Monte Carlo methods for American options

Ahmed Kebaier
kebaier@math.univ-paris13.fr

HEC, Paris

Outline

1 Pricing American Options

The Market model

- Let $(W_t)_{t \geq 0}$ be a brownian motion under $\mathbb{P}^*$ the risk neutral probability measure
- Under $\mathbb{P}^*$ the Black-Scholes model is solution to

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad \text{where } r \geq 0, \sigma > 0 \text{ and } S_0 > 0$$

with explicit solution given by

$$S_t = S_0 \exp\left((r - \frac{\sigma^2}{2}) T + \sigma W_T \right)$$

- Let $(\mathcal{F}_t)_{t \geq 0}$ denotes the natural filtration associated to W.

European option setting

- A European option with maturity T and with payoff $f(S_T)$ may be exercised only at the expiration date T.
- The price of this European option at time $t \in [0, T]$ is equal to the value at time t of the associated hedging portfolio given by

$$V_t = H_t^0 e^{rt} + H_t S_t.$$

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- Then

$$V_t = e^{-r(T-t)} \mathbb{E}^*(f(S_T)|\mathcal{F}_t).$$

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- at date $T = t_n$ we have $V_{t_n} = f(S_{t_n})$
- at date $T = t_{n-1}$ we have

$$\begin{aligned} V_{t_{n-1}} &= \max\left(f(S_{t_{n-1}})\ ;\ e^{-r(t_n - t_{n-1})}\mathbb{E}^*[f(S_{t_n})|\mathcal{F}_{t_{n-1}}]\right) \\ &= \max\left(f(S_{t_{n-1}})\ ;\ e^{-r(t_n - t_{n-1})}\mathbb{E}^*[V_{t_n}|\mathcal{F}_{t_{n-1}}]\right) \end{aligned}$$

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- In the same way we get $\forall k \in \{0, \cdots, n-1\}$

$$\begin{cases} V_T &= f(S_T) \\ V_{t_k} &= \max\left(f(S_{t_k})\ ;\ e^{-r(t_{k+1} - t_k)}\mathbb{E}^*[V_{t_{k+1}}|\mathcal{F}_{t_k}]\right), \end{cases}$$

Remarks

- Note that as the process $(S_t)_{0 \leq t \leq T}$ is Markovian then

$$\mathbb{E}^* \left[V_{t_{k+1}} | \mathcal{F}_{t_k} \right] = \mathbb{E}^* \left[V_{t_{k+1}} | \mathcal{S}_{t_k} \right].$$

- The price of a Bermudan option is more expensive than the price of an European option.

- For $f(x) = (x - K)_+$ we have

$$e^{-r(t_{k+1} - t_k)} \mathbb{E}^* [V_{t_{k+1}} | \mathcal{F}_{t_k}] \geq f(S_{t\,k})$$

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- If we let $n \to \infty$ than the price of the Bermudan option tends to the price of the American option.

Theorem

- Now if we consider $\forall k \in \{0, \cdots, n-1\}$

$$\begin{cases} V_T &= f(S_T) \\ V_{t_k} &= \max\left(f(S_{t_k})\ ;\ e^{-r(t_{k+1}-t_k)}\mathbb{E}^*[V_{t_{k+1}}|\mathcal{F}_{t_k}]\right), \end{cases}$$

Then,

$$V_{t_k} = \sup_{\tau \in \{t_k, \cdots, t_n\}} e^{-r(\tau - t_k)}\mathbb{E}^*\left(f(S_\tau)|\mathcal{S}_{t_k}\right)$$

- The stopping time

$$\tau_k^* = \inf\left\{t_i \in \{t_k, \cdots, t_n\} \mid f(S_{t_i}) \geq e^{-r(t_{i+1}-t_i)}\mathbb{E}^*\left[f(S_{t_{i+1}})|\mathcal{S}_{t_k}\right]\right\}$$

satifies

$$V_{t_k} = e^{-r(\tau_k^* - t_k)}\mathbb{E}^*\left[f(S_{\tau_k^*})|\mathcal{S}_{t_k}\right].$$

Longstaff Schwarz algorithm

- We consider the sequence $(\tau_k^*)_{0\le k\le n}$ where

$$\tau_k^* = \inf\left\{t_i \in \{t_k, \cdots, t_n\} \mid f(S_{t_i}) \ge e^{-r(t_{i+1}-t_i)}\mathbb{E}^*\left[f(S_{t_{i+1}})|\mathcal{S}_{t_k}\right]\right\}$$

- The sequence $(\tau_k^*)_{0\le k\le n}$ satisfies a dynamic programming principle

$$\begin{cases} \tau_n^* &= T \\ \tau_k^* &= t_k \mathbf{1}_{B_k} + \tau_{k+1}^* \mathbf{1}_{B_k^c}, \quad \text{for} \quad 0 \le k \le n-1 \end{cases}$$

where

$$B_k = \left\{f(S_{t_k}) \ge \mathbb{E}^*[e^{-r(\tau_{k+1}^*-t_k)} f(S_{\tau_{k+1}^*})|\mathcal{S}_{t_k}]\right\}$$

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$$B_k = \left\{ f(S_{t_k}) \geq \mathbb{E}^* [e^{-r(\tau_{k+1}^* - t_k)} f(S_{\tau_{k+1}^*}) | \mathcal{S}_{t_k}] \right\}$$

- How to approximate $\mathbb{E}^* [e^{-r(\tau_{k+1}^* - t_k)} f(S_{\tau_{k+1}^*}) | \mathcal{S}_{t_k}]$?

Regression

- We write $\mathbb{E}^*[e^{-r(\tau^*_{k+1}-t_k)} f(S_{\tau^*_{k+1}})|\mathcal{S}_{t_k}] = \sum_{\ell\geq 1} \alpha_{k,\ell} P_\ell(S_{t_k})$ where $(P_\ell)_{\ell\geq 1}$ is a basis function.

- Using the definition of the conditional expectation, the sequence $(\alpha_{k,\ell})_{\ell\geq 1}$ is the sequence that minimizes the distance

$$\mathbb{E}^* \left[\left(e^{-r(\tau^*_{k+1}-t_k)} f(S_{\tau^*_{k+1}}) - \sum_{\ell\geq 1} \alpha_{k,\ell} P_\ell(S_{t_k}) \right)^2 \right]$$

- In practice we need to truncate the sum $\sum_{\ell\geq 1} \alpha_{k,\ell} P_\ell(S_{t_k})$ and approximate it by

$$\sum_{\ell=1}^{L} \alpha_{k,\ell} P_\ell(S_{t_k}), \text{ where } L > 1.$$

Longstaff Schwarz algorithm

1. Simulate $(S^j_{t_0}, \cdots, S^j_{t_n})_{1\le j\le M}$ M copies of $(S_{t_0}, \cdots, S_{t_n})$

2. For all $1 \le j \le M$ we set $\tau_{j,n} = t_n = T$
3. Then compute the sequence $(\alpha^j_{k,\ell})_{1\le \ell\le L}$ that minimizes

$$\frac{1}{M}\sum_{j=1}^{M}\left[\left(e^{-r(\tau_{j,k+1}-t_k)}f(S^j_{\tau_{j,k+1}}) - \sum_{\ell=1}^{L}\alpha_{k,\ell}P_\ell(S^j_{t_k})\right)^2\right]$$

4. For all $j \in \{1, \cdots, M\}$ we define

$$\tau_{j,k} = t_k \mathbf{1}_{A_{j,k}} + \tau_{j,k+1}\mathbf{1}_{A^c_{j,k}}, \quad \text{for} \quad 0 \le k \le n-1$$

where

$$A_{j,k} = \left\{ f(S^j_{t_k}) \ge \sum_{\ell=1}^{L}\alpha^j_{k,\ell}P_\ell(S^j_{t_k}) \right\}$$

Price approximation

- For $k = 0$ we the price of the Bermudan option is approximated by

$$\frac{1}{M}\sum_{j=1}^{M} e^{-r\tau_{j,0}} f(S^j_{\tau_{j,0}})$$

- The Longstaff Schwarz algorithm converges in L^2 as $L \to \infty$ and for fixed L converges almost surely as $M \to \infty$.

Computing the coordinates $(\alpha_{k,\ell})_{1\leq \ell \leq L}$

A Basic approach

For a fixed time step t_k, this approach consists on simply solving the following system

$$\begin{pmatrix} P_1(S^1_{t_k}) & \cdots & P_L(S^1_{t_k}) \\ \vdots & \vdots & \vdots \\ P_1(S^M_{t_k}) & \cdots & P_L(S^M_{t_k}) \end{pmatrix} \begin{pmatrix} \alpha_{k,1} \\ \vdots \\ \alpha_{k,L} \end{pmatrix} = \begin{pmatrix} e^{-r(\tau_{1,k+1}-t_k)} f(S^1_{\tau_{1,k+1}}) \\ \vdots \\ e^{-r(\tau_{M,k+1}-t_k)} f(S^M_{\tau_{M,k+1}}) \end{pmatrix}$$

- Advantage: easy to implement
- Drawback: not a high accuracy.

Computing the coordinates $(\alpha_{k,\ell})_{1\leq \ell \leq L}$

An optimal approach

- For a fixed time step t_k we aim at computing the sequence $(\alpha^j_{k,\ell})_{1\leq \ell\leq L}$ that minimizes

$$\sum_{j=1}^{M}\left[\left(e^{-r(\tau_{j,k+1}-t_k)}f(S^j_{\tau_{j,k+1}})-\sum_{\ell=1}^{L}\alpha_{k,\ell}P_\ell(S^j_{t_k})\right)^2\right]$$

- we differentiate the above quantity with respect to α_{k,ℓ_0} and solve

$$\sum_{j=1}^{M}\left(e^{-r(\tau_{j,k+1}-t_k)}f(S^j_{\tau_{j,k+1}})-\sum_{\ell=1}^{L}\alpha_{k,\ell}P_\ell(S^j_{t_k})\right)P_{\ell_0}(S^j_{t_k})=0$$

An optimal approach

- This is equivelent to solve

$$\sum_{\ell=1}^{L}\left(\sum_{j=1}^{M} P_\ell(S_{t_k}^j)P_{\ell_0}(S_{t_k}^j)\right)\alpha_{k,\ell} = \sum_{j=1}^{M} e^{-r(\tau_{j,k+1}-t_k)} f(S_{\tau_{j,k+1}}^j)P_{\ell_0}(S_{t_k}^j)$$

- Let $H_{\ell_0,\ell} = \sum_{j=1}^{M} P_\ell(S_{t_k}^j)P_{\ell_0}(S_{t_k}^j)$ we need to solve

$$\sum_{\ell=1}^{L} H_{\ell_0,\ell}\alpha_{k,\ell} = \sum_{j=1}^{M} e^{-r(\tau_{j,k+1}-t_k)} f(S_{\tau_{j,k+1}}^j)P_{\ell_0}(S_{t_k}^j)$$

- We can write this in a matrix equation given by

$$H\alpha_k = \sum_{j=1}^{M} e^{-r(\tau_{j,k+1}-t_k)} f(S_{\tau_{j,k+1}}^j)P(S_{t_k}^j)$$

where

$$\alpha_k = (\alpha_{k,1},\cdots,\alpha_{k,L}) \text{ and } P(S_{t_k}^j) = (P_1(S_{t_k}^j),\cdots,P_L(S_{t_k}^j))$$