Lecture 3: Monte Carlo methods

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Outline of The Talk

Change of variable method

Lemma 1

Let ϕ denote a $\mathcal{C}^1$-diffeomorphism from $\mathcal{O} \subset \mathbb{R}^d$ to $\mathcal{O}' \subset \mathbb{R}^d$ and $g : \mathcal{O} \to \mathbb{R}$ an integrable function.

$$\int_{\mathcal{O}} g(v)dv = \int_{\mathcal{O}'} g \circ \phi^{-1}(u) |\det(\nabla \phi^{-1}(u))| du$$

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Theorem 2 (Box-Müller)

Let U and V be two independent uniform random variables on $[0, 1]$. We set

$$X = \sqrt{-2\ln(U)} \cos(2\pi V), \qquad Y = \sqrt{-2\ln(U)} \sin(2\pi V).$$

Then,

$$(X, Y) \sim \mathcal{N}(0, I_2).$$

Box-Müller on scilab

```
function [y1,y2] = randnorm(n) // assumes n even
m = n/2
x1 = rand(1,m)
x2 = rand(1,m)
y1 = sin(2*%pi*x1).*sqrt(-2*log(x2));
y2 = cos(2*%pi*x1).*sqrt(-2*log(x2));
endfunction

-->x = randnorm(10000);
-->histplot(100,x)
```

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We can compare the histogram to exact density:

```
-->xx = -4:0.01:4;
-->yy = exp(-(xx.^ 2)/2)/sqrt(2*%pi);
-->plot2d(xx, yy)
```

Simulation of Gaussian vectors

Theorem 3

Let $\mu \in \mathbb{R}^d$ and $\Gamma \in S_d^+$ a $d \times d$ definite positive symmetric matrix.

- *By Cholesky's algorithm, there exists a lower triangular matrix $A \in \mathbb{R}^{d \times d}$ such that $\Gamma = AA^T$*
- *If $G \sim \mathcal{N}(0, I_d)$, then $\mu + AG \sim \mathcal{N}(\mu, \Gamma)$*

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Exercise

For $\rho \in [-1, 1]$, simulate a sample of size n of centred Gaussian couple (G_1, G_2) with covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.

Solution

```
function [] = gaussian_vector(rho)
if abs(rho) >1
disp('the correlation must be between -1 and 1 !')
disp('aborting...')
return
end
n=1000;
[g1,g2]=randnorm(n) ;
Gamma=[1,rho;rho,1];
A=chol(Gamma);
z=A*[g1;g2];
x=z(1,:);
y=z(2,:);
plot2d(x,y,-1)
endfunction
```

Outline

Principle of the method

Monte Carlo method is based on the Large Law numbers

Theorem 4

SLLN Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with the same law as X. If $\mathbb{E}|X| < \infty$ then

$$S_n := \frac{1}{n}\sum_{i=1}^{n} X_i \underset{n\to\infty}{\longrightarrow} \mathbb{E}X, \quad a.s.$$

The convergence holds also in L^1

Let us assume that $\mathbb{E}X = 0$ and $\mathbb{E}|X|^4 < \infty$.

Proof.

$$\begin{aligned}
\mathbb{E}S_n^4 &= \frac{1}{n^4}\left(\sum_{k=1}^{n}\mathbb{E}(X_k^4) + 6\sum_{i<j}\mathbb{E}(X_i^2)\mathbb{E}(X_j^2)\right)\\
&= \frac{1}{n^4}\left(n\mathbb{E}(X^4) + 3n(n-1)\mathbb{E}(X^2)^2\right)\\
&\leq \frac{3\mathbb{E}(X^4)}{n^2}
\end{aligned}$$

By the monotone convergence Theorem we deduce $\mathbb{E}(\sum_{n=1}^{\infty} S_n^4) < \infty$ which implies that $\sum_{n=1}^{\infty} S_n^4 < \infty$ a.s. This completes the proof. □

Central Limit Theorem

Theorem 5

Let $(X_n)_{n\geq 1}$ be a sequence of independent random copies of X. If $\mathbb{E}(|X|^3) < \infty$, then

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}X\right) \Rightarrow \mathcal{N}(0,\sigma^2), \qquad \text{as } n \to \infty$$

where $\sigma^2 = \mathtt{Var}(X)$.

Sketch of the proof

- Without a loss of generality we can consider the case where $\mathbb{E}(X) = 0$ and $d = 1$. This implies that $\mathrm{Var}(X) = \mathbb{E}(X^2) = \sigma^2$.

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- For all $u \in \mathbb{R}$

$$\psi_{\sqrt{n}S_n}(u) := \mathbb{E}\left[\exp(iu\sqrt{n}S_n)\right] = \left(\mathbb{E}\left[\exp\left(\frac{iuX}{\sqrt{n}}\right)\right]\right)^n$$

- By Taylor expansion

$$\left|e^{iy} - 1 - iy + \frac{y^2}{2}\right| \leq \min\left(\frac{|y|^3}{6}, y^2\right).$$

- Then it follows that

$$\exp\left(\frac{iuX}{\sqrt{n}}\right) = 1 + \frac{iuX}{\sqrt{n}} - \frac{u^2X^2}{2n} + h_n(X), \text{ where}$$

$$|h_n(X)| \leq \frac{u^2}{n} \min\left(\frac{u|X|^3}{6\sqrt{n}}, X^2\right)$$

- Note that the sequence $(nh_n(X))_n$ is uniformly dominated by u^2X^2 and

$$\mathbb{E}\left[\exp\left(\frac{iuX}{\sqrt{n}}\right)\right] = 1 - \frac{u^2\sigma^2}{2n} + \mathbb{E}[h_n(X)],$$

- Thnaks to the Dominated convergence theorem we get

$$\lim_{n\to\infty} \mathbb{E}[nh_n(X)] = 0$$

- We deduce that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[\exp\left(\frac{iuX}{\sqrt{n}}\right)\right] &= \lim_{n\to\infty}\left(1 - \frac{u^2\sigma^2}{2n} + o(n^{-1})\right)^n \\ &= \exp\left(-\frac{u^2\sigma^2}{2}\right) \\ &= \mathbb{E}[\exp(iuG)]\end{aligned}$$

where $G \sim \mathcal{N}(0, 1)$

Exercise

We consider a sample $(Z_1, \ldots, Z_n)$ with Z_i sont i.i.d random variables with the same law as $\sqrt{12p}\left(\frac{1}{p}\sum_{i=1}^{p} U_i - \frac{1}{2}\right)$, the random variables $(U_i, i \leq p)$ are i.i.d with uniform distribution on the interval $[0, 1]$. Use the following function to plot the histogram of $(Z_i, i \leq n)$ with nc classes. Vary n, p, nc we can take $p = 1$, $p = 12$with large and small values of nc. We will take n of order 1000.

solution

```
function [] = tcl(n, p, nc)
X=rand(n,p);
Z=sqrt(12/p)*(sum(X,'c') - p/2); // sum of columns,
centred and renormalised
histplot(nc,Z)
C=[-5:1/1000:5];
plot2d(C,exp(-C.^ 2/2)/sqrt(2*%pi),3) // plobability
density of standad distribution
endfunction
```