UNIQUENESS OF "MILD" SOLUTIONS FOR THE NAVIER–STOKES EQUATIONS IN $\mathrm{L}^3(\mathbb{R}^3)$ AND OTHER LIMIT SPACES

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Abstract: We prove uniqueness for mild solutions of the Navier–Stokes equations in $\mathrm{L}^3(\mathbb{R}^3)$ (and more general limit spaces).

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We are going to prove uniqueness of mild solutions of the Navier–Stokes equations in $\mathrm{L}^3(\mathbb{R}^3)$ and in other limit spaces. We define the operator defined on $B \times B$ vector functions :

$$\begin{aligned}\vec{T}(\vec{F})(t) &= \int_0^t \mathbb{P} \exp((t-s)\Delta)\vec{\nabla} \cdot \vec{F}(s)ds \\ &= \int_0^t \Big(\sum_{i=1}^{3} \frac{1}{(t-s)^{3/2}} g_i \left(\frac{x}{\sqrt{t-s}} \right) * F_{i,j} \Big)_{1 \le j \le 3} \frac{ds}{\sqrt{t-s}}\end{aligned}$$

where $\mathbb{P}$ is the orthogonal projection operator on divergence–free vectors fields.

A mild solution in a function space B for the Navier–Stokes equations is a continuous path $\vec{u}$ in B^3, $\vec{u}(t) \in C\left([0, t_0[, B^3\right)$, such that $\vec{T}(\vec{u} \otimes \vec{u})(t)$ is a continuous path in $(\mathcal{S}'(\mathbb{R}^3))^3$ (with initial value 0) and such that $\vec{u}(t) = \exp(t\Delta)\vec{u}_0 - \vec{T}(\vec{u} \otimes \vec{u})(t)$.

Definition. A *limit space* for the Navier–Stokes equations is a Banach function space B on $\mathbb{R}^3$ such that :

i) $\mathcal{S}(\mathbb{R}^3)$ is continuously and densely imbedded in B;

ii) B is continuously imbedded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$;

iii) $\forall\, x_0 \in \mathbb{R}^3, \forall\, f \in B \quad \|f(x - x_0)\|_B = \|f(x)\|_B$;

iv) $\forall \lambda > 0, \forall\, f \in B \quad \|\lambda f(\lambda x)\|_B = \|f(x)\|_B$.

Exemple : $B = \mathrm{L}^3(\mathbb{R}^3)$.

If B is a limit space, then $\vec{T}(\vec{u} \otimes \vec{u})$, when $\vec{u} \in C\left([0, t_0[, B^3\right)$, is a continuous path in E^3, where $E = \left\{ f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^3) \text{ t.q. } \sup_{R>0} \sup_{x_0 \in \mathbb{R}^3} R^2 \int_{|x-x_0|<1} |f(Rx)|dx < \infty \right\}$, and $\left\| \vec{T}(\vec{u} \otimes \vec{u})(t) \right\|_{E^3} \le C_0 \sqrt{t} \,\left(\sup_{0<s<t} \|\vec{u}(s)\|_{B^3}\right)^2$.

Theorem. *If $p > 2$ and if B is a limit space continuously imbedded in $\mathrm{L}^{\mathrm{p}}_{\mathrm{loc}}(\mathbb{R}^3)$, then mild solutions for the Navier–Stokes equations are unique in B : if $\vec{u}_0 \in B^3$, if $\vec{u}, \vec{v} \in C\left([0, t_0[, B^3\right)$ are such that $\vec{u}(t) = \exp(t\Delta)\vec{u}_0 - \vec{T}(\vec{u} \otimes \vec{u})(t)$, $\vec{v}(t) = \exp(t\Delta)\vec{u}_0 - \vec{T}(\vec{v} \otimes \vec{v})(t)$, then $\vec{u} = \vec{v}$.*

The proof will be detailed and references will be given in [FUR2]. The case of $B = \mathrm{L}^3(\mathbb{R}^3)$ is proved in [FUR1]. In this paper we just give the sketch of the proof.

Basic idea.

We introduce the Littlewood–Paley decomposition $f = \sum_{j \in \mathbb{Z}} \Delta_j f$ where $\widehat{\Delta_j f}(\xi) = \omega(\frac{\xi}{2^j})\hat{f}(\xi)$, for $\xi \neq 0$ $\sum_{j \in \mathbb{Z}} \omega(\frac{\xi}{2^j}) = 1$, $\omega \in C^\infty$, $\mathrm{Supp}\omega \subset \{\xi \text{ t.q. } \frac{1}{2} \le |\xi| \le 2\}$ and we notice that $\Delta_j = \sum_{|k-j| \le 2} \Delta_j \Delta_k$.

The basic idea which we shall use in the proof is the following one. If A is a Banach function space such that

$$(1) \qquad \forall f \in A,\ \forall g \in \mathrm{L}^1(\mathbb{R}^3) \quad \|f * g\|_A \le C_1 \|f\|_A \|g\|_{\mathrm{L}^1}$$

then we have :

$$\left\|\vec{T}(\vec{F})(t)\right\|_{A^3} \leq 6\sqrt{t}\ C_2 \sup_{0<s<t} \left\|\vec{F}(t)\right\|_{A^{3\times 3}} \tag{2}$$

$$\left\|\Delta_j\vec{T}(\vec{F})(t)\right\|_{A^3} \leq 3\pi\ C_3 \sup_{0<s<t} 2^{-j} \left\|\Delta_j\vec{F}(t)\right\|_{A^{3\times 3}} \tag{3}$$

$$\left\|\Delta_j\vec{T}(\vec{F})(t)\right\|_{A^3} \leq 3\int_0^1 \frac{d\sigma}{(1-\sigma)^{7/8}\sigma^{1/8}}\ C_3 \sup_{0<s<t} 2^{-j}(2^j\sqrt{s})^{1/4} \left\|\Delta_j\vec{F}(t)\right\|_{A^{3\times 3}} \tag{4}$$

(where $\|\vec{u}\|_{A^3} = \mathrm{Max}_{1\leq i\leq 3}\|u_i\|_A$, $\left\|\vec{M}\right\|_{A^{3\times 3}} = \mathrm{Max}_{1\leq i\leq 3, 1\leq l\leq 3}\|m_{i,l}\|_A$).

The proof of these estimates is straightforward. Indeed, it is enough to check that the functions $G_i(x,t-s) = \frac{1}{(t-s)^2} g_i\left(\frac{x}{\sqrt{t-s}}\right)$ which are used in $\vec{T}$ satisfy :

$$\|G_i\|_{\mathrm{L}^1} \leq \frac{C_2}{\sqrt{t-s}}, \qquad \left\|\sum_{k=j-2}^{j+2}\Delta_k G_i\right\|_{\mathrm{L}^1} \leq \frac{C_3}{\sqrt{t-s}}\frac{1}{1+4^j(t-s)}$$

and to check that :

$$\int_0^t \frac{ds}{\sqrt{t-s}} = 2\sqrt{t}, \qquad \int_0^t \frac{1}{\sqrt{t-s}}\frac{2^j\, ds}{1+4^j(t-s)} \leq \pi,$$

$$\int_0^t \frac{1}{\sqrt{t-s}}\frac{2^j\, ds}{(2^j\sqrt{s})^{1/4}(1+4^j(t-s))} \leq \int_0^1 \frac{d\sigma}{(1-\sigma)^{7/8}\sigma^{1/8}}$$

Uniqueness in $\mathrm{L}^3(\mathbb{R}^3)$ whith small–normed initial data.

If $\vec{v} \in C\left([0,t_0[, (\mathrm{L}^3(\mathbb{R}^3))^3\right)$, then we notice that :

$$\|\Delta_j(fg)\|_{\mathrm{L}^2} \leq C_4 2^{j/2}\|f\|_{\mathrm{L}^3}\|g\|_{\mathrm{L}^3}\,, \tag{5}$$

hence

$$\sup_j 2^{j/2}\left\|\Delta_j\vec{T}(\vec{v}\otimes\vec{v})(t)\right\|_{(\mathrm{L}^2)^3} \leq 3\pi\ C_3C_4\left(\sup_{0<s<t}\|\vec{v}(s)\|_{(\mathrm{L}^3)^3}\right)^2 \tag{6}$$

Moreover, it is easy to see (as we shall prove below) that :

$$\|\Delta_j(fg)\|_{\mathrm{L}^2} \leq C_5\ 2^{j/2}\|f\|_{\mathrm{L}^3}\sup_{k\in\mathbb{Z}} 2^{k/2}\|\Delta_k g\|_{\mathrm{L}^2} \tag{7}$$

hence, writing for our two solutions $\vec{u}, \vec{v}$, $\quad \vec{w} = \vec{u} - \vec{v} = \vec{T}(\vec{v} \otimes \vec{v}) - \vec{T}(\vec{u} \otimes \vec{u}) =$
$\vec{T}(-\vec{w} \otimes \vec{v}) + \vec{T}(\vec{u} \otimes -\vec{w}) = \vec{T}(-\vec{w} \otimes -\vec{w}) + \vec{T}(\vec{u} \otimes -\vec{w}) + \vec{T}(-\vec{w} \otimes \vec{u})$, we get :

$$\begin{aligned} &\sup_{0<s<t} \sup_j 2^{j/2} \|\Delta_j \vec{w}(s)\|_{(\mathrm{L}^2)^3} \leq \\ &\leq 3\pi\ C_3 C_5 \sup_{0<s<t} \sup_j 2^{j/2} \|\Delta_j \vec{w}(s)\|_{(\mathrm{L}^2)^3} \left(2 \sup_{0<s<t} \|\vec{u}(s)\|_{(\mathrm{L}^3)^3} + \sup_{0<s<t} \|\vec{w}(s)\|_{(\mathrm{L}^3)^3}\right). \end{aligned} \tag{8}$$

We consider $\bar{t} = \sup\{t \in [0, t_0] \text{ s.t. } \vec{u} = \vec{v} \text{ on } [0, t]\ \}$; we suppose $\bar{t} < t_0$.

Since $\vec{w} = 0$ in $t = 0$ and $\vec{w}$ is continuous from $[0, t_0] \to (\mathrm{L}^3(\mathbb{R}^3))^3$, we find $t_1 > \bar{t}$ s.t. $\sup_{0<s<t_1} \|\vec{w}\|_{(\mathrm{L}^3)^3} \leq \frac{1}{12\pi C_3 C_5}$.

We conclude that $\vec{w}(s) = 0$ on $[0, t_1]$ provided that $\sup_{0<s<t_0} \|\vec{u}(s)\|_{(\mathrm{L}^3)^3} \leq \frac{1}{12\pi C_3 C_5}$, which we may assume when $\|\vec{u}_0\|_{(\mathrm{L}^3)^3}$ is small enough by choosing the solution $\vec{u}$ which is given by the formalism of T.Kato [KAT]. Thus $\vec{u} = \vec{v}$.

The proof of (7) is quite easy. It's enough to write $\Delta_j(fg) = \alpha_j + \beta_j + \gamma_j$ with

$$\begin{aligned} \alpha_j &= \sum_{|j-l|\leq 3} \Delta_l g \sum_{k<l-3} \Delta_k f \\ \beta_j &= \sum_{|j-l|\leq 3} \Delta_l f \sum_{k<l-3} \Delta_k g \\ \gamma_j &= \sum_{l\geq j-3} \Delta_l g \sum_{|k-l|\leq 3} \Delta_k f. \end{aligned}$$

For estimating $\|\alpha_j\|_{\mathrm{L}^2}$, we write $\|\Delta_l g\|_{\mathrm{L}^2} \leq 2^{-\frac{l}{2}} \sup_{k\in\mathbb{Z}} 2^{\frac{k}{2}} \|\Delta_k g\|_{\mathrm{L}^2}$ and $\|\Delta_k f\|_{\mathrm{L}^\infty} \leq C_6 2^k \|f\|_{\mathrm{L}^3}$.

For estimating $\|\beta_j\|_{\mathrm{L}^2}$, we write $\|\Delta_l f\|_{\mathrm{L}^3} \leq C_7 \|f\|_{\mathrm{L}^3}$ and $\|\Delta_k g\|_{\mathrm{L}^6} \leq C_8 2^k \|\Delta_k g\|_{\mathrm{L}^2} \leq C_8 2^{\frac{k}{2}} \sup_{p\in\mathbb{Z}} 2^{\frac{p}{2}} \|\Delta_p g\|_{\mathrm{L}^2}$.

For estimating $\|\gamma_j\|_{\mathrm{L}^2}$, we write $\|\gamma_j\|_{\mathrm{L}^2} \leq C_9 2^j \|\gamma_j\|_{\mathrm{L}^{\frac{6}{5}}}$, $\|\Delta_k f\|_{\mathrm{L}^3} \leq C_7 \|f\|_{\mathrm{L}^3}$ and $\|\Delta_l g\|_{\mathrm{L}^2} \leq 2^{-\frac{l}{2}} \sup_{p\in\mathbb{Z}} 2^{\frac{p}{2}} \|\Delta_p g\|_{\mathrm{L}^2}$. Thus (7) is proved.

Uniqueness in L^3 (general case).

For the general case we replace (7) by

$$\|\Delta_j(fg)\|_{\mathrm{L}^2} \leq C_{10} 2^{\frac{3}{4}j} \|f\|_{\mathrm{L}^4} \sup_{k\in\mathbb{Z}} 2^{\frac{k}{2}} \|\Delta_k g\|_{\mathrm{L}^2} \tag{9}$$

which is proved in the same way as (7). Then we get, writing

$$\begin{aligned} \vec{w} &= \vec{T}(-\vec{w} \otimes \vec{v}) + \vec{T}(\vec{u} \otimes -\vec{w}) \\ &= \vec{T}(-\vec{w} \otimes \exp(t\Delta)\vec{u}_0) + \vec{T}(\exp(t\Delta)\vec{u}_0 \otimes -\vec{w}) \\ &+ \vec{T}(-\vec{w} \otimes (\vec{v} - \exp(t\Delta)\vec{u}_0)) + \vec{T}(\vec{u} - \exp(t\Delta)\vec{u}_0) \otimes -\vec{w}), \end{aligned}$$

$$
\sup_{0<s<t}\sup_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\,\|\Delta_j\vec{w}(s)\|_{(\mathrm{L}^2)^3} \leq \tag{10}
$$

$$
\leq 3C_3C_{10}\int_0^1 \frac{d\sigma}{(1-\sigma)^{\frac{7}{8}}\sigma^{\frac{1}{8}}}\sup_{0<s<t}\sup_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\,\|\Delta_j\vec{w}(s)\|_{(\mathrm{L}^2)^3}\Big(2\sup_{0<s<t}(\sqrt{s})^{\frac{1}{4}}\,\|\exp(s\Delta)\vec{u}_0\|_{(\mathrm{L}^4)^3}
$$

$$
+\sup_{0<s<t}\|\vec{u}(s)-\exp(s\Delta)\vec{u}_0\|_{(\mathrm{L}^3)^3}+\sup_{0<s<t}\|\vec{v}(s)-\exp(s\Delta)\vec{u}_0\|_{(\mathrm{L}^3)^3}\Big)
$$

$$
= C_{11}\sup_{0<s<t}\sup_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\,\|\Delta_j\vec{w}(s)\|_{(\mathrm{L}^2)^3}\,A(t,\vec{u}_0,\vec{u},\vec{v})
$$

Since $\lim_{t\to 0} A(t,\vec{u}_0,\vec{u},\vec{v}) = 0$, we find $\vec{w}(t)=\vec{0}$ on $[0,\epsilon)$ for some positive ϵ. But if $\vec{u}$ is a mild solution on $[0,t_0)$ then $\vec{u}(t_1+t)=\exp(t\Delta)\vec{u}(t_1)-\vec{T}\,(\vec{u}(t_1+s)\otimes\vec{u}(t_1+s))$ on $[0,t_0-t_1)$ hence local uniqueness implies uniqueness on the whole interval where $\vec{u}$ is defined, since $\vec{u}$ remains a mild solution when evolving in time.

Uniqueness in B.

The proof follows the same lines, replacing

$$
\sup_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\,\|\Delta_j f\|_{\mathrm{L}^2} \quad\text{by}\quad \sup_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\sup_{R>0}\sup_{x_0\in\mathbb{R}^3} R^{\frac{3}{2}}\left\|1_{|x-x_0|<1}\Delta_j f(Rx)\right\|_{\mathrm{L}^{\frac{4}{3}}},
$$

$$
\|f\|_{\mathrm{L}^3} \quad\text{by}\quad \sup_{R>0}\sup_{x_0\in\mathbb{R}^3} R\left\|1_{|x-x_0|<1} f(Rx)\right\|_{\mathrm{L}^p} \quad \text{(which is controlled by } \|f\|_B)
$$

and

$$
\|f\|_{\mathrm{L}^4} \quad\text{by}\quad \sup_{R>0}\sup_{x_0\in\mathbb{R}^3} R^{\frac{3}{4}}\left\|1_{|x-x_0|<1} f(Rx)\right\|_{\mathrm{L}^{\frac{8}{3}}}.
$$

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