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Optimal portfolio management with American capital guarantee

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Abstract

The aim of the paper is to investigate finite horizon portfolio strategies which maximize a utility criterion when a constraint is imposed on a terminal date (European guarantee) or on every intermediate date (American Guarantee). We prove the optimality of the Option Based Portfolio Insurance method for both European and American cases, when an expected CRRA utility function is maximized. The American OBPI is fully described in a Black–Scholes environment as well as in the more general case of complete markets using the Gittins index techniques developed by El-Karoui and Karatzas (1995). Optimality results are extended to general utility functions.

JEL classification: C61; D81; G11; G13

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1. Introduction

Portfolio managers have available a large class of investment strategies. The most celebrated and the simplest one is the Buy and Hold strategy, where in reference to an investment horizon, a well-diversified portfolio (for example an Index portfolio) is maintained without any rebalancing to the maturity date. The portfolio performance can in general be improved using a dynamic strategy. However in practice the portfolio can sustain large losses if the market turns sharply down as it did, for example in October 1987 or 1998. In order to avoid large losses the manager may decide to 'insure'

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a specified-in-advance minimum value for the portfolio, which results in the sacrifice of potential gains. There may be constraints which require that the liquidation value of the portfolio never drops below a certain threshold during the life of the contract. The problem of portfolio selection is of both theoretical and practical interest.

Leland and Rubinstein (1976) introduced the concept of Option Based Portfolio Insurance based on dynamic replication using either synthetic options or traded options. Subsequently, Black and Jones (1987) and Perold and Sharpe (1988) developed automated strategies including the Constant Proportion Portfolio Insurance method[1] which has become popular with practitioners. Both these approaches guarantee that the current portfolio value dominates the discounted value of the pre-specified final floor.

More generally, practitioners have well understood the separation principle which was introduced by Markowitz (1959) and extended by Merton (1971, 1973): the asset allocation is made optimal through the use of two separate funds, the first one being a combination of basic securities, the second one being the money market account. Later papers (see for example Cox and Huang, 1989) have extended this result to the three fund separation principle, when an extra insurance is required by the investor. A third fund is added to the basic Markowitz portfolio corresponding to a derivative which is based on the portfolio of risky securities. The OBPI method is an application of such a separation principle.

More recently, various authors have proposed dynamic fund strategies in the case of American protection (see Gerber and Pafumi, 2000; Imai and Boyle, 2001). The main results of the present paper are first to extend the three fund separation principle to an American constraint and second to fully describe the optimal dynamic portfolio in a general framework. Throughout the paper, we follow the methodology developed by El-Karoui and Jeanblanc (1998) who derive the optimal wealth under a non-negativity constraint for an investor receiving an income. Their work, as well as ours, is set in a complete market framework. Other related papers are Grossman and Zhou (1993) and Cvitanić and Karatzas (1996) where the lower bound for the investor's wealth at time t is a fraction of its maximum value over time $[0,t]$. Cvitanić and Karatzas (1999) also study the European constraint where the final wealth satisfies lower bounds for a given level of probability.

In our paper, the investor chooses in the first step an *unconstrained* (or *uninsured*) *allocation* using traded assets and without taking care of the guarantee. In the second step, he insures his portfolio by specifying a dynamic strategy which we call the *insured allocation*. Our unconstrained allocation is related to but not equivalent to the tactical allocation introduced by Brennan et al. (1997). In Brennan et al., "the tactical asset allocation is essentially a myopic strategy" and is static. We emphazise that, in our framework, the unconstrained allocation might be a dynamic strategy, depending on the form of the utility function to be maximized. We also prove that the choice of our insured allocation is optimal among all strategies which satisfy the guarantee.

The paper is organized as follows: the second section presents the classical OBPI method and focuses on the maximization of an expected utility criterion, over all self-financing strategies whose value satisfies a European constraint. The Put Based

[1] Also known as the *cushion method*.

Strategy (referred to as the *insured allocation*) written on the optimal portfolio solving the unconstrained problem is shown to be optimal for CRRA utility functions. In such a case, the three fund separation principle appears to be strictly valid. The third section extends the OBPI method for the fund to satisfy an American rather than a European constraint: the strategy is then based on American put options. In order to remain self-financing, we introduce a path dependent gearing parameter which specifies the current nominal amount to be invested in the unconstrained allocation. This path dependent parameter is shown to optimally increase when the value of the tactical allocation drops below a given exercise boundary. The analysis is conducted within a Black–Scholes environment. The fourth section extends the latter result to the more general case of complete markets using the properties of American options developed by El-Karoui and Karatzas (1995) using the Gittins index methodology. The optimality of the strategy is then proven for a CRRA utility function criterion. In the fifth section, we extend our optimality results to a general class of utility functions. It is shown that despite the non-linearity of the unconstrained optimal portfolio with respect to the initial wealth, the OBPI is still optimal, but the unconstrained allocation to be insured is changed depending on the initial cost of the protection.

Throughout the paper we assume a framework of complete, arbitrage free, frictionless markets.

2. Option Based Portfolio Insurance in the European case

The present section describes one of the most popular insurance strategies, known as the *Option Based Portfolio Insurance* introduced by Leland and Rubinstein (1976).

2.1. Unconstrained allocation

The first step in the management of investment funds is to define an *unconstrained allocation* related to a finite horizon. According to the investor's risk aversion, the manager decides the proportion of indexes, securities, coupon bonds, to be held in a well-diversified portfolio with positive value. An example would be the efficient portfolio in the Markovitz setting, or the portfolio constructed using the mutual fund result (see Merton 1971, 1973, or Section 2.4 of this paper).

In what follows we denote by S_t the time-t value of *one unit* of the unconstrained allocation. Without loss of generality, we assume that $S_0 = 1$. Consequently, an initial amount $\lambda > 0$ invested at date 0 evolves in the future following $(\lambda S_t)_{t\geqslant 0}$.

At this step, we do not need to specify the dynamics of $(S_t)_{t\geqslant 0}$. Recall that we have assumed the market to be *complete*, *arbitrage free* and *frictionless*. The following assumptions give the restrictions we impose on $(S_t)_{t\geqslant 0}$:

Assumptions. A1. $(S_t)_{t\geqslant 0}$ follows a continuous diffusion process, $\mathbb{R}^+$-valued.

A2. All dividends and coupons are assumed to be reinvested, in such a way that the allocation is self-financing.

A3. At any date $t \geqslant 0$ all zero-coupon bonds with maturities larger than t are tradable assets.

Throughout the paper, we use the following general characterization for self-financing strategies:

(SF) $(X_t,\ t \geqslant 0)$ is the value process of a self-financing strategy if and only if the process RX is a Q-martingale, where R is the discount factor, i.e., $R_t = \exp(-\int_0^t r(s)\,\mathrm{d}s)$, $r(s)$ being the time-s instantaneous rate. Q is the risk-neutral probability measure.

Indeed, the discounted value of a self-financing portfolio is a martingale under the risk-neutral probability measure due to the arbitrage free assumption. Conversely, if RX is an adapted martingale process, the completion of the market gives the existence of a real number h and an adapted process π such that $R_T X_T = h + \int_0^T \pi_s d(R_s S_s)$. The process π represents the hedging portfolio of X and can be shown to be self-financing (see for example Karatzas and Shreve, 1998, for full details).

2.2. European versus American guarantee

We now focus on the second step for the manager which is to define the *insured allocation*, that is to manage dynamically the unconstrained allocation to fulfill the guarantee.

We assume that the manager desires protection against declines in the value of the unconstrained portfolio. The horizon date is T and we denote by $(K_t)_{0 \leqslant t \leqslant T}$ the current floor value.

One example is to define a minimal final value for the fund, denoted by $K_T = K$, where K is a percentage of the initial fund value. When the guarantee holds only for the terminal date T, the protection is said to be European. American type funds offer the same guarantee for any intermediate date between 0 and T. In such a case the floor value is defined as a time dependent function (or process) $(K_t)_{0 \leqslant t \leqslant T}$.

Remark 2.1. In the case where the current floor K_t equals the discounted value of the final strike price K:

$$K_t = K B_{t,T},$$

where $B_{t,T}$ denotes the time-t value of a zero-coupon bond paying \$1 at time T, a simple arbitrage argument implies that the American guarantee reduces to a European one.

The present section focuses on the European case.

2.3. European OBPI Strategy

The option based portfolio insurance (OBPI) pioneered by Leland and Rubinstein (1976), has the following two attributes: first it protects the portfolio value at maturity and second it takes advantage of rises in the underlying portfolio.

The put based strategy combines a long position in the underlying unconstrained allocation $(S_t)_{t\geqslant 0}$ with a put option.

The initial capital invested in the fund, supposed to be normalized at 1 (and strictly larger than $KB_{0,T}$) is then split into two parts, say λ and $1-\lambda$, where λ lies between 0 and 1. With the first part, the manager buys at date 0 a fraction λ of the unconstrained allocation, and with the remaining part, he insures his position with a put written on $(\lambda S_t)_{t\geqslant 0}$. The final payoff of the put is

$$\Psi_T = \max(K - \lambda S_T; 0),$$

where the strike price K is the final floor value for the fund.

Remark 2.2. The put option can also be written as a fraction λ of a put option on $(S_t)_{t\geqslant 0}$ with strike price K/λ following: $(K - \lambda S_T)^+ = \lambda(K/\lambda - S_T)^+$. We note that the maturity value of the unconstrained portfolio has to exceed K/λ for the investor to receive more than the guaranteed amount.

Remark 2.3. Note that we have to require $1 \geqslant KB_{0,T}$ in order to get the existence of a portfolio satisfying the terminal constraint (see following Proposition 2.1).

Notation. Let us denote by $P^e_S(t,T,K)$ (resp. $P^e_{\lambda S}(t,T,K)$) the time-t value of a European Put with maturity T and strike K written on one unit (resp. on λ units) of the unconstrained allocation, where the superscript e stands for European. We suppose that such options are traded in the market for every strike K. Alternatively, since the market is complete and frictionless such options can be replicated dynamically.

At maturity, the manager obtains a final value for the fund:

$$V_T(\lambda) = \lambda S_T + (K - \lambda S_T)^+ = \sup(\lambda S_T, K), \tag{1}$$

which is superior to K.

In Eq. (1) the parameter λ is usually called the gearing or leverage of the fund. The value of λ must be defined at the inception date 0. It has to satisfy the initial *budget constraint*:

$$V_0(\lambda) = \lambda + P^e_{\lambda S}(0,T,K) = 1.$$

Let us note that λ depends critically on the level of volatility, through the initial price of the put option.

Proposition 2.1. *There exists a unique constant* λ, *with* $0 < \lambda < 1$, *such that*:

$$\lambda + P^e_{\lambda S}(0,T,K) = 1.$$

Proof. The liquidation value of the fund at maturity T: $V_T(\lambda) = \sup(\lambda S_T, K)$, is a non-decreasing function with respect to λ valued in $[K, +\infty)$ for $\lambda \in \mathbb{R}_+$. For any $\lambda' < \lambda$:

$$0 \leqslant V_T(\lambda) - V_T(\lambda') \leqslant (\lambda - \lambda')S_T.$$

Discounting and taking the expectation we see that the initial value of the fund $V_0(\lambda)$ is a non-decreasing and Lipschitzian function with respect to λ with a Lipschitz constant equal to S_0.

From the no-arbitrage assumption, and recalling that S_0 is positive $V_0(\lambda)$ is valued in $]KB_{0,T}, +\infty)$ for $\lambda \in \mathbb{R}_+$.

Considering that $KB_{0,T} < 1$ the existence and uniqueness of λ satisfying $V_0(\lambda) = 1$ follows.

Finally, $V_0(1) > S_0 = 1$ gives $\lambda < 1$. □

Notation. When the dynamics of $(S_t)_{t \geqslant 0}$ are assumed to be Markovian, the value $P^e_S(t, T, K)$ of a European put on S turns out to be a deterministic function of time t and current value of the underlying S_t. In such a case (see Section 4) we denote this value by $P^e(t, S_t, T, K)$. The function $P^e(\cdot, \cdot, T, K)$ then solves the classical valuation Partial Differential Equation.

Remark 2.4. The function $P^e_{\lambda S}(t, T, K)$ generally differs from $\lambda P^e_S(t, S_t, T, K/\lambda)$ which represents the price of λ put options written on S with strike K/λ. The equality holds when the Put price is homogeneous, in particular in the Black and Scholes framework. For more general volatility models, in particular when fitting the volatility function to smile patterns (see for example Dupire, 1994), the homogeneity property no longer holds.

2.4. OBPI Optimality for a European Guarantee

In this subsection, we study the optimal portfolio strategy in the case of a European guarantee for which the constraint holds only at the terminal date T.

We consider an investor who maximizes an expected utility criterion where the expectation is taken under the subjective probability P. Given a *utility function* u (concave, strictly increasing, defined on $\mathbb{R}^+$) we solve the following maximization problem:

$$\max \mathrm{E}[u(V_T)]; \quad \text{under the constraints } V_T \geqslant K, \text{ and } V_0 = 1, \tag{2}$$

over all self-financing portfolios.

We prove the optimality of the Put Based Strategy written on an appropriate unconstrained allocation in the case of a constant relative risk aversion (CRRA) utility function u defined as $u(x) = x^{1-\gamma}/(1-\gamma)$, for all $x \in \mathbb{R}^+$, with $\gamma \in (0, 1)$. The result is extended to a general class of utility functions in Section 5.1.

2.4.1. Choice of the unconstrained allocation

We first characterize the solution to the problem without the European constraint:

$$\max \mathrm{E}[u(X^\lambda_T)]; \quad \text{under the budget constraint } X^\lambda_0 = \lambda, \tag{3}$$

where λ denotes the initial wealth. It is well known (see for example Karatzas and Shreve, 1998) that the terminal value $\hat{X}^\lambda_T$ of the optimal strategy with initial wealth λ

satisfies the first order condition:[2]

$$\mathrm{E}(u'(\hat{X}_T^\lambda)(X_T^\lambda - \hat{X}_T^\lambda)) = 0, \tag{4}$$

for any X_T^λ, terminal value of a self-financing portfolio with initial wealth λ.

Moreover, in the CRRA case, the optimal terminal value $\hat{X}_T^\lambda$ is linear with respect with the initial wealth, i.e., $\hat{X}_T^\lambda = \lambda S_T$, where $S_T = \hat{X}_T^1$ is the optimal strategy with initial wealth 1.

Remark 2.5. The linearity property of the solution with respect to the initial wealth for a CRRA utility function criterion allows one to describe the optimal solution as a proportion of *one unit* of the optimal tactical allocation. This justifies the fact that the initial fund value has been previously chosen equal to 1.

Referring to S as the *unconstrained allocation*, we have: $u'(\hat{X}_T^\lambda) = \lambda^{-\gamma} u'(S_T)$. Hence, the first order condition (4) can be re-written as

$$\mathrm{E}(u'(S_T)(X_T^\lambda - \hat{X}_T^\lambda)) = 0. \tag{5}$$

2.4.2. Choice and optimality of the insured allocation

We now consider the Put Based strategy written on $S_T = \hat{X}_T^1$.

Let us denote by $\hat{V}_t$ the value at time $t \leqslant T$ of the self-financing strategy with initial value 1 and terminal value:

$$\hat{V}_T = \max(\lambda S_T, K) \geqslant K, \tag{6}$$

where λ is defined in Proposition 2.1. We refer to $\hat{V}$ as the *insured allocation*.

Proposition 2.2. *The put based strategy written on the optimal portfolio with no constraint solves the optimization problem with a European constraint for CRRA utility functions.*

More precisely, if V_T is the terminal value of a self-financing portfolio with initial value 1 such that $V_T \geqslant K$, and $\hat{V}_T$ is the terminal value of the Put Based Strategy defined in (6), then

$$\mathrm{E}[u(\hat{V}_T)] \geqslant \mathrm{E}[u(V_T)].$$

Proof. The concavity of u yields to

$$u(V_T) - u(\hat{V}_T) \leqslant u'(\hat{V}_T)(V_T - \hat{V}_T).$$

From equality (6) and the CRRA property of u we get

$$u'(\hat{V}_T) = u'(\lambda S_T) \wedge u'(K) = [\lambda^{-\gamma} u'(S_T)] \wedge u'(K).$$

[2] If $\hat{X}_T^\lambda$ is optimal, consider the portfolio with terminal value $V_T(\varepsilon) = \varepsilon \hat{X}_T^\lambda + (1-\varepsilon) X_T^\lambda$ and write that $\varepsilon = 1$ is the maximum of $\mathrm{E}[u(V_T(\varepsilon))] : (\partial/\partial\varepsilon \mathrm{E}[u(V_T(\varepsilon))])_{\varepsilon=1} = 0$.

Since $u'(\hat{V}_T) \geqslant u'(K)$ is equivalent to $\hat{V}_T = K$ due to the constraint $\hat{V}_T \geqslant K$ and the decreasing property of u', we get

$$[[\lambda^{-\gamma}u'(S_T)] \wedge u'(K)](V_T - \hat{V}_T) = \lambda^{-\gamma}u'(S_T)(V_T - \hat{V}_T) \\ -[\lambda^{-\gamma}u'(S_T) - u'(K)]^+(V_T - K).$$

On one hand, from the first order condition (5) written for $\lambda = 1$, we have

$$\mathrm{E}[u'(S_T)(V_T - \hat{V}_T)] = \mathrm{E}[u'(S_T)(V_T - \hat{X}_T)] + \mathrm{E}[u'(S_T)(\hat{X}_T - \hat{V}_T)] = 0,$$

and on the other hand, from the terminal constraint on V_T, the following inequality holds:

$$-\mathrm{E}([\lambda^{-\gamma}u'(S_T) - u'(K)]^+(V_T - K)) \leqslant 0.$$

Hence, $\mathrm{E}(u(\hat{V}_T)) \geqslant \mathrm{E}(u(V_T))$. $\square$

3. American case in the Black and Scholes framework

We now consider the problem of an American guarantee.

We first show how to construct *self-financing* put-based strategies when $(S_t)_{t\geqslant 0}$ is known to follow Black–Scholes dynamics. Optimality is proven for CRRA utility function in Section 4.3 and the result is extended to a general class of utility functions in Section 5.2.

In the present section, we describe a path dependent self-financing strategy under the assumption that the dynamics of the unconstrained allocation $(S_t)_{t\geqslant 0}$ are given by

$$\mathrm{d}S_t = S_t(r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t), \quad S_0 = 1, \tag{7}$$

where $(W_t)_{t\geqslant 0}$ is a Brownian motion under the risk-neutral probability Q. The interest rate r is assumed to be constant, as well as the volatility σ.

3.1. American Put Based strategy

By analogy with the European case, we introduce an American put on $(\lambda S_t)_{t\geqslant 0}$, where λ is now to be adjusted such that

$$1 = \lambda + P^{\mathrm{a}}_{\lambda S}(0), \tag{8}$$

where $P^{\mathrm{a}}_{\lambda S}(0)$ is the price at time 0 of the American put on the underlying $(\lambda S_t; t \geqslant 0)$, with strike K and maturity T. Note that by definition of an American contract $P^{\mathrm{a}}_{\lambda S}(t) \geqslant (K - \lambda S_t)^+$ for all $t \leqslant T$. Hence, defining X_t as the time-t value of a portfolio composed of positions in both the unconstrained allocation and the American derivative:

$$X_t \overset{\mathrm{def}}{=} \lambda S_t + P^{\mathrm{a}}_{\lambda S}(t) \geqslant K, \quad \forall t \leqslant T.$$

However, the part of the portfolio which consists in the American put is not self-financing. The put delivers a continuous dividend to the writer of the option as soon as the stopping region is attained, that is after time $\sigma(\lambda)$ where

$$\sigma(\lambda) = \inf\{t : P^{a}_{\lambda S}(t) = K - \lambda S_t\}.$$

3.2. Properties of the American put price

We recall some well known properties of the American put price in the Black–Scholes framework (we refer to Musiela and Rutkowski, 1998, Chapter 8, for a detailed presentation of American options and references).

The price $P^{a}_{S}(t)$ of an American option on the underlying S is a deterministic function of time t and current value S_t of the underlying. We denote by $P^{a}(t,x)$ such a function. Hence the price of a put on the underlying $(\lambda S_t,\ t \geqslant 0)$ is $P^{a}(t, \lambda S_t)$. By definition:

$$P^{a}(t,x) = \operatorname*{essup}_{\tau \in \mathscr{T}_{t,T}} \mathrm{E}_Q((K - S^{t,x}_{\tau})^{+} \mathrm{e}^{-r(\tau - t)}),$$

where $\mathscr{T}_{t,T}$ is the set of stopping times taking values in the interval $[t,T]$ and $S^{t,x}_{s}$ denotes the s-time value of the solution of the Black–Scholes equation (7) which equals x at time t. Let us denote by $\mathscr{C}$ the continuation region defined as $\mathscr{C} = \{(t,x) | P^{a}(t,x) > (K - x)^{+}\}$. In the Black and Scholes framework, the continuation region is also described via the increasing exercise boundary $(b(t),\ t \geqslant 0)$ where b is a deterministic function defined as

$$b(t) = \sup\{x : P^{a}(t,x) = (K - x)^{+}\}.$$

Therefore

$$\mathscr{C} = \{(t,x) : x > b(t)\}.$$

The function $P^{a}(t,x)$ is C^2 with respect to x in the interior of the continuation region. It is affine outside, equal to $K - x$. It is C^1 over $\mathbb{R}^{+}$ and (smooth pasting principle) $\partial_x P^{a}(t, b(t)) = -1$. The second derivative admits only one discontinuity at the point $x = b(t)$.

Introducing the operator $\mathscr{L}$ defined as

$$\mathscr{L} = \partial_t + \frac{1}{2}\sigma^2 x^2 \partial_{xx} + rx\partial_x,$$

$P^{a}(t,x)$ satisfies

$$\begin{aligned} &P^{a}(t,x) = K - x, \quad \forall (t,x) \notin \mathscr{C}, \\ &\mathscr{L}P^{a}(t,x) - rP^{a}(t,x) = 0, \quad \forall (t,x) \in \mathscr{C}, \\ &\mathscr{L}P^{a}(t,x) = -1, \quad \forall (t,x) \notin \mathscr{C}. \end{aligned} \tag{9}$$

Let us now introduce the function $A(t,x)$:

$$A(t,x) \overset{\text{def}}{=} x + P^{a}(t,x).$$

From the properties of $P^{a}(t,x)$ we deduce that $A(t,x)$ is C^2 with respect to x in the interior of the continuation region, and constant outside equal to K. $A(t,x)$ is C^1 and $\partial_x A(t, b(t)) = 0$. The second derivative admits only one discontinuity at point $x = b(t)$.

Using the operator $\mathscr{L}$ we have

$$
\begin{aligned}
&A(t,x) = K, \quad \forall (t,x) \notin \mathscr{C}, \\
&\mathscr{L}A(t,x) = \mathscr{L}P^{\mathrm{a}}(t,x) + rx = rA(t,x), \quad \forall (t,x) \in \mathscr{C}, \\
&\mathscr{L}A(t,x) = 0, \quad \forall (t,x) \notin \mathscr{C}.
\end{aligned} \tag{10}
$$

Let us consider $A(t,\lambda S_t)$, which is the value of a long position in $(\lambda S_t)_{t\geqslant 0}$ and an American Put on the underlying λS with strike K:

$$A(t,\lambda S_t) = \lambda S_t + P^{\mathrm{a}}(t,\lambda S_t).$$

The generalized Itô's formula holds since A is C^1 and its second derivative admits only one discontinuity. Therefore the process $A(t,\lambda S_t)$ follows:

$$
\begin{aligned}
\mathrm{d}A(t,\lambda S_t) &= \lambda \partial_x A(t,\lambda S_t)(\mathrm{d}S_t - rS_t\,\mathrm{d}t) + \mathscr{L}A(t,\lambda S_t)\,\mathrm{d}t \\
&= \lambda \partial_x A(t,\lambda S_t)(\mathrm{d}S_t - rS_t\,\mathrm{d}t) + rA(t,\lambda S_t)\mathbb{1}_{\{(t,\lambda S_t)\in\mathscr{C}\}}\,\mathrm{d}t \\
&= \lambda \partial_x A(t,\lambda S_t)(\mathrm{d}S_t - rS_t\,\mathrm{d}t) + rA(t,\lambda S_t)\mathbb{1}_{\{\lambda S_t > b(t)\}}\,\mathrm{d}t \\
&= rA(t,\lambda S_t)\,\mathrm{d}t + \lambda \partial_x A(t,\lambda S_t)(\mathrm{d}S_t - rS_t\,\mathrm{d}t) - rK\mathbb{1}_{\{\lambda S_t \leqslant b(t)\}}\,\mathrm{d}t.
\end{aligned} \tag{11}
$$

Therefore, from the (SF) condition, the process $(A(t,\lambda S_t),\ t\geqslant 0)$ is the value of a self-financing portfolio *up to the hitting time of the boundary*. If the exercise boundary is reached before maturity, the portfolio generates a continuous dividend rate rK, which must be re-invested for the portfolio to remain self-financing.

3.3. An adapted self-financing strategy

Our next task id to find a continuous and adapted non-negative process $(\lambda_t,\ t\geqslant 0)$ such that the portfolio we denote by $(V_t)_{t\geqslant 0}$:

$$V_t \overset{\mathrm{def}}{=} \lambda_t S_t + P^{\mathrm{a}}(t,\lambda_t S_t) = A(t,\lambda_t S_t)$$

is self-financing. We recall that the self-financing property holds in the continuation region $\mathscr{C}$. Therefore, we choose $(\lambda_t,\ t\geqslant 0)$ such that λ is constant as long as $\lambda_t S_t \in \mathscr{C}$ and such that $\lambda_t S_t \geqslant b(t)$ in order to remain within the continuation region or at the boundary. Hence, the choice of an increasing process[3] for λ leads to

$$\lambda_t = \sup_{u\leqslant t}\left(\lambda_0, \frac{b(u)}{S_u}\right) = \lambda_0 \vee \sup_{u\leqslant t}\left(\frac{b(u)}{S_u}\right), \tag{12}$$

where λ_0 is to be adjusted to satisfy the budget constraint.

Proposition 3.1. *Let $(S_t)_{t\geqslant 0}$ follow the Black–Scholes dynamics (7). The strategy*

$$V_t = \lambda_t S_t + P^{\mathrm{a}}(t,\lambda_t S_t),$$

[3] The choice of λ as an increasing process is justified while dealing with optimality. Intuitively, λ is increasing because outside the continuation region the dividend rate rK can be reinvested in buying more stocks.

where λ_t is given by (12) *and λ_0 is adjusted by the budget constraint:*

$$V_0 = \lambda_0 + P^{\mathrm{a}}(0, \lambda_0) = 1,$$

is self-financing and satisfies $V_t \geqslant K$, $\forall t \leqslant T$.

Remark 3.1. Let us emphasize two features of strategy V_t: (i) when the constraint is active, i.e. $V_t = K$, the strategy V_t gives a zero investment in the tactical allocation S_t; (ii) the terminal value of the strategy is

$$V_T = \lambda_T S_T + (K - \lambda_T S_T)^+ = \sup\left(K, \sup_{u \leqslant T}\left(\lambda_0, \frac{b(u)}{S_u}\right) S_T\right). \tag{13}$$

The terminal value V_T (which can be viewed as a terminal European payoff) therefore has a path dependent lookback feature.

Proof. Since $(\lambda_t)_{t \geqslant 0}$ is a continuous bounded variation process Itô's formula implies

$$\mathrm{d}V_t = [\mathrm{d}A(t, \lambda S_t)]_{\lambda = \lambda_t} + S_t \partial_x A(t, \lambda_t S_t)\,\mathrm{d}\lambda_t.$$

Noting that λ increases only at the boundary and that the smooth pasting principle implies $\partial_x A(t, b(t)) = 0$, we get as in (11)

$$\mathrm{d}V_t = rA(t, \lambda_t S_t)\,\mathrm{d}t + \lambda_t \partial_x A(t, \lambda_t S_t)(\mathrm{d}S_t - rS_t\,\mathrm{d}t) - rK\mathbb{1}_{\{\lambda_t S_t \leqslant b(t)\}}\,\mathrm{d}t$$
$$+ S_t \partial_x A(t, \lambda_t S_t)\mathbb{1}_{\{\lambda_t S_t = b(t)\}}\,\mathrm{d}\lambda_t.$$

We have noted that $\partial_x A(t, \lambda_t S_t) = 0$ on the set $\{\lambda_t S_t = b(t)\}$. Therefore

$$\mathrm{d}V_t = rA(t, \lambda_t S_t)\,\mathrm{d}t + \lambda_t \partial_x A(t, \lambda_t S_t)[\mathrm{d}S_t - rS_t\,\mathrm{d}t] - Kr\mathbb{1}_{\{\lambda_t S_t \leqslant b(t)\}}\,\mathrm{d}t.$$

The set $\{(t, \omega) : \lambda_t S_t \leqslant b(t)\}$ is equal to the set $\{(t, \omega) : S_t = b(t)/\lambda_t\}$ and has a zero $\mathrm{d}P \otimes \mathrm{d}t$ measure, since the process $b(t)/\lambda_t$ has bounded variation. Hence, from (SF), the portfolio $(V_t,\ t \geqslant 0)$ is self-financing. □

The put based strategy described in Proposition 3.1 which we have just shown to be self-financing appears to be a good candidate for optimality. This will be established in Proposition 4.3.

Remark 3.2. Our strategy is similar to that proposed by Gerber and Pafumi (2000). In their seminal paper, the authors propose a protected level given by

$$\lambda_t^G = \sup_{0 \leqslant u \leqslant t}\left(\lambda_0^G, \frac{K}{S_u}\right).$$

They derive a closed formed formula for the price of this guarantee in the Black and Scholes framework. The main difference with our approach is that we use the American boundary instead of the strike level. We also prove our strategy to maximize an expected utility criterion as soon as the unconstrained allocation is chosen in an optimal way.

4. American case for general complete markets

In the current section we first describe the put based self-financing strategies in the general case of complete markets, and secondly we prove optimality for CRRA utility functions when the uninsured allocation solves the maximization problem (3). The result is extended to the case of general utility functions in the last section of this paper.

In order to construct put based self-financing strategies in the general setting of complete markets, we use the Gittins index methodology, and we follow the ideas of El-Karoui and Karatzas (1995). The starting point is that when a family of processes indexed by a parameter are martingales then the processes of the differentiated family with respect to the index parameter are also martingales. The remarkable result is that we obtain the same representation for the put based strategy as in the Markovian case (compare Eqs. (13) and (19)).

4.1. Price of an American put

We now suppose that S is an arbitrary continuous, strictly positive process, which represents the value of a self-financing strategy. We assume, without loss of generality, that $S_0 = 1$. We denote by Q the risk-neutral probability measure.

We introduce $P_t^{\mathrm{a}}(\lambda)$, the American put price on the underlying $(\lambda S_t,\ t \geqslant 0)$ and strike K, defined as

$$P_t^{\mathrm{a}}(\lambda) = \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_{t,T}} \mathrm{E}_Q(R_\tau^t (K - \lambda S_\tau)^+ \mid \mathscr{F}_t),$$

where $R_s^t = R_s / R_t$, R being the discount factor (see definition SF) and $\mathscr{T}_{t,T}$ is the set of stopping times taking values in $(t, T]$. Let us remark that $P_t^{\mathrm{a}}(\lambda)$ is decreasing with respect to λ, $P_t^{\mathrm{a}}(0) = K$, $P_T^{\mathrm{a}}(\lambda) = (K - \lambda S_T)^+$ and that $P_t^{\mathrm{a}}(\lambda) \geqslant (K - \lambda S_t)^+$. We denote by $\sigma(\lambda)$ the associated optimal stopping time

$$\sigma(\lambda) \stackrel{\text{def}}{=} \inf\{u \geqslant 0 : P_u^{\mathrm{a}}(\lambda) = (K - \lambda S_u)^+\}.$$

The map $\lambda \to \sigma(\lambda)$ is non-decreasing and right-continuous. From the value of $P_T^{\mathrm{a}}(\lambda)$, we observe that $\sigma(\lambda) \leqslant T$. We define, for $t < T$, the stochastic critical price b_t by

$$\frac{b_t}{S_t} \stackrel{\text{def}}{=} \sup\{\lambda,\ P_t^{\mathrm{a}}(\lambda) = (K - \lambda S_t)^+\},$$

and we note $\gamma_t \stackrel{\text{def}}{=} b_t / S_t$. We define the so-called Gittins index $(G_t,\ t \geqslant 0)$ as the right-continuous inverse of σ, for $t < T$:

$$\{G_t < \lambda\} = \{\sigma(\lambda) > t\},$$

hence

$$G_t = \sup_{0 \leqslant u < t} \gamma_u \quad \text{for } t < T,$$

and we set

$$G_T^+ = \left(\sup_{0 \leqslant u < T} \gamma_u\right) \vee \frac{K}{S_T}. \tag{14}$$

Let us remark that

$$\{G_T^+ < \lambda\} \subseteq \{\{\sigma(\lambda) = T\} \cap \{K < \lambda S_T\}\} \subseteq \{G_T^+ \leqslant \lambda\}. \tag{15}$$

Proposition 4.1. *The price of the American put is*

$$P_0^{\mathrm{a}}(\lambda) = \mathrm{E}_Q(R_T S_T (G_T^+ - \lambda)^+). \tag{16}$$

Proof. From the envelope theorem[4] the supremum and the differential can be interchanged (see El-Karoui and Karatzas, 1995, for details), hence

$$\frac{\partial P^{\mathrm{a}}}{\partial \lambda}(\lambda) = -\mathrm{E}_Q(R_{\sigma(\lambda)} S_{\sigma(\lambda)} \mathbb{1}_{\{K \geqslant \lambda S_{\sigma(\lambda)}\}}).$$

From the Q-martingale property of RS, we prove that the right-hand side equals

$$-\mathrm{E}_Q(R_T S_T) + \mathrm{E}_Q(R_{\sigma(\lambda)} S_{\sigma(\lambda)} \mathbb{1}_{\{K < \lambda S_{\sigma(\lambda)}\}}).$$

On the set $\{K < \lambda S_{\sigma(\lambda)}\} \cap \{T > \sigma(\lambda)\}$, we would get $P^{\mathrm{a}}_{\sigma(\lambda)}(\lambda) = 0$, which is absurd, therefore $\sigma(\lambda)$ is equal to T on $\{K < \lambda S_{\sigma(\lambda)}\}$ and

$$\mathrm{E}_Q(R_{\sigma(\lambda)} S_{\sigma(\lambda)} \mathbb{1}_{\{K < \lambda S_{\sigma(\lambda)}\}}) = \mathrm{E}_Q(R_T S_T \mathbb{1}_{\{K < \lambda S_T\}} \mathbb{1}_{\{\sigma(\lambda) = T\}}).$$

From (15), we deduce

$$\mathrm{E}_Q(R_T S_T \mathbb{1}_{\{G_T^+ > \lambda\}}) \leqslant -\frac{\partial P^{\mathrm{a}}}{\partial \lambda}(\lambda) \leqslant \mathrm{E}_Q(R_T S_T \mathbb{1}_{\{G_T^+ \geqslant \lambda\}}), \tag{17}$$

and by integrating with respect to λ it follows that the price of the American put can be written as in (16). □

Remark 4.1. The value of the American Put at any time t can be obtained using the same ideas, with the help of

$$\sigma_t(\lambda) \stackrel{\text{def}}{=} \inf\{u \geqslant t : P_u^{\mathrm{a}}(\lambda) = (K - \lambda S_u)^+\},$$

and the Gittins index:

$$G_{t,u} = \sup_{t \leqslant \theta < u} \gamma_\theta, \quad \text{for } u < T, \quad G_{t,T}^+ = \left(\sup_{t \leqslant \theta < T} \gamma_\theta\right) \vee \frac{K}{S_T}.$$

With this notation,

$$P_t^{\mathrm{a}}(\lambda) = \mathrm{E}_Q(R_T^t S_T (G_{t,T}^+ - \lambda)^+ \mid \mathscr{F}_t) = \mathrm{E}_Q(R_T^t (S_T G_{t,T}^+ - \lambda S_T)^+ \mid \mathscr{F}_t). \tag{18}$$

Remark 4.2. As suggested by a referee, Eq. (16) gives a description of the American put as the expectation of a final European path-dependent payoff in a most general framework. Therefore it may be useful in numerical computation of the American put price, by Monte Carlo methods for example. It is noticeable though that the boundary b_t needs to be computed before running simulations, and, as it is well known, this is the most difficult numerical problem to solve.

[4] The envelope theorem states that, if $a^*(\lambda) = \operatorname{argmax} f(a, \lambda)$, then $\partial_\lambda f(a^*(\lambda), \lambda) = \sup \partial_\lambda f(a, \lambda)$.

4.2. Self-financing strategy

Proposition 4.2. *The strategy*

$$V_t = S_t\lambda_t + P^{\mathrm{a}}_t(\lambda_t)$$

is self-financing with terminal value

$$V_T = K \vee S_T\lambda_T = S_T(G^+_T \vee \lambda_0),$$

and satisfies $V_t \geqslant K$, $\forall t \leqslant T$ when choosing λ_t such that

$$\lambda_t = G_t \vee \lambda_0 = \left(\sup_{0\leqslant u\leqslant t} \frac{b_u}{S_u}\right) \vee \lambda_0, \tag{19}$$

where λ_0 is to be adjusted by means of the budget constraint $\lambda_0 + P^{\mathrm{a}}_0(\lambda_0) = 1$. When the constraint is active, i.e. $V_t = K$, we have a zero position in the unconstrained allocation.

Proof. From the $\mathscr{F}_t$-measurability of λ_t and equality (18)

$$V_t = S_t\lambda_t + P^{\mathrm{a}}_t(\lambda_t) = \mathrm{E}_Q(S_t\lambda_t + R^t_T S_T(G^+_{t,T} - \lambda_t)^+ \mid \mathscr{F}_t).$$

Using the martingale property of RS,

$$\begin{aligned} V_t &= \mathrm{E}_Q(R^t_T S_T[\lambda_t + (G^+_{t,T} - \lambda_t)\mathbb{1}_{G^+_{t,T} > \lambda_t}] \mid \mathscr{F}_t) \\ &= \mathrm{E}_Q(R^t_T S_T[G^+_{t,T} \vee \lambda_t] \mid \mathscr{F}_t). \end{aligned} \tag{20}$$

From

$$\sup_{0\leqslant u<t} (\gamma_u) \vee \sup_{t\leqslant u<T} (\gamma_u) = \sup_{0\leqslant u<T} (\gamma_u),$$

we obtain $G^+_{t,T} \vee \lambda_t = G^+_{0,T} \vee \lambda_0$. Therefore

$$V_t = \mathrm{E}_Q(R^t_T S_T[G^+_{0,T} \vee \lambda_0] \mid \mathscr{F}_t),$$

and the process RV is a Q-martingale; hence V_t is the value of a self-financing strategy. In particular,

$$V_T = S_T[G^+_{0,T} \vee \lambda_0] = S_T\lambda_T \vee K.$$

From the definition of $G^+_{0,T} = \sup_{t\in[0,T[} (b_t/S_t) \vee K/S_T$, we recover the lookback feature noticed in Eq. (13). $\square$

At the boundary, i.e. when $P^{\mathrm{a}}_t(\lambda_t) = (K - \lambda_t S_t)$, we obtain $V_t = K$. Moreover, the Gittins index is increasing only at the boundary. Therefore, as in the Black and Scholes framework, the process λ is increasing with support included in the set $\{V_t = K\}$.

4.3. Optimality for CRRA utility functions

Let u be a CRRA utility function, and S the optimal strategy for the unconstrained problem (3) with initial wealth 1 as defined in Section 3.1.

In this subsection we prove that the process $(V_t,\ t \geqslant 0)$ defined in Proposition 4.2 solves the maximization problem with American constraint:

$$\max \mathrm{E}[u(V_T)]; \quad \text{under the constraints } V_t \geqslant K,\ \forall t \in [0,T] \quad \text{and} \quad V_0 = 1. \tag{21}$$

In order to give a precise proof, let us introduce the state price process $(H_t,\ t \geqslant 0)$, such that for all self-financing portfolios with value $(X_t,\ t \geqslant 0)$, the budget constraint can be written:

$$X_0 = \mathrm{E}[H_T X_T]. \tag{22}$$

We recall that H_t is the product of the discount factor $\exp(-\int_0^t r_s\,\mathrm{d}s)$ and the Radon–Nikodym density of the equivalent risk-neutral martingale measure Q with respect to the subjective probability P. We also know that the Market Numeraire $M_t = H_t^{-1}$ is a portfolio, namely M_t is the optimal portfolio to hold for an unconstrained log-utility agent, as deduced from Eq. (23) (see for example Long, 1990, or Bajeux-Besnainou and Portait, 1998, for more details).

From optimization theory (see Karatzas and Shreve, 1998), we know that the solution for the unconstrained problem (3) with initial wealth λ, which we denote by $\hat{X}_T^\lambda$, satisfies the marginal utility condition

$$u'(\hat{X}_T^\lambda) = (yM_T)^{-1}, \tag{23}$$

where y is a parameter (the inverse of the Lagrange multiplier) to be adjusted as a function of the initial wealth λ by means of the budget constraint $\mathrm{E}(H_T \hat{X}_T^\lambda) = \lambda$.

In the case of CRRA utility functions, it is well known that $\hat{X}^z = z\hat{X}^1$.

Proposition 4.3. *Let u be a CRRA utility function and $S = \hat{X}^1$ be the optimal strategy for the unconstrained problem (3) with initial wealth equal to* 1 *and λ_t the gearing parameter described in Proposition* 4.2. *The self-financing strategy:*

$$\hat{V}_t = \lambda_t S_t + P_t^{\mathrm{a}}(\lambda_t)$$

is the optimal strategy for the problem with American guarantee (21).

Proof. Let $(V_t,\ t \geqslant 0)$ be any self-financing portfolio such that $V_t \geqslant K,\ \forall t$. From the concavity of u:

$$u(V_T) - u(\hat{V}_T) \leqslant u'(\hat{V}_T)(V_T - \hat{V}_T).$$

The same arguments as in the European case lead to

$$u'(\hat{V}_T)(V_T - \hat{V}_T) = u'(S_T\lambda_T)(V_T - \hat{V}_T) - [u'(S_T\lambda_T) - u'(\hat{V}_T)]^+(V_T - K).$$

Then using the property of CRRA functions that: $u'(xy) = u'(x)u'(y)$, and that $u'(S_T) = vH_T$, we obtain

$$u'(S_T\lambda_T)(V_T - \hat{V}_T) = vH_T u'(\lambda_T)(V_T - \hat{V}_T).$$

An integration by parts formula, and the fact that the process $(u'(\lambda_t),\ t \geqslant 0)$ is a decreasing process provide:

$$\mathrm{E}(H_T u'(\lambda_T)(V_T - \hat{V}_T)) = \mathrm{E}\left(\int_0^T u'(\lambda_s)\,\mathrm{d}(H_s(V_s - \hat{V}_s)) + \int_0^T H_s(V_s - \hat{V}_s)\,\mathrm{d}u'(\lambda_s)\right). \tag{24}$$

From the martingale property of $H\hat{V}$ and HV and the upper bound $u'(\lambda_s) \leqslant u'(\lambda_0)$, the first term in the right-hand side of (24) is equal to 0. The process $(u'(\lambda_t),\ t \geqslant 0)$ is decreasing with support $\{\omega, t : \hat{V}_t(\omega) = K\}$, therefore

$$\mathrm{E}\left(\int_0^T H_s(K - \hat{V}_s)\,\mathrm{d}u'(\lambda_s)\right) = 0.$$

It follows that

$$\mathrm{E}\left(\int_0^T H_s(V_s - \hat{V}_s)\,\mathrm{d}u'(\lambda_s)\right) = \mathrm{E}\left(\int_0^T H_s(V_s - K)\,\mathrm{d}u'(\lambda_s)\right).$$

The process λ is increasing, hence $\mathrm{E}(\int_0^T H_s(V_s - K)\,\mathrm{d}u'(\lambda_s))$ is non-positive. Putting all the inequalities together, we establish that

$$\mathrm{E}(u(\hat{V}_T)) \geqslant \mathrm{E}(u(V_T)). \qquad \square$$

5. Optimality results for general utility functions

5.1. European guarantee

For general utility functions, the linear property of the solution to problem (3) fails to be true: the optimal solution depends on the value of the initial wealth. The concept of one *unit* of unconstrained allocation is no longer helpful, nor the concept of one unit of fund. We therefore introduce a new parameter $x > 0$ being the initial fund value. Consequently we now consider the maximization problem

$$\max \mathrm{E}[u(V_T)]; \quad \text{under the constraints } V_T \geqslant Kx, \text{ and } \quad V_0 = x, \tag{25}$$

and the unconstrained problem

$$\max \mathrm{E}[u(X_T^{\lambda x})]; \quad \text{under the budget constraint } X_0^{\lambda x} = \lambda x. \tag{26}$$

We prove in this section that, even though the tactical allocation cannot be defined independently, a separation principle similar to Proposition 3.1 still applies: the optimal policy consists in investing an initial amount λx in an optimal unconstrained portfolio, and protecting the fund by buying a put with strike Kx on that position. Let us remark that λ still represents the fraction of initial capital invested in risky assets at date 0.

Recall that the solution of the unconstrained problem is (see Eq. (23)) $u'(\hat{X}_T^{\lambda x}) = (yM_T)^{-1}$. As we show in our proofs, it is better to parameterize S with y rather than λ and to refer to the unconstrained allocation as

$$S_T(y) = (u')^{-1}(H_T/y). \tag{27}$$

The *modified unconstrained allocation* is given by

$$S_t(y) = \mathrm{E}_Q(R_T^t[S_T(y)] \mid \mathscr{F}_t) = \mathrm{E}_Q(R_T^t(u')^{-1}[H_T/y] \mid \mathscr{F}_t), \tag{28}$$

where Q is the risk-neutral probability and y is adjusted by means of the budget constraint:

$$\mathrm{E}_Q(R_T S_T(y)) + P_0^e(S(y)) = x,$$

which is not linear with respect to y. Here $P^e(S(y))$ is the price of the European put with strike Kx written on the underlying $(S_t(y),\ t \geqslant 0)$. Therefore the first order condition (4) takes the form

$$\mathrm{E}[H_T(X_T - S_T(y))] = 0, \tag{29}$$

for any X_T, terminal value of a self-financing portfolio such that $X_0 = S_0(y)$. Let us now consider the put based strategy described in Section 2.4 associated $S_T(y)$ and terminal wealth

$$\hat{V}_T^x = \max(S_T(y), Kx).$$

Proposition 5.1. *The put based strategy written on the optimal portfolio with no constraint solves the optimization problem with a European constraint for any utility function.*

Proof. The proof is similar to the CRRA case. Indeed, for any V_T terminal value of an admissible strategy with initial wealth x satisfying the constraint $V_T > Kx$, we get from the concavity of u and the definition of $S_T(y)$:

$$u(V_T) - u(\hat{V}_T^x) \leqslant u'(\hat{V}_T^x)(V_T - \hat{V}_T^x) = [y^{-1}H_T \wedge u'(Kx)](V_T - \hat{V}_T^x).$$

The right-hand side of this last equation is equal to

$$y^{-1}H_T(V_T - \hat{V}_T^x) - [u'(S_T(y)) - u'(Kx)]^+(V_T - Kx).$$

As before, from the first order condition (29):

$$\mathrm{E}[H_T(V_T - \hat{V}_T^x)] = 0,$$

and using the terminal constraint on V_T, we deduce

$$\mathrm{E}[u(V_T) - u(\hat{V}_T^x)] = -\mathrm{E}([u'(S_T(y)) - u'(Kx)]^+(V_T - Kx)) \leqslant 0. \qquad \square$$

5.2. American guarantee

We now deal with the maximization problem with an American constraint. Again we follow closely the method exposed in the CRRA case. The main difference resides in the choice of the parametrization: we now refer to the parametrized unconstrained allocation (see Eq. (28)) of the form

$$S_t(y) = \mathrm{E}_Q(R_T^t(u')^{-1}[H_T/y] \mid \mathscr{F}_t).$$

Let us remark that $S_t(y)$ is increasing with respect to y.

The process $(R_tS_t(y),\ t \geqslant 0)$ being a martingale, the process $(R_t\partial_yS_t(y),\ t \geqslant 0)$ is also a martingale. The price of an American put on $(S_t(y),\ t \geqslant 0)$ with strike $k = Kx$ is $P^{\mathrm{a}}(y) = \sup_\tau \mathrm{E}_Q(R_\tau(k - S_\tau(y))^+)$ and is decreasing with respect to y. As before, we have also $P_u^{\mathrm{a}}(y) \geqslant (k - S_u(y))^+$ and $P_u^{\mathrm{a}}(0) = k$.

Let $\sigma(y)$ be the optimal stopping time

$$\sigma(y) = \inf\{u, P_u^{\mathrm{a}}(y) = (k - S_u(y))^+\},$$

and note that σ is increasing with respect to y and that $\sigma(y) \leqslant T$. Then

$$P_0^{\mathrm{a}}(y) = \mathrm{E}_Q(R_{\sigma(y)}(k - S_{\sigma(y)}(y))^+).$$

For $t < T$ let us note $\gamma_t = \sup\{y : P_t(y) = (k - S_t(y))^+\}$ and G the right continuous inverse of $\sigma(y)$ such that

$$\{\sigma(y) > t\} = \{G_t < y\}.$$

Note that we obtain as before: $G_t = \sup_{u<t} \gamma_u$.

Proposition 5.2. *The value of the American put is*

$$P^{\mathrm{a}}(y) = \mathrm{E}_Q[R_T(S_T(G_T^+) - S_T(y))^+],$$

where G_T^+ is defined in the proof below by (30).

Proof. The derivative with respect to y of the American put price is

$$\begin{aligned}\frac{\partial P^{\mathrm{a}}(y)}{\partial y} &= -\mathrm{E}_Q\left(R_{\sigma(y)}\frac{\partial S_{\sigma(y)}(y)}{\partial y}\mathbb{1}_{\{k > S_{\sigma(y)}(y)\}}\right)\\ &= -\mathrm{E}_Q\left(R_T\,\frac{\partial S_T(y)}{\partial y}\right) + \mathrm{E}_Q\left(R_{\sigma(y)}\frac{\partial S_{\sigma(y)}(y)}{\partial y}\,\mathbb{1}_{\{S_{\sigma(y)} > k\}}\right).\end{aligned}$$

We remark that on the set $\{S_{\sigma(y)} > k\}$ the stopping time $\sigma(y)$ equals T, therefore

$$\frac{\partial P^{\mathrm{a}}(y)}{\partial y} = -E\left(R_T\frac{\partial S_T(y)}{\partial y}\right) + E\left(R_T\,\frac{\partial S_T(y)}{\partial y}\mathbb{1}_{\{k < S_T(y)\}}\mathbb{1}_{\{\sigma(y) = T\}}\right).$$

Setting

$$G_T^+ = \sup_{t<T} \gamma_t \vee \kappa(k), \tag{30}$$

where $\kappa(k)$ is defined via the increasing property of $S_T(\cdot)$ as

$$\kappa(k) = \sup_y \{k \geqslant S_T(y)\},$$

(i.e., $y < \kappa(k)$ if and only if $k \geqslant S_T(y)$), we get

$$\begin{aligned}-\mathrm{E}\left(R_T\,\frac{\partial S_T(y)}{\partial y}\right) + \mathrm{E}\left(R_T\,\frac{\partial S_T(y)}{\partial y}\mathbb{1}_{\{G_T^+ < y\}}\right) &\leqslant \frac{\partial P^{\mathrm{a}}(y)}{\partial y}\\ &\leqslant -\mathrm{E}\left(R_T\,\frac{\partial S_T(y)}{\partial y}\right) + \mathrm{E}\left(R_T\,\frac{\partial S_T(y)}{\partial y}\,\mathbb{1}_{\{G_T^+ \leqslant y\}}\right).\end{aligned}$$

Therefore, by integration with respect to y:

$$P^{\mathrm{a}}(y) = \mathrm{E}_Q[R_T(S_T(G_T^+) - S_T(y))^+].$$

Starting at time t, and working with $G_{t,u} = \sup_{t \leqslant \theta < u} G_\theta$ and $G_{t,T}^+ = \sup_{u<T} G_{t,u} \vee \kappa(k)$ leads to

$$P_t^{\mathrm{a}} = \mathrm{E}_Q(R_T^t(S_T(G_{t,T}^+) - S_T(G_t))^+ \mid \mathscr{F}_t). \qquad \square$$

Let us now define $\widetilde{G}_t = G_t \vee y_0$, where y_0 is to be adjusted by means of the budget constraint:

$$x = \mathrm{E}_Q(S_T(y_0)) + P_0^{\mathrm{a}}(y_0).$$

Proposition 5.3. (i) *The strategy*

$$V_t = S_t(\widetilde{G}_t) + P_t^{\mathrm{a}}(\widetilde{G}_t)$$

is self-financing, with terminal value $V_T = S_T(\widetilde{G}_T) \vee k$.

(ii) *The latter strategy, based on the optimal strategy for the unconstrained problem, is optimal for the constrained problem.*

Proof. We follow the proof of Proposition 3.1. Using Proposition 5.2, and the fact that, for any t, the process $(R_u^t S_u(\widetilde{G}_t),\ t \leqslant u)$ is a Q-martingale, we can write, as in (20):

$$V_t = \mathrm{E}_Q(S_t(\widetilde{G}_t) + R_T^t[S_T(G_{t,T}^+) - S_T(\widetilde{G}_t)]^+) \mid \mathscr{F}_t) = \mathrm{E}(R_T^t[S_T(G_{t,T}^+) \vee S_T(\widetilde{G}_t)] \mid \mathscr{F}_t).$$

On the set $S_T(G_{t,T}^+) > S_T(\widetilde{G}_t)$, the increasing property of $S_T(\cdot)$ implies that $G_{t,T}^+ \geqslant \widetilde{G}_t$, hence $G_{t,T}^+ = G_T^+ \vee y_0$. On the complementary set $S_T(G_{t,T}^+) \leqslant S_T(\widetilde{G}_t)$, the equality $G_{t,T}^+ = G_T^+ \vee y_0$ still holds. Finally,

$$V_t = \mathrm{E}(R_T^t S_T(G_T^+ \vee y_0) \mid \mathscr{F}_t),$$

and result (i) follows.

The proof of optimality is the same as in Proposition 4.3. The only change to make is to replace $u'(S_T \lambda_T)$ by

$$u'(S_T(\widetilde{G}_T)) = H_T / \widetilde{G}_T,$$

and to work with the decreasing process $(\widetilde{G}_t)^{-1}$ rather than $u'(\lambda_t)$. This process decreases only at the boundary, i.e. when $V_t = k$, and we are done. □

6. Conclusion

The present paper solves the problem of optimizing dynamic portfolio management when a constraint is imposed on the liquidation value. We first considered the simplest situation where the utility maximization problem is linear with respect to the initial wealth and the unconstrained allocation is driven by Black–Scholes dynamics. We then moved from this simple case to more complex settings, first extending our results to the general framework of complete markets, and then considering general utility functions to be maximized. Both European and American constraints are studied. In particular, we developed in the American case a path dependent self financing strategy based on American puts, which we showed to be optimal under fairly general assumptions.

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References

Bajeux-Besnainou, I., Portait, R., 1998. Dynamic asset allocation in a mean-variance framework. Management Science 44 (11), S79–S95.

Black, F., Jones, R., 1987. Simplifying portfolio insurance. Journal of Portfolio Management 14, 48–51.

Brennan, M.J., Schwartz, E.S., Lagnado, R., 1997. Tactical asset allocation. Journal of Economics Dynamics and Control 21, 1377–1403.

Cox, J.C., Huang, C.-F., 1989. Optimal consumption and portfolio policies when asset prices follow a diffusion process. Journal of Economic Theory 49, 33–83.

Cvitanić, J., Karatzas, I., 1996. On portfolio optimization under drawdown constraints. In: Davis, M.H., et al. (Eds.), Mathematical Finance, IMA Volumes in Mathematics and its Applications, Vol. 65. Springer, New York, pp. 35–45.

Cvitanić, J., Karatzas, I., 1999. On dynamic measures of risk. Finance and Stocks 3, 451–482.

Dupire, B., 1994. Pricing with a smile. Risk 7 (1), 18–20.

El-Karoui, N., Jeanblanc-Picqué, M., 1998. Optimization of consumption with labor income. Finance and Stocks 2, 409–440.

El-Karoui, N., Karatzas, I., 1995. The optimal stopping problem for a general American put-option. in: Davis, M.H., et al. (Eds.), Mathematical Finance. IMA Volumes in Mathematics and its Applications, Vol. 65, pp. 63–74.

Gerber, H., Pafumi, G., 2000. Pricing dynamic investment fund protection. North American Actuarial Journal 4, 28–41.

Grossman, S.J., Zhou, Z., 1993. Optimal investment strategies for controlling drawdowns. Mathematical Finance 3, 241–276.

Imai, J., Boyle, P.E., 2001. Dynamic fund protection. North American Actuarial Journal 5 (3), 31–51.

Karatzas, I., Shreve, S., 1998. Methods of Mathematical Finance. Springer, Berlin.

Leland, H.E., Rubinstein, M., 1976. The evolution of portfolio insurance. In: Luskin, D.L. (Ed.), Portfolio Insurance: A Guide to Dynamic Hedging. Wiley, New York.

Long, J.B., 1990. The numeraire portfolio. Journal of Financial Economics 26, 29–69.

Markowitz, H.M., 1959. Portfolio Selection. Yale University Press, New Haven, CT.

Merton, R., 1971. Optimum consumption and portfolio rules in a continuous time model. Journal of Economic Theory 3, 373–413.

Merton, R., 1973. An intertemporal capital asset pricing model. Econometrica 41, 867–888.

Musiela, M., Rutkowski, M., 1998. Martingale Methods in Financial Modelling. Springer, Berlin.

Perold, A., Sharpe, W., 1988. Dynamic strategies for asset allocation. Financial Analysts Journal 51 (January-February), 16–27.