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More Regular Wavelets

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We introduce a new class of compactly supported orthonormal wavelets which are more regular than the Daubechies wavelets.

INTRODUCTION

Let ψ be a *compactly supported wavelet;* i.e., ψ is a real-valued square-integrable compactly supported function such that the family

$$(\psi_{j,k} = 2^{j/2}\psi(2^j x - k))_{j\in\mathbb{Z},k\in\mathbb{Z}}$$

is an Hilbertian basis of $L^2(\mathbb{R})$. It is known that (at least in the case when ψ is slightly better than L^2:$\psi \in H^\epsilon$ for some positive ϵ) the basis $(\psi_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is derived from a multiresolution analysis associated to a compactly supported *scaling function* φ [5] and a *scaling filter* m_0, which is a trigonometric polynomial. We may choose m_0 such that the lower frequency in m_0 is $k = 0$,

$$m_0(\xi) = \sum_{k=0}^{k_1} a_k e^{-ik\xi}, \quad a_{k_1} \neq 0, \quad a_0 \neq 0, \quad m_0(0) = 1, \tag{1}$$

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and φ is defined by

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0\left(\frac{\xi}{2^j}\right), \tag{2}$$

and then we have

$$\hat{\psi}(\xi) = e^{-iK\xi} e^{-i\xi/2} \overline{m}_0\left(\frac{\xi}{2} + \pi\right) \hat{\varphi}\left(\frac{\xi}{2}\right), \tag{3}$$

where $K + \frac{1}{2}$ is the midpoint of Supp ψ,

$$\text{Supp } \varphi = [0, k_1], \quad \text{Supp } \psi = \left[K + \frac{1}{2} - \frac{k_1}{2}, K + \frac{1}{2} + \frac{k_1}{2}\right],$$

k_1 is always an odd integer, and we write $k_1 = 2N - 1$.

The length of the filter m_0 is then the number $l = 2N$. It is a measurement of the complexity of the fast wavelet transform [6] associated to ψ. In this paper, we are interested in the regularity ratio $\sigma / l = \sigma / 2N$, where σ is the Sobolev regularity exponent of ψ:

$$\sigma = \text{Sup}\{s / |\xi|^s \hat{\psi} \in L^2\}. \tag{4}$$

More precisely, we are going to improve the ratio $\sigma/2N$ when N goes to infinity by introducing a new class of wavelets which are more regular than the Daubechies wavelets.

THEOREM. *There exists a family* $(\psi_N)_{N \geq 0}$ *of compactly supported orthonormal wavelets* (*of Sobolev exponent* σ_N) *such that:*

(i) *for all* N, *Supp* $\psi_N = [0, 2N - 1]$
(ii) $\liminf_{N \to +\infty} (\sigma_N / 2N) > 1/2 - \ln 3/4 \ln 2$.

We will begin by some elementary lemmas on wavelets, then recall the proof that if ψ_N is a Daubechies wavelet of length $2N$ and regularity exponent σ_N, then $\lim_{N \to +\infty} \sigma_N / 2N = 1/2 - \ln 3/4 \ln 2$. We slightly modify the classical proof of Cohen and Conze [2] and Volkmer [8] in order to generalize the proof to other families of wavelets.

1. BASIC RESULTS ON WAVELETS

How can we construct wavelets of length $2N$? The procedure is given by formulas (2) and (3) provided that we may construct a good polynomial m_0. Such polynomials are characterized by the Cohen criterion [1]:

LEMMA 1. *Let $Q \in \mathbb{R}[X]$, $Q(1) = 1$. Then the following assertions are equivalent:*

(A1) *$Q(\cos\xi)$ is the square modulus $|m_0(\xi)|^2$ of an orthonormal scaling filter of length $2N$.*

(A2) *Q satisfies:*

(i) $\deg Q = 2N - 1$

(ii) $Q(X) + Q(-X) = 1$

(iii) $Q(X) \geq 0$ *for* $X \in [-1, 1]$

(iv) *if* $Q(\cos(2^k\xi_0 + \pi)) = 0$ *for all* $k \in \mathbb{N}$ *then* $\xi_0 \in 2\pi\mathbb{Z}$.

Condition (iv) is satisfied as soon as Q does not vanish on $[\frac{1}{2}, 1]$. Of course, Q does not determine an unique m_0 (it determines only the modulus $|m_0(\xi)|$), and we have to use a Riesz factorization [allowed by the condition (A2) (iii)] for the extraction of a ''square root'' m_0. But, the knowledge of Q is enough for describing the regularity of the wavelets we may produce from it (since σ depends only on $|\hat{\psi}|$, hence on $|m_0|$).

LEMMA 2. (i) $\tilde{\sigma} - 1/2 \leq \sigma \leq \tilde{\sigma}$, where $\tilde{\sigma} = \mathrm{Max}\{\alpha / |\xi|^\alpha \hat{\varphi} \in L^\infty\}$.

(ii) *if* $Q(X) = ((1 + X)/2)^L A(X)$ *then*

$$L - \frac{1}{2\ln 2}\ln\rho_1 \leq \tilde{\sigma} \leq L - \frac{1}{2\ln 2}\ln\rho_2, \tag{5}$$

where

$$\rho_1 = \mathrm{Max}\{\sup_{-1 \leq X \leq -1/2}\sqrt{A(X)A(2X^2 - 1)}, \sup_{X \geq -1/2} A(X)\}$$

$$\rho_2 = A(-1/2) \quad \left(\text{so that } L - \frac{1}{2\ln 2}\ln\rho_2 = -\ln Q(-1/2)\right).$$

Notice that φ and ψ have the same Sobolev exponent. Lemma 2 is classical [4] ($\sigma \leq \tilde{\sigma}$ is not classical; one finds often $\sigma \leq \tilde{\sigma} + 1/2$; but it is known that if $\varphi \in H^{s+\epsilon}$, then $(\varphi(x - k))_{k\in\mathbb{Z}}$ is a Riesz family in H^s, hence $\sum_{k\in\mathbb{Z}} |\xi + 2k\pi|^{2s}|\hat{\varphi}(\xi + 2k\pi)|^2 \in L^\infty$, which proves $\sigma \leq \tilde{\sigma}$).

Our last lemma solves $Q(X) + Q(-X) = 1$, with the assumption that $\deg Q \leq 2N - 1$ and Q is a multiple of $(1 + X)^L$:

LEMMA 3. *The following assertions are equivalent:*

(B1) *$Q \in \mathbb{R}_{2N-1}[X]$, $Q(1) = 1$, $Q(X) + Q(-X) = 1$ and Q is a multiple of $(1 + X)^L$.*

(B2) $Q = \sum_{k=0}^{2N-1} \epsilon_{N,k}\binom{2N-1}{k}((1+X)/2)^k((1-X)/2)^{2N-1-k}$ *with* $\epsilon_{N,k} + \epsilon_{N,2N-1-k} = 1$ *for* $0 \leq k \leq 2N - 1$ *and* $\epsilon_{N,k} = 0$ *for* $0 \leq k \leq L - 1$.

(B3) *there is a polynomial $A \in \mathbb{R}_{N-L}[X]$ such that $Q(1) = 1$ and*

$$Q(X) = \int_{-1}^{X}(1 - t^2)^{L-1}A(t^2)\,dt.$$

(B2) is called the *Bernstein representation* of Q and (B3) its *integral representation.* Lemma 3 is obvious, but very useful.

2. DAUBECHIES WAVELETS

Daubechies wavelets ψ_N are defined as wavelets of length $2N$ with optimal power of approximation. We say that a wavelet ψ (associated to a scaling function φ) has *power of approximation* $\lambda \in \mathbb{N}$ if

$$\forall f \in H^\lambda \quad \lim_{j \to +\infty} 2^{\lambda j} \| f - P_j f \|_2 = 0, \tag{6}$$

where P_j is the orthogonal projection operator on the closed linear span of the $\varphi_{j,k} = 2^{j/2}\varphi(2^j x - k)$, $k \in \mathbb{Z}$. The following lemma is classical.

LEMMA 4. *The following assertions are equivalent:*

(C1) ψ *has power of approximation* λ
(C2) $\forall k \in \{0, \dots, \lambda\}$, $P_j(x^k) = x^k$ (*Strang–Fix condition*)
(C3) $\forall k \in \{0, \dots, \lambda\}$, $((d^k/d\xi^k)m_0)(\pi) = 0$
(C4) Q *is a multiple of* $(1 + X)^{\lambda+1}$ (*Daubechies condition*).

Thus, looking at (B3), we see that if ψ has power of approximation λ, then its length is at least $2\lambda + 2$. The *Daubechies wavelets* ψ_N are precisely the wavelets of length $2N$ and power of approximation $N - 1$. (B2) gives us that Q_N is given by

$$Q_N = \sum_{N}^{2N-1} \binom{2N-1}{k} \left(\frac{1+X}{2}\right)^k \left(\frac{1-X}{2}\right)^{2N-1-k} \tag{7}$$

and (B3) gives us that

$$Q_N = \frac{N}{2^{2N-1}} \binom{2N-1}{N} \int_{-1}^{X} (1 - t^2)^{N-1} dt \tag{8}$$

(where the constant $N/2^{2N-1}\binom{2N-1}{N}$ can be determined by differentiating (7)). We are going to prove

$$\sigma_N = N\left(1 - \frac{\ln 3}{2 \ln 2}\right) + 0\left(\frac{\ln N}{N}\right) \tag{9}$$

by proving that, in Lemma 2,

$$\rho_1 \leq C_0 \sqrt{N} 3^N$$
$$\rho_2 \geq C_1\, 3^N \sqrt{N}.$$

The proof is very easy. First, we use the Stirling formula to get

$$\binom{2N-1}{N} \sim \frac{2^{2N-1}}{\sqrt{N\pi}}.$$

Thus, we have, for $X \leq 0$

$$Q_N(X) \leq \gamma_0\sqrt{N}\int_{-1}^{X}(1-t^2)^{N-1}dt \leq \gamma_0\sqrt{N}(1-X^2)^{N-1}\int_{-1}^{X}dt$$
$$\leq \gamma_0\sqrt{N}\left(\frac{1+X}{2}\right)^N(2(1-X))^N.$$

Thus, $A_N(X) \leq \gamma_0\sqrt{N}(2(1-X))^N$ if $X \leq 0$, $\leq (2/(1+X))^N$ if $X \geq 0$. Hence,

- if $X \geq 0$, $A_N(X) \leq 2^N \leq 3^N$
- if $-\frac{1}{2} \leq X \leq 0$, $A_N(X) \leq \gamma_0\sqrt{N}(2(1-X))^N \leq \gamma_0\sqrt{N}3^N$
- if $-\sqrt{2}/2 \leq X \leq \frac{1}{2}$, $A_N(X)A_N(2X^2-1) \leq \gamma_0^2 N(8(1-X)(1-X^2))^N \leq \gamma_0^2 N 9^N$
- if $X \leq -\sqrt{2}/2$, $A_N(X)A_N(2X^2-1) \leq \gamma_0\sqrt{N}(2(1-X))^N \cdot 2^N \leq \gamma_0\sqrt{N}8^N \leq \gamma_0\sqrt{N}9^N$,

thus we have a good estimate for ρ_1.

For estimating ρ_2, it is enough to write

$$Q_N\left(-\frac{1}{2}\right) \geq \binom{2N-1}{N}\left(\frac{1-1/2}{2}\right)^N\left(\frac{1+1/2}{2}\right)^{N-1} \geq C\frac{1}{\sqrt{N}}\left(\frac{3}{4}\right)^N.$$

Thus (9) is proved.

3. THE RESTRICTED BERNSTEIN CLASS

Our first attempt to improve (9) was to replace Q_N by another polynomial Q given by

$$Q(X) = \sum_{0}^{2N-1}\epsilon_{N,k}\binom{2N-1}{k}\left(\frac{1+X}{2}\right)^k\left(\frac{1-X}{2}\right)^{2N-1-k}$$

with $\epsilon_{N,k} \geq 0$ for all k, $\epsilon_{N,k} + \epsilon_{2N-1-k} = 1$, and $\epsilon_{N,0} = 0$. This polynomial is of degree $\leq 2N-1$ and satisfies obviously conditions (ii), (iii), (iv) of (A2) and thus defines a wavelet. Such polynomials will be said to belong to the *restricted Bernstein class*. But, we cannot improve (9).

PROPOSITION 1. *If* deg $Q \leq 2N-1$ *and Q belongs to the restricted Bernstein class, then* $\sigma \leq N(1 - \ln 3/2\ln 2) + O(\ln N/N)$.

Proof. It is enough to estimate $Q(-\frac{1}{2})$. We have

$$Q\left(-\frac{1}{2}\right) \geq \binom{2N-1}{N-1}\left\{\epsilon_{N,N-1}\frac{3^N}{4^{2N-1}} + (1-\epsilon_{N,N-1})\frac{3^{N-1}}{4}\right\}$$

$$\geq \binom{2N-1}{N-1}\frac{3^{N-1}}{4^{2N-1}}. \quad \blacksquare$$

4. THE MATZINGER WAVELETS

After the failure of the restricted Bernstein class, we had to introduce some negative coefficients in the Bernstein representation; hence we encountered a problem for keeping $Q(X)$ positive-valued. Therefore, we changed our plan and turned to the integral representation. Moreover, we decided to impose $Q(-\frac{1}{2}) = 0$, since $Q(-\frac{1}{2})$ was the main obstacle for better regularity in the case of the restricted Bernstein class. (After completing this work, we learned that H. Volkmer constructed regular wavelets using the integral representation [9], but in a different way than ours; his polynomials $Q(X)$ are increasing on $[-1, 1]$.)

Thus, we asked E. Matzinger to study the filters m_0 of minimal length such that m_0 has L zeros at π and $2M$ zeros at $2\pi/3$ [7]. The result we obtained with Matzinger is the following.

THEOREM 2. (*a*) *m_0 has L zeros at π and $2M$ zeros at $2\pi/3$ if and only if:*

$$Q(-1/2) = 0$$

and for some $A \in \mathbb{R}[X]$

$$Q(X) = \int_{-1}^{X} (1-t^2)^{L-1}(1-4t^2)^{2M-1}A(t^2)\,dt.$$

Moreover, we must have $\deg A \geq 1$.

(*b*) *If* $2M \leq (L-1)(\ln 5/\ln 4)$ *then the polynomial*

$$Q_{M,L}(X) = \alpha_{M,L}\int_{-1}^{X} (1-t^2)^{L-1}(1-4t^2)^{2M-1}(1-\beta_{M,L}t^2)\,dt,$$

where $\beta_{M,L}$ *is determined by* $Q_{M,L}(-\frac{1}{2}) = 0$ *and* $\alpha_{M,L}$ *by* $Q_{M,L}(1) = 1$, *satisfies the Cohen criterion* (A2) (ii)–(iv).

Proof. (a) is obvious, since dQ/dX is even, has $L-1$ zeros at -1 and $2M-1$ zeros at $-1/2$. Moreover, $A(t^2)$ vanishes between -1 and $-1/2$ by Rolle's theorem; hence $\deg A \geq 1$.

(b) is easy. First, we see that $\beta_{M,L}$ and $\alpha_{M,L}$ are well defined. $\beta_{M,L}$ is the root of $F(\beta) = 0$, where

$$F(\beta) = \int_{-1}^{-1/2} (1 - t^2)^{L-1}(1 - 4t^2)^{2M-1}(1 - \beta t^2)dt = A_{M,L} - B_{M,L}\beta.$$

Obviously $F(1) < 0$ and $F(4) > 0$; hence, $1 < \beta_{M,L} < 4$.

Now, $\alpha_{M,L}$ is defined by

$$\alpha_{M,L} \int_{-1/2}^{1/2} (1 - t^2)^{L-1}(1 - 4t^2)^{2M-1}(1 - \beta_{M,L}t^2)dt = 1$$

and the multiplicand of $\alpha_{M,L}$ is clearly positive.

In order to prove that $Q_{M,L}$ satisfies the Cohen criterion, we will prove more precisely that $Q_{M,L}$ is nonnegative on $[-1, 1]$ and that its roots on $[-1, 1]$ are just -1 and $-1/2$. Since $Q_{M,L}$ increases on $[-1/2, 1/2]$ and since $Q_{M,L}(X) + Q_{M,L}(-X) = 1$, it is enough to show that for $-1 < X < -1/2$ we have $0 < Q_{M,L}(X) < 1$. $Q_{M,L}$ increases between -1 and $\gamma_{M,L} = -1/\sqrt{\beta_{M,L}}$ and decreases between $\gamma_{M,L}$ and $-1/2$, so we have just to prove

$$Q_{M,L}(\gamma_{M,L}) < Q_{M,L}(1/2) = 1,$$

or equivalently, to prove $I_{M,L} < J_{M,L}$, where

$$I_{M,L} = \int_{-1}^{\gamma_{M,L}} (1 - t^2)^{L-1}(1 - 4t^2)^{2M-1}(1 - \beta_{M,L}t^2)dt$$

and

$$J_{M,L} = \int_{-1/2}^{1/2} (1 - t^2)^{L-1}(1 - 4t^2)^{2M-1}(1 - \beta_{M,L}t^2)dt.$$

But, on $[-1, \gamma_{M,L}]$ we have

$$(1 - 4t^2)^{2M-1}(1 - \beta_{M,L}t^2) = (4t^2 - 1)^{2M-1}(\beta_{M,L}t^2 - 1) \leq (4t^2 - 1)^{2M};$$

hence

$$I_{M,L} \leq \int_{-1}^{-1/2} (1 - t^2)^{L-1}(1 - 4t^2)^{2M}dt = \tilde{I}_{M,L},$$

while, on $[-1/2, 1/2]$, we have $1 - \beta_{M,L}t^2 \geq 1 - 4t^2$; hence

$$J_{M,L} \geq \int_{-1/4}^{1/4} (1 - t^2)^{L-1}(1 - 4t^2)^{2M}dt = \tilde{J}_{M,L}.$$

Now, if we look at $f(\theta) = (1 - \theta^2)|1 - 4\theta^2|$, then $f(\theta) \leq 36/64$ on $[-1, -1/2]$ and $f(\theta) \geq 45/64$ on $[-1/4, 1/4]$; hence

TABLE 1
The Matzinger Scaling Filter for $L = 3$ and $M = 1$

	k	$a_k^{M,L}$
$L = 3$	0	0.01387003
	1	−0.03351638
	2	−0.01732329
	3	0.10179320
	4	−0.01710161
	5	−0.16606030
	6	0.00383799
	7	0.38395131
	8	0.51671690
	9	0.21383210

- if $2M \le L - 1$,

$$\tilde{I}_{M,L} \le \left(\frac{36}{64}\right)^{2M} \int_{-1}^{-1/2} (1 - t^2)^{L-1-2M} dt \le \frac{1}{2}\left(\frac{36}{64}\right)^{2M}\left(\frac{3}{4}\right)^{L-1-2M}$$

and

$$\tilde{J}_{M,L} \ge \left(\frac{45}{64}\right)^{2M} \int_{-1/4}^{1/4} (1 - t^2)^{L-1-2M} dt \ge \frac{1}{2}\left(\frac{45}{64}\right)^{2M}\left(\frac{15}{16}\right)^{L-1-2M};$$

hence

$$J_{M,L} \ge \left(\frac{45}{36}\right)^{2M}\left(\frac{15}{12}\right)^{L-1-2M} I_{M,L} > I_{M,L};$$

- if $2M \ge L$, we get similarly

$$\tilde{I}_{M,L} \le \frac{1}{2}\left(\frac{36}{64}\right)^{L-1} 3^{2M-L+1}, \quad \tilde{J}_{M,L} \ge \frac{1}{2}\left(\frac{45}{64}\right)^{L-1}\left(\frac{3}{4}\right)^{2M-L+1};$$

hence

$$J_{M,L} \ge \left(\frac{45}{36}\right)^{L-1} 4^{-2M+L-1} I_{M,L} = \frac{5^{L-1}}{4^{2M}} I_{M,L} > I_{M,L}$$

if $5^{L-1} > 4^{2M}$. Hence, Theorem 2 is proved. ■

5. MATZINGER WAVELETS: TABLES AND FIGURES

In this section, we give the Matzinger filters for some small values of L and M (choosing for the computations the minimum-phased square root of $Q_{M,L}(\cos(\xi))$) and we plot the associated scaling functions.

The Matzinger wavelet for L = 3 and M = 1 are shown in Table 1 and Fig. 1. The Matzinger wavelet for L = 4 and M = 1 are shown in Table 2 and Fig. 2.

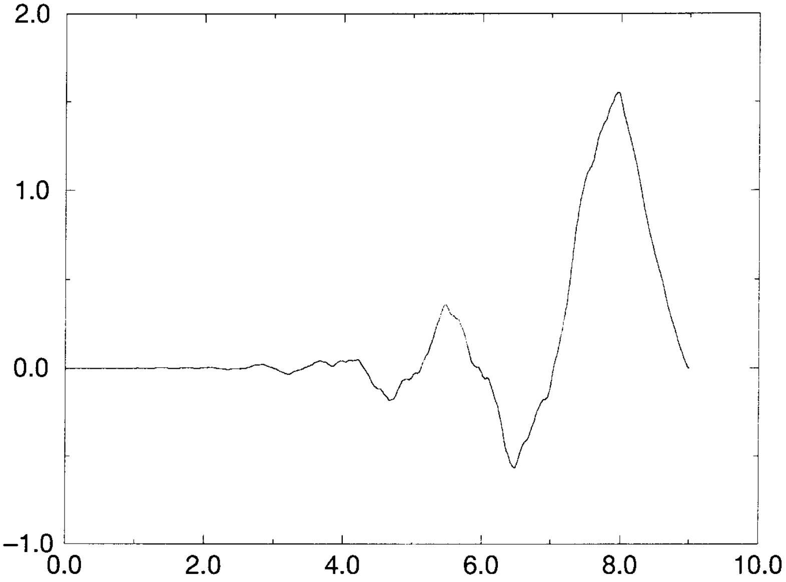

FIG. 1. Matzinger scaling function for $L = 3$ and $M = 1$.

We now give the Sobolev regularity exponent σ for these Matzinger wavelets,

$$\sigma = \max\{s/\phi \in H^s\},$$

which can be computed exactly through the spectral analysis of a transition operator on a finite-dimensional space [3]. We recall in our tables the length of the scaling filter (Tables 3–5).

In Table 5 we write $\sigma_{L,M}$ and σ_{L+2M} for the Sobolev exponent of the Matzinger and the Daubechies scaling functions $\varphi_{L,M}$ and φ_{L+2M} (they have the same support).

TABLE 2
The Matzinger Scaling Filter for $L = 4$ and M = 1

	k	$a_k^{M,L}$
$L = 4$	0	−0.00466166
	1	0.01430884
	2	0.00423221
	3	−0.05451530
	4	0.02704138
	5	0.09655332
	6	−0.06914172
	7	−0.16778700
	8	0.08275964
	9	0.46165201
	10	0.45977020
	11	0.14978820

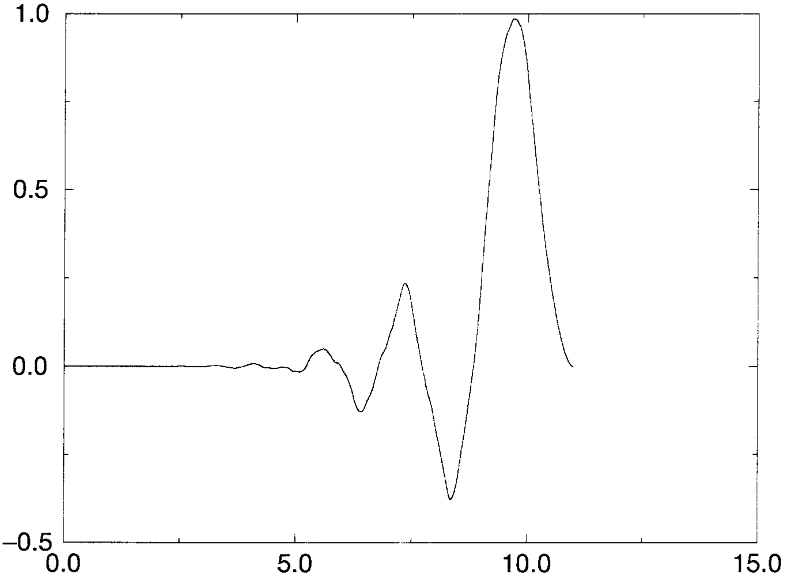

FIG. 2. Matzinger scaling function for $L = 4$ and $M = 1$.

Thus, we can see that for $L \geq 8$ and $M = 1$ the Matzinger wavelet is more regular than the Daubechies wavelet ϕ_{L+2M} which has a support of the same size. This is a general feature of the Matzinger wavelets for big enough values of L as we shall see in the next section.

6. REGULARITY OF THE MATZINGER WAVELETS

We are now able to prove Theorem 1 in the following way.

THEOREM 3. *Let* $\lambda > 0$. *For* $N \geq 1/\lambda$, *let* $\psi_N = \psi_{M,L}$ *be a Matzinger wavelet* (*associated to a Q which has L zeros at* -1 *and* $2M$ *zeros at* $-\frac{1}{2}$) *of length* $2N = 2L$

TABLE 3
Sobolev Exponent of the Matzinger Scaling Functions for $M = 1$

L	Length	σ
3	10	1.411
4	12	1.867
5	14	2.297
6	16	2.711
7	18	3.114
8	20	3.508
9	22	3.897
10	24	4.482
11	26	4.661

TABLE 4
Sobolev Exponent of the Matzinger Scaling Functions for $M = 2$

L	Length	σ
3	14	1.000
4	16	1.600
5	18	2.046
6	20	2.472
7	22	2.883
8	24	3.258
9	26	3.682
10	28	4.076
11	30	4.467

$+ 4M$ *with* $M = [\lambda N]$. *Then, if* λ *is small enough, we have* $\liminf_{N\to+\infty} (\sigma_N/2N) > \frac{1}{2} - \ln 3/4 \ln 2$ (*where* σ_N *is the Sobolev exponent of* ψ_N).

Proof. We are going to prove more precisely: $\sigma_N/2N \geq \frac{1}{2} - \ln 3/4 \ln 2 + \mu + O(\text{Log } n/n)$, where μ is a positive constant.

We will always assume $2M \leq L - 1$, hence $\lambda < \frac{1}{4}$. Then we may prove the following estimate.

LEMMA 5. *There is a constant* $C(\lambda)$ *depending only on* λ *such that:*

$$\forall N \geq \frac{1}{\lambda}, \forall x \in [-1, 0] \quad Q_{M,L}(x) \leq C(\lambda)\sqrt{N}(1 - x^2)^L(1 - 4x^2)^{2M}.$$

TABLE 5
Comparison of Sobolev Exponent of the Matzinger and the Daubechies Scaling Functions

L	Length	$\sigma_{L,M}$	σ_{L+2M}
3	9	1.411	2.096
4	11	1.807	2.469
5	13	2.297	2.701
6	15	2.711	2.914
7	17	3.114	3.161
8	19	3.508	3.402
9	21	3.897	3.639
10	23	4.482	3.875
11	25	4.661	4.106
12	27	5.036	4.336
13	29	5.405	4.566
14	31	5.771	4.792
15	33	6.131	5.019

The lemma is easy to check; we have

$$Q_{M,L}(x) \leq 3\alpha_{M,L} \inf \left(\int_{-1}^{x} (1-t^2)^{L-1} |1-4t^2|^{2M-1} dt, \left| \int_{-1/2}^{x} (1-t^2)^{L-1} |1-4t^2|^{2M-1} dt \right| \right).$$

Let us estimate $\alpha_{M,L}$,

$$\begin{aligned} \frac{1}{\alpha_{M,L}} &= \int_{-1/2}^{1/2} (1-t^2)^{L-1}(1-4t^2)^{2M-1}(1-\beta_{M,L}t^2)\,dt \\ &\geq \int_{-1/2}^{1/2} (1-4t^2)^{2M+L-1} dt \\ &= \frac{1}{2}\int_{-1}^{1} (1-t^2)^{N-1} dt \\ &= \frac{1}{2} \frac{2^{2N-1}}{N\binom{2N-1}{N}} \\ &\geq C\frac{1}{\sqrt{N}} \end{aligned}$$

and thus $\alpha_{M,L} \leq C\sqrt{N}$.

Now, let us look at $P(x) = |(1-x^2)^{L-1}(1-4x^2)^{2M-1}|$.

Clearly, P is increasing on $[-1, t_{M,L}]$, decreasing on $[t_{M,L}, -\frac{1}{2}]$ and increasing on $[-\frac{1}{2}, 0]$, where $t_{M,L}$ is the negative root of

$$(2M-1)(1-t^2) + 4(L-1)(1-4t^2) = 0.$$

Thus,

$$Q_{M,L}(X) \leq 3\alpha_{M,L}(1+X)(1-X^2)^{L-1}|1-4X^2|^{2M-1} \quad \text{if } -1 \leq X \leq t_{M,L}$$

and

$$Q_{M,L}(X) \leq 3\alpha_{M,L}\left|\frac{1}{2}+X\right|(1-X^2)^{L-1}|1-4X^2|^{2M-1} \quad \text{if } t_{M,L} \leq X \leq 0,$$

and we get, finally,

$$Q_{M,L}(X) \leq 3\alpha_{M,L}(1-X^2)^{L}(1-4X^2)^{2M} \mathrm{Max}\left(\frac{1}{|1-4t_{M,L}^2|}, \frac{1}{2|1-t_{M,L}^2|} \right)$$

on $[-1, 0]$. But $t_{M,L}^2 \sim (1+6\lambda)/4$. Thus, Lemma 5 is proved.

We are now going to estimate σ_N by estimating ρ_1 in Lemma 2 in a way very similar to the case of Daubechies filters (the Daubechies filters correspond to the value $M = 0$ of a Matzinger wavelet). As a matter of fact, $\rho_1 \leq \delta_{M,L} = \operatorname{Max}_{-1\leq x\leq 1} \alpha(x)$, where $\alpha(x)$ is defined by

- for $0 \leq x \leq 1$, $\alpha(x) = (2/(1+x))^L$
- for $-\frac{1}{2} \leq x < 0$, $\alpha(x) = C(\lambda)\sqrt{N}(2(1-x))^L(1-4x^2)^{2M}$
- for $-\sqrt{2}/2 \leq x < -\frac{1}{2}$,

$\alpha(x) = C(\lambda)\sqrt{N}(8(1-x)(1-x^2))^{L/2}(1-4x^2)^{2M}|3-4x^2|^M$

- for $-1 \leq x < -(-\sqrt{2}/2)$, $\alpha(x) = \sqrt{C(\lambda)}N^{1/4}\left(\frac{2(1-x)}{x^2}\right)^{L/2}(1-4x^2)^M$

(we have just used $Q_{M,L}(x) = ((1+x)/2)^L A_{M,L}(x)$ and Lemma 5 and $Q_{M,L}(x) \leq 1$ in order to estimate the size of $A_{M,L}(x)$ and of $A_{M,L}(x)A_{M,L}(2x^2-1)$). Thus, we find

$$\frac{\sigma_N}{2N} \geq \frac{1}{2N}\frac{1}{2\ln 2}\ln\frac{4^L}{\delta_{M,L}} = \frac{1}{4\ln 2}\ln\left(\frac{4^L}{\delta_{M,L}}\right)^{1/N}.$$

For $M = 0$ (the Daubechies wavelets), we find again $\sigma_N/2N \sim (1/4\ln 2)\ln\frac{4}{3} = 0.103\cdots$. Numerically, we find that for $\lambda = \frac{1}{20}$, $\sigma_N/2N \geq 0.128\cdots$.

We are going to prove that for $0 < \lambda < \ln(9/8)/2\ln 6$ we have indeed that lim inf$(\sigma_N/2N) > (1/4\ln 2)\ln\frac{4}{3}$. We are going to prove more precisely that $\delta_{M,L} \leq C(\lambda)3^L(\frac{3}{4})^{2M}\theta^N\sqrt{N}$, where $C(\lambda)$ and θ depends only on λ and where $0 < \theta < 1$:

- For $x \geq 0$, $\alpha(x) \leq 2^L = 3^L(\frac{3}{4})^{2M}\{(\frac{2}{3})^{1-2\lambda}(\frac{4}{3})^{2\lambda}\}^N$.
- For $-\frac{1}{3} \leq x < 0$, we write $|1-4x^2| \leq 1$; hence,

$$\alpha(x) \leq C(\lambda)\sqrt{N}\left(\frac{8}{3}\right)^L = C(\lambda)\sqrt{N}3^L\left(\frac{3}{4}\right)^{2M}\left\{\left(\frac{8}{9}\right)^{1-2\lambda}\left(\frac{4}{3}\right)^{2\lambda}\right\}^N.$$

- For $-\frac{1}{2} \leq x < -\frac{1}{3}$, we write $|1-4x^2| \leq \frac{5}{9}$; hence

$$\alpha(x) \leq C(\lambda)\sqrt{N}3^L\left(\frac{5}{9}\right)^{2M} = C(\lambda)\sqrt{N}3^L\left(\frac{3}{4}\right)^{2M}\left\{\left(\frac{20}{27}\right)^{2\lambda}\right\}^N.$$

- For $-\frac{5}{8} \leq x < -\frac{1}{2}$ we write $(1-4x^2)^{2M}|3-4x^2|^M \leq ((\frac{207}{256}) \times (\frac{9}{16}))^M$

$$\alpha(x) \leq C(\lambda)\sqrt{N}3^L\left(\frac{9\times 207}{16\times 256}\right)^M = C(\lambda)\sqrt{N}3^L\left(\frac{3}{4}\right)^{2M}\left\{\left(\frac{207}{256}\right)^{\lambda}\right\}^N.$$

- For $-\sqrt{2}/2 \le x < -\frac{5}{8}$, we write $(1-4x^2)^{2M}|3-4x^2|^M \le 1$

$$\alpha(x) \le C(\lambda)\sqrt{N}\left(\frac{337}{64}\right)^{L/2} = C(\lambda)\sqrt{N}3^L\left(\frac{3}{4}\right)^{2M}\left\{\left(\frac{337}{576}\right)^{1/2-\lambda}\left(\frac{4}{3}\right)^{2\lambda}\right\}^N.$$

- For $-1 \le x < -\sqrt{2}/2$ we write $|1-4x^2| \le 3$; hence

$$\alpha(x) \le \sqrt{C(\lambda)}N^{1/4}3^M 8^{L/2} \le C(\lambda)\sqrt{N}3^L\left(\frac{3}{4}\right)^{2M}\left\{\left(\frac{8}{9}\right)^{1/2-\lambda}\left(\frac{16}{3}\right)^{\lambda}\right\}^N.$$

Thus, we have proved $\delta_{M,L} \le C(\lambda)\sqrt{N}3^L(\frac{3}{4})^{2M}\theta^N$ with $\theta = \text{Max}\{\sqrt{\frac{8}{9}}\cdot 6^\lambda, (\frac{207}{256})^\lambda\}$, and $\theta < 1$ as soon as $\lambda \ln 6 < \frac{1}{2}\ln\frac{9}{8}$. Theorem 3 is proved.

CONCLUSION

We have slightly improved the ratio between regularity and complexity. The natural question which remains open is then the following one: which is the best asymptotic ratio.

Remark. H. Volkmer achieved the construction of wavelets with the regularity ratio $0.175\cdots$ [9]; hence, they are better than the Matzinger wavelets.

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