# MULTILEVEL MONTE CARLO FOR ASIAN OPTIONS AND LIMIT THEOREMS

Mohamed Ben Alaya[*1] & Ahmed Kebaier[†‡1]

[1] Université Paris 13
Sorbonne Paris Cité, LAGA, CNRS (UMR 7539)
99, av. J.B. Clément 93430 Villetaneuse, France
mba@math.univ-paris13.fr kebaier@math.univ-paris13.fr

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**Abstract**

The purpose of this paper is to study the problem of pricing Asian options using the multilevel Monte Carlo method recently introduced by Giles [8] and to prove a central limit theorem of Lindeberg Feller type for the obtained algorithm. Indeed, the implementation of such a method requires first a discretization of the integral of the payoff process. To do so, we use two well-known second order discretization schemes, namely, the Riemann scheme and the trapezoidal scheme. More precisely, for each one of these schemes we prove a stable law convergence result for the error on two consecutive levels of the algorithm. This allows us to go further and prove two central limit theorems on the multilevel algorithm providing us a precise description on the choice of the associated parameters with an explicit representation of the limiting variance. For this setting of second order schemes, we give new optimal parameters leading to the convergence of the central limit theorem. A complexity of the multilevel Monte Carlo algorithm is carried out.

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# 1 Introduction

Let $S$ be the Black and Scholes model on a probability space $\mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying

$$dS_t = S_t(rdt + \sigma dW_t), \qquad \text{with } t \in [0,T], \;\; T > 0,$$

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where $\sigma$ and $r$ are real constants, with $\sigma > 0$ and $(W_t)_{t\in[0,T]}$ is a standard Brownian motion on $\mathcal{B}$. The solution of the last equation is given by $S_t = S_0 \exp\big((r - \frac{\sigma^2}{2})t + \sigma W_t\big)$ where $S_0 > 0$. The payoff of an Asian option is related to the integral of the asset price process $I_T = \frac{1}{T}\int_0^T S_u\, du$. Our aim is to evaluate $e^{-rT}\mathbb{E}\, f(I_T)$, where $f$ is a given real valued function. In a financial setting, if $f(x) = (x - K)_+$ then this last quantity stands for the price of an Asian call option with fixed strike $K$. In this case there is no explicit formula that gives the real price. So, in order to compute this price by a probabilistic method, we need first to approach the integral $I_T$ using a discretization scheme $(I_t^n)_{0\le t\le T}$, with time step $T/n$. Then, we approximate $\mathbb{E}\, f\,(I_T^n)$ by $\frac{1}{N}\sum_{i=1}^N f(I_{T,i}^n)$, where $f(I_{T,i}^n)_{1\le i\le N}$ is a sample of $N$ independent copies of $f(I_T^n)$. This approximation is affected respectively by a discretization error and a statistical error

$$\varepsilon_n := \mathbb{E}\left(f(I_T^n) - f(I_T)\right) \text{ and } \frac{1}{N}\sum_{i=1}^N f(I_{T,i}^n) - \mathbb{E} f(I_T^n).$$

On one hand, Lapeyre and Temam [19] have proved that $\varepsilon_n \sim c/n$, $c \in \mathbb{R}$, for either Riemann or trapezoidal scheme. On the other hand, the statistical error is controlled by the central limit theorem with order $1/\sqrt{N}$. Further, the optimal choice of the sample size $N$ in the classical Monte Carlo method mainly depends on the order of the discretization error. More precisely, it turns out that for order $1/n$ the optimal choice of $N$ is $n^2$. This leads to a total complexity in the Monte Carlo method of order $n^3$.

Recently, Giles [8] has introduced a new multilevel approach which reduces the complexity to $O(n^2)$ when using a second order scheme which is the case for the Riemann and trapezoidal schemes in the setting of pricing Asian options. This multilevel approach is an extension of the two-level method of Kebaier [17] known as the statistical Romberg method and reducing the complexity of the crude Monte Carlo method to $O(n^{7/3})$, for the same setting above. There are also similarities to Heinrich's multilevel method approach for parametric integration [12] (see also Ben Alaya and Kebaier [1], Creutzig, Dereich, Müller-Gronbach and Ritter [4], Dereich [5], Giles [7], Giles, Higham and Mao [9], Giles and Szpruch [10], Heinrich [11], Heinrich and Sindambiwe [13] and Hutzenthaler, Jentzen and Kloeden [14] for related results).

Using the telescoping series representation with decreasing step sizes

$$\mathbb{E}(f(I_T^n)) = \mathbb{E}\left(f(I_T^1)\right) + \sum_{\ell=1}^L \mathbb{E}\left(f(I_T^{m^\ell}) - f(I_T^{m^{\ell-1}})\right)$$

the multilevel Monte Carlo method consists in computing the $L+1$ expectations by $L+1$ independent empirical means and approximating the quantity $\mathbb{E}f(I_T)$ by

$$Q_n = \frac{1}{N_0}\sum_{k=1}^{N_0} f(I_{T,k}^1) + \sum_{\ell=1}^L \frac{1}{N_\ell}\sum_{k=1}^{N_\ell}\left(f(I_{T,k}^{\ell,m^\ell}) - f(I_{T,k}^{\ell,m^{\ell-1}})\right),\ m \in \mathbb{N}\setminus\{0,1\},$$

where the fine discretization step is equal to $T/n$ thereby $L = \frac{\log n}{\log m}$. For $\ell \in \{1,\cdots,L\}$ and $k \in \{1,\cdots,N_\ell\}$ the processes $(I_{t,k}^{\ell,m^\ell}, I_{t,k}^{\ell,m^{\ell-1}})_{0\le t\le T}$ are independent copies of $(I_t^{\ell,m^\ell}, I_t^{\ell,m^{\ell-1}})_{0\le t\le T}$ whose components denote the discretization schemes with time steps $m^{-\ell}T$ and $m^{-(\ell-1)}T$.

However, for fixed $\ell$, the simulation of $(I_t^{\ell,I^\ell})_{0\le t\le T}$ and $(I_t^{\ell,m^{\ell-1}})_{0\le t\le T}$ have to be based on the same Brownian path. Concerning the first empirical mean, for $k\in\{1,\cdots,N_0\}$ the processes $(I^1_{t,k})_{0\le t\le T}$ are independent copies of $(I^1_t)_{0\le t\le T}$ which denotes the discretization scheme with time step $T$. Here, it is important to point out that all these $L+1$ Monte Carlo estimators have to be based on different independent samples. Due to the above independence assumption for the paths, the variance of the multilevel estimator is given by

$$\sigma^2 := Var(Q_n) = N_0^{-1} Var(f(I^1_T)) + \sum_{\ell=1}^{L} N_\ell^{-1}\sigma_\ell^2,$$

where $\sigma_\ell^2 = Var\left(f(I_T^{\ell,m^\ell}) - f(I_T^{\ell,m^{\ell-1}})\right)$. If we assume that the function $f$ is Lipschitz continuous and if we have an approximation scheme of second order satisfying $\mathbb{E}(I_T^n - I_T)^2 = O(1/n^2)^*$ then

$$\sigma^2 \le c\sum_{\ell=0}^{L} N_\ell^{-1} m^{-2\ell}$$

for some positive constant $c$. In order to minimize the time complexity $\sum_{\ell=0}^{L} N_\ell^{-1} m^{-\ell}$, for a given root mean square error (RMSE) of order $1/n$, Giles obtains an optimal choice of the parameters given by (see Theorem 3.1 of [8])

$$N_\ell = 2cn^2\sqrt{T}\left(\frac{\sqrt{m}-1}{\sqrt{m}}\right)\left(\frac{T}{m^\ell}\right)^{3/2} \quad \text{for } \ell\in\{0,\cdots,L\} \quad \text{and} \quad L = \frac{\log n}{\log m}. \tag{1}$$

This choice leads to an optimal complexity for the multilevel Monte Carlo proportional to $n^2$.

More recently, in a previous work we have proved a central limit theorem for the multilevel approach when using an Euler path discretization$^\dagger$ for an European option [1]. So, in that setting we have studied a first order scheme for a non-path-dependent option. However, in the present paper we are rather interested by studying a second order scheme for a path-dependent option. Actually, we investigate the central limit theorem for the multilevel method when using the Riemann scheme or the trapezoidal scheme for an Asian option. Our main results are two Lindeberg Feller central limit theorems for the multilevel Monte Carlo algorithm associated to both Riemann and trapezoidal schemes. To do so, we first prove a stable law convergence theorem for each approximation error on two consecutive levels $m^{\ell-1}$ and $m^\ell$. More precisely, we study the asymptotic behavior of the distribution of the error $\sqrt{\frac{m^\ell}{(m-1)T}}(I^{\ell,m^\ell} - I^{\ell,m^{\ell-1}})$ associated to both Riemann and trapezoidal schemes, as $\ell$ tends to $\infty$. (See Theorem 3 and Theorem 4). These results are stated and proved in section 3.

In section 4, we take advantage of this study to establish two new Lindeberg Feller central limit theorems (see Theorem 5 and Theorem 6 ). We also obtain a Berry-Essen type Bound on these central limit theorems. These results provide us a precise description for the choice of the parameters in the multilevel Monte Carlo method when used to price Asian options.

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$^*$This is the case for the Riemann and the trapezoidal schemes (see subsection 2.2 for more details).

$^\dagger$For more details on the Euler discretization scheme see e.g. Bouleau and Lépingle [3], Kloeden and Platen [18] and Talay and Tubaro [20].

Section 5 is devoted to the complexity analysis of the algorithm. The optimal sequence of sample sizes $(N_\ell)_{0\le\ell\le L}$ is given by

$$N_\ell = \frac{m^2-1}{m^{3\ell/2}(\sqrt{m}-1)} n^2 \left(1-\frac{1}{\sqrt{n}}\right), \quad \text{for } \ell \in \{0,\cdots,L\} \quad \text{and} \quad L = \frac{\log n}{\log m}. \tag{2}$$

By comparing relations (1) and (2), we note that our optimal sequence of sample sizes $(N_\ell)_{0\le\ell\le L}$ is in line with the choice proposed by Giles [8] in the context of second order schemes. Nevertheless, this choice given by (2) does not satisfy the so called Lyapunov assumption of the Lindeberg Feller central limit theorem (see section 4 and 5 for more details). Finally, we provide three possible choices of sample sizes $(N_\ell)_{0\le\ell\le L}$ satisfying this assumption and for which the optimal complexity can be closer to the order $n^2$. Section 2 below is devoted to recall some useful stochastic limit theorems and to introduce our notations. The final section indicates the direction of future research.

# 2 General framework

## 2.1 Preliminaries

Let $(X_n)$ be a sequence of random variables with values in a Polish space $E$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be an extension of $(\Omega, \mathcal{F}, \mathbb{P})$, and let $X$ be an $E$-valued random variable on the extension. We say that $(X_n)$ converges in law to $X$ stably and write $X_n \Rightarrow^{stably} X$, if

$$\mathbb{E}(Uh(X_n)) \to \tilde{\mathbb{E}}(Uh(X))$$

for all $h : E \to \mathbb{R}$ bounded continuous and all bounded random variable $U$ on $(\Omega, \mathcal{F})$ . This convergence is obviously stronger than convergence in law that we will denote here by "$\Rightarrow$". According to section 2 of Jacod [15] and Lemma 2.1 of Jacod and Protter [16], we have the following result.

**Lemma 1** *let $V_n$ and $V$ be defined on $(\Omega, \mathcal{F})$ with values in another metric space $E'$.*

$$\text{if} \quad V_n \xrightarrow{\mathbb{P}} V, \;\; X_n \Rightarrow^{stably} X \quad \text{then} \quad (V_n, X_n) \Rightarrow^{stably} (V, X).$$

*Conversely, if $(V, X_n) \Rightarrow (V, X)$ and $V$ generates the $\sigma$-field $\mathcal{F}$, we can realize this limit as $(V, X)$ with $X$ defined on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$ and $X_n \Rightarrow^{stably} X$.*

Now, we recall a result on the convergence of stochastic integrals formulated from Theorem 2.3 in Jacod and Protter [16]. This is a simplified version but it is sufficient for our study. Let $X^n = (X^{n,i})_{1\le i\le d}$ be a sequence of $\mathbb{R}^d$-valued continuous semimartingales with the decomposition

$$X_t^{n,i} = X_0^{n,i} + A_t^{n,i} + M_t^{n,i}, \quad 0 \le t \le T$$

where, for each $n \in \mathbb{N}$ and $1 \le i \le d$, $A^{n,i}$ is a predictable process with finite variation, null at 0 and $M^{n,i}$ is a martingale null at 0.

**Theorem 1** *Assume that the sequence $(X^n)$ is such that*

$$\langle M^{n,i}\rangle_T + \int_0^T \left|dA_s^{n,i}\right|$$

*is tight. Let $H^n$ and $H$ be a sequence of adapted, right-continuous and left-hand limited processes all defined on the same filtered probability space. If $(H^n, X^n) \Rightarrow (H, X)$ then $X$ is a semimartingale with respect to the filtration generated by the limit process $(H, X)$, and we have $(H^n, X^n, \int H^n dX^n) \Rightarrow (H, X, \int H dX)$.*

We recall also the following Lindeberg Feller central limit theorem that will be used in the sequel (see for instance Theorem 7.2 and 7.3 in [2]).

**Theorem 2 (central limit theorem for triangular array)** *Let $(k_n)_{n\in\mathbb{N}}$ be a sequence such that $k_n \longrightarrow \infty$ as $n \longrightarrow \infty$. For each $n$, let $X_{n,1}, \cdots, X_{n,k_n}$ be $k_n$ independent random variables with finite variance such that $\mathbb{E}(X_{n,k}) = 0$ for all $k \in \{1, \cdots, k_n\}$. Suppose that the following conditions hold.*

*A1.* $\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbb{E}|X_{n,k}|^2 = \sigma^2, \sigma > 0.$

*A2. Lindeberg's condition: for all $\varepsilon > 0$, $\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbb{E}\left(|X_{n,k}|^2 1_{\{|X_{n,k}|>\varepsilon\}}\right) = 0$. Then*

$$\sum_{k=1}^{k_n} X_{n,k} \Rightarrow \mathcal{N}(0, \sigma^2) \qquad \textit{as} \;\; n \to \infty.$$

*Moreover, if the $X_{n,k}$ have moments of order $p > 2$, then the Lindeberg's condition can be obtained by the following one*

*A3 Lyapunov's condition:* $\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbb{E}|X_{n,k}|^p = 0.$

## 2.2 Asian Option

There are several approximation schemes used in practice and one can consider either Riemann scheme or the trapezoidal scheme. We have

$$I_T^n = \frac{1}{n}\sum_{k=0}^{n-1} S_{k\delta}, \quad \text{and} \quad J_T^n = \frac{1}{n}\sum_{k=0}^{n-1} \frac{S_{k\delta} + S_{(k+1)\delta}}{2}, \quad \text{where } \delta = \frac{T}{n}.$$

We recall some results proved by Lapeyre and Temam [19] on the expansions for the strong and weak errors associated with both Riemann and trapezoidal schemes. Indeed, concerning the strong errors, see the proposition on page 98 of [19], they prove that for $p \geq 1$, there exist $K_p(T) > 0$ and $\tilde{K}_p(T) > 0$ such that

$\mathcal{PR}\mathbf{1})$ $\mathbb{E} \sup_{t\in[0,T]} |I_t^n - I_t|^{2p} \leq \dfrac{K_p(T)}{n^{1/2p}},$

$\mathcal{PT}\mathbf{1})$ $\mathbb{E} \sup_{t\in[0,T]} |J_t^n - I_t|^{2p} \leq \dfrac{\tilde{K}_p(T)}{n^{1/2p}}.$

Hence, it is obvious that both schemes are of second order. Concerning the weak errors, see the theorem on page 108 of [19], they prove that for any $\mathbb{R}$-valued function $f$ satisfying condition

$$(\mathcal{H}_{\mathbf{f}}) \quad |f(x)-f(y)| \leq C(1+|x|^p+|y|^p)|x-y|, \quad \text{for some } C, p>0,$$

if $\mathbb{P}(I_T \notin \mathcal{D}_f) = 0$, where $\mathcal{D}_f := \{x \in \mathbb{R}^d; f \text{ is differentiable at } x\}$, then there exist real constants $C_f^I$ and $C_f^J$ such that

$\mathcal{PR}\mathbf{2}$) $\lim_{n\to\infty} n\big(\mathbb{E}f(I_T^n) - \mathbb{E}f(I_T)\big) = C_f^I$,

$\mathcal{PT}\mathbf{2}$) $\lim_{n\to\infty} n\big(\mathbb{E}f(J_T^n) - \mathbb{E}f(I_T)\big) = C_f^J$.

# 3 Stable limit theorems

In order to obtain central limit theorems for the multilevel Monte Carlo method associated with both Riemann and Trapezoidal schemes, we study the asymptotic behavior of the distribution errors. We establish a stable convergence theorem for each scheme on two consecutive levels $m^{\ell-1}$ and $m^{\ell}$.

## 3.1 Stable convergence of the Riemann scheme error

The Riemann approximation of the process is given by

$$I_t^n = \frac{1}{T}\int_0^t S_{\eta_n(u)} du = \frac{1}{n}\sum_{k=0}^{[t/\delta]-1} S_{k\delta} + \frac{t-\eta_n(t)}{T} S_{\eta_n(t)}$$

with $\eta_n(t) = [t/\delta]\delta$. Note that the study of the error $I^{m^\ell} - I^{m^{\ell-1}}$ as $\ell \to \infty$ can be reduced to the study of the error $I^{mn} - I^n$ as $n \to \infty$.

**Theorem 3** *We have the following result*

$$\frac{mn}{\sqrt{m^2-1}}(I^{mn} - I^n) \Rightarrow^{stably} \xi$$

*where $\xi$ is the process defined by*

$$\xi_t := \sqrt{\frac{m-1}{m+1}}\frac{S_t - S_0}{2} + \frac{1}{2\sqrt{3}}\int_0^t \sigma S_u dB_u,$$

*with $B$ a standard Brownian motion on an extension $\hat{\mathcal{B}}$ of $\mathcal{B}$, which is independent of $W$.*

**Proof** The error, $\mathcal{E}_t^{mn,n}$, is given by

$$\mathcal{E}_t^{mn,n} := I_t^{mn} - I_t^n = \frac{1}{T}\int_0^t \big(S_{\eta_{mn}(s)} - S_{\eta_n(s)}\big)\, ds.$$

Noting that the integrand vanishes on the interval $[\eta_n(s), \eta_n(s) + \frac{1}{mn}[$, this error can be written as follows

$$\begin{aligned}
\mathcal{E}_t^{mn,n} &= \frac{1}{T} \int_0^t \left( S_{\eta_{mn}(s)} - S_{\eta_n(s)} \right) \mathbf{1}_{\{\eta_n(s) + \frac{1}{mn} \le s < \eta_n(s) + \frac{1}{n}\}} ds + R_t^{mn,n} \\
&= \frac{1}{mn} \sum_{k=0}^{[t/\delta]-1} \sum_{\ell=1}^{m-1} \left( S_{(mk+\ell)\delta/m} - S_{k\delta} \right) + R_t^{mn,n},
\end{aligned}$$

where $R_t^{mn,n} = \dfrac{1}{T} \displaystyle\int_{\eta_n(t)}^t \left( S_{\eta_{mn}(s)} - S_{\eta_n(s)} \right) ds$. Now, using the dynamic of $S_t$ we get

$$\begin{aligned}
\mathcal{E}_t^{mn,n} &= \frac{1}{mn} \sum_{k=0}^{[t/\delta]-1} \sum_{\ell=1}^{m-1} \int_{k\delta}^{(mk+\ell)\delta/m} r S_u du + \frac{1}{mn} \sum_{k=0}^{[t/\delta]-1} \sum_{\ell=1}^{m-1} \int_{k\delta}^{(mk+\ell)\delta/m} \sigma S_u dW_u + R_t^{mn,n} \\
&= \frac{1}{mn} \sum_{k=0}^{[t/\delta]-1} \sum_{\ell=1}^{m-1} \int_{k\delta}^{(k+1)\delta} S_u \mathbf{1}_{\{k\delta \le u < (mk+\ell)\delta/m\}} (r du + \sigma dW_u) + R_t^{mn,n} \\
&= \frac{1}{mn} \int_0^{\eta_n(t)} r S_u \sum_{\ell=1}^{m-1} d_\ell^{mn,n}(u) du + \frac{1}{mn} \int_0^{\eta_n(t)} \sigma S_u \sum_{\ell=1}^{m-1} d_\ell^{mn,n}(u) dW_u + R_t^{mn,n}
\end{aligned}$$

with the digital function defined, for $\ell \in \{1, \cdots, m-1\}$, by

$$d_\ell^{mn,n}(u) := \mathbf{1}_{\{\eta_n(u) \le u < \eta_n(u) + \ell\delta/m\}}.$$

Hence we get

$$mn\mathcal{E}_t^{mn,n} = \int_0^t r S_u dD_u^{mn,n} + \int_0^t \sigma S_u dM_u^{mn,n} + mnR_t^{mn,n}$$

with the martingale integrand

$$M_t^{mn,n} := \int_0^{\eta_n(t)} \sum_{\ell=1}^{m-1} d_\ell^{mn,n}(u) dW_u,$$

and a drift term with bounded variation

$$D_t^{mn,n} := \int_0^{\eta_n(t)} \sum_{\ell=1}^{m-1} d_\ell^{mn,n}(u) du.$$

To study the convergence of the martingale, we compute its quadratic variation

$$\langle M^{mn,n} \rangle_t = \int_0^{\eta_n(t)} \sum_{\ell=1}^{m-1} \left( d_\ell^{mn,n}(u) \right)^2 du + 2 \int_0^{\eta_n(t)} \sum_{1 \le \ell < \ell' \le m-1} d_\ell^{mn,n}(u) d_{\ell'}^{mn,n}(u) du \tag{3}$$

For the first term, we note

$$\sum_{\ell=1}^{m-1} \left( d_\ell^{mn,n}(u) \right)^2 = \sum_{\ell=1}^{m-1} 1_{\{\eta_n(u) \le u < \eta_n(u) + \ell\delta/m\}}.$$

Concerning the second integral, since for $1 \leq \ell < \ell' \leq m-1$ we have

$$\eta_n(u) \leq \eta_n(u) + \ell\delta/m \leq \eta_n(u) + \ell'\delta/m \leq \eta_n(u) + \delta,$$

the expansion of $d_\ell^{mn,n}(u) d_{\ell'}^{mn,n}(u) = d_\ell^{mn,n}(u)$. Therefore, coming back to the bracket (3), we get after computation

$$\langle M^{mn,n} \rangle_t = \frac{[t/\delta]\delta}{m} \sum_{\ell=1}^{m-1} \ell + \frac{2[t/\delta]\delta}{m} \sum_{1 \leq \ell < \ell' \leq m-1} \ell \underset{n\to\infty}{\longrightarrow} \frac{(m-1)(2m-1)}{6} t.$$

Furthermore, by simple computation we get

$$\langle M^{mn,n}, W \rangle_t = \int_0^{\eta_n(t)} \sum_{\ell=1}^{m-1} d_\ell^{mn,n}(u) du = \frac{[t/\delta]\delta}{m} \sum_{\ell=1}^{m-1} \ell = \frac{m-1}{2} [t/\delta]\delta \underset{n\to\infty}{\longrightarrow} \frac{m-1}{2} t.$$

Besides, using this last computation it is easy to check that

$$\sup_{0 \leq t \leq T} \left| D_t^{mn,n} - \frac{m-1}{2} t \right| = \frac{m-1}{2} \sup_{0 \leq t \leq T} \left| [t/\delta]\delta - t \right| \underset{n\to\infty}{\longrightarrow} 0.$$

By virtue of Theorem 2-1 in Jacod [15] we obtain the stable convergence of

$$r D_t^{mn,n} + \sigma M_t^{mn,n} \Rightarrow^{stably} \frac{m-1}{2} rt + \frac{m-1}{2} \sigma W_t + \sqrt{\frac{m^2-1}{12}} \sigma B_t$$

where $(B_t)_{t \geq 0}$ is a Brownian motion independent of $(W_t)_{t \geq 0}$. Moreover, according to the above computations, it is easy to check the tightness of $\langle M^{mn,n} \rangle_T + \int_0^T d|D_t^{mn,n}|$ and thanks to Lemma 1 and Theorem 1 we get

$$\int_0^t r S_u dD_u^{mn,n} + \int_0^t \sigma S_u dM_u^{mn,n} \Rightarrow^{stably} (m-1) \frac{S_t - S_0}{2} + \sqrt{\frac{m^2-1}{12}} \int_0^t \sigma S_u dB_u.$$

Now, it remains to prove the convergence of $\sup\limits_{0 \leq t \leq T} |mn R_t^{mn,n}|$ in probability to zero. This rest term is bounded up to a constant factor by $\sup_{0 \leq t \leq T} |S_{\eta_{mn}(t)} - S_{\eta_n(t)}|$. Finally, the proof is completed using the Hölder regularity of the process $S$.

□

The subsection below is devoted to the study of the trapezoidal scheme error.

## 3.2 Stable convergence of the trapezoidal scheme error

The trapezoidal approximation of the process is given by

$$J_t^n = \frac{1}{T} \int_0^t \frac{S_{\eta_n(u)} + S_{(\eta_n(u)+\delta)\wedge t}}{2} du = \frac{1}{n} \sum_{k=0}^{[t/\delta]-1} \frac{S_{k\delta} + S_{(k+1)\delta}}{2} + (t - \eta_n(t)) \frac{S_{\eta_n(t)} + S_t}{2T}$$

with $\eta_n(t) = [t/\delta]\delta$. One have to study the error process given by $J^{mn} - J^n$.

**Theorem 4** *We have the following result*

$$\frac{mn}{\sqrt{m^2-1}}(J^{mn}-J^n) \Rightarrow^{stably} \chi$$

*where $\chi$ is the process defined by*

$$\chi_t := \frac{1}{2\sqrt{3}}\int_0^t \sigma S_u dB_u,$$

*with $B$ a standard Brownian motion on an extension $\hat{\mathcal{B}}$ of $\mathcal{B}$, which is independent of $W$.*

**Remark 1** *The process $\chi$ above is the same limit process given in Theorem 4.1 of Kebaier [17]. In fact, he proves that*

$$m^\ell(J^{m^\ell}-J) \Rightarrow^{stably} \chi, \quad as \quad \ell \to \infty.$$

*This latter convergence can not be used to prove our Theorem 6 below, since the multilevel Monte Carlo method involves the error process $J^{m^\ell} - J^{m^{\ell-1}}$ rather than $J^{m^\ell} - J$. For this reason, we are led to make a further study adapted to our setting.*

**Proof:** Considering the trapezoidal scheme, for the fine time discretization step $\delta/m$, we can write it as follows

$$\begin{aligned} J_t^{mn} &= \frac{1}{2T}\int_0^{\eta_n(t)}(S_{\eta_{mn}(u)} + S_{(\eta_{mn}(u)+\delta/m)})du + \frac{1}{2T}\int_{\eta_n(t)}^t (S_{\eta_{mn}(u)} + S_{(\eta_{mn}(u)+\delta/m)\wedge t})du \\ &= \frac{1}{2mn}\sum_{\ell=0}^{m-1}\sum_{k=0}^{[t/\delta]-1}\left(S_{(mk+\ell)\delta/m} + S_{(mk+\ell+1)\delta/m}\right) + \frac{1}{2T}\int_{\eta_n(t)}^t (S_{\eta_{mn}(u)} + S_{(\eta_{mn}(u)+\delta/m)\wedge t})du. \end{aligned}$$

The first term in the right-hand side, can be arranged as follows

$$\frac{1}{2mn}\sum_{k=0}^{[t/\delta]-1}\left(S_{k\delta} + S_{(k+1)\delta}\right) + \frac{1}{mn}\sum_{\ell=1}^{m-1}\sum_{k=0}^{[t/\delta]-1} S_{(mk+\ell)\delta/m}.$$

So that, the error, $\mathcal{E}_t^{mn,n}$, can be arranged as follows

$$\mathcal{E}_t^{mn,n} := J_t^{mn} - J_t^n = \frac{1-m}{2mn}\sum_{k=0}^{[t/\delta]-1}(S_{k\delta} + S_{(k+1)\delta}) + \frac{1}{mn}\sum_{\ell=1}^{m-1}\sum_{k=0}^{[t/\delta]-1} S_{(mk+\ell)\delta/m} + R_t^{mn,n},$$

with

$$R_t^{mn,n} = \frac{1}{2T}\int_{\eta_n(t)}^t (S_{\eta_{mn}(u)} + S_{(\eta_{mn}(u)+\delta/m)\wedge t} - S_{\eta_n(u)} - S_{(\eta_n(u)+\delta)\wedge t})du.$$

Furthermore, we rewrite the error as

$$\mathcal{E}_t^{mn,n} = -\frac{1}{2mn}\sum_{k=0}^{[t/\delta]-1}\sum_{\ell=1}^{m-1}\left(\left(S_{(k+1)\delta} - S_{(mk+\ell)\delta/m}\right) - \left(S_{(mk+\ell)\delta/m} - S_{k\delta}\right)\right) + R_t^{mn,n}.$$

Now, using the dynamic of $S_t$ we get

$$\begin{aligned}
\mathcal{E}_t^{mn,n} &= -\frac{1}{2mn}\sum_{k=0}^{[t/\delta]-1}\sum_{\ell=1}^{m-1}\int_{k\delta}^{(k+1)\delta} rS_u\left(\mathbf{1}_{\{(mk+\ell)\delta/m\le u<(k+1)\delta\}}-\mathbf{1}_{\{k\delta\le u<(mk+\ell)\delta/m\}}\right)du \\
&\quad -\frac{1}{2mn}\sum_{k=0}^{[t/\delta]-1}\sum_{\ell=1}^{m-1}\int_{k\delta}^{(k+1)\delta} \sigma S_u\left(\mathbf{1}_{\{(mk+\ell)\delta/m\le u<(k+1)\delta\}}-\mathbf{1}_{\{k\delta\le u<(mk+\ell)\delta/m\}}\right)dW_u + R_t^{mn,n} \\
&= -\frac{1}{2mn}\int_0^{\eta_n(t)} rS_u\sum_{\ell=1}^{m-1} d_\ell^{mn,n}(u)du - \frac{1}{2mn}\int_0^{\eta_n(t)} \sigma S_u\sum_{\ell=1}^{m-1} d_\ell^{mn,n}(u)dW_u + R_t^{mn,n}
\end{aligned}$$

where the digital function defined, for $\ell \in \{1,\cdots,m-1\}$, by

$$d_\ell^{mn,n}(u) := \mathbf{1}_{\{\eta_n(u)+\ell\delta/m\le u<\eta_n(u)+\delta\}} - \mathbf{1}_{\{\eta_n(u)\le u<\eta_n(u)+\ell\delta/m\}}.$$

Hence, we get

$$mn\mathcal{E}_t^{mn,n} = \int_0^t rS_u dD_u^{mn,n} + \int_0^t \sigma S_u dM_u^{mn,n} + mnR_t^{mn,n}$$

with the martingale integrand

$$M_t^{mn,n} := -\frac{1}{2}\int_0^{\eta_n(t)}\sum_{\ell=1}^{m-1} d_\ell^{mn,n}(u)dW_u,$$

and a drift term

$$D_t^{mn,n} := -\frac{1}{2}\int_0^{\eta_n(t)}\sum_{\ell=1}^{m-1} d_\ell^{mn,n}(u)du.$$

To study the convergence of the martingale, we compute its quadratic variation

$$4\langle M^{mn,n}\rangle_t = \int_0^{\eta_n(t)}\sum_{\ell=1}^{m-1}\left(d_\ell^{mn,n}(u)\right)^2 du + 2\int_0^{\eta_n(t)}\sum_{1\le\ell<\ell'\le m-1} d_\ell^{mn,n}(u)d_{\ell'}^{mn,n}(u)du \tag{4}$$

For the first term, we note that

$$\begin{aligned}
\sum_{\ell=1}^{m-1}\left(d_\ell^{mn,n}(u)\right)^2 &= \sum_{\ell=1}^{m-1}\mathbf{1}_{\{\eta_n(u)+\ell\delta/m\le u<\eta_n(u)+\delta\}} + \mathbf{1}_{\{\eta_n(u)\le u<\eta_n(u)+\ell\delta/m\}} \\
&= \sum_{\ell=1}^{m-1}\mathbf{1}_{\{\eta_n(u)\le u<\eta_n(u)+\delta\}} = (m-1).
\end{aligned}$$

Concerning the second integral, since for $1\le\ell<\ell'\le m-1$ we have

$$\eta_n(u)\le\eta_n(u)+\ell\delta/m\le\eta_n(u)+\ell'\delta/m\le\eta_n(u)+\delta,$$

the expansion of $d_\ell^{mn,n}(u)d_{\ell'}^{mn,n}(u)$ is equal to

$$\mathbf{1}_{\{\eta_n(u)+\ell'\delta/m\le u<\eta_n(u)+\delta\}} - \mathbf{1}_{\{\eta_n(u)+\ell\delta/m\le u<\eta_n(u)+\ell'\delta/m\}} + \mathbf{1}_{\{\eta_n(u)\le u<\eta_n(u)+\ell\delta/m\}}$$

that we rewrite as $1 - 2 \times \mathbf{1}_{\{\eta_n(u)+\ell\delta/m \le u < \eta_n(u)+\ell'\delta/m\}}$. Coming back to the bracket (4), we get after computation

$$4\langle M^{mn,n}\rangle_t = (m-1)^2 t - \frac{4[t/\delta]\delta}{m} \sum_{1\le\ell<\ell'\le m-1} (\ell'-\ell) \underset{n\to\infty}{\longrightarrow} \frac{m^2-1}{3} t.$$

Furthermore, by simple computation we get

$$-2\langle M^{mn,n}, W\rangle_t = \int_0^{\eta_n(t)} \sum_{\ell=1}^{m-1} d_\ell^{mn,n}(u)du = \frac{[t/\delta]\delta}{m} \sum_{\ell=1}^{m-1}(m-\ell) - \frac{[t/\delta]\delta}{m} \sum_{\ell=1}^{m-1} \ell = 0.$$

Finally, We can proceed analogously to the Riemann case to achieve the proof. $\square$

We can now formulate our main results for both Riemann and trapezoidal schemes.

# 4 Central Limit Theorems

It is worth to note that the advantage of the central limit theorem is to construct a more accurate confidence interval. In fact, for a given root mean square error (RMSE), the radius of the 90%-confidence interval by the central limit theorem is 1.64×RMSE. However, if we just have a control of the variance without any central limit theorem, we can use only Chebyshev's inequality which yields a radius equal to 3.16×RMSE. To do so, we consider a real sequence $(a_\ell)_{\ell\ge1}$ of positive terms satisfying

$$(\mathcal{W}) \qquad \lim_{n\to\infty} \sum_{\ell=1}^{L} a_\ell = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\left(\sum_{\ell=1}^{L} a_\ell\right)^{p/2}} \sum_{\ell=1}^{L} a_\ell^{p/2} = 0, \text{ for } p>2. \tag{5}$$

Let us assume that the sample sizes $N_\ell$, for $\ell \in \{1,\cdots,L\}$, for the multilevel Monte Carlo method, have the following form

$$N_\ell = \frac{n^2(m^2-1)}{m^{2\ell} a_\ell} \sum_{\ell=1}^{L} a_\ell, \quad \ell \in \{0,\cdots,L\} \quad \text{and} \quad L = \frac{\log n}{\log m}. \tag{6}$$

## 4.1 Riemann Scheme

Now, we consider the Riemann scheme

$$\mathbb{E}(f(I_T^n)) = \mathbb{E}\left(f(I_T^1)\right) + \sum_{\ell=1}^{L} \mathbb{E}\left(f(I_T^{m^\ell}) - f(I_T^{m^{\ell-1}})\right). \tag{7}$$

It is worth to note that $f(I_T^1)$ is deterministic equal to $f(s_0)$. Hence, the multilevel method in this case can be written as

$$Q_n = f(s_0) + \sum_{\ell=1}^{L} \frac{1}{N_\ell} \sum_{k=1}^{N_\ell} \left(f(I_{T,k}^{\ell,m^\ell}) - f(I_{T,k}^{\ell,m^{\ell-1}})\right). \tag{8}$$

We can now state the Central limit theorem in this setting.

**Theorem 5** *Let $f$ be a $\mathbb{R}$-valued function satisfying condition $(\mathcal{H}_{\mathbf{f}})$ and such that $\mathbb{P}(I_T \notin \mathcal{D}_{\dot{f}}) = 0$, where $\mathcal{D}_{\dot{f}} := \{x \in \mathbb{R}^d; f$ is differentiable at $x\}$. We have*

$$n\big(Q_n - \mathbb{E}\left(f(I_T)\right)\big) \Rightarrow \mathcal{N}\left(C_f^I, \sigma^2\right)$$

*where $\sigma^2 = \tilde{V}ar\big(f'(I_T)\xi_T\big)$ and $C_f^I$ is given by property $\mathcal{PR}\mathbf{2}$). Here $\xi$ is the limit process in Theorem 3.*

**Proof :** Combining relations (7) and (8) we obtain

$$Q_n - \mathbb{E}\left(f(I_T)\right) = \hat{Q}_n + \mathbb{E}\left(f(I_T^n)\right) - \mathbb{E}\left(f(I_T)\right),$$

where

$$\hat{Q}_n = \sum_{\ell=1}^{L} \frac{1}{N_\ell} \sum_{k=1}^{N_\ell} \Big(f(I_{T,k}^{\ell,m^\ell}) - f(I_{T,k}^{\ell,m^{\ell-1}}) - \mathbb{E}\left(f(I_T^{m^\ell}) - f(I_T^{m^{\ell-1}})\right)\Big)$$

Using assumption $\mathcal{PR}\mathbf{2}$), we obviously obtain the term $C_f^I$ in the limit. So, we have only to establish

$$n\hat{Q}_n \Rightarrow \mathcal{N}\Big(0, \tilde{V}ar\big(f'(I_T)\xi_T\big)\Big).$$

To do so, we plan to use Theorem 2 with Lyapunov condition and we set

$$X_{n,\ell} := \frac{n}{N_\ell} \sum_{k=1}^{N_\ell} Z_{T,k}^{m^\ell,m^{\ell-1}} \text{ and } Z_{T,k}^{m^\ell,m^{\ell-1}} := f(I_{T,k}^{\ell,m^\ell}) - f(I_{T,k}^{\ell,m^{\ell-1}}) - \mathbb{E}\left(f(I_{T,K}^{m^\ell}) - f(I_{T,k}^{m^{\ell-1}})\right), \quad (9)$$

and we have only to check the following conditions :
- $\lim_{n\to\infty} \sum_{\ell=1}^{L} \mathbb{E}(X_{n,\ell})^2 = \tilde{V}ar\big(f'(I_T)\xi_T\big)$
- (Lyapunov condition) there exists $p > 2$ such that $\lim_{n\to\infty} \sum_{\ell=1}^{L} \mathbb{E}\left|X_{n,\ell}\right|^p = 0$.

For the first one, we have

$$\begin{aligned}\sum_{\ell=1}^{L} \mathbb{E}(X_{n,\ell})^2 = \sum_{\ell=1}^{L} Var(X_{n,\ell}) = \sum_{\ell=1}^{L} \frac{n^2}{N_\ell} Var\left(Z_{T,1}^{m^\ell,m^{\ell-1}}\right) \\ = \frac{1}{\sum_{\ell=1}^{L} a_\ell} \sum_{\ell=1}^{L} a_\ell \frac{m^{2\ell}}{m^2-1} Var\left(Z_{T,1}^{m^\ell,m^{\ell-1}}\right). \quad (10)\end{aligned}$$

Otherwise, since $\mathbb{P}(I_T \notin \mathcal{D}_f) = 0$, applying the Taylor expansion theorem twice we get

$$\begin{aligned}f(I_T^{\ell,m^\ell}) - f(I_T^{\ell,m^{\ell-1}}) = f'(I_T)(I_T^{\ell,m^\ell} - I_T^{\ell,m^{\ell-1}}) + \\ (I_T^{\ell,m^\ell} - I_T)\varepsilon(I_T, I_T^{\ell,m^\ell} - I_T) - (I_T^{\ell,m^{\ell-1}} - I_T)\varepsilon(I_T, I_T^{\ell,m^{\ell-1}} - I_T).\end{aligned}$$

The function $\varepsilon$ is given by the Taylor-Young expansion, so it satisfies $\varepsilon(I_T, I_T^{\ell,m^\ell} - I_T))\xrightarrow[\ell\to\infty]{\mathbb{P}}0$ and $\varepsilon(I_T, I_T^{\ell,m^{\ell-1}} - I_T))\xrightarrow[\ell\to\infty]{\mathbb{P}}0$. By property $\mathcal{PR}\mathbf{1}$), we get the tightness of

$\frac{m^\ell}{\sqrt{m^2-1}}(I_T^{\ell,m^\ell} - I_T)$ and $\frac{m^\ell}{\sqrt{m^2-1}}(I_T^{\ell,m^{\ell-1}} - I_T)$ and we deduce

$$\frac{m^\ell}{\sqrt{m^2-1}}\Big((I_T^{\ell,m^\ell} - I_T)\varepsilon(I_T, I_T^{\ell,m^\ell} - I_T) - (I_T^{\ell,m^{\ell-1}} - I_T)\varepsilon(I_T, I_T^{\ell,m^{\ell-1}} - I_T)\Big) \xrightarrow[\ell\to\infty]{\mathbb{P}} 0.$$

So, according to Lemma 1 and Theorem 3 we conclude that

$$\frac{m^\ell}{\sqrt{m^2-1}}\Big(f(I_T^{\ell,m^\ell}) - f(I_T^{\ell,m^{\ell-1}})\Big) \Rightarrow^{stably} f'(I_T)\xi_T, \quad \text{as} \quad \ell \to \infty. \tag{11}$$

Now, using $(\mathcal{H}_\mathbf{f})$ it follows from property $\mathcal{PR}\mathbf{1}$) that

$$\forall\varepsilon > 0, \quad \sup_\ell \mathbb{E}\left|\frac{m^\ell}{\sqrt{m^2-1}}\Big(f(I_T^{m^\ell}) - f(I_T^{m^{\ell-1}})\Big)\right|^{2+\varepsilon} < \infty.$$

We deduce using (11) that

$$\mathbb{E}\left(\frac{m^\ell}{\sqrt{m^2-1}}\Big(f(I_T^{m^\ell}) - f(I_T^{m^{\ell-1}})\Big)\right)^k \to \tilde{\mathbb{E}}\Big(f'(I_T)\xi_T\Big)^k < \infty \quad \text{with} \quad k \in \{1,2\}.$$

Consequently,

$$\frac{m^{2\ell}}{m^2-1}Var(Z_{T,1}^{m^\ell,m^{\ell-1}}) \longrightarrow \tilde{V}ar\,(f'(I_T)\xi_T) < \infty.$$

Combining this last convergence with relation (10), we obtain the first condition a using Toeplitz lemma. Concerning the second one, by Burkholder's inequality and elementary computations, we get for $p > 2$

$$\mathbb{E}|X_{n,\ell}|^p = \frac{n^p}{N_\ell^p}\mathbb{E}\left|\sum_{\ell=1}^{N_\ell} Z_{T,1}^{m^\ell,m^{\ell-1}}\right|^p \le C_p\frac{n^p}{N_\ell^{p/2}}\mathbb{E}\left|Z_{T,1}^{m^\ell,m^{\ell-1}}\right|^p, \tag{12}$$

where $C_p$ is a numerical constant that depends on $p$ only. Otherwise, property $\mathcal{PR}\mathbf{1}$) ensures the existence of a constant $K_p > 0$ such that

$$\mathbb{E}|Z_{T,1}^{m^\ell,m^{\ell-1}}|^p \le \frac{K_p}{m^{p\ell}}.$$

Therefore,

$$\sum_{\ell=1}^L \mathbb{E}\,|X_{n,\ell}|^p \le \tilde{C}_p\sum_{\ell=1}^L \frac{n^p}{N_\ell^{p/2}m^{p\ell}} \le \frac{\tilde{C}_p}{\Big(\sum_{\ell=1}^L a_\ell\Big)^{p/2}}\sum_{\ell=1}^L a_\ell^{p/2} \underset{n\to\infty}{\longrightarrow} 0. \tag{13}$$

This completes the proof. □

**Remark 2** *From Theorem 2 page 544 in [6], we prove a Berry-Essen type bound on our central limit theorem. This improves the relevance of the above result. Indeed, put*

$$s_n^2 = \sum_{\ell=1}^L \mathbb{E}|X_{n,\ell}|^2, \quad \rho_n = \sum_{\ell=1}^L \mathbb{E}|X_{n,\ell}|^3$$

*with $X_{n,\ell}$ given by relation (9), $\ell \in \{1,\cdots,L\}$, and denote by $F_n$ the distribution function of $n(Q_n - \mathbb{E}f(X_T^n))/s_n$. Then for all $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$*

$$|F_n(x) - G(x)| \leq 6\frac{\rho_n}{s_n^3}, \tag{14}$$

*where $G$ is the distribution function of a standard Gaussian random variable. According to the above proof, it is clear that $s_n$ behaves like a constant and for $\rho_n$, taking $p=3$ in both inequalities (12) and (13) gives us an upper bound. In fact, when $f$ is Lipschitz, there exists a positive constant $C$ depending on $b$, $\sigma$, $T$ and $f$ such that*

$$\rho_n \leq \frac{C}{\left(\sum_{\ell=1}^L a_\ell\right)^{3/2}} \sum_{\ell=1}^L a_\ell^{3/2}.$$

*Hence, the order of the Berry-Essen type bound depend on the choice of $a_\ell$. For example, for $a_\ell = 1$ the obtained Berry-Essen type bound is of order $1/\sqrt{\log n}$ and for $a_\ell = 1/\ell$ the bound is of order $1/(\log\log n)^{\frac{3}{2}}$.*

## 4.2 Trapezoidal Scheme

The multilevel method in this case can be written as

$$Q_n = \frac{1}{N_0}\sum_{k=1}^{N_0} f\left(\frac{S_0 + S_{T,k}}{2}\right) + \sum_{\ell=1}^L \frac{1}{N_\ell}\sum_{k=1}^{N_\ell}\left(f(J_{T,k}^{\ell,m^\ell}) - f(J_{T,k}^{\ell,m^{\ell-1}})\right).$$

In the following we consider the same real sequence $(a_\ell)_{\ell\geq 1}$ of positive terms given by relation (5) and the sequence of sample sizes $(N_\ell)_{0\leq\ell\leq L}$ given by relation (6). We can now state the Central limit theorem for the trapezoidal scheme.

**Theorem 6** *Let $f$ be a $\mathbb{R}$-valued function satisfying condition $(\mathcal{H}_\mathbf{f})$ and such that $\mathbb{P}(I_T \notin \mathcal{D}_{\dot f}) = 0$, where $\mathcal{D}_{\dot f} := \{x \in \mathbb{R}^d; f \text{ is differentiable at } x\}$. We have*

$$n\big(Q_n - \mathbb{E}\left(f(I_T)\right)\big) \Rightarrow \mathcal{N}\left(C_f^J, \sigma^2\right)$$

*where $\sigma^2 = \tilde{V}ar\big(f'(I_T)\chi_T\big)$ and $C_f^J$ is given by property $\mathcal{PT}\mathbf{2}$). Here, $\chi$ is the limit process in Theorem 4.*

**Proof:** We can write

$$Q_n - \mathbb{E}\left(f(I_T)\right) = \hat{Q}_n^1 + \hat{Q}_n^2 + \mathbb{E}\left(f(J_T^n)\right) - \mathbb{E}\left(f(I_T)\right),$$

where

$$\begin{aligned}
\hat{Q}_n^1 &= \frac{1}{N_0}\sum_{k=1}^{N_0}\left(f(J_{T,k}^{m^0}) - \mathbb{E}\left(f(J_T^{m^0})\right)\right) \\
\hat{Q}_n^2 &= \sum_{\ell=1}^L \frac{1}{N_\ell}\sum_{k=1}^{N_\ell}\left(f(J_{T,k}^{\ell,m^\ell}) - f(J_{T,k}^{\ell,m^{\ell-1}}) - \mathbb{E}\left(f(J_T^{m^\ell}) - f(J_T^{m^{\ell-1}})\right)\right).
\end{aligned}$$

Using assumption $\mathcal{PT}\mathbf{2}$) we obviously obtain the term $C_f^J$ in the limit. Afterward, one can apply the classical central limit theorem for the quantity $\hat{Q}_n^1$ to get $n\hat{Q}_n^1 \xrightarrow{\mathbb{P}} 0$.

On the other hand, the convergence of $n\hat{Q}_n^2$ is obtained by following the proof steps of the Central Limit Theorem for the Riemann scheme, Theorem 5. Using this approach, we have only to use respectively property $\mathcal{PT}\mathbf{1}$) and Theorem 4 instead of $\mathcal{PR}\mathbf{1}$) and Theorem 3. Hence, we obtain the following convergence

$$n\hat{Q}_n^2 \Rightarrow \mathcal{N}\left(0, \tilde{V}ar\big(f'(I_T)\chi_T\big)\right).$$

This completes the proof.

$\square$

**Remark 3** *As in remark 2, we have also a Berry-Essen type bound. Indeed, let* $X_{n,0} = n\hat{Q}_n^1$, $X_{n,\ell} = \frac{1}{N_\ell}\sum_{k=1}^{N_\ell}\left(f(J_{T,k}^{\ell,m^\ell}) - f(J_{T,k}^{\ell,m^{\ell-1}}) - \mathbb{E}\left(f(J_T^{m^\ell}) - f(J_T^{m^{\ell-1}})\right)\right)$, $\ell \in \{1,\cdots,L\}$, $s_n^2 = \sum_{\ell=0}^L \mathbb{E}|X_{n,\ell}|^2$ *and* $\rho_n = \sum_{\ell=0}^L \mathbb{E}|X_{n,\ell}|^3$. *If we denote by* $F_n$ *the distribution function of* $n(Q_n - \mathbb{E}f(J_T^n))/s_n$, *then relation (14) and the analysis on factors* $s_n$ *and* $\rho_n$, *given in remark 2, remain valid.*

# 5 The complexity

The following complexity analysis is available for any second order discretization scheme. Thanks to Theorem 5 and 6, we note that for a total error of order $1/n$ the computational effort necessary to run the multilevel algorithm applied to the Riemann or trapezoidal scheme, with step numbers $m^\ell$, $(m,\ell) \in \mathbb{N}\setminus\{0,1\}\times\{1,\cdots,L\}$, corresponds to the sequence of sample sizes $(N_\ell)_{0\le\ell\le L}$ given by relation (6). Consequently, the time complexity in the multilevel Monte Carlo method for these second order schemes is given by

$$\begin{aligned} C_{MMC} &= C \times \sum_{\ell=1}^{L} N_\ell(m^\ell + m^{\ell-1}) \quad \text{with} \quad C>0 \\ &= C \times \frac{(m+1)^2(m-1)}{m} n^2 \sum_{\ell=1}^{L}\frac{1}{m^\ell a_\ell}\sum_{\ell=1}^{L} a_\ell. \end{aligned}$$

The minimum of this complexity is reached for the choice of weights $a_\ell^* = m^{-\ell/2}$, $\ell \in \{1,\cdots,L\}$, since the Cauchy-Schwartz inequality ensures that $\left(\sum_{\ell=1}^L m^{-\ell/2}\right)^2 \le \sum_{\ell=1}^L \frac{1}{m^\ell a_\ell}\sum_{\ell=1}^L a_\ell$, and the optimal complexity for the multilevel Monte Carlo method for this choice is given by

$$C_{MMC}^{a_\ell^*} = C \times \frac{(m+1)^2(m-1)}{m} n^2 \left(\sum_{\ell=1}^{L} m^{-\ell/2}\right)^2 = O\left(n^2\right).$$

Note that this optimal choice $a_\ell^* = m^{-\ell/2}$ corresponds to the sample size

$$N_\ell = \frac{m^2-1}{m^{3\ell/2}(\sqrt{m}-1)} n^2 \left(1 - \frac{1}{\sqrt{n}}\right) \tag{15}$$

of the $\ell^{th}$ level in the multilevel algorithm, which is consistent with the complexity analysis given by Giles. More precisely, by taking $\beta = 2$ in Theorem 3.1 of [8] we recover the same complexity as well as the same order of sample sizes $(N_\ell)_{0\leq\ell\leq L}$ (see also relation (1) ).

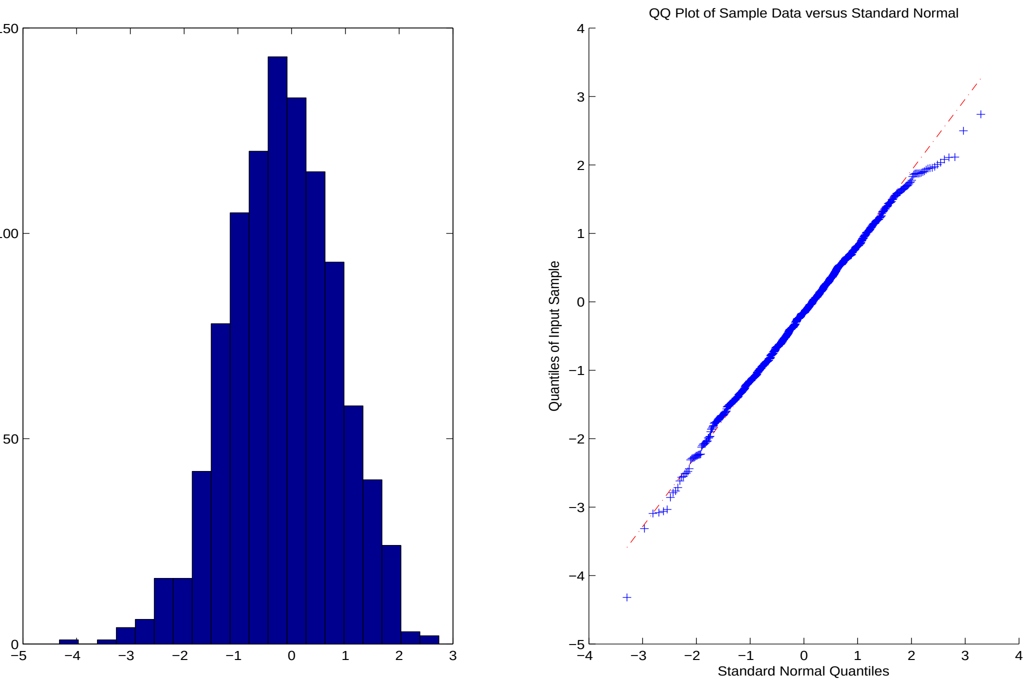


Figure 1: Numerical tests for the optimal choice $a^*_\ell = m^{-\ell/2}$.

However, this optimal choice $a^*_\ell$, leading to the complexity $n^2$, does not satisfy condition $(\mathcal{W})$ and even the Lyapunov condition. Then, it seems natural to try to check experimentally if the central limit theorem is satisfied or not and we proceed to some numerical tests. In Figure 1 we plot at the left the data histogram of 1000 samples of $Q_n$ correctly renormalized and at the right we proceed to the quantile-quantile test where the horizontal axis means quantiles of a standard normal distribution and the vertical axis indicates the empirical quantiles of the same data. According to these numerical tests, the central limit theorem seems to be true despite the lack of theoretical proof.

Now, we shall exhibit three sequences $(a_l)_{1\leq\ell\leq L}$ satisfying our condition $(\mathcal{W})$ and reducing significantly the complexity and for which the complexity is explicit.

a) The choice $a_{\ell,1} = 1$, corresponds to the sample size $N_{\ell,1} = \dfrac{m^2-1}{m^{2\ell}} n^2 L$, $\ell \in \{1, \cdots, L\}$. This leads to a complexity $C^{a_{\ell,1}}_{MMC} = C \times \left( \dfrac{(m+1)^2}{m \log m} n^2 \log n \right) = O\left(n^2 \log n\right)$. In this case, the optimal choice of the parameter $m$ is equal to 4 and $N_0 = n^2 \frac{\log n}{\log m}$.

b) For $a_{\ell,2} = 1/\ell$, we get $N_{\ell,2} = \dfrac{(m^2-1)\ell}{m^{2\ell}} n^2 \sum_{\ell=1}^{L} \frac{1}{\ell}$. This leads to a complexity

$$C_{MMC}^{a_{\ell,2}} = C \times \left( \frac{(m+1)^2(m-1)}{m} n^2 \sum_{\ell=1}^{L} \frac{\ell}{m^\ell} \sum_{\ell=1}^{L} \frac{1}{\ell} \right)$$
$$\sim C \times \frac{(m+1)^2}{m-1} n^2 \log\log n = O\left(n^2 \log\log n\right)$$

and the optimal choice of the parameter $m$ is equal to 3.

c) For $a_{\ell,3} = 1/(\ell\log\ell)$, we get $N_{\ell,3} = \dfrac{(m^2-1)\ell\log\ell}{m^{2\ell}} n^2 \sum_{\ell=1}^{L} \frac{1}{\ell\log\ell}$ and a complexity

$$C_{MMC}^{a_{\ell,3}} = C \times \left( \frac{(m+1)^2(m-1)}{m} n^2 \sum_{\ell=1}^{L} \frac{\ell\log\ell}{m^\ell} \sum_{\ell=1}^{L} \frac{1}{\ell\log\ell} \right)$$
$$\sim C \times \frac{(m+1)^2(m-1)}{m} n^2 \sum_{\ell=1}^{\infty} \frac{\ell\log\ell}{m^\ell} n^2 \log\log\log n = O\left(n^2 \log\log\log n\right).$$

In this last case, the factor depending on $m$, in the above complexity, can be interpreted as $\frac{(m+1)^2}{m} \mathbb{E}\left(G_m \log(G_m)\right)$, where $G_m \overset{law}{=} \text{Geometric}(1-1/m)$. So, a simple Monte Carlo approximation yields the optimal choice of the parameter $m$ which is equal to 5.

Through these examples, we note that the central limit theorem is conserved and the complexity can be very close to the order $n^2$ which is clearly better than the complexity $n^3$ achieved by a crude Monte Carlo method for the same error of order $1/n$.

# 6 Conclusion

The central limit theorems derived in this paper confirm the superiority of the multilevel Monte Carlo approach even when using second order schemes with a path-dependent payoff and fills the gap in the literature for this setting. A next natural question consists on studying central limit theorems for multilevel Monte Carlo when using high order discretization schemes for a general setting of stochastic differential equation for a given payoff function.

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