Lecture 4: Option Pricing

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Outline of The Talk

1. Confidence Interval
2. Simulation of Brownian Motion
3. Black-Scholes Model

Outline

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2. Simulation of Brownian Motion
3. Black-Scholes Model

An other version of the CLT

Theorem 1

Let (X_n) *be a sequence of independents copies of* X *such that* $\mathbb{E}|X|^2 < \infty$ *and* $\mathrm{Var}(X) = \sigma^2 > 0$. *Let*

$$\varepsilon_n = \frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X)$$

$$\sigma_n^2 = \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right).$$

Then,

$$\sqrt{n}\frac{\varepsilon_n}{\sigma_n} \Rightarrow \mathcal{N}(0,1).$$

Confidence interval

Our aim is to evaluate $\mathbb{E}f(X)$.

- We simulate a sample $(X_1, \ldots, X_n)$ of independent copies of X and let

$$S_n = \frac{1}{n}\sum_{i=1}^{n} f(X_i)$$

$$\sigma_n^2 = \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i)^2 - \left(\frac{1}{n}\sum_{i=1}^{n} f(X_i)\right)^2\right).$$

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- Applying the above CLT we get

$$\sqrt{n}\frac{S_n - \mathbb{E}f(X)}{\sigma_n} \Rightarrow \mathcal{N}(0,1).$$

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- This yields

$$\mathbb{P}\left(\left|\sqrt{n}\frac{S_n - \mathbb{E}f(X)}{\sigma_n}\right| \le a\right) \underset{n\to\infty}{\longrightarrow} \mathbb{P}\left(|G| \le a\right), \quad G \sim \mathcal{N}(0,1).$$

- If we set $\mathbb{P}(|G| \leq a) = \alpha$, then we say that with a level of confidence equal to α our target

$$\mathbb{E}(f(X)) \in \left[S_n - \frac{a\sigma_n}{\sqrt{n}}, S_n + \frac{a\sigma_n}{\sqrt{n}} \right]$$

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- **Example:**

For a level of confidence equal to 95% we have that

$$\mathbb{E}(f(X)) \in \left[S_n - \frac{1.96\sigma_n}{\sqrt{n}}, S_n + \frac{1.96\sigma_n}{\sqrt{n}}\right]$$

Outline

Brownian Motion

- A Brownian motion is a continuous process with independent and stationary increments such that $W_t \sim \mathcal{N}(0, t)$

```
m=5;
n=300;
t=linspace(0,1,n+1)';
h=diff(t(1:2)); // step size
dw=sqrt(h)*rand(n,m,'normal');
w=cumsum([zeros(1,m);dw]);
plot(t,w)
```

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European options

- Vanilla options
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- exotic options
  - Barrier $(S_T - K)_+ \mathbf{1}_{\{\max_{t \leq T} S_t > B\}}$

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  - Asian $(\frac{1}{T} \int_0^T S_u du - K)_+$

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  - Basket Call $(\sum_{i=1}^d S_T^i - K)_+$

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  - Basket Call $(\sum_{i=1}^d S_T^i - K)_+$
  - options on several assets $(S_T^1 \wedge \dots S_T^d - K)_+$
- American Option
  - Call $(S_\tau - K)_+$ where τ is a stopping time.

Outline

Black-Scholes model

In the Black-Scholes model, the risky asset satisfies the SDE

$$dS_t = rS_t dt + \sigma S_t dWt,$$

under the martingale measure $\mathbb{Q}$. The solution S follows a geometric Brownian motion

$$S_t = S_0 \exp\left(\sigma W_t + (r - \frac{\sigma^2}{2})t\right).$$

The price of a call option with payoff $(S_T - K)_+$ is

$$\pi = e^{-rT}\mathbb{E}(S_T - K)_+ = e^{-rT}\mathbb{E}(S_T \mathbf{1}_{\{S_T > K\}}) - Ke^{-rT}\mathbb{E}(\mathbf{1}_{\{S_T > K\}})$$

Black-Scholes model: Call option

$$\begin{aligned}\{S_T > K\} &= \left\{\log(S_0) + \sigma W_T + (r - \frac{\sigma^2}{2})T > \log(K)\right\} \\ &= \left\{W_T > \frac{1}{\sigma}\left(\log\left(\frac{K}{S_0}\right) - (r - \frac{\sigma^2}{2})T\right)\right\}\end{aligned}$$

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$$

and that $W_T = \sqrt{T}G$ where $G \sim \mathcal{N}(O, 1)$. Let us introduce $\phi(x) = \mathbb{P}(G \leq x)$.

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and that $W_T = \sqrt{T}G$ where $G \sim \mathcal{N}(O, 1)$. Let us introduce $\phi(x) = \mathbb{P}(G \leq x)$.Now, use that $1 - \phi(x) = \phi(-x)$ to deduce that

$$\mathbb{E}(\mathbf{1}_{\{S_T>K\}}) = \mathbb{P}(S_T > K) = \phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{S_0}{K}\right) + (r - \frac{\sigma^2}{2})T\right)\right)$$

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Applying Girsanov's theorem, we get

$$\pi = S_0\phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{S_0}{K}\right) + (r + \frac{\sigma^2}{2})T\right)\right)$$
$$- Ke^{-rT}\phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{S_0}{K}\right) + (r - \frac{\sigma^2}{2})T\right)\right)$$

Black-Scholes Formula

```
function y=BScall(S0,K,T,r,sigma)
tic();
d1=(log(S0/K)+(r+sigma^ 2/2)*T)/(sigma*sqrt(T));
d2=(log(S0/K)+(r-sigma^ 2/2)*T)/(sigma*sqrt(T));
price=S0*cdfnor("PQ",d1,0,1)-K*exp(-r*T)*cdfnor("PQ",d2,0,1);
time=toc();
y=[price time]
endfunction
```

Exercise

- Create a function to evaluate the price of an European Call option with maturity T and strike K, on the Black-Sholes model $(S_t)_{0 \leq t \leq T}$ with a Monte Carlo method.

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- Create a function to evaluate the price of an European Call option with maturity T and strike K, on the Black-Sholes model $(S_t)_{0\leq t\leq T}$ with a Monte Carlo method.

- In other words approximate $e^{-rT}\mathbb{E}(S_T - K)_+$ with

$$\frac{e^{-rT}}{N}\sum_{i=1}^{N}(S_{T,i} - K)_+$$

where $(S_{T,i})_{1\leq i\leq N}$ are i.i.d copies of S_T.

Solution

```
function y=BSMCcall(S0,K,T,r,sigma,M) stacksize('max')
tic();
X=rand(1,M,'normal');
S=S0*exp(sigma*sqrt(T)*X+(r-sigma^ 2/2)*T);
C=exp(-r*T)*max(S-K,0);
price=sum(C)/M;
VarEst=sum((C-price)^ 2)/(M-1);
RMSE=sqrt(VarEst)/sqrt(M);
CI95=[price-1.96*RMSE,price+1.96*RMSE];
CI99=[price-2.58*RMSE,price+2.58*RMSE];
time=toc();
y=[price time RMSE; CI95 0; CI99 0]
endfunction
```

Black-Scholes model: Asian option

- The price of an Asian option at $t = 0$ is given by

$$\pi := e^{-rT}\mathbb{E}\left(\frac{1}{T}\int_0^T S_u du - K\right)_+$$

- This quantity have no explicit formula, so we need to approximate π. How to proceed ?

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  2. Then, approximate π by a Monte Carlo method

$$\pi \sim \frac{e^{-rT}}{N}\sum_{j=1}^{N}(\bar{S}_{n,j} - K)_+,$$

where $(\bar{S}_{n,j})_{1\leq j\leq N}$ are i.i.d copies of $\bar{S}_n$.

```
// S0:  the spot price of the underlying
// K: the strike price of the option
// T: the maturity of the option
// n:  the number of time intervals
// r:  the risk free interest rates
// sigma:  the volatility of the underlying
// N: the number of Monte Carlo iterations
function [price ] = AsianCall(S0, K, T, n, r, sigma, N)
z = rand(N,n,'norm');
dt = T/n;
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function [price ] = AsianCall(S0, K, T, n, r, sigma, N)
z = rand(N,n,'norm');
dt = T/n;
LogPaths= cumsum([log(S0)*ones(N,1),(r-0.5*sigma^ 2)*dt +
sigma*sqrt(dt)*z],'c');
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sigma*sqrt(dt)*z],'c');
S = exp(LogPaths);
payoff = max(mean(S, 'c')-K, 0);
price=exp(-r*T)*mean(payoff);
```

Exercise

Create a function to compute the price a Barrier option with maturity T, strike K and barrier B given by

$$\pi := e^{-rT}\mathbb{E}\left((S_T - K)_+ \mathbf{1}_{\left\{\max\limits_{0\leq t\leq T} S_t > B\right\}}\right)$$

Solution

```
function [price] = BarrierUpInCall(S0, K, T, n, r,
sigma,B, N)
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