_0\sqrt{\frac{4t}{T_0}}\|u_0\|_{\dot{M}^{2,3}}^2.$$

10

Thus, we have proved

(40)
$$\sup_{\epsilon>0} \sup_{x_0\in\mathbb{R}^3,\ R>0,\ t>0} \frac{1}{R+\sqrt{t}} \int_{\|x-x_0\|<R} |\vec{u}_\epsilon(t,x)|^2 \ dx < \infty.$$

and we get similarly

(41)
$$\sup_{\epsilon>0} \sup_{x_0\in\mathbb{R}^3,\ t>0} \frac{1}{\sqrt{t}} \int_0^t \int_{\|x-x_0\|<\sqrt{t}} |\vec{\nabla}\otimes\vec{u}_\epsilon(s,x)|^2 \ ds \ dx < \infty.$$

Those estimates then allows one to use the limiting process of Leray [LER 34] to extract a subsequence $\vec{u}_{\epsilon_n}$ that is convergent to a solution $\vec{u}$ of the Navier-Stokes equations associated to $\vec{u}_0$. More precisely, when ϵ_n converges to 0, we have for all $\phi \in \mathcal{D}((0,T_0)\times\mathbb{R}^3)$ strong convergence of $\phi\vec{u}_\epsilon$ in $L^p((0,T_0),(L^2)^3)$ for all $p < \infty$ and weak convergence in $L^2((0,T_0),(H^1)^3))$. The details of the proof (and the proof of the local energy inequality) are exactly similar to the case of weak solutions in L^2_{uloc} [LEM 02].

5. Comparison of weak and mild solutions.

If $\vec{u}_0 \in (\tilde{M}^{2,3})^3$, we have a maximal regular solution which is described by Point (A) of Theorem 1 and Proposition 1, and global weak solutions which are described by Point (C) if Theorem 1. As a matter of fact, those solutions coincide on the domain of definition of the regular solution. We have more precisely the following convergence theorem :

Theorem 2 :
    Let $\vec{u}_0 \in (\tilde{M}^{2,3})^3$ and let $\vec{u}$ be the maximal continuation in $L^\infty_{loc}((0,T^*),(L^\infty)^3)$ of the mild solution associated to $\vec{u}_0$ by Theorem 1, and let $\vec{u}_\epsilon$ be the solution of the mollified equations (9). Then, for every $T \in (0,T^*)$, there exists a positive ϵ_T such that, for every $\epsilon \in (0,\epsilon_T)$, $\vec{u}_\epsilon \in \mathcal{C}([0,T],(\tilde{M}^{2,3})^3)$ and moreover, we have

(42)
$$\lim_{\epsilon\to 0} \|\vec{u} - \vec{u}_\epsilon\|_{\mathcal{C}([0,T],(\tilde{M}^{2,3})^3)} = 0.$$

Indeed, we have, for any positive θ,

(43)
$$\|B(\vec{u},\vec{v})\|_{\mathcal{E}_\theta} \leq C_0\|\vec{u}\|_{\mathcal{E}_\theta}\|\vec{v}\|_{\mathcal{E}_\theta}$$

and

(44)
$$\|B(\vec{u}*\omega_\epsilon,\vec{v})\|_{\mathcal{E}_\theta} \leq C_0\|\vec{u}*\omega_\epsilon\|_{\mathcal{E}_\theta}\|\vec{v}\|_{\mathcal{E}_\theta} \leq C_0\|\vec{u}\|_{\mathcal{E}_\theta}\|\vec{v}\|_{\mathcal{E}_\theta}$$

where the positive constant C_0 does not depend on θ. If $\vec{v}_0 \in (\tilde{M}^{2,3})^3)$ satisfies the inequality $\|e^{t\Delta}\vec{v}_0\|_{\mathcal{E}_\theta} \leq \frac{1}{4C_0}$, then the Kato algorithm provides a mild solution in $\mathcal{C}([0,\theta],(\tilde{M}^{2,3})^3)$ of the Navier-Stokes equations or of the mollified Navier–Stokes equations associated to

11

the initial value $\vec{v}_0$. By compactness of $[0, T]$, hence of the set $\{\vec{u}(t, .) \, / \, 0 \leq t \leq T\}$ in $(\tilde{M}^{2,3})^3$, there exists a positive θ such that

$$(45) \qquad \sup_{0 \leq t \leq T} \|e^{s\Delta}\vec{u}(t, .)\|_{\mathcal{E}_\theta} \leq \frac{1}{16C_0}$$

Now, we are going to show that, if $t_0 \in [0, T]$, if $\vec{u}_\epsilon(t_0, .) \in (\tilde{M}^{2,3})^3$ for $0 < \epsilon < \epsilon(t_0)$ and $\lim_{\epsilon \to 0} \|\vec{u}(t_0, .) - \vec{u}_\epsilon(t_0, .)\|_{\dot{M}^{2,3}} = 0$, then there exists a positive $\eta(t_0)$ such that, for $0 < \epsilon < \eta(t_0)$, $\vec{u}_\epsilon \in \mathcal{C}([t_0, t_0 + \theta], (\tilde{M}^{2,3})^3)$ and

$$\lim_{\epsilon \to 0} \|\vec{u} - \vec{u}_\epsilon\|_{\mathcal{C}([t_0, t_0+\theta], (\tilde{M}^{2,3})^3)} = 0.$$

Since θ does not depend on t_0, we shall have proved the theorem.

Since $\lim_{\epsilon \to 0} \|\vec{u}(t_0, .) - \vec{u}_\epsilon(t_0, .)\|_{\dot{M}^{2,3}} = 0$, we find that

$$\lim_{\epsilon \to 0} \|e^{t\Delta}\vec{u}(t_0, .) - e^{t\Delta}\vec{u}_\epsilon(t_0, .)\|_{\mathcal{E}_\theta} = 0.$$

Thus, there exists a positive $\eta(t_0)$ such that, for $0 < \epsilon < \eta(t_0)$, $\|e^{t\Delta}\vec{u}_\epsilon(t_0, .)\|_{\mathcal{E}_\theta} \leq \frac{1}{8C_0}$. This proves that, for $0 < \epsilon < \eta(t_0)$, $\vec{u}_\epsilon \in \mathcal{C}([t_0, t_0 + \theta], (\tilde{M}^{2,3})^3)$, with $\|\vec{u}_\epsilon(t_0 + t, .)\|_{\mathcal{E}_\theta} \leq \frac{1}{4C_0}$. Now, for $0 \leq t \leq \theta$, we define $\vec{w}(t, .) = \vec{u}(t_0 + t, .)$ and $\vec{w}_\epsilon(t, .) = \vec{u}_\epsilon(t_0 + t, .)$; we have

$$\vec{w}_\epsilon(t, .) - \vec{w}(t, .) = e^{t\Delta}(\vec{u}_\epsilon(t_0, .) - \vec{u}(t_0, .)) \\ + B(\vec{w}, \vec{w} - \vec{w}_\epsilon) + B((\vec{w} - \vec{w}_\epsilon) * \omega_\epsilon, \vec{w}_\epsilon) + B(\vec{w} - (\vec{w} * \omega_\epsilon), \vec{w}_\epsilon)$$

which gives

$$
\begin{aligned}
\|\vec{w}_\epsilon(t, .) - \vec{w}(t, .)\|_{\mathcal{E}_\theta} &\leq \|e^{t\Delta}(\vec{u}_\epsilon(t_0, .) - \vec{u}(t_0, .))\|_{\mathcal{E}_\theta} \\
&+ C_0\|\vec{w}\|_{\mathcal{E}_\theta}\|\vec{w} - \vec{w}_\epsilon\|_{\mathcal{E}_\theta} + C_0\|(\vec{w} - \vec{w}_\epsilon) * \omega_\epsilon\|_{\mathcal{E}_\theta}\|\vec{w}_\epsilon\|_{\mathcal{E}_\theta} \\
&+ C_0\|\vec{w} - \vec{w} * \omega_\epsilon\|_{\mathcal{E}_\theta}\|\vec{w}_\epsilon\|_{\mathcal{E}_\theta} \\
&\leq C\|\vec{u}_\epsilon(t_0, .) - \vec{u}(t_0, .)\|_{\dot{M}^{2,3}} \\
&+ C_0\frac{1}{4C_0}\|\vec{w} - \vec{w}_\epsilon\|_{\mathcal{E}_\theta} + C_0\|\vec{w} - \vec{w}_\epsilon\|_{\mathcal{E}_\theta}\frac{1}{4C_0} \\
&+ C_0\|\vec{w} - \vec{w} * \omega_\epsilon\|_{\mathcal{E}_\theta}\frac{1}{4C_0}
\end{aligned}
$$

and finally

$$(46) \qquad \|\vec{w}_\epsilon(t, .) - \vec{w}(t, .)\|_{\mathcal{E}_\theta} \leq 2C\|\vec{u}_\epsilon(t_0, .) - \vec{u}(t_0, .)\|_{\dot{M}^{2,3}} + \frac{1}{2}\|\vec{w} - \vec{w} * \omega_\epsilon\|_{\mathcal{E}_\theta}.$$

The operators $f \mapsto f * \omega_\epsilon$ are equicontinuous on $\mathcal{E}_\theta$ and when $\varphi \in \mathcal{D}([0, \theta] \times \mathbb{R}^3)$ the functions $\varphi * \omega_\epsilon$ converge to φ in $\mathcal{D}$ (hence in $\mathcal{E}_\theta$) as ϵ goes to O. Moreover, $\vec{w}$ belongs to the closure of $(\mathcal{D}([0, \theta] \times \mathbb{R}^3))^3$ in $(\mathcal{E}_\theta)^3$. Thus, we have $\lim_{\epsilon \to 0} \|\vec{w} - \vec{w} * \omega_\epsilon\|_{\mathcal{E}_\theta} = 0$. Finally, we get $\lim_{\epsilon \to 0} \|\vec{w}_\epsilon(t, .) - \vec{w}(t, .)\|_{\mathcal{E}_\theta} = 0$, which proves that $\lim_{\epsilon \to 0} \|\vec{u} - \vec{u}_\epsilon\|_{\mathcal{C}([t_0, t_0+\theta], (\tilde{M}^{2,3})^3)} = 0$.

6. The condition $\lim_{t \to 0} \sqrt{t}\|\vec{u}\|_\infty = 0$.

The results described above may be partially extended to the case when $\vec{u}_0 \notin (\tilde{M}^{2,3})^3$, provided that we assume that $\vec{u}_0 \in (\dot{M}^{2,3})^3$ and $\lim_{t\to 0} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$. Under those assumptions, we have the following results :

i) local existence : there exists a positive $T = T(\vec{u}_0)$ such that the sequence $\vec{u}^{(n)}$ defined by

$$\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \text{ and } \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)})$$

remains bounded in the space $(\mathcal{E}_T)^3$ where $\mathcal{E}_T$ is defined as

$$f \in \mathcal{E}_T \Leftrightarrow f \in L^2_{loc}((0,T)\times\mathbb{R}^3), \sup_{0<t<T} t^{1/4}\|f(t,.)\|_{\dot{M}^{4,6}} < \infty \text{ and } \sup_{0<t<T} t^{1/2}\|f(t,.)\|_\infty < \infty$$

Moreover, the sequence $\vec{u}^{(n)}$ converges in $(\mathcal{E}_T)^3$ to a solution $\vec{u}$ of (7) which belongs to $\mathcal{C}([0,T],(\dot{M}^{2,3})^3)$ and satisfies $\lim_{t\to 0} \sqrt{t}\|\vec{u}\|_\infty = 0$.

ii) uniqueness : if $\vec{u}_1$ and $\vec{u}_2$ are two solutions of (7) (associated to the same initial value $\vec{u}_0$) which satisfy (for $j = 1, 2$) $\vec{u}_j \in L^\infty([0,T],(\dot{M}^{2,3})^3)$ and $\lim_{t\to 0} \sqrt{t}\|\vec{u}_j\|_\infty = 0$, then $\vec{u}_1 = \vec{u}_2$.

iii) maximal solutions : let $\vec{u}$ be the maximal continuation in $L^\infty_{loc}((0,T^*),(L^\infty)^3)$ of the mild solution associated to $\vec{u}_0$ by Point i). Then, $\vec{u} \in \mathcal{C}((0,T^*),(\dot{M}^{2,3})^3)$. More precisely, $B(\vec{u},\vec{u})$ is locally Hölderian (of exponent $1/2$) [MEY 99] [LEM 02] and is continuous at $t = 0$. However, $t \mapsto e^{t\Delta}\vec{u}_0$ may be not (strongly) continous at $t = 0$.

iv) weak solutions : we cannot identify the weak solutions given by Point C) in Theorem 1 to the maximal solution given by Point iii). However, if $\vec{u}$ is such a weak solution and if moreover $\vec{u} \in L^\infty_{loc}((0,T^*),(L^\infty)^3)$ with $\lim_{t\to 0} \sqrt{t}\|\vec{u}\|_\infty = 0$, then $\vec{u}$ is the maximal solution given by Point iii). This is a direct consequence of Point ii), since in that case $\vec{u}$ obviously belongs to $L^\infty([0,T],(\dot{M}^{2,3})^3)$ for every $T < T^*$: indeed, we already know that $\int_{B(x_0,R)} |\vec{u}(t,x)|^2 \, dx \leq C(R + \sqrt{t})$; moreover, we have $|\vec{u}(t,x)| \leq C_T t^{-1/2}$, hence $\int_{B(x_0,R)} |\vec{u}(t,x)|^2 \, dx \leq C'_T R^3 t^{-1}$, and we conclude since $\min(R + \sqrt{t}, 2R^3 t^{-1}) = 2R$.

7. Comparison of mild solutions.

When $\vec{u}_0 \in (\dot{M}^{2,3})^3$, Kato's algorithm may converge to a solution of the Navier–Stokes equations on other norms than the norm of $\mathcal{E}_T$. Indeed, we have seen that there exists a positive constant C_0 such that

(47) $$\|B(\vec{u},\vec{v})\|_{\mathcal{E}_T} \leq C_0\|\vec{u}\|_{\mathcal{E}_T}\|\vec{v}\|_{\mathcal{E}_T};$$

Thus, the sequence $\vec{u}^{(n)}$ defined by

$$\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \text{ and } \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)})$$

will converge in $\mathcal{E}_T^3$ to a solution $\vec{u}$ of (7), as soon as $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} \leq \frac{1}{4C_0}$.

On the other hand, we may use the embedding $\dot{M}^{2,3} \subset BMO^{(-1)}$. Koch and Tataru [KOCT 01] have proved that there exists a positive constant C_1 such that

(48) $$\|B(\vec{u},\vec{v})\|_{\mathcal{G}_T} \leq C_0\|\vec{u}\|_{\mathcal{G}_T}\|\vec{v}\|_{\mathcal{G}_T}$$

13

where $\mathcal{G}_T$ is defined as

$$f \in \mathcal{G}_T \Leftrightarrow \begin{cases} f \in L^2_{loc}((0,T) \times \mathbb{R}^3) \\ \sup_{0<t<T, \ x_0 \in \mathbb{R}^3} t^{-3/2} \int_0^t \int_{B(x_0,\sqrt{t})} |f(s,x)|^2 \ dx \ ds < \infty \\ \sup_{0<t<T} t^{1/2} \|f(t,.)\|_\infty < \infty \end{cases}$$

Thus, the sequence $\vec{u}^{(n)}$ will converge in $\mathcal{G}_T^3$ to a solution $\vec{u}$ of (7), as soon as $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{G}_T} \leq \frac{1}{4C_1}$. Due to Lemma 3, this sequence will then satisfy

$$(49) \qquad \sup_{0<t<T} \sqrt{t}\|\vec{u}^{(n+1)} - \vec{u}^{(n)}\|_\infty \leq C\frac{1}{(n+1)^2}.$$

We may as well use the embedding $L^{3,\infty} \subset \dot{M}^{2,3}$. Meyer [MEY 99] has proved that there exists a positive constant C_2 such that

$$(50) \qquad \|B(\vec{u},\vec{v})\|_{\mathcal{H}_T} \leq C_2 \|\vec{u}\|_{\mathcal{H}_T} \|\vec{v}\|_{\mathcal{H}_T}$$

where $\mathcal{H}_T$ is defined as $\mathcal{H}_T = \mathcal{C}_w([0,T], L^{3,\infty}(\mathbb{R}^3))$ ($f \in \mathcal{H}_T$ means that f is continuous and bounded from $(0,T]$ to $L^{3,\infty}$ and is weakly continuous at $t = 0$). Thus, the sequence $\vec{u}^{(n)}$ will converge in $\mathcal{H}_T^3$ to a solution $\vec{u}$ of (7), as soon as $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{H}_T} \leq \frac{1}{4C_2}$. Due to Lemma 3, this sequence will then satisfy

$$(51) \qquad \sup_{0<t<T} \|\vec{u}^{(n+1)} - \vec{u}^{(n)}\|_{\dot{M}^{2,3}} \leq C\frac{1}{(n+1)^2}.$$

A frequently asked question on mild solutions is the regularity of those solutions. Since Meyer's solutions do not use the smoothing properties of the heat kernel, one may wonder if the solution $\vec{u}$ obtained in $(\mathcal{H}_T)^3$ will satisfy $\sup_{0<t<T} \sqrt{t}\|\vec{u}(t,.)\|_\infty < \infty$. Such a question is raised for instance in [GRU 06]. As a matter of fact, the answer is positive, due to the persistency formalism developed in [FURLZZ 00] [LEM 02] [LEM 05] which gives a more precise answer :

Theorem 3 :
Let $\vec{u}_0 \in (\dot{M}^{2,3}(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Let the sequence $\vec{u}^{(n)}$ be defined by

$$(52) \qquad \vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \ and \ \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)}).$$

Let $T \in (0, +\infty]$. Then the following assertions are equivalent :

$$(A) \qquad \sum_{n \in \mathbb{N}} \sup_{0<t<T} \|\vec{u}^{(n+1)}(t,.) - \vec{u}^{(n)}(t,.)\|_{\dot{M}^{2,3}} < \infty$$

14

(B)
$$\sum_{n \in \mathbb{N}} \sup_{0<t<T} \sqrt{t} \|\vec{u}^{(n+1)}(t,.) - \vec{u}^{(n)}(t,.)\|_\infty < \infty$$

Let A_T and B_T be the norms

$$A_T(f) = \sup_{0<t<T} \|f(t,.)\|_{\dot{M}^{2,3}} \text{ and } B_T(f) = \sqrt{t}\|f(t,.)\|_\infty.$$

We easily get

$(53) \quad A_T(B(\vec{u},\vec{v})) \leq C \min(A_T(\vec{u})B_T(\vec{v}), B_T(\vec{u})A_T(\vec{v})) \leq C\sqrt{A_T(\vec{u})B_T(\vec{v})B_T(\vec{u})A_T(\vec{v})}.$

On the other hand, we have, for $0 < \tau < t < T$

$$\|B(\vec{u},\vec{v}))\|_\infty \leq C \int_0^\tau \frac{ds}{(t-s)^{3/2}} A_T(\vec{u})A_T(\vec{v}) + C \int_\tau^t \frac{ds}{s\sqrt{t-s}} B_T(\vec{u})B_T(\vec{v}).$$

Hence, we have

$$\|B(\vec{u},\vec{v}))\|_\infty \leq C(t-\tau)^{-1/2} A_T(\vec{u})A_T(\vec{v}) + C\tau^{-1}\sqrt{t-\tau} B_T(\vec{u})B_T(\vec{v}).$$

For
$$\tau = t \frac{B_T(\vec{u})B_T(\vec{v})}{A_T(\vec{u})A_T(\vec{v}) + B_T(\vec{u})B_T(\vec{v})}$$

we find

$(54) \qquad B_T(B(\vec{u},\vec{v})) \leq C\sqrt{A_T(\vec{u})A_T(\vec{v})(A_T(\vec{u})A_T(\vec{v}) + B_T(\vec{u})B_T(\vec{v}))}.$

(53) and (54) then give

$(55) \quad A_T(B(\vec{u},\vec{v})) + B_T(B(\vec{u},\vec{v})) \leq C\sqrt{A_T(\vec{u})A_T(\vec{v})(A_T(\vec{u}) + B_T(\vec{u}))(A_T(\vec{v}) + B_T(\vec{v}))}.$

From (55) (to get $(A) \Rightarrow (B)$) and (53) (to get $(B) \Rightarrow (A)$), we see that Theorem 3 is a direct consequence of the following lemma :

Lemma 4 :
    Let N_1 and N_2 be two norms on a vector space E and T a bilinear operator on E such that there exists a positive constant C_0 such that

$(56) \qquad \forall x, y \in E \quad N_2(T(x,y)) \leq C_0 \sqrt{N_1(x)N_2(x)N_1(y)N_2(y)}.$

Let $x_0 \in E$ and let the sequence (x_n) be defined by

$(57) \qquad\qquad\qquad x_{n+1} = T(x_n, x_n).$

If $\sum_{n \in \mathbb{N}} N_1(x_{n+1} - x_n) < \infty$, then $\sum_{n \in \mathbb{N}} N_2(x_{n+1} - x_n) < \infty$.

15

Indeed, let $\alpha_0 = N_1(x_0)$, $\alpha_{n+1} = N_1(x_{n+1} - x_n)$ and, similarly, $\beta_0 = N_2(x_0)$, $\beta_{n+1} = N_2(x_{n+1} - x_n)$. We write

$$x_{n+2} - x_{n+1} = T(x_{n+1}, x_{n+1}) - T(x_n, x_n) = T(x_{n+1} - x_n, x_{n+1}) + T(x_n, x_{n+1} - x_n),$$

hence

$$(58) \qquad \beta_{n+2} \leq C_0 \sqrt{2\alpha_{n+1}\beta_{n+1}(\sum_{k=0}^{n+1} \alpha_k)(\sum_{j=0}^{n+1} \beta_j)} \leq \frac{1}{2}\beta_{n+1} + C_0^2 \alpha_{n+1}(\sum_{k=0}^{n+1} \alpha_k)(\sum_{j=0}^{n+1} \beta_j).$$

Now, if N_0 is big enough to grant that

$$C_0^2 (\sum_{k=N_0+1}^{+\infty} \alpha_{k+1})(\sum_{j=0}^{+\infty} \alpha_j) \leq \frac{1}{4}$$

we get that, for $N \geq N_0 + 2$ we have

$$\sum_{j=0}^{N} \beta_j \leq \beta_0 + \beta_1 + \frac{1}{2}\sum_{j=1}^{N-1} \beta_j + \sum_{j=1}^{N_0} C_0^2 \alpha_j (\sum_{k=0}^{j} \alpha_k)(\sum_{l=0}^{j} \beta_l) + \frac{1}{4}\sum_{j=0}^{N-1} \beta_j$$

and thus

$$(59) \qquad \sum_{j=0}^{\infty} \beta_j \leq 4\big(\beta_0 + \beta_1 + \sum_{j=1}^{N_0} C_0^2 \alpha_j (\sum_{k=0}^{j} \alpha_k)(\sum_{l=0}^{j} \beta_l)\big)$$

8. Serrin's uniqueness criterion.

Recall that we consider the Navier-Stokes equations on the whole space $\mathbb{R}^3$:

$$(60) \qquad \begin{cases} \exists p \in \mathcal{D}'((0,T) \times \mathbb{R}^3) \quad \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

We shall speak of weak solutions when the derivatives in (60) are taken in the sense of distributions theory.

Leray [LER 34] studied the Cauchy initial value problem for equations (60) with a square-integrable initial value. He proved the existence of weak solutions, which satisfy moreover an energy inequality :

Definition 2 : (Leray solutions)
A Leray solution on $(0,T)$ for the Navier-Stokes equations with initial value $\vec{u}_0 \in (L^2)^3$ is a solution $\vec{u}$ such that
i) $t \mapsto \vec{u}(t,.)$ is weakly continuous from $(0,T)$ to $(L^2)^3$

16

ii) $\vec{u}(t,.)$ converges weakly to $\vec{u}_0$ as $t \to 0^+$,
iii) $\vec{u} \in L^\infty((0,T),(L^2)^3) \cap L^2((0,T),(\dot{H}^1)^3)$,
iv) $\vec{u}$ satisfies the Leray energy inequality

(61) $\qquad$ *for all $t \in (0,T)$, $\|\vec{u}(t,.)\|_2^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\vec{\nabla} \otimes \vec{u}|^2 \ dx \ ds \le \|\vec{u}_0\|_2^2.$*

Weak continuity of (a representant of) $\vec{u}$ is a consequence of the Navier–Stokes equations and of the hypothesis iii). An easy consequence of inequality (61) is then the strong continuity at $t = 0$:

$$\lim_{t \to 0^+} \|\vec{u} - \vec{u}_0\|_2 = 0.$$

But it is still not known whether we have continuity for all time t and whether we have uniqueness in the class of Leray solutions. Serrin's theorem [SER 62] gives a criterion for uniqueness :

Proposition 3 : (Serrin's uniqueness theorem)
Let $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^3$ (for some $T \in (0,+\infty]$) with initial value $\vec{u}_0$ such that :
i) $\vec{u} \in L^\infty((0,T),(L^2(\mathbb{R}^3)^3)$;
ii) $\vec{u} \in L^2((0,T),(\dot{H}^1(\mathbb{R}^3)^3)$;
iii) For some $r \in [0,1)$, $\vec{u}$ belongs to $(L^\sigma((0,T),L^{3/r}))^3$ with $2/\sigma = 1 - r$.
Then, $\vec{u}$ satisfies the Leray energy inequality and it is the unique Leray solution associated to $\vec{u}_0$ on $(0,T)$.

The limit case $r = 1$ is dealt with Von Wahl's theorem [WAH 85] :

Proposition 4 : (Sohr and Von Wahl's uniqueness theorem)
Let $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^3$ (for some $T \in (0,+\infty]$) with initial value $\vec{u}_0$ such that :
i) $\vec{u} \in L^\infty((0,T),(L^2(\mathbb{R}^3)^3)$;
ii) $\vec{u} \in L^2((0,T),(H^1(\mathbb{R}^3)^3)$;
iii) $\vec{u}$ belongs to $(\mathcal{C}([0,T],L^3))^3$.
Then, $\vec{u}$ satisfies the Leray energy inequality and it is the unique Leray solution associated to $\vec{u}_0$ on $(0,T)$.

The theorem of Sohr and Von Wahl has been generalized to the case of a solution $\vec{u} \in (L^\infty([0,T],L^3))^3$ (instead of $(\mathcal{C}([0,T],L^3))^3$) by Kozono and Sohr [KOZS 96].

We sketch the proof of those well-known propositions. Let $\vec{v}$ be another solution associated to $\vec{u}_0$ on $(0,T)$ (with associated pressure q) such that $\vec{v} \in L^\infty((0,T),(L^2(\mathbb{R}^3)^3) \cap L^2((0,T),(\dot{H}^1(\mathbb{R}^3)^3)$. The main point is to check the validity of the formula

(62) $\qquad \partial_t \int \vec{u}.\vec{v} \ dx = -2 \int \vec{\nabla} \otimes \vec{u}.\vec{\nabla} \otimes \vec{v} \ dx + \int \vec{u}.(\vec{u}.\vec{\nabla})\vec{v} \ dx - \int \vec{u}.(\vec{v}.\vec{\nabla})\vec{v} \ dx.$

17

This is checked by regularizing $\vec{u}$ and $\vec{v}$: we use a smoothing function $\theta(t,x) = \alpha(t)\beta(x)$ $\in \mathcal{D}(\mathrm{I\!R}^{3+1})$, where α is supported in $[-1,1]$, with $\int\int \theta\ dx\ dt = 1$, and define, for $\epsilon > 0$, $\theta_\epsilon(t,x) = \frac{1}{\epsilon^{3+1}}\theta(\frac{t}{\epsilon}, \frac{x}{\epsilon})$. Then, $\theta_\epsilon * \vec{u}$ and $\theta_\epsilon * \vec{v}$ are smooth functions on $(\epsilon, T-\epsilon) \times \mathrm{I\!R}^3$ and we may write $\partial_t\big((\theta_\epsilon * \vec{u}).(\theta_\epsilon * \vec{v})\big) = (\theta_\epsilon * \partial_t\vec{u}).(\theta_\epsilon * \vec{v}) + (\theta_\epsilon * \vec{u}).(\theta_\epsilon * \partial_t\vec{v})$. We then get by an integration with respect to x :

(63)
$$
\begin{aligned}
\partial_t \int (\theta_\epsilon * \vec{u}).(\theta_\epsilon * \vec{v})\ dx = \quad &-2\int (\theta_\epsilon * [\vec{\nabla}\otimes\vec{u}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{v}])\ dx \\
&+ \int (\theta_\epsilon * [\vec{u}\otimes\vec{u}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{v}])\ dx \\
&+ \int (\theta_\epsilon * [\vec{v}\otimes\vec{v}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{u}])\ dx
\end{aligned}
$$

We may rewrite the last summand in

(64)
$$
\begin{aligned}
\int (\theta_\epsilon * [\vec{v}\otimes\vec{v}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{u}])\ dx \quad &= -\int (\theta_\epsilon * [\vec{\nabla}.(\vec{v}\otimes\vec{v})]).(\theta_\epsilon * \vec{u})\ dx \\
&= -\int (\theta_\epsilon * [\{\vec{v}.\vec{\nabla}\}\vec{v}]).(\theta_\epsilon * \vec{u})\ dx
\end{aligned}
$$

To deal with $\theta_\epsilon * [\vec{u}\otimes\vec{u}]$, we write that the pointwise product maps $L^{2/r}\dot{H}^r \times L^{2/(1-r)}L^{3/r}$ to $L^2 L^2$, hence $\theta_\epsilon * [\vec{u}\otimes\vec{u}]$ converges strongly to $\vec{u}\otimes\vec{u}$ in $(L^2((0,T)\times\mathrm{I\!R}^3))^{3\times 3}$. To deal with $\theta_\epsilon * [\{\vec{v}.\vec{\nabla}\}\vec{v}]$, we write that the pointwise product maps $\dot{H}^r \times L^2$ to the pre-dual $L^{\frac{3}{3-r}}$ of $L^{3/r}$ and that smooth functions are dense in $L^{3/r}$; thus, $\theta_\epsilon * [(\vec{v}.\vec{\nabla})\vec{v}]$ converges strongly to $(\vec{v}.\vec{\nabla})\vec{v}$ in $(L^{\frac{2}{1+r}}L^{\frac{3}{3-r}})^3$ while $\theta_\epsilon * \vec{u}$ converges weakly to $\vec{u}$ in $(L^{\frac{2}{1-r}}L^{3/r})^3$. This proves (62).

Since $\vec{u}\otimes\vec{u} \in (L^2((0,T)\times\mathrm{I\!R}^3))^{3\times 3}$ and $\vec{u} = e^{t\Delta}\vec{u}(0) - \mathrm{I\!P}\int_0^t e^{(t-s)\Delta}\vec{\nabla}.(\vec{u}\otimes\vec{u})\ ds$, $t\mapsto\vec{u}$ is continuous from $[0,T]$ to $(L^2(dx))^3$ and (since $t\mapsto\vec{v}$ is weakly continuous from $[0,T]$ to $(L^2(dx))^3$) $t\mapsto\int\vec{u}.\vec{v}\ dx$ is continuous. Thus, we may integrate equality (62) and obtain

(65)
$$
\begin{aligned}
&\int \vec{u}(t,x).\vec{v}(t,x)\ dx + 2\int_0^t\int_{\mathrm{I\!R}^3} \vec{\nabla}\otimes\vec{u}.\vec{\nabla}\otimes\vec{v}\ dx\ ds = \\
&\|\vec{u}_0\|_2^2 + \int_0^t\int_{\mathrm{I\!R}^3}\vec{u}.(\vec{u}.\vec{\nabla})\vec{v}\ dx\ ds - \int_0^t\int_{\mathrm{I\!R}^3}\vec{u}.(\vec{v}.\vec{\nabla})\vec{v}\ dx\ ds
\end{aligned}
$$

Of course, this equality holds as well for $\vec{v} = \vec{u}$.

Now, if we assume moreover that $\vec{v}$ satisfies the Leray inequality

(66)
$$
\|\vec{v}(t)\|_2^2 + 2\int_0^t \|\vec{\nabla}\otimes\vec{v}\|_2^2\ ds \leq \|\vec{u}_0\|_2^2,
$$

we get the following inequality for $\vec{u} - \vec{v}$:

(67)
$$
\|\vec{u}(t,.) - \vec{v}(t,.)\|_2^2 \leq -2\int_0^t\int_{\mathrm{I\!R}^3} |\vec{\nabla}\otimes(\vec{u}-\vec{v})|^2\ dx\ ds - 2\int_0^t\int_{\mathrm{I\!R}^3}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)\vec{v}\ dx\ ds
$$

Moreover, we have the antisymmetry property $\int_0^t\int_{\mathrm{I\!R}^3}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)\vec{u}\ dx\ ds = 0$.
We then write

(68)
$$
\begin{aligned}
&|\int_\tau^t\int_{\mathrm{I\!R}^3}\vec{u}.\big((\vec{u}-\vec{v}).\vec{\nabla}\big)(\vec{v}-\vec{u})\ dx\ ds| \\
&\leq C_r(\int_\tau^t \|\vec{u}\|_{L^{3/r}}^{\frac{2}{1-r}}\ ds)^{(1-r)/2}(\int_0^t \|\vec{v}-\vec{u}\|_{\dot{H}^1}^2\ ds)^{1/2}(\int_\tau^t \|\vec{v}-\vec{u}\|_{\dot{H}^r}^{\frac{2}{r}}\ ds)^{\frac{r}{2}} \\
&\leq C_r'(\int_\tau^t \|\vec{u}\|_{L^{3/r}}^{\frac{2}{1-r}}\ ds)^{(1-r)/2}(\int_0^t \|\vec{v}-\vec{u}\|_{\dot{H}^1}^2\ ds)^{(1+r)/2}(\sup_{\tau<s<t} \|\vec{v}-\vec{u}\|_2^2)^{\frac{(1-r)}{2}}
\end{aligned}
$$

18

If $r < 1$, we may easily conclude : we write with help of the Young inequality

$$(69) \qquad C'_r a^{(1-r)/2} b^{(1+r)/2} \leq \frac{1-r}{2} C'^{\,2/(1-r)}_r a + \frac{1+r}{2} b$$

Thus, if $\vec{u} = \vec{v}$ on $[0, \tau]$, we find from (67) and (68) that

$$(70) \qquad \sup_{0 < s \leq t} \|\vec{u} - \vec{v}\|_2^2 \leq C''_r \Big(\int_\tau^t \|\vec{u}\|_{L^{3/r}}^{\frac{2}{1-r}} \, ds \Big)^{1/2} \sup_{0 < s \leq t} \|\vec{u} - \vec{v}\|_2^2$$

and uniqueness is valid on a bigger interval. By weak continuity of $t \mapsto \vec{v}$, we find $\vec{u} = \vec{v}$.

If $r = 1$, $\vec{u}$ belongs to $(\mathcal{C}([0, T], L^3))^3$; if $T_0 < T$, then for each $\epsilon > 0$ we may split $\vec{u}$ on $[0, T_0]$ in $\vec{u} = \vec{\alpha} + \vec{\beta}$ with $\|\vec{\alpha}\|_{L^\infty L^3} < \epsilon$ and $\vec{\beta} \in (L^\infty((0, T_0) \times \mathbb{R}^3))^3$. Then we write

$$(71) \qquad \begin{aligned} & \Big| \int_0^t \!\! \int_{\mathbb{R}^3} \vec{u}.\big((\vec{u} - \vec{v}).\vec{\nabla}\big)(\vec{v} - \vec{u}) \, dx \, ds \Big| \\ & \leq C \|\vec{\alpha}\|_{L^\infty L^3} \int_0^t \|\vec{v} - \vec{u}\|_{\dot{H}^1}^2 \, ds \\ & + \|\vec{\beta}\|_\infty \big(\int_0^t \!\! \int_{\mathbb{R}^3} |\vec{\nabla} \otimes (\vec{v} - \vec{u})|^2 \, dx \, ds \big)^{1/2} \big(\int_0^t \!\! \int_{\mathbb{R}^3} |\vec{v} - \vec{u}|^2 \, dx \, ds \big)^{1/2} \\ & \leq 2C\epsilon \int_0^t \!\! \int_{\mathbb{R}^3} |\vec{\nabla} \otimes (\vec{v} - \vec{u})|^2 \, dx \, ds + \big(\tfrac{4}{C\epsilon} \|\vec{\beta}\|_\infty^2 + C\epsilon \big) \int_0^t \!\! \int_{\mathbb{R}^3} |\vec{v} - \vec{u}|^2 \, dx \, ds. \end{aligned}$$

Choosing ϵ such that $2C\epsilon < 1$, we get that

$$(72) \qquad \|\vec{v}(t, .) - \vec{u}(t, .)\|_2^2 \leq \Big(\frac{4}{C\epsilon} \|\vec{\beta}\|_\infty^2 + C\epsilon \Big) \int_0^t \|\vec{v}(s, .) - \vec{u}(s, .)\|_2^2 \, ds.$$

The Gronwall lemma gives then that $\vec{u} = \vec{v}$.

Thus, the main tool in proving Propositions 3 and 4 is the facts that when $f \in L^\infty L^2 \cap L^2 \dot{H}^1$, then f belongs to $L^{2/r} \dot{H}^r$ and that the pointwise product is bounded from $\dot{H}^r \times L^{3/r}$ to L^2. Considering the space $\dot{X}^r$ of pointwise multipliers from $\dot{H}^r$ to L^2 then gives a direct generalization of Propositions 3 and 4, as it has been observed in [LEM 02] [LEMM 06] :

Definition 3 : (Pointwise multipliers of negative order.)
For $0 \leq r < 3/2$, we define the space $\dot{X}_r(\mathbb{R}^3)$ as the space of functions which are locally square integrable on $\mathbb{R}^3$ and such that pointwise multiplication with these functions maps boundedly the Sobolev space $\dot{H}^r(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. The norm in $\dot{X}_r$ is given by the operator norm of pointwise multiplication:

$$(73) \qquad \|f\|_{\dot{X}_r} = \sup\{ \|fg\|_2 \, / \, \|g\|_{\dot{H}^r} \leq 1 \}.$$

The closure of the space $\mathcal{D}$ of smooth test functions in $\dot{X}_r$ will be denoted by $\tilde{X}_r$.

The spaces $\dot{X}_r$ have been characterized by Maz'ya [MAZ 64] in terms of Sobolev capacities. A weaker result establishes a comparison between the spaces $\dot{X}_r$ and the Morrey–Campanato spaces $\dot{M}^{2,p}$ [FEF 83] [LEM 02].

Lemma 5: (Comparison theorem)
   For $2 < p \le 3/r$ and $0 < r$ we have $\dot{M}^{p,3/r} \subset \dot{X}_r \subset \dot{M}^{2,3/r}$.

   Another easy result is the embedding $L^{3/r,\infty} \subset \dot{X}_r$ for $r < 3/2$.

   We may now state the generalization of Propositions 3 and 4 :

Proposition 5 :
   Let $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^3$ (for some $T \in (0, +\infty]$) with initial value $\vec{u}_0$ such that :
   i) $\vec{u} \in L^\infty((0,T), (L^2(\mathbb{R}^3)^3)$;
   ii) $\vec{u} \in L^2((0,T), (\dot{H}^1(\mathbb{R}^3)^3)$;
   iii) For some $r \in [0,1)$, $\vec{u}$ belongs to $(L^\sigma((0,T), \dot{X}_r))^3$ with $2/\sigma = 1 - r$.
   Then, $\vec{u}$ satisfies the Leray energy inequality and it is the unique Leray solution associated to $\vec{u}_0$ on $(0,T)$.

   A similar results holds for $r = 1$ when iii) is replaced by
   iii') $\vec{u}$ belongs to $(\mathcal{C}([0,T], \dot{X}_1))^3$.

   The structure of the multiplier spaces $\dot{X}_r$ is not easy to describe. However, when $r < 1$, we may replace the space $\dot{X}_r$ by the (greater) Morrey–Campanato space $\dot{M}^{2,3/r}$:

Theorem 4 :
   Let $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^3$ (for some $T \in (0, +\infty]$) with initial value $\vec{u}_0$ such that :
   i) $\vec{u} \in L^\infty((0,T), (L^2(\mathbb{R}^3)^3)$;
   ii) $\vec{u} \in L^2((0,T), (\dot{H}^1(\mathbb{R}^3)^3)$;
   iii) For some $r \in [0,1)$, $\vec{u}$ belongs to $(L^\sigma((0,T), \dot{M}^{2,3/r}))^3$ with $2/\sigma = 1 - r$.
   Then, $\vec{u}$ satisfies the Leray energy inequality and it is the unique Leray solution associated to $\vec{u}_0$ on $(0,T)$.

   For $r = 0$, we have $\dot{M}^{2,\infty} = \dot{X}_0 = L^\infty$, and this is Serrin's theorem. When $0 < r < 1$, we use the fact that $L^2 \cap \dot{H}^1 \subset \dot{B}_2^{r,1} \subset \dot{H}^r$. Thus, in generalizing Serrin's theorem, we may replace the pointwise multipliers from $\dot{H}^r$ to L^2 by the pointwise multipliers from the Besov space $\dot{B}_2^{r,1}$ to L^2. We then conclude with the following lemma :

Lemma 6 :

For $0 \leq r \leq 3/2$, we define the space $\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2)$ as the space of functions which are locally square integrable on $\mathbb{R}^3$ and such that pointwise multiplication with these functions maps boundedly the Besov space $\dot{B}_2^{r,1}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. The norm in $\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2)$ is given by the operator norm of pointwise multiplication:

$$(74) \qquad \|f\|_{\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2)} = \sup\{ \|fg\|_2 \ / \ \|g\|_{\dot{B}_2^{r,1}} \leq 1\}.$$

Then, f belongs to $\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2)$ if and only if f belongs to $\dot{M}^{2,3/r}$ (with equivalence of norms).

The embedding $\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2) \subset \dot{M}^{2,3/r}$ is obvious : if $\omega \in \mathcal{D}(\mathbb{R}^3)$ is equal to 1 on $B(0,1)$, then we find that

$$\int_{B(x_0,R)} |f(x)|^2 \ dx \leq \|f\|^2_{\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2)} \|\omega(\frac{x-x_0}{R})\|^2_{\dot{B}_1^{r,2}} = R^{3-2r} \|f\|^2_{\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2)} \|\omega\|^2_{\dot{B}_1^{r,2}}.$$

Conversely, if $f \in \dot{M}^{2,3/r}$ and $g \in \dot{B}_1^{2,3/r}$, then we use the decomposition of g in a regular enough Daubechies basis of compactly supported wavelets [MEY 92] [KAHL 95] [LEM 02]. The wavelet basis is an orthonormal basis of $L^2(\mathbb{R}^3)$ which is given as a family of functions $(\psi_{\epsilon,j,k})_{1 \leq \epsilon \leq 7, j \in \mathbb{Z}, k \in \mathbb{Z}^3}$ derived through dyadic dilations and translations from a finite set of functions $(\psi_\epsilon)_{1 \leq \epsilon \leq 7}$:

$$(75) \qquad \psi_{\epsilon,j,k}(x) = 2^{3j/2} \psi_\epsilon(2^j x - k)$$

where the functions ψ_ϵ are compactly supported and of class $\mathcal{C}^2$. Then for $0 \leq r < 3/2$, the family $(\psi_{\epsilon,j,k})_{1 \leq \epsilon \leq 7, j \in \mathbb{Z}, k \in \mathbb{Z}^3}$ is a Riesz basis of $\dot{B}_2^{r,1}(\mathbb{R}^3)$; more precisely, there exists two positive constants $0 < A_r \leq B_r < \infty$ such that for all $g \in \dot{B}_2^{r,1}(\mathbb{R}^3)$ we have

$$(76) \qquad A_r \|g\|_{\dot{B}_2^{r,1}} \leq N_r(g) \leq B_r \|g\|_{\dot{B}_2^{r,1}(\mathbb{R}^3)}$$

with

$$(77) \qquad N_r(g) = \sum_{j \in \mathbb{Z}} 2^{jr} \sqrt{\sum_{k \in \mathbb{Z}^3} \sum_{1 \leq \epsilon \leq 7} |\langle g|\psi_{\epsilon,j,k}\rangle|^2}.$$

Now, we write

$$(78) \qquad \|fg\|_2 \leq \sum_{j \in \mathbb{Z}} \| \sum_{k \in \mathbb{Z}^3} \sum_{1 \leq \epsilon \leq 7} \langle g|\psi_{\epsilon,j,k}\rangle \ \psi_{\epsilon,j,k} f\|_2;$$

since the family $(\psi_{\epsilon,j,k} f)_{k \in \mathbb{Z}^3, \ 1 \leq \epsilon \leq 7}$ is (uniformly) locally finite, we find that

$$(79) \qquad \| \sum_{k \in \mathbb{Z}^3} \sum_{1 \leq \epsilon \leq 7} \langle g|\psi_{\epsilon,j,k}\rangle \ \psi_{\epsilon,j,k} f\|_2^2 \leq C \sum_{k \in \mathbb{Z}^3} \sum_{1 \leq \epsilon \leq 7} |\langle g|\psi_{\epsilon,j,k}\rangle|^2 \|\psi_{\epsilon,j,k} f\|_2^2$$

21

and, since ψ_ϵ is bounded and compactly supported,

$$\tag{80} \|\psi_{\epsilon,j,k}f\|_2^2 \leq C2^{3j}\|f\|_{\dot{M}^{2,3/r}}^2 2^{-3j(1-\frac{2r}{3})}.$$

Thus, we get

$$\tag{81} \|fg\|_2 \leq C\sum_{j\in\mathbb{Z}}\sqrt{\sum_{k\in\mathbb{Z}^3}\sum_{1\leq\epsilon\leq 7}|\langle g|\psi_{\epsilon,j,k}\rangle|^2\|f\|_{\dot{M}^{2,3/r}}^2 2^{2jr}} \leq C'\|g\|_{\dot{B}_2^{r,1}}\|f\|_{\dot{M}^{2,3/r}}.$$

Thus, Lemma 6 is proved.

Theorem 4 does not include the limit case $r = 1$, which is still an open question :

Open question 1 :
    In Theorem 4, does a similar results holds for $r = 1$ when iii) is replaced by iii') $\vec{u}$ belongs to $(\mathcal{C}([0,T],\tilde{M}^{2,3}))^3$?

We end this section with two further remarks :
i) the condition $\vec{u} \in (L^2((0,T),L^\infty))^3$ in the limit case $r = 0$ may be modified in $\vec{u} \in (L^2((0,T),BMO))^3$ [KOZT 00]. In order to prove this, one replaces poinwise multiplication with the paraproduct operator. Using paramultipliers instead of multipliers, Germain has recently extended Proposition 5 to negative values of r [GER 05].
ii) If $\vec{u}$ is a solution to the Navier–Stokes equations such that $\vec{u} \in (L^\sigma((0,\infty),\dot{M}^{2,3/r}))^3$ with $2/\sigma = 1 - r$ and $0 < r < 1$, then it is easy to see that $\vec{u}_0$ belongs to the Besov space $(\dot{B}_{\dot{M}^{2,3/r}}^{r-1,\sigma})^3$ based on the Morrey–Campanato space $\dot{M}^{2,3/r}$. Conversely, if $\vec{u}_0$ belongs to the Besov space $(\dot{B}_{\dot{M}^{2,3/r}}^{r-1,\sigma})^3$ and has a small enough norm in this space, then one can construct a solution $\vec{u} \in (L^\sigma((0,\infty),\dot{M}^{2,3/r}))^3$ [KOZY 97] [LEM 02].

9. Uniqueness theorems.

In 1997, Furioli, Lemarié-Rieusset and Terraneo [FURLT 00] proved uniqueness of mild solutions in $\mathcal{C}([0,T^*),(L^3)^3)$. They extended their proof to the case of Morrey-Campanato spaces by using the Besov spaces over Morrey-Campanato spaces described by Kozono and Yamazaki [KOZY 97] and found that uniqueness holds as well in the class $\mathcal{C}([0,T^*),(\tilde{M}^{p,3})^3)$ for $p > 2$, where $\tilde{M}^{p,3}$ is the closure of the smooth compactly supported functions in the Morrey-Campanato space $\dot{M}^{p,3}$. In his thesis dissertation, May [MAY 02] [LEMM 06] proved a slightly more general result by extending the approach of Monniaux (i.e. by using the maximal L^pL^q property of the heat kernel) :

Proposition 6 :
    If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier-Stokes equations on $(0,T^*)\times\mathbb{R}^3$ such that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0,T^*),(\tilde{X}_1)^3)$ and have the same initial value, then $\vec{u} = \vec{v}$.

May's result generalizes the results of Furioli, Lemarié–Rieusset and Terraneo, but leaves open the limit case of $\dot{M}^{2,3}$:

Open question 2 :
  Does uniqueness holds in $(\mathcal{C}([0,T^*), \tilde{M}^{2,3}))^3$?

  The problem of uniqueness we may consider in a more general approach is the following one :

Definition 3 : (Regular critical space)
  A regular critical space is a Banach space X such that we have the continuous embeddings $\mathcal{D}(\mathbb{R}^3) \subset X \subset L^2_{loc}(\mathbb{R}^3)$ and such that moreover :
  (a) for all $x_0 \in \mathbb{R}^3$ and for all $f \in X$, $f(x - x_0) \in X$ and $\|f\|_X = \|f(x - x_0)\|_X$.
  (b) for all $\lambda > 0$ and for all $f \in X$, $f(\lambda x) \in X$ and $\lambda\|f(\lambda x)\|_X = \|f\|_X$.
  (c) $\mathcal{D}(\mathbb{R}^3)$ is dense in X.

  We have the obvious result :

Lemma 7 :
  Let X be a regular critical space. Then X is continuously embedded in $\tilde{M}^{2,3}$.

  We shall then consider the problem of uniqueness in $\mathcal{C}([0,T^*), X^3)$:

Uniqueness problem :
  Let X be a regular critical space. If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier-Stokes equations on $(0,T^*) \times \mathbb{R}^3$ such that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0,T^*), X^3)$ and have the same initial value, then do we have $\vec{u} = \vec{v}$?

  We may do the following remarks on this problem :
i) We can write the Navier–Stokes equations as

$$(82) \qquad \begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla} \cdot (\vec{u} \otimes \vec{u}) \\[2mm] \vec{\nabla}.\vec{u} = 0 \end{cases}$$

since a solution $\vec{u} \in \mathcal{C}([0,T^*), X^3)$ vanishes at infinity in the sense of [FURLT 00] (see also [LEM 02]). Another way to write the equations is then

$$(83) \qquad \vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$$

where B is the bilinear operator

$$(84) \qquad B(\vec{u}, \vec{v})(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{v}) \, ds.$$

ii) It is easy to check that, in a regular critical space, local uniqueness implies global uniqueness. Local uniqueness means that, if $T^* > 0$ and if $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier-Stokes equations on $(0,T^*) \times \mathbb{R}^3$ such that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0,T^*), X^3)$

and have the same initial value, then there exists a positive ϵ such that we have $\vec{u} = \vec{v}$ on $[0, \epsilon] \times \mathbb{R}^3$. Global uniqueness then means that we must have $\vec{u} = \vec{v}$ on $[0, T^*)$.

iii) The basic idea in Furioli, Lemarié-Rieusset and Terraneo [FURLT 00] is to split the solutions in tendency and fluctuation, and to use different estimates on each term. More precisely, we consider two mild solutions $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u}) = e^{t\Delta}\vec{u}_0 - \vec{w}_1$ and $\vec{v} = e^{t\Delta}\vec{u}_0 - B(\vec{v}, \vec{v}) = e^{t\Delta}\vec{u}_0 - \vec{w}_2$ in $\mathcal{C}([0, T^*), X^3)$ and write $\vec{w} = \vec{u} - \vec{v} = \vec{w}_2 - \vec{w}_1 = -B(\vec{w}, \vec{v}) - B(\vec{u}, \vec{w})$, and finally

$$(85) \qquad \vec{w} = B(\vec{w}_1, \vec{w}) + B(\vec{w}, \vec{w}_2) - B(e^{t\Delta}\vec{u}_0, \vec{w}) - B(\vec{w}, e^{t\Delta}\vec{u}_0).$$

Thus we see clearly the role of the fluctuations : they control the behaviour of $\vec{w}$. We shall use the regularization properties of the heat kernel for the term $e^{t\Delta}\vec{u}_0$ (mainly, that $\lim_{t\to 0} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$), while we shall use the fact that, for $i = 1$ or 2, we have $\lim_{\epsilon \to 0} \sup_{0 \le t \le \epsilon} \|\vec{w}_i(t)\|_X = 0$, thus we shall assume that the norm of $\vec{w}_i$ is very small.

Those remarks give us a simple way for proving uniqueness of some regular solutions. First, we define fully adapted critical Banach spaces for the Navier–Stokes equations. The notion of adapted spaces was introduced by Cannone in his book [CAN 95] : Cannone studied Banach spaces X such that the bilinear operator B defined by

$$B(\vec{f}, \vec{g})(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f} \otimes \vec{g} \, ds$$

is bounded from $L^\infty((0, T), X^3) \times L^\infty((0, T), X^3)$ to $L^\infty((0, T), X^3)$. According to Cannone, a Banach space X is *adapted* to the Navier-Stokes equations if the following assertions are satisfied:

a) X is a shift-invariant Banach space of distributions.

b) the pointwise product between two elements of X is still well defined as a tempered distribution

c) there is a sequence of real numbers $\eta_j > 0$, $j \in \mathbb{Z}$, such that

$$\sum_{j \in \mathbb{Z}} 2^{-|j|} \eta_j < \infty$$

and such that

$$\forall j \in \mathbb{Z}, \forall f \in X, \forall g \in X \ \|\Delta_j(fg)\|_X \le \eta_j \ \|f\|_X \ \|g\|_X$$

If X is a Banach space adapted (according to Cannone) to the Navier-Stokes equations, then the bilinear transform B is continuous on $L^\infty((0, T), X^3)$. But this definition doesn't work in the case of critical spaces : if the norm of X is invariant under the dilations $f \mapsto \lambda f(\lambda x)$ and if we have the inequalities $\|\Delta_j(fg)\|_X \le \eta_j \ \|f\|_X \ \|g\|_X$, then we find that $\eta_j = 2^j \eta_0$ and thus $\sum_{j \in \mathbb{Z}} 2^{-|j|} \eta_j = \infty$.

Other definitions of adapted spaces have been proposed by Meyer and Muschetti [MEY 99] or Auscher and Tchamitchian [AUST 99]. While those definitions are introduced to deal

24

with critical spaces, they don't allow to prove the boundedness of B on $L^\infty((0,T), X^3)$, but on a smaller space of smooth trajectories (as in the case of Theorem 1, points (A) and (B)).

However, there are some critical shift–invariant spaces E for which the boundedness of B on $L^\infty((0,T), E^3)$ is known : the first instance was given by Le Jan and Sznitman [LEJS 97] and is known since the works of Cannone as the Besov space $\dot{B}_{PM}^{2,\infty}$ based on pseudo-measures [CANK 04] [LEM 02]; another instance was then given by Yves Meyer [MEY 99] : the Lorentz space $L^{3,\infty}$. All those examples can be dealt with with the following notion of fully adapted Banach spaces :

Definition 4 : (Fully adapted critical space)

A fully adapted critical Banach space for the Navier–Stokes equations is a Banach space E such that we have the continuous embeddings $\mathcal{D}(\mathbb{R}^3) \subset E \subset L_{loc}^2(\mathbb{R}^3)$ and such that moreover :
(a) for all $x_0 \in \mathbb{R}^3$ and for all $f \in E$, $f(x - x_0) \in E$ and $\|f\|_E = \|f(x - x_0)\|_E$.
(b) for all $\lambda > 0$ and for all $f \in E$, $f(\lambda x) \in E$ and $\lambda\|f(\lambda x)\|_E = \|f\|_E$.
(c) The closed unit ball of E is a metrizable compact subset of $\mathcal{S}'(\mathbb{R}^3)$.
(d) e^Δ maps boundedly E to the space $\mathcal{M}$ of pointwise multipliers of E
(e) Let F be the Banach space

$$F = \{f \in L_{loc}^1 \ / \ \exists (f_n), (g_n) \in E^{\mathbb{N}} \ \ s.t. \ f = \sum_{n \in \mathbb{N}} f_n g_n \ and \ \sum_{n \in \mathbb{N}} \|f_n\|_E \|g_n\|_E < \infty$$

(normed with $\|f\|_F = \min_{f = \sum_{n \in \mathbb{N}} f_n g_n} \sum_{n \in \mathbb{N}} \|f_n\|_E \|g_n\|_E$). There exists a Banach space of tempered distributions G such that
i) e^Δ maps boundedly F to G
ii) the real interpolation space $[F, G]_{1/2,\infty}$ is continuously embedded into E
iii) for all $\lambda > 0$ and for all $f \in G$, $f(\lambda x) \in G$ and $\|f(\lambda x)\|_G = \|f\|_G$.

Hypothesis (c) (together with (a)) shows that E is invariant under convolution with an integrable kernel :

(86) $$\forall f \in E \ \forall g \in L^1 \ f * g \in E \text{ and } \|f * g\|_E \leq \|f\|_E \|g\|_1.$$

This hypothesis (c) is fulfilled in the case where E is the dual space of a separable Banach space containing $\mathcal{S}$ as a dense subspace.

The following proposition shows why those spaces are called adapted to the Navier–Stokes equations :

Proposition 7 :

Let E be a fully adapted critical space and let $\mathcal{M} = \mathcal{M}(E \mapsto E)$ be the space of pointwise multipliers of E. For $T \in (0, +\infty)$, let $\mathcal{A}_T$ and $\mathcal{B}_T$ be the spaces defined by

$$f \in \mathcal{A}_T \Leftrightarrow f \in L_{loc}^2((0,T) \times \mathbb{R}^3), \ \sup_{0 < t < T} \|f(t,.)\|_E < \infty$$

25

and

$$f \in \mathcal{B}_T \Leftrightarrow f \in L^1_{loc}((0,T) \times \mathbb{R}^3), \quad \sup_{0<t<T} t^{1/2} \|f(t,.)\|_{\mathcal{M}} < \infty.$$

Then B is bounded from $(\mathcal{A}_T)^3 \times (\mathcal{A}_T)^3$ to $(\mathcal{A}_T)^3$ and from $(\mathcal{A}_T)^3 \times (\mathcal{B}_T)^3$ or $(\mathcal{B}_T)^3 \times (\mathcal{A}_T)^3$ to $(\mathcal{A}_T)^3$. More precisely, there exists a constant C_E such that, for all $T \in (0,+\infty]$, all $\vec{u}_0 \in E^3$, all $\vec{f}, \vec{g} \in (\mathcal{A}_T)^3$ and all $\vec{h} \in (\mathcal{A}_T)^3$ we have

$$(87) \qquad \sup_{t>0} \sqrt{t} \|e^{t\Delta} \vec{u}_0\|_{\mathcal{M}} \leq C_E \|\vec{u}_0\|_E$$

$$(88) \qquad \|B(\vec{f}, \vec{g})\|_{\mathcal{A}_T} \leq C_E \|\vec{f}\|_{\mathcal{A}_T} \|\vec{g}\|_{\mathcal{A}_T}$$

and

$$(89) \qquad \|B(\vec{f}, \vec{h})\|_{\mathcal{A}_T} + \|B(\vec{h}, \vec{f})\|_{\mathcal{A}_T} \leq C_E \|\vec{f}\|_{\mathcal{A}_T} \|\vec{h}\|_{\mathcal{B}_T}.$$

Since e^{Δ} maps E to $\mathcal{M}$, (87) is a direct consequence of the homogeneity of the norm of E (and therefore of the norm of $\mathcal{M}$). (89) is a direct consequence of the convolution inequality (86). We now prove (88). We want to estimate the norm $\|B(\vec{f}, \vec{g})\|_E$. We split the integral $I = \int_0^t e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{f}) \, ds$ into $G_A + H_A$ with

$$\begin{cases} G_A = \int_0^A e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g}) \, ds \\ H_A = \int_A^t e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g}) \, ds \end{cases}$$

We then use the inequalities

$$\begin{cases} \|e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\|_F \leq C \frac{1}{\sqrt{t-s}} \|\vec{f}\|_E \|\vec{g}\|_E \\ \|e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\|_G \leq C \frac{1}{(t-s)^{3/2}} \|\vec{f}\|_E \|\vec{g}\|_E \end{cases}$$

We obtain

$$\|H_A\|_F \leq C\sqrt{t-A} \sup_{0<s<t} \|\vec{f}\|_E \sup_{0<s<t} \|\vec{g}\|_E$$

and

$$\|G_A\|_G \leq C \frac{1}{\sqrt{t-A}} \sup_{0<s<t} \|\vec{f}\|_E \sup_{0<s<t} \|\vec{g}\|_E.$$

If $\lambda \geq \sqrt{t}$, we define $A(\lambda) = 0$ so that $I = H_A$ and $G_A = 0$; if $0 < \lambda < \sqrt{t}$, we define $A(\lambda) = t - \lambda^2$. Thus, we find that, for every $\lambda > 0$, $I = G_{A(\lambda)} + H_{A(\lambda)}$ with

$$\|H_{A(\lambda)}\|_F \leq C\lambda \sup_{0<s<t} \|\vec{f}\|_E \sup_{0<s<t} \|\vec{g}\|_E$$

26

and

$$\|G_{A(\lambda)}\|_G \leq C \frac{1}{\lambda} \sup_{0<s<t} \|\vec{f}\|_E \sup_{0<s<t} \|\vec{g}\|_E.$$

Since such a splitting may be done for any positive λ, we obtain

$$\|I\|_{[F,G]_{1/2,\infty}} \leq C \sup_{0<s<t} \|\vec{f}\|_E \sup_{0<s<t} \|\vec{g}\|_E.$$

Then, one easily finishes the proof of Proposition 7.

Combining (85) and Proposition 7, we easily get the following uniqueness result :

Theorem 5 :
  If X is a regular critical space such that X is boundedly embedded into a a fully adapted critical space E, then uniqueness holds in $(\mathcal{C}([0,T^*), X))^3$.

Indeed, we have, with the notations of Proposition 7 and of formula (85)

$$\|\vec{w}\|_{\mathcal{A}_T} \leq C_E \|\vec{w}\|_{\mathcal{A}_T} (2\|e^{t\Delta} \vec{u}_0\|_{\mathcal{B}_T} + \|\vec{w}_1\|_{\mathcal{A}_T} + \|\vec{w}_2\|_{\mathcal{A}_T})$$

with

$$\lim_{T\to 0} \|e^{t\Delta} \vec{u}_0\|_{\mathcal{B}_T} = \lim_{T\to 0} \|\vec{w}_1\|_{\mathcal{A}_T} = \lim_{T\to 0} \|\vec{w}_2\|_{\mathcal{A}_T} = 0$$

so that $\vec{w} = 0$ on $(0,T)$ for T small enough. Thus, local (hence global) uniqueness holds in $(\mathcal{C}([0,T^*), X))^3$.

Examples of fully adapted spaces :
i) the space of Le Jan and Sznitman

$$E = \dot{B}_{PM}^{2,\infty} = \{f \in \mathcal{S}'(\mathbb{R}^3) \; / \; \hat{f} \in L_{loc}^1 \text{ and } \xi^2 \hat{f}(\xi) \in L^\infty\}$$

with $F \subset \dot{B}_{PM}^{1,\infty}$ and $G = \dot{B}_{PM}^{3,1}$
ii) the homogeneous Besov space

$$E = \dot{B}_p^{3/p-1,\infty} \text{ where } 1 \leq p < 3$$

with $F \subset \dot{B}_p^{3/p-2,\infty}$ and $G = \dot{B}_p^{3/p,1}$
iii) the Lorentz space
$$E = L^{3,\infty}$$

with $F = L^{3/2,\infty}$ and $G = L^\infty$
iv) the homogeneous Morrey–Campanato spaces based on Lorentz spaces :

$$E = \dot{M}_*^{p,3} \text{ where } 2 < p \leq 3$$

27

with $F = \dot{M}_*^{p/2,3/2}$ and $G = L^\infty$. The space $\dot{M}_*^{p,q}(\mathbb{R}^3)$ is defined for $1 < p \le q < \infty$ as the space of locally integrable functions f such that

$$\sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{3(1/q - 1/p)} \|1_{B(x_0,R)} f\|_{L^{p,\infty}} < \infty;$$

the predual of $\dot{M}_*^{p,q}(\mathbb{R}^3)$ is then the space of functions f which may be decomposed as a series $\sum_{n \in \mathbb{N}} \lambda_n f_n$ with f_n supported by a ball $B(x_n, R_n)$ with $R_n > 0$, $f_n \in L^{p/(p-1),1}$, $\|f_n\|_{L^{p/(p-1),1}} \le R_n^{3(1/q - 1/p)}$ and $\sum_{n \in \mathbb{N}} |\lambda_n| < \infty$.

All those examples however give no new information on the uniqueness problem, since we have the embeddings (for $2 \le p < 3$ and $2 < q \le 3$)

(90) $$\dot{B}_{PM}^{2,\infty} \subset \dot{B}_p^{3/p-1,\infty} \subset L^{3,\infty} \subset \dot{M}_*^{q,3} \subset \dot{X}_1$$

and thus uniqueness may be dealt with by using May's theorem (Proposition 6).

We finish this section with an example of a regular space where uniqueness holds but which cannot be dealt with by using either Theorem 5 or Proposition 6 :

Theorem 6 :
   Let X be defined as the space of locally integrable functions f such that

$$\sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{-1/2} \|1_{B(x_0,R)} f\|_{L^{2,1}} < \infty$$

and let $\tilde{X}$ be the closure of $\mathcal{D}$ in X. Then
a) Uniqueness holds in $(\mathcal{C}([0,T^*), \tilde{X}))^3$.
b) $\tilde{X}$ is not included in the multiplier space $\dot{X}_1 = \mathcal{M}(\dot{H}^1 \mapsto L^2)$
c) there is no fully adapted critical space E such that $\tilde{X} \subset E$.

The proof of (b) and (c) is easy. Indeed, let β be the bilinear operator $\beta(u,v) = \frac{1}{\sqrt{-\Delta}}(uv)$. Then β is continuous from $\dot{X}_1 \times \dot{X}_1$ to $\dot{X}_1$ (this is a direct consequence of the characterization by Maz'ya and Verbitsky of $\mathcal{M}(\dot{H}^1 \mapsto H^{-1})$ [MAZV 02]) and from $E \times E$ to E for a fully adapted critical space E (as is checked by writing $\frac{1}{\sqrt{-\Delta}} = \int_0^\infty e^{t\Delta} \sqrt{-\Delta} \, dt$ and adapting the proof of Proposition 7). Thus, we shall prove (b) and (c) by checking that β is not bounded from $\tilde{X} \times \tilde{X}$ to $\dot{M}^{2,3}$. Let $\omega \in \mathcal{D}(\mathbb{R})$ with $\omega = 1$ on $[-1/4, 1/4]$ and $\int_\mathbb{R} \omega(t) \, dt = 1$. We define

(91) $$u_R(x) = R\omega(Rx_1)\omega(Rx_2)\omega(x_3/R)$$

$u_R \in \tilde{X}$ and $\|u_R\|_X = \|u_1\|_X$. If β were bounded from $\tilde{X} \times \tilde{X}$ to $\dot{M}^{2,3}$, we would have that $\lim_{R \to \infty} \beta(u_R, u_R) \in \dot{M}^{2,3} \subset L^2_{loc}$ (the limit being taken in the sense of distributions). But

(92) $$\lim_{R \to \infty} \beta(u_R, u_R) = \frac{c_0}{|x|^2} * (\delta(x_1) \otimes \delta(x_2) \otimes 1) = \frac{\pi c_0}{\sqrt{x_1^2 + x_2^2}}$$

for a positive constant c_0. Since $(\sqrt{x_1^2 + x_2^2})^{-1}$ is not locally square integrable, we proved (b) and (c).

In order to prove (a), we use again the equality

$$(93) \qquad \vec{w} = B(\vec{w}_1, \vec{w}) + B(\vec{w}, \vec{w}_2) - B(e^{t\Delta}\vec{u}_0, \vec{w}) - B(\vec{w}, e^{t\Delta}\vec{u}_0).$$

where we have $\vec{w} \in \mathcal{C}([0, T^*), (\tilde{X})^3)$, $\vec{w}_i \in \mathcal{C}([0, T^*), (\tilde{X})^3)$ with $\vec{w}_i(0, .) = 0$ for $i \in \{1, 2\}$, $\sup \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty < \infty$ and $\lim_{t\to 0} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$. We then use the inclusion $X \subset \dot{M}_*^{2,3}$. Since the pointwise product maps boundedly $X \times \dot{M}_*^{2,3}$ to $\dot{M}^{1,3/2}$ and since we have

$$(94) \qquad \dot{M}_*^{2,3} = [\dot{M}^{1,3/2}, L^\infty]_{1/2,\infty}$$

we find that

$$(96) \quad \sup_{0<t<T} \|\vec{w}\|_{\dot{M}_*^{2,3}} \leq C \sup_{0<t<T} \|\vec{w}\|_{\dot{M}_*^{2,3}} \Big(\sup_{0<t<T} \|\vec{w}_1\|_X + \sup_{0<t<T} \|\vec{w}_2\|_X + \sup_{0<t<T} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty \Big).$$

Thus, Theorem 6 is proved.

References.

[AUST 99] AUSCHER, P. & TCHAMITCHIAN, P., Espaces critiques pour le système des équations de Navier–Stokes, Preprint, 1999.

[CAN 95] CANNONE, M., *Ondelettes, paraproduits et Navier–Stokes* , Diderot Editeur, Paris, 1995.

[CANK 04] CANNONE, M. & KARCH, G., Smooth or singular solutions to the Navier–Stokes system ?, *J. Diff. Eq.* 197 (2004), pp. 247–274.

[FED 93] FEDERBUSH, P., Navier and Stokes meet the wavelet, *Comm. Math. Phys.* 155 (1993), pp. 219–248.

[FEF 83] FEFFERMAN, C. The uncertainty principle. *Bull. Amer. Math. Soc.* 9 (1983), pp. 129–206.

[FURLT 00] FURIOLI, G., LEMARIÉ-RIEUSSET, P.G. & TERRANEO, E., Unicité dans $L^3(\mathbb{R}^3)$ et d'autres espaces limites pour Navier–Stokes. *Revista Mat. Iberoamer* 16 (2000), pp. 605–667.

[FURLZZ 00] FURIOLI, G., LEMARIÉ-RIEUSSET, P.G., ZAHROUNI, E. & ZHIOUA, A., Un théorème de persistance de la régularité en norme d'espaces de Besov pour les solutions de Koch et Tataru des équations de Navier–Stokes dans $\mathbb{R}^3$, *C. R. Acad. Sci. Paris* (Série I) 330 (2000) 339–342.

[GER 05] GERMAIN, P., *Solutions fortes, solutions faibles d'équations aux dérivées partielles d'évolution*, Thèse, Ecole Polytechnique, 2005.

[GRU 06] GRUJIĆ, Z., Regularity of forward-in-time self-similar solutions to the 3D Navier–Stokes equations, *Discrete Cont. Dyn. Systems* 14 (2006), pp. 837–843.

[KAHL 95] KAHANE, J.P. & LEMARIÉ-RIEUSSET, P.G., *Fourier series and wavelets*, Gordon & Breach, London, 1995.

[KAT 84] KATO, T., Strong L^p solutions of the Navier–Stokes equations in $\mathbb{R}^m$ with applications to weak solutions, *Math. Zeit.* 187 (1984), pp. 471–480.

[KAT 92] KATO, T., Strong solutions of the Navier–Stokes equations in Morrey spaces, *Bol. Soc. Brasil. Math.* 22 (1992), pp. 127–155.

[KOCT 01] KOCH, H. & TATARU, D., Well-posedness for the Navier–Stokes equations, *Advances in Math.* 157 (2001) 22–35.

[KOZS 96] KOZONO, H. & SOHR, H., Remark on uniqueness of weak solutions to the Navier–Stokes equations, *Analysis* 16 (1996), pp. 255–271.

[KOZT 00] KOZONO, H. & TANIUCHI, Y., Bilinear estimates in BMO and the Navier–Stokes equations, *Math. Z.* 235 (2000), pp. 173–194.

[KOZY 97] KOZONO, H. & YAMAZAKI, Y., Semilinear heat equations and the Navier–Stokes equations with distributions in new function spaces as initial data, *Comm. P.D. E.* 19 (1994), pp. 959–1014.

[LEJS 97] LE JAN, Y. & SZNITMAN, A. S., Cascades aléatoires et équations de Navier–Stokes, *C. R. Acad. Sci. Paris* 324 Série I (1997), pp. 823–826.

[LEM 02] LEMARIÉ-RIEUSSET, P.G., *Recent developments in the Navier–Stokes problem*, Chapman & Hall/CRC, 2002.

[LEM 05] LEMARIÉ-RIEUSSET, P.G., Point fixe d'une application non contractante., to appear in *Rev. Mat. Iberoamer.*.

[LEMM 06] LEMARIÉ-RIEUSSET, P.G. & MAY, R., Uniqueness for the Navier–Stokes equations and multiplier between Sobolev spaces, to appear in *Nonlinear Analysis*.

[LER 34] LERAY, J., Essai sur le mouvement d'un fluide visqueux emplissant l'espace, *Acta Math.* 63 (1934), pp. 193–248.

[MAY 02] MAY, R., *Régularité et unicité des solutions milds des équations de Navier–Stokes*, Thèse, Université d'Evry, 2002.

[MAZ 64] MAZ'YA, V.G., On the theory of the n-dimensional Schrödinger operator, *Izv. Akad. Nauk SSSR*, ser. Mat., 28 (1964), 1145–1172. (in Russian)

[MAZS 85] MAZ'YA, V.G. & SHAPOSHNIKOVA, T.O., *The theory of multipliers in spaces of differentiable functions*, Pitman, New-York, 1985.

[MAZV 02] MAZ'YA, V.G. & VERBITSKY, I.E., The Schrödinger operator on the energy space : boundedness and compactness criteria, *Acta Math.* 188 (2002), 263–302.

[MEY 92] MEYER, Y., *Wavelets and operators* , Cambridge University Press, 1992.

[MEY 99] MEYER, Y., *Wavelets, paraproducts and Navier–Stokes equations*, Current developments in mathematics 1996; International Press: PO Box 38-2872, Cambridge, MA 02238-2872, 1999.

[ORU 98] ORU, F., *Rôle des oscillations dans quelques problèmes d'analyse non linéaire*. Thèse, École Normale Supérieure de Cachan (1998).

[SER 62] SERRIN, J., On the interior regularity of weak solutions of the Navier–Stokes equations, *Arch. Rat. Mech. Anal.*, 9 (1962), pp. 187–195.

[TAY 92] TAYLOR, M.E., Analysis on Morrey spaces and applications to Navier–Stokes equations and other evolution equations, *Comm. P. D. E.* 17 (1992), pp. 1407–1456.

[WAH 85] VON WAHL, W., *The equations of Navier–Stokes and abstract parabolic equations.* Vieweg & Sohn, Wiesbaden, 1985.

[WEI 81] WEISSLER, F.B., The Navier–Stokes initial value problem in L^p, *Arch. Rational Mech. Anal.* 74 (1981) pp. 219–230.