Lecture 7: Computation of Greeks

Ahmed Kebaier
kebaier@math.univ-paris13.fr

HEC, Paris

Outline

Motivation

- The pathwise method requires some restrictive regularity assumptions on the payoff function of the option price, at least one time differentiable. This is not the case of some particular payoffs such as the digital options with a payoff function

$$h_T = h(X_T^x) \quad \text{where} \quad h(x) = \mathbf{1}_{\{x \geq K\}}.$$

- The Greek Δ of such option cannot be evaluated using the pathwise method.

- Note also that for the same arguments we cannot use the pathwise method to evaluate the Greek payoff Γ of a given vanilla Call option.

General idea

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ii) $\exists g : \mathbb{R}^d \mapsto \mathbb{R}$ *a measurable function such that* $\int_{\mathbb{R}^d} \phi(y)g(y)dy < \infty$ *and* $\forall \theta \in \Theta$ $|\partial_\theta p(\theta, y)| \leq g(y)$,

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$$\forall \theta \in \Theta, \quad f'(\theta) = \mathbb{E}\left(\phi(X(\theta)) \frac{\partial \log p}{\partial \theta}(\theta, X(\theta))\right)$$

Proof.

- According to assumptions i) and ii) we have

$$f'(\theta) = \frac{\partial}{\partial\theta}\mathbb{E}\phi(X(\theta)) = \int_{\mathbb{R}^d} \phi(y)\frac{\partial}{\partial\theta}p(\theta, y)dy.$$

- Rewriting the above expression we get

$$f'(\theta) = \int_{\mathbb{R}^d} \phi(y)\frac{\frac{\partial}{\partial\theta}p(\theta, y)}{p(\theta, y)}p(\theta, y)dy = \mathbb{E}\left(\phi(X(\theta))\frac{\frac{\partial}{\partial\theta}p(\theta, X(\theta))}{p(\theta, X(\theta))}\right),$$

which completes the proof.

Black Scholes Delta

- Recall that the Black-Scholes model with parameters r, σ, T is given by the explicit solution

$$S_T = S_0 \exp\left((r - \sigma^2/2)T + \sigma\sqrt{T}G\right), \quad \text{where} \quad G \sim \mathcal{N}(0,1).$$

- Consequently, for any given measurable function we get

$$\begin{aligned} \mathbb{E}f(S_T) &= \mathbb{E}f\left(S_0 \exp\left((r - \sigma^2/2)T + \sigma\sqrt{T}G\right)\right) \\ &= \int_{\mathbb{R}} f\left(S_0 \exp\left((r - \sigma^2/2)T + \sigma\sqrt{T}y\right)\right) \\ &\qquad \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy. \end{aligned}$$

Black Scholes Delta

- Using the following change of variable

$$u = S_0 \exp\left((r - \sigma^2/2)T + \sigma\sqrt{T}y\right) \quad \text{with}$$

$$du = \sigma\sqrt{T}S_0 \exp\left((r - \sigma^2/2)T + \sigma\sqrt{T}y\right) dy = u\sigma\sqrt{T}dy$$

and $y = \zeta(u)$ where $\zeta(u) = \left(\log(u/S_0) - (r - \sigma^2/2)T\right)/\sigma\sqrt{T}$

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- we get

$$\mathbb{E}f(S_T) = \int_{\mathbb{R}} f(y) \frac{1}{u\sigma\sqrt{T}\sqrt{2\pi}} \exp\left(-\frac{\zeta(u)^2}{2}\right) du.$$

Black Scholes Delta

- Therefore we deduce that the density of the random variable S_T is given by

$$g(x) = \frac{1}{x\sigma\sqrt{T}\sqrt{2\pi}} \exp\left(-\frac{\zeta(x)^2}{2}\right).$$

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- Then

$$\frac{\partial \log g(x)}{\partial S_0} = \frac{\frac{\partial g}{\partial S_0}(x)}{g(x)} = \frac{\log(x/S_0) - (r - \sigma^2/2)T}{S_0\sigma^2 T}.$$

Black Scholes Delta

Hence, according to the above theorem

$$
\begin{aligned}
\Delta &= \frac{\partial}{\partial S_0} e^{-rT}\mathbb{E}(S_T - K)_+ \\
&= e^{-rT}\mathbb{E}\left((S_T - K)_+ \frac{\log(S_T/S_0) - (r - \sigma^2/2)T}{S_0\sigma^2 T}\right) \\
&= e^{-rT}\mathbb{E}\left((S_T - K)_+ \frac{G}{S_0\sigma\sqrt{T}}\right).
\end{aligned}
$$

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Example: Path-dependent deltas

- Let us consider the case of an Asian option. Since the associated payoff involves the vector $(S_{t_1}, S_{t_2}, \cdots, S_{t_n})$

- we need to make explicit the joint density of this vector if we aim to apply the log-likelihood ratio method for the computation of sensitivities associated to the Asian option price.
- Using the Markovian property of the Brownian motion we can rewrite the density associated to the above vector as follows

$$g(x_1, \cdots, x_n) = g_1(x_1|S_0)g_2(x_2|x_1)\cdots g_n(x_n|x_{n-1}), \text{ with}$$

$$g_j(x_j|x_{j-1}) = \frac{1}{x_i\sigma\sqrt{t_j - t_{j-1}}}\varphi(\zeta_j(x_j|x_{j-1})), \quad \text{where}$$

$$\varphi(u) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)$$

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$$\zeta_j(x_j|x_{j-1}) = \frac{\log(x_j/x_{j-1}) - (r - \sigma^2/2)(t_j - t_{j-1})}{\sigma\sqrt{t_j - t_{j-1}}}.$$

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- Consequently,

$$\frac{\partial \log g}{\partial S_0}(S_{t_1}, \cdots, S_{t_n}) = \frac{\partial \log g_1}{\partial S_0}(S_{t_1}|S_0) = \frac{G_1}{S_0\sigma\sqrt{t_1}},$$

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- Finally, the value of the delta Asian call option using the Log-likelihood ratio method is given by

$$\Delta = e^{-rT}\mathbb{E}^*\left((\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K)_+ \frac{G_1}{S_0\sigma\sqrt{t_1}} \right).$$