Lecture 5: Variance Reduction

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Outline of The Talk

Outline

Antithetic Variables

Assume that we aim at computing $\pi = \mathbb{E}(g(U))$, where $U \sim \mathcal{U}([0,1])$.

- We simulate a sample $(U_1, \ldots, U_n)$ of indepedentent copies of U and we approximate target π by

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- Noticing that $U \overset{Law}{=} 1 - U$, suppose n is even, then one can consider the new estimator

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- Compute $\mathrm{Var}(\hat{S}_n)$ and compare with $\mathrm{Var}(S_n)$?

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Theorem 1

Let X be a random variable, f and g be two non-decreasing functions. Then

$$\mathrm{Cov}\left(f(X), g(X)\right) \geq 0.$$

- Answer: $\mathrm{Var}(\hat{S}_n) = \mathrm{Var}(S_n) + \frac{1}{n}\mathrm{Cov}\,(g(U), g(1-U))$

Theorem 1

Let X be a random variable, f and g be two non-decreasing functions. Then

$$\mathrm{Cov}\,(f(X), g(X)) \geq 0.$$

Hint: Consider Y an independent copy of X and use that $(f(X) - f(Y))(g(X) - g(Y)) \geq 0$

Exercise

The aim of the following exercise is to use antithetic variables when computing the price of a call option in the Black-Scholes model with maturity T and strike K. More precisely, let

$$\psi(x) = e^{-rT}(\lambda e^{\sigma\sqrt{T}x} - K)_+, \quad \text{where } \lambda = S_0 e^{(r-\frac{\sigma^2}{2})T}.$$

For $G \sim \mathcal{N}(0,1)$, our aim is to compute

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$$\mathbb{E}(\psi(G)).$$

Noticing that $G \stackrel{Law}{=} -G$ use a variance reduction method based on antithetic variables and compare the variances.

Solution

```
function y=BSCallAntithetic(S0,K,T,r,sigma,M)
tic();
X=rand(1,M,'normal');
G=X(1:M/2);
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hatC=exp(-r*T)*max(S0*exp(sigma*sqrt(T)*G+(r-sigma^ 2/2)*T)-K,0)
+exp(-r*T)*max(S0*exp(-sigma*sqrt(T)*G+(r-sigma^ 2/2)*T)-K,0);
price=sum(C)/M;
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price=sum(C)/M;
hatprice=sum(hatC)/M;
VarEst=sum((C-price).^ 2)/(M-1);
hatVarEst=sum((hatC-hatprice).^ 2)/(M-1);
RMSE=sqrt(VarEst)/sqrt(M);
hatRMSE=sqrt(hatVarEst)/sqrt(M);
time=toc();
y=[price hatprice RMSE hatRMSE]
endfunction
```

Importance Sampling

Importance sampling involves a change of probability measure. Instead of taking X from a distribution with density $p(x)$ we instead take it from a different distribution Y with density $\tilde{p}(x)$. We can write

$$\begin{aligned} \mathbb{E}(f(X)) &= \int f(x) \frac{p(x)}{\tilde{p}(x)} \tilde{p}(x) dx \\ &= \mathbb{E}\left(f(Y) \frac{p(Y)}{\tilde{p}(Y)} \right) \end{aligned}$$

We want the new variance $\mathrm{Var}\left(f(Y) \frac{p(Y)}{\tilde{p}(Y)} \right) << \mathrm{Var}(f(X))$.

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The optimal choice is then

$$\tilde{p}(x) = \frac{p(x)f(x)}{\mathbb{E}f(X)}$$

Gaussian variables

Our aim is to compute $\pi = \mathbb{E}(f(G))$ where $G \sim \mathcal{N}(0, I_d)$. Then, prove that for all $\mu \in \mathbb{R}^d$ we have

$$\pi = \mathbb{E}(f(G)) = \mathbb{E}\left(f(G + \mu) e^{-\frac{|\mu|^2}{2} - \mu \cdot G} \right).$$

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The variance associated to the estimation of the term on r.h.s f the above relation is of order

$$\mathbb{E}\left(f^2(G+\mu) e^{-|\mu|^2 - 2\mu \cdot G} \right) = \mathbb{E}\left(f^2(G) e^{\frac{|\mu|^2}{2} - \mu \cdot G} \right)$$

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For $G \sim \mathcal{N}(0,1)$, our aim is to compute $\mathbb{E}(\psi(G))$.

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For $G \sim \mathcal{N}(0,1)$, our aim is to compute $\mathbb{E}(\psi(G))$. Noticing that for $\lambda \in \mathbb{R}$

$$\pi = \mathbb{E}(\psi(G)) = \mathbb{E}\left(f(G+\lambda)e^{-\frac{\lambda^2}{2}-\lambda G} \right).$$

1. Use this last relation to implement a Monte Carlo methods and compute the associated variance $v(\lambda)$.
2. Prove tha $\lambda \mapsto v(\lambda)$ is a deacrising function on the interval $(-\infty, \log(K/\lambda)/\sigma]$.

Generalisation

Let $(\Omega, \mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtred probability space with finite Horizon T. Assume that $(B)_{0 \leq t \leq T}$ is a standard $\mathcal{F}$-Brownian motion and $\theta \in \mathbb{R}^d$ be an $\mathcal{F}$-adapted process such that $\int_0^T |\theta_s|^2 ds < \infty$ a.s. If the process

$$L_t^\theta = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^T |\theta_s|^2 ds\right)$$

is a martingale then under the probability $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L_T^\theta$$

then the process

$$W_t = B_t + \int_0^t \theta_s ds$$

is a Brownian motion under $\mathbb{Q}$.