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Pricing and Hedging of Credit Risk: Replication and Mean-Variance Approaches

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Abstract. The paper presents some methods and results related to the valuation and hedging of defaultable claims (credit-risk sensitive derivative instruments). Both the exact replication of attainable defaultable claims and the mean-variance hedging of non-attainable defaultable claims are examined. For the sake of simplicity, the general methods are then applied to simple cases of defaultable equity derivatives, rather than to the more complicated examples of real-life credit derivatives.

Introduction

The goal of this paper is to present some methods and results related to the valuation and hedging of credit derivatives (defaultable claims). For the most part in this paper we adhere to the so-called *reduced-form approach* to modelling of credit risk. Therefore, in the next section we briefly describe one of the main relevant concepts – the *hazard process* of a random time (for more details, we refer to Elliott et al. (2000) or Jeanblanc and Rutkowski (2002)).

In Section 2, we formulate the basic set-up of the paper. In particular, we provide a model for the primary (default-free) market underlying our valuation and hedging results.

Section 3 deals with valuation and hedging issues in the situation when there are liquid instruments available for trading that are sensitive to the same risks as the claims that we want to price and hedge. In other words, this section deals with the situation when the perfect hedging (i.e., the exact replication) is possible. In contrast to some other related works, in which this issue was addressed by invoking a suitable version of the martingale representation theorem (see, for instance, Bélanger et al. (2001) or Blanchet-Scalliet and Jeanblanc (2000)), we shall directly analyse the possibility of replication of a given defaultable contingent claim by means of a trading strategy based on default-free and defaultable securities. We believe that such an approach, motivated by the working paper by Vaillant (2001),

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is much more intuitive and it leads directly to explicit expressions for replicating strategies. Moreover, the issue of the judicious choice of tradeable (i.e., liquid) securities is emphasized in a natural way.

Finally, in Section 4 we present an approach to pricing default risk that can not be perfectly hedged. Our approach here is rooted in the classical Markowitz mean-variance portfolio optimization methodology.

1. Hazard process of a default time

Let τ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbf{Q}^*)$, referred to as the *default time.* We introduce the jump process $H_t = \mathbb{1}_{\{\tau \le t\}}$ and we denote by $\mathbb{H}$ the filtration generated by this process. We now assume that some reference filtration $\mathbb{F}$ is given. We set $\mathbb{G} := \mathbb{F} \vee \mathbb{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for $t \in \mathbb{R}_+$. $\mathbb{G}$ is referred to as to the *full filtration*: it includes the observations of default event. Of course, τ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time (but not necessarily an $\mathbb{F}$-stopping time). The concept of the *hazard process* of a random time τ is closely related to the conditional distribution function F_t of τ, defined as

$$F_t = \mathbf{Q}^*(\tau \le t \,|\, \mathcal{F}_t), \qquad \forall\, t \in \mathbb{R}_+.$$

We also set $G_t = 1 - F_t = \mathbf{Q}^*(\tau > t \,|\, \mathcal{F}_t)$, and we postulate that $G_0 = 1$ and $G_t > 0$ for $t \in \mathbb{R}_+$. Hence, we exclude the case where τ is an $\mathbb{F}$-stopping time. The process $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula

$$\Gamma_t = -\ln(1 - F_t) = -\ln G_t$$

is called the *hazard process* of a random time τ with respect to the reference filtration $\mathbb{F}$, or briefly, the $\mathbb{F}$-*hazard process.*

Explicit construction of a default time. We shall now briefly describe the most commonly used construction of a default time associated with a given hazard process Γ. It should be stressed that the random time obtained through this particular method – which will be called the *canonical construction* in what follows – has certain specific features that are not necessarily shared by all random times with a given $\mathbb{F}$-hazard process Γ. We start by assuming that we are given an $\mathbb{F}$-adapted, right-continuous, increasing process Γ defined on a filtered probability space $(\tilde{\Omega}, \mathbb{F}, \mathbf{P}^*)$. As usual, we postulate that $\Gamma_0 = 0$ and $\Gamma_\infty = +\infty$. In many instances, the process Γ is given by the equality

$$\Gamma_t = \int_0^t \gamma_u \, du,$$

for some non-negative, $\mathbb{F}$-progressively measurable *intensity process* γ.

To construct a random time τ such that Γ is the $\mathbb{F}$-hazard process of τ, we need to enlarge the underlying probability space $\tilde{\Omega}$. This also means that Γ is not the $\mathbb{F}$-hazard process of τ under $\mathbf{P}^*$, but rather the $\mathbb{F}$-hazard process of τ under a suitable extension $\mathbf{Q}^*$ of the probability measure $\mathbf{P}^*$. Let ξ be a random variable defined on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbf{Q}})$, uniformly distributed on the interval $[0,1]$ under $\hat{\mathbf{Q}}$. We consider the product space $\Omega = \tilde{\Omega} \times \hat{\Omega}$, endowed with the product σ-field $\mathcal{G} = \mathcal{F}_\infty \otimes \hat{\mathcal{F}}$ and the product probability measure $\mathbf{Q}^* = \mathbf{P}^* \otimes \hat{\mathbf{Q}}$. The latter equality means that for arbitrary events $A \in \mathcal{F}_\infty$ and $B \in \hat{\mathcal{F}}$ we have $\mathbf{Q}^*(A \times B) = \mathbf{P}^*(A)\hat{\mathbf{Q}}(B)$. For the sake of simplicity, we denote by $\mathbb{F}$ the natural extension of the original filtration $\mathbb{F}$ to the product space.

We define the random time $\tau : \Omega \to \mathbb{R}_+$ by setting

$$\tau = \inf \{ t \in \mathbb{R}_+ : e^{-\Gamma_t} \leq \xi \} = \inf \{ t \in \mathbb{R}_+ : \Gamma_t \geq \eta \}$$

where the random variable $\eta = -\ln \xi$ has a unit exponential law under $\mathbf{Q}^*$ and is independent of $\mathbb{F}$. It is not difficult to find $F_t = \mathbf{Q}^*(\tau \leq t \,|\, \mathcal{F}_t)$. Indeed, since clearly $\{\tau > t\} = \{\xi < e^{-\Gamma_t}\}$ and the random variable Γ_t is $\mathcal{F}_\infty$-measurable, we obtain

$$\mathbf{Q}^*(\tau > t \,|\, \mathcal{F}_\infty) = \mathbf{Q}^*(\xi < e^{-\Gamma_t} \,|\, \mathcal{F}_\infty) = \hat{\mathbf{Q}}(\xi < e^{-x})_{x=\Gamma_t} = e^{-\Gamma_t}.$$

Consequently, we have

$$1 - F_t = \mathbf{Q}^*(\tau > t \,|\, \mathcal{F}_t) = \mathbf{E}_{\mathbf{Q}^*}\big(\mathbf{Q}^*\{\tau > t \,|\, \mathcal{F}_\infty\} \,|\, \mathcal{F}_t\big) = e^{-\Gamma_t},$$

and so F is an $\mathbb{F}$-adapted, right-continuous, increasing process. It is also clear that the process Γ represents the $\mathbb{F}$-hazard process of τ under $\mathbf{Q}^*$. We shall assume from now on that the default time τ is constructed through the canonical construction.

2. The underlying primary market model

In this section we set the stage for further developments. For the sake of simplicity, we focus in the present paper on simple cases defaultable equity derivatives, rather than on more sophisticated credit derivatives encountered in the market practice (cf. Lando (1998), Schönbucher (1998), Greenfield (2000), Jamshidian (2002), or Jeanblanc and Rutkowski (2003a)).

We shall examine the possibility of replication of defaultable claims within various models of defaultable markets. We argue that the choice of a particular setting (in particular, of default-free and defaultable tradeable assets) is an essential step in model building. Let us stress that from the practical perspective the valuation and hedging of credit derivatives should always be done with respect to liquid credit-risk-sensitive financial instruments of a similar nature (that is, of a similar exposure with respect to relevant risk factors) if possible. This kind of situation is studied in Section 3.

However, the rationale formulated in the previous paragraph may not be implementable, if, for example, there are no liquid instruments available for trading that would be sensitive to the same credit/default risk factors as the claim that we want to price and hedge. In such cases, the perfect replication will typically be impossible, and one needs to decide on some other pricing and hedging criteria. We present a possible approach in Section 4.

Default-free market. Consider an economy in continuous time, with the time parameter $t \in \mathbb{R}_+$. We are given a filtered probability space $(\Omega, \mathbb{F}, \mathbf{P}^*)$ endowed with a one-dimensional standard Brownian motion W^*. It is convenient to assume that $\mathbb{F}$ is the $\mathbf{P}^*$-augmented and right-continuous version of the natural filtration generated by W^*. As we shall see in what follows, the filtration $\mathbb{F}$ will also play the role of the *reference filtration* for the default intensity. It is important to notice that all martingales with respect to a Brownian filtration $\mathbb{F}$ are necessarily continuous processes; this well-known property will be of frequent use in what follows.

In the first step, we shall introduce an arbitrage-free market model for default-free securities. Notice that all price processes introduced in this subsection are $\mathbb{F}$-adapted and continuous. In the *default-free market* we have the following primary tradeable assets:

- a *money market account* B satisfying
$$dB_t = r_t B_t \, dt, \qquad B_0 = 1,$$
where r is an $\mathbb{F}$-adapted stochastic process, or, equivalently,
$$B_t = \exp\left(\int_0^t r_u \, du\right),$$
- a *default-free discount bond* (also known at the *Treasury zero-coupon bond*) with the price process
$$B(t,T) = B_t \, \mathbf{E}_{\mathbf{P}^*}(B_T^{-1} \,|\, \mathcal{F}_t), \qquad \forall \, t \in [0,T],$$
where T is the bond's maturity date,
- a *risky asset* whose price dynamics under $\mathbf{P}^*$ are

$$dS_t = S_t \left(r_t \, dt + \sigma_t \, dW_t^*\right), \qquad S_0 > 0, \tag{2.1}$$

for some $\mathbb{F}$-progressively measurable volatility process σ.

For the sake of expositional simplicity, we shall make an assumption that our model of default-free market is complete. The probability $\mathbf{P}^*$ is thus the unique martingale measure for the default-free market model. Let us finally notice that we may equally well assume that the Wiener process W^* is d-dimensional and $S = (S^1, \ldots, S^d)$ is the vector of cash prices of d risky assets. It is well known that the completeness of such a market model is essentially equivalent to the non-degeneracy of the volatility matrix σ_t.

3. Replication of defaultable claims

In this section, we analyze the valuation and replication of defaultable claims within the reduced-form set-up. We present here only the basic results under several simplifying assumptions. For more general results, the interested reader is referred to Jeanblanc and Rutkowski (2003b).

A generic defaultable claim (Y, Z, τ) consists of:

- the *promised contingent claim* Y, representing the payoff received by the owner of the claim at time T, if there was no default prior to or at time T,
- the *recovery process* Z, representing the recovery payoff at time of default, if default occurs prior to or at time T,
- the *default time* τ specifying, in particular, the default event $\{\tau \le T\}$.

It is convenient to introduce the *dividend process* D representing all cash flows associated with a defaultable claim (Y, Z, τ). The process D is given by the formula
$$D_t = Y \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T,\infty[}(t) + \int_{]0,t]} Z_u \, dH_u$$
where both integrals are (stochastic) Stieltjes integrals.

Definition 3.1. The *ex-dividend price process* U of a defaultable claim of the form (Y, Z, τ) which settles at time T is given as
$$U_t = B_t \, \mathbf{E}_{\mathbf{Q}^*}\left(\int_{]t,T]} B_u^{-1} \, dD_u \,\Big|\, \mathcal{G}_t\right)$$
where $\mathbf{Q}^*$ is the *spot martingale measure* and the process B represents the savings account.

Observe that $U_t = U_t(Y) + U_t(Z)$ where the meaning of $U_t(Y)$ and $U_t(Z)$ is clear. Recall also that the filtration $\mathbb{G}$ models the full information, that is, the default-free market and the default event.

Let us assume that we are given the $\mathbb{F}$-adapted, right-continuous, increasing process Γ on $(\Omega, \mathbb{F}, \mathbf{P}^*)$. The default time τ and the probability measure $\mathbf{Q}^*$ are assumed to be constructed as in Section 1. The probability $\mathbf{Q}^*$ will play the role of the *martingale probability* for the defaultable market. It is essential to observe that

- the Wiener process W^* is also a Wiener process with respect to $\mathbb{G}$ under the probability measure $\mathbf{Q}^*$ (recall that the default time τ is constructed through the canonical construction),
- we have $\mathbf{Q}^*{}_{|\mathcal{F}_t} = \mathbf{P}^*{}_{|\mathcal{F}_t}$ for every $t \in [0, T]$.

3.1. Risk-neutral valuation of defaultable claims. We shall now present the basic valuation formulae for defaultable claims within the reduced-form approach.

3.1.1. *Terminal payoff.* The valuation of the terminal payoff is based on the following well-known result (see, for instance, Corollary 5.1.1 in Bielecki and Rutkowski (2002)).

LEMMA 3.2. *For any $\mathcal{F}_T$-measurable, $\mathbf{Q}^*$-integrable, random variable Y and any $t \le T$ we have*

$$\mathbf{E}_{\mathbf{Q}^*}(\mathbb{1}_{\{\tau > T\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \, \mathbf{E}_{\mathbf{Q}^*}(e^{\Gamma_t - \Gamma_T} Y \,|\, \mathcal{F}_t).$$

Let Y be an $\mathcal{F}_T$-measurable random variable representing the promised payoff at maturity date T. We consider a defaultable claim of the form $X = \mathbb{1}_{\{\tau > T\}} Y$ with zero recovery in case of default (i.e., we set $Z = 0$). Using the definition of the ex-dividend price of a defaultable claim, we get the following *risk-neutral valuation formula*

$$U_t(Y) = B_t \, \mathbf{E}_{\mathbf{Q}^*}(B_T^{-1} X \,|\, \mathcal{G}_t)$$

which holds for any $t < T$. The next result is an immediate consequence of Lemma 3.2.

PROPOSITION 3.1. *The ex-dividend price of the promised payoff Y satisfies for every $t < T$*

$$U_t(Y) = B_t \, \mathbf{E}_{\mathbf{Q}^*}(B_T^{-1} Y \mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \tilde{U}_t(Y) \tag{3.1}$$

where

$$\tilde{U}_t(Y) = B_t \, \mathbf{E}_{\mathbf{Q}^*}(B_T^{-1} e^{\Gamma_t - \Gamma_T} Y \,|\, \mathcal{F}_t) = \hat{B}_t \, \mathbf{E}_{\mathbf{Q}^*}(\hat{B}_T^{-1} Y \,|\, \mathcal{F}_t)$$

and the credit-risk-adjusted savings account $\hat{B}$ satisfies $\hat{B}_t = B_t e^{\Gamma_t}$ for every $t \in \mathbb{R}_+$.

The process $\tilde{U}(Y)$ represents the *pre-default value* of Y at time t. Notice that the process $\tilde{U}_t(Y)/\hat{B}_t$ is a continuous $\mathbb{F}$-martingale (thus $\tilde{U}(Y)$ is a continuous $\mathbb{F}$-semimartingale).

REMARK 3.3. The valuation formula (3.1), as well as the concept of the pre-default value, should be supported by replication arguments. To this end, we need first to construct a suitable model of a defaultable market with prespecified liquid instruments. In fact, if we wish to use formula (3.1) for explicit calculations, we need to know the joint law of all random variables involved, and this appears to be a non-trivial issue, in general.

3.1.2. *Recovery payoff.* The following result appears to be useful in the valuation of the recovery payoff Z_τ which occurs at time τ. The process $\tilde{U}(Z)$ introduced below represents the pre-default value of the recovery payoff.

PROPOSITION 3.2. *Let Γ be a continuous process and let Z be an $\mathbb{F}$-predictable, bounded process. Then for any $t \le T$ we have*

$$\begin{aligned} U_t(Z) &= B_t \, \mathbf{E}_{\mathbf{Q}^*}(B_\tau^{-1} Z_\tau 1\!\!1_{\{t<\tau\le T\}} \,|\, \mathcal{G}_t) \\ &= 1\!\!1_{\{\tau>t\}} B_t \, \mathbf{E}_{\mathbf{Q}^*}\Big(\int_t^T Z_u B_u^{-1} e^{\Gamma_t - \Gamma_u} \, d\Gamma_u \,\Big|\, \mathcal{F}_t \Big) \\ &= 1\!\!1_{\{\tau>t\}} \hat{B}_t \, \mathbf{E}_{\mathbf{Q}^*}\Big(\int_t^T Z_u \hat{B}_u^{-1} d\Gamma_u \,\Big|\, \mathcal{F}_t \Big) = 1\!\!1_{\{\tau>t\}} \tilde{U}_t(Z). \end{aligned}$$

PROOF. For the proof of Proposition 3.2 we refer, for instance, to Bielecki and Rutkowski (2002) (see Propositions 5.1.1 and 8.2.1 therein). □

3.2. Defaultable term structure. For a *defaultable discount bond* with zero recovery[1] it is natural to adopt the following definition of the price process

$$D^0(t,T) = B_t \, \mathbf{E}_{\mathbf{Q}^*}(B_T^{-1} 1\!\!1_{\{\tau>T\}} \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \tilde{D}^0(t,T)$$

where the auxiliary process $\tilde{D}^0(t,T)$ represents the *pre-default value* of the bond. By virtue of Proposition 3.1, it is given by the following equality

$$\tilde{D}^0(t,T) = \hat{B}_t \, \mathbf{E}_{\mathbf{Q}^*}(\hat{B}_T^{-1} \,|\, \mathcal{F}_t) \tag{3.2}$$

where, as before, $\hat{B}_t = B_t e^{\Gamma_t}$. Since the process $\tilde{D}^0(t,T)/\hat{B}_t$ is a continuous, strictly positive, $\mathbb{F}$-martingale, it is clear that the pre-default value $\tilde{D}^0(t,T)$ of a defaultable discount bond is a continuous, strictly positive, $\mathbb{F}$-semimartingale. In the very special case when r and γ are constants, we get the following simple representation:

$$\tilde{D}^0(t,T) = e^{-(r+\gamma)(T-t)} = e^{-\gamma(T-t)} B(t,T).$$

REMARK 3.4. The probability measure $\mathbf{Q}^*$ introduced above is an essential input in the specification of defaultable term structure. It is essential to stress that we deal here with the modelling of bond prices $D^0(t,T)$, rather than with the arbitrage valuation of contingent claims. In this sense, the probability measure $\mathbf{Q}^*$ is "given by the liquid market of corporate bonds", rather than derived using some formal mathematical arguments.

3.3. Self-financing trading strategies: default-free case. Our goal in this section is to present some auxiliary results related to the concept of a self-financing trading strategy for a market model involving default-free and defaultable securities.

For the sake of the reader's convenience, we shall first discuss briefly the classic concepts of self-financing cash and futures strategies in the context of default-free market model. It will soon appear that in case of defaultable securities only minor adjustments of definitions and results are needed (see, Blanchet-Scalliet and Jeanblanc (2000) or Vaillant (2001)).

[1] A defaultable (corporate) discount bond is also known as a *corporate zero-coupon bond*. Note that the superscript 0 refers to the postulated zero recovery scheme.

3.3.1. *Cash strategies.* Let Z_t^1 and Z_t^2 be the cash prices at time $t \in [0,T]$ of two default-free tradeable assets, where Z^1 and Z^2 are continuous semimartingales. We assume, in addition, that the process Z^1 is strictly positive. We denote by V_t the wealth of the cash strategy (ϕ^1, ϕ^2) at time $t \in [0,T]$, so that

$$V_t = \phi_t^1 Z_t^1 + \phi_t^2 Z_t^2.$$

We say that the cash strategy (ϕ^1, ϕ^2) is *self-financing* if

$$dV_t = \phi_t^1\, dZ_t^1 + \phi_t^2\, dZ_t^2.$$

This yields

$$dV_t = (V_t - \phi_t^2 Z_t^2)(Z_t^1)^{-1}\, dZ_t^1 + \phi_t^2\, dZ_t^2.$$

Let us introduce the relative values: $\tilde{V}_t = V_t/Z_t^1$ and $\tilde{Z}_t^2 = Z_t^2/Z_t^1$. Using Itô's lemma, we get

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \phi_u^2\, d\tilde{Z}_u^2.$$

A similar result holds for any finite number of assets. Let $Z_t^1, Z_t^2, \dots, Z_t^k$ be cash prices at time t of k assets. We postulate that Z^1, $Z^2, \dots, Z^k$ are continuous semimartingales, and the process Z^1 is strictly positive. Then the wealth process equals

$$V_t = \phi_t^1 Z_t^1 + \phi_t^2 Z_t^2 + \cdots + \phi_t^k Z_t^k$$

and the strategy $(\phi^1, \phi^2, \dots, \phi^k)$ is said to be self-financing if

$$dV_t = \phi_t^1\, dZ_t^1 + \phi_t^2\, dZ_t^2 + \cdots + \phi_t^k\, dZ_t^k.$$

By combining the last two formulae, we obtain

$$dV_t = \Big(V_t - \sum_{i=2}^k \phi_t^i Z_t^i\Big)(Z_t^1)^{-1}\, dZ_t^1 + \sum_{i=2}^k \phi_t^i\, dZ_t^i.$$

Choosing Z^1 as a numeraire and denoting $\tilde{V}_t = V_t/Z_t^1$, $\tilde{Z}_t^i = Z_t^i/Z_t^1$, we get the standard result.

LEMMA 3.5. *We have for any $t \in [0,T]$*

$$\tilde{V}_t = \tilde{V}_0 + \sum_{i=2}^k \int_0^t \phi_u^i\, d\tilde{Z}_u^i.$$

3.3.2. *Futures strategies.* Now let Z_t^1 and Z_t^2 represent the cash and futures prices at time $t \in [0,T]$ of some default-free assets, respectively. As before, we assume that Z^1 and Z^2 are continuous semimartingales. Moreover, Z^1 is assumed to be a strictly positive process. In view of specific features of futures contracts, it is natural to postulate that the wealth V satisfies[2]

$$V_t = \phi_t^1 Z_t^1 + \phi_t^2 0 = \phi_t^1 Z_t^1.$$

The futures strategy (ϕ^1, ϕ^2) is self-financing if

$$dV_t = \phi_t^1\, dZ_t^1 + \phi_t^2\, dZ_t^2. \tag{3.3}$$

We thus have

$$dV_t = V_t (Z_t^1)^{-1}\, dZ_t^1 + \phi_t^2\, dZ_t^2.$$

[2]Let us recall that the futures price Z_t^2 (that is, the quotation at time t of a futures contract) has different practical features than the cash price of an asset. We make here the standard assumption that it is possible to enter a futures contract at no initial cost.

Lemma 3.6. *The process* $\tilde{V}_t = V_t/Z^1_t$ *satisfies for* $t \in [0,T]$

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \hat{\phi}^2_u \, d\hat{Z}^2_u$$

where $\hat{\phi}^2_t = \phi^2_t e^{\alpha_t}/Z^1_t$, $\hat{Z}^2_t = Z^2_t e^{-\alpha_t}$ *and*

$$\alpha_t = \langle \ln Z^1, \ln Z^2 \rangle_t = \int_0^t (Z^1_u)^{-1}(Z^2_u)^{-1} \, d\langle Z^1, Z^2\rangle_u.$$

Proof. The Itô formula, combined with (3.3), yield

$$\begin{aligned} d\tilde{V}_t &= (Z^1_t)^{-1} dV_t + V_t \, d(Z^1_t)^{-1} + d\langle (Z^1)^{-1}, V\rangle_t \\ &= \phi^1_t (Z^1_t)^{-1} \, dZ^1_t + \phi^2_t (Z^1_t)^{-1} \, dZ^2_t + \phi^1_t Z^1_t \, d(Z^1_t)^{-1} \\ &\quad -\phi^1_t (Z^1_t)^{-2} \, d\langle Z^1, Z^1\rangle_t - \phi^2_t (Z^1_t)^{-2} \, d\langle Z^1, Z^2\rangle_t \\ &= \phi^2_t (Z^1_t)^{-1} dZ^2_t - \phi^2_t (Z^1_t)^{-2} \, d\langle Z^1, Z^2\rangle_t \\ &= \phi^2_t e^{\alpha_t} (Z^1_t)^{-1} \big(e^{-\alpha_t} dZ^2_t - Z^2_t e^{-\alpha_t} d\alpha_t \big) \end{aligned}$$

and the result follows. □

3.3.3. *Special cash strategies.* Assume now that three default-free assets are traded on the market and let (ϕ^1, ϕ^2, ϕ^3) be a self-financing trading strategy. The processes Z^1, Z^2 and Z^3 are assumed to be continuous semimartingales and we postulate that Z^1 and Z^3 are positive processes. We shall first consider two particular cases.

Zero net investment in Z^2 *and* Z^3*.* Assume first that at any time there is zero net investment in Z^2 and Z^3 so that

$$\phi^2_t Z^2_t + \phi^3_t Z^3_t = 0$$

or, equivalently, $\phi^3_t = -\phi^2_t Z^2_t / Z^3_t$. Then, from $V_t = \phi^1_t Z^1_t$ and

$$dV_t = \phi^1_t \, dZ^1_t + \phi^2_t \, dZ^2_t + \phi^3_t \, dZ^3_t$$

we get

$$dV_t = V_t (Z^1_t)^{-1} \, dZ^1_t + \phi^2_t \left(dZ^2_t - Z^2_t (Z^3_t)^{-1} \, dZ^3_t \right). \tag{3.4}$$

Let us denote $\bar{Z}^1_t = Z^1_t / Z^3_t$, $\bar{Z}^2_t = Z^2_t / Z^3_t$. The following result extends Lemma 3.6.

Proposition 3.3. *Assume that* (ϕ^1, ϕ^2, ϕ^3) *is a self-financing strategy such that* $\phi^2_t Z^2_t + \phi^3_t Z^3_t = 0$ *for every* $t \in [0,T]$*. The process* $\tilde{V}_t = V_t/Z^1_t$ *satisfies for* $t \in [0,T]$

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \hat{\phi}^2_u \, d\hat{Z}^2_u$$

where

$$\hat{\phi}^2_t = \phi^2_t e^{\bar{\alpha}_t}/\bar{Z}^1_t, \ \hat{Z}^2_t = \bar{Z}^2_t e^{-\bar{\alpha}_t}, \ \bar{\alpha}_t = \langle \ln \bar{Z}^1, \ln \bar{Z}^2\rangle_t.$$

Proof. It suffices to consider the relative values of all considered processes, with the price Z^3 being chosen as a numeraire. Then equation (3.4) becomes

$$d\bar{V}_t = \bar{V}_t (\bar{Z}^1_t)^{-1} \, d\bar{Z}^1_t + \phi^2_t d\bar{Z}^2_t$$

where $\bar{V}_t = V_t / Z^3_t$. To conclude, it suffices to apply Lemma 3.6, and to note that $\tilde{V}_t = V_t / Z^1_t = \bar{V}_t / \bar{Z}^1_t$. □

REMARK 3.7. Suppose that $\bar{\sigma}^1$ and $\bar{\sigma}^2$ are the volatilities of $\bar{Z}^1$ and $\bar{Z}^2$ respectively, so that

$$d\bar{Z}^i_t = \bar{Z}^i_t\big(\bar{\mu}^i_t\,dt + \bar{\sigma}^i_t\,dW^*_t\big)$$

for $i = 1, 2$. Then clearly

$$\bar{\alpha}_t = \int_0^t \bar{\sigma}^1_u \cdot \bar{\sigma}^2_u\,du$$

where $\cdot$ denotes the inner product. In a typical application, the volatilities $\bar{\sigma}^1$ and $\bar{\sigma}^2$ (and thus also $\bar{\alpha}_t$) will be deterministic functions of the time parameter.

Non-zero net investment in Z^2 and Z^3. Assume now that

$$\phi^2_t Z^2_t + \phi^3_t Z^3_t = Z_t$$

for a given continuous, $\mathbb{F}$-adapted process Z. Proceeding as in the proof of Lemma 3.6, it is easy to establish the following result.

PROPOSITION 3.4. *Assume that (ϕ^1, ϕ^2, ϕ^3) is a self-financing strategy such that $\phi^2_t Z^2_t + \phi^3_t Z^3_t = Z_t$ for every $t \in [0,T]$. Then the relative wealth process $\tilde{V}_t = V_t/Z^1_t$ satisfies for $t \in [0,T]$*

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \hat{\phi}^2_u\,d\hat{Z}^2_u + \int_0^t \bar{Z}_u\,d(\bar{Z}^1_u)^{-1}$$

where $\bar{Z}_t = Z_t/Z^3_t$ and

$$\hat{\phi}^2_t = \phi^2_t e^{\bar{\alpha}_t}/\bar{Z}^1_t,\ \hat{Z}^2_t = \bar{Z}^2_t e^{-\bar{\alpha}_t},\ \bar{\alpha}_t = \langle \ln \bar{Z}^1, \ln \bar{Z}^2\rangle_t.$$

3.4. Self-financing trading strategies with defaultable assets. We shall first examine basic properties of general financial models involving default-free and defaultable securities. At this stage, our goal is to derive fundamental relationships. Subsequently, we shall be more specific about the nature of these securities, and we shall furnish closed-form solutions for specific defaultable claims.

3.4.1. *Case A. Single defaultable tradeable asset and two default-free assets.* For the sake of simplicity, we assume the zero recovery scheme for the defaultable tradeable (i.e., liquid) asset with the price process Z^1.

Zero recovery. First, we assume zero recovery for the defaultable contingent claim (i.e., we set $Z = 0$). Thus, at time τ the wealth process of any strategy that replicates X should necessarily jump to zero. The process Z^1 vanishes at time of default (and thus also after this date). Nevertheless, it can be used as a numeraire prior to τ. Indeed, we have

$$Z^1_t = \mathbb{1}_{\{\tau > t\}}\tilde{Z}^1_t$$

for some $\mathbb{F}$-adapted process $\tilde{Z}^1$. We assume that $\tilde{Z}^1$ is a strictly positive, continuous, $\mathbb{F}$-semimartingale (clearly $\tilde{Z}^1 = \tilde{U}$ for some defaultable claim which settles at T). On the other hand, it is obvious that the price process Z^1 jumps from $\tilde{Z}^1_{\tau-}$ to 0 at default time τ.

REMARK 3.8. Continuous $\mathbb{F}$-semimartingales Z^2 and Z^3 are assumed to model cash prices of liquid default-free securities. We postulate that Z^3 is a strictly positive process. It is convenient to assume, in addition, that the processes Z^2 and Z^3 are stopped at τ. Since we are going to deal with defaultable claims that are subject to the zero recovery scheme, it will be sufficient to examine replicating strategies on the random interval $[\![0, \tau \wedge T]\!]$. For this reason, we shall postulate throughout that the processes ϕ^1, ϕ^2 and ϕ^3 are $\mathbb{F}$-predictable, rather than $\mathbb{G}$-predictable. In fact, it can be formally shown that for any $\mathbb{G}$-predictable process ϕ there exists a unique $\mathbb{F}$-predictable process ψ such that $\mathbb{1}_{\{\tau \geq t\}}\phi_t = \mathbb{1}_{\{\tau \geq t\}}\psi_t$ for every $t \in \mathbb{R}_+$.

We consider a self-financing cash strategy (ϕ^1, ϕ^2, ϕ^3) such that at any time $t \in [0,T]$ there is zero net investment in default-free assets Z^2 and Z^3, namely,

$$\phi_t^2 Z_t^2 + \phi_t^3 Z_t^3 = 0. \tag{3.5}$$

We thus have $\phi_t^3 = -\phi_t^2 Z_t^2 / Z_t^3$. Moreover $V_t = \phi_t^1 Z_t^1$ and (cf. (3.4))

$$dV_t = V_{t-}(\tilde{Z}_t^1)^{-1}\, dZ_t^1 + \phi_t^2 \left(dZ_t^2 - Z_t^2 (Z_t^3)^{-1}\, dZ_t^3 \right).$$

Let us denote $\bar{Z}_t^1 = \tilde{Z}_t^1 / Z_t^3$, $\bar{Z}_t^2 = Z_t^2 / Z_t^3$. The next result is a direct counterpart of Proposition 3.3.

Proposition 3.5. *Assume that $\phi_t^2 Z_t^2 + \phi_t^3 Z_t^3 = 0$ for every $t \in [0,T]$. Then the wealth process satisfies for $t \in [0,T]$*

$$V_t = Z_t^1 \Big(\tilde{V}_0 + \int_0^t \hat{\phi}_u^2\, d\hat{Z}_u^2 \Big)$$

where $\tilde{V}_0 = V_0 / \tilde{Z}_0^1 = V_0 / Z_0^1$ and

$$\hat{\phi}_t^2 = \phi_t^2 e^{\bar{\alpha}_t} / \bar{Z}_t^1,\ \hat{Z}_t^2 = \bar{Z}_t^2 e^{-\bar{\alpha}_t},\ \bar{\alpha}_t = \langle \ln \bar{Z}^1, \ln \bar{Z}^2 \rangle_t.$$

Non-zero recovery. Assume now that for every $t \in [0,T]$

$$\phi_t^2 Z_t^2 + \phi_t^3 Z_t^3 = Z_t$$

for a given continuous, $\mathbb{F}$-adapted process Z. Thus prior to default time we have $V_t = \phi_t^1 \tilde{Z}_t^1 + Z_t$ and

$$dV_t = \phi_t^1 d\tilde{Z}_t^1 + \phi_t^2 dZ_t^2 + \phi_t^3 dZ_t^3.$$

Notice that at default time τ we have $V_\tau = Z_\tau$ on the set $\{\tau \leq T\}$. Using Proposition 3.4 we immediately obtain the following result.

Proposition 3.6. *Assume that $\phi_t^2 Z_t^2 + \phi_t^3 Z_t^3 = Z_t$ for every $t \in [0,T]$. Then the process V satisfies on $\{\tau > t\}$*

$$V_t = \tilde{Z}_t^1 \Big(\tilde{V}_0 + \int_0^t \hat{\phi}_u^2\, d\hat{Z}_u^2 + \int_0^t \bar{Z}_u\, d(\bar{Z}_u^1)^{-1} \Big)$$

where $\bar{Z}_t = Z_t / Z_t^3$ and, as before,

$$\hat{\phi}_t^2 = \phi_t^2 e^{\bar{\alpha}_t} / \bar{Z}_t^1,\ \hat{Z}_t^2 = \bar{Z}_t^2 e^{-\bar{\alpha}_t},\ \bar{\alpha}_t = \langle \ln \bar{Z}^1, \ln \bar{Z}^2 \rangle_t.$$

3.4.2. *Case B. Two defaultable tradeable assets.* Assume that Z^1 and Z^2 are defaultable tradeable assets with zero recovery, and a common default time τ. Then $Z_t^1 = \mathbb{1}_{\{\tau > t\}} \tilde{Z}_t^1$, $Z_t^2 = \mathbb{1}_{\{\tau > t\}} \tilde{Z}_t^2$ for some processes $\tilde{Z}^1, \tilde{Z}^2$, that are assumed to be strictly positive, continuous, $\mathbb{F}$-semimartingales. In this case, we postulate zero recovery for a defaultable claim so that we shall consider trading strategies (ϕ^1, ϕ^2) for which

$$V_t = \phi_t^1 Z_t^1 + \phi_t^2 Z_t^2 = 0$$

on the set $\{\tau \leq t\}$, that is, after default. We say that a strategy (ϕ^1, ϕ^2) is self-financing provided that

$$dV_t = \phi_{t-}^1\, dZ_t^1 + \phi_{t-}^2\, dZ_t^2.$$

Simple considerations show that the wealth process V satisfies

$$dV_t = (V_{t-} - \phi^2_{t-} Z^2_{t-})(Z^1_{t-})^{-1}\, dZ^1_t + \phi^1_{t-}\, dZ^2_t$$

or, equivalently,

$$dV_t = (V_{t-} - \phi^2_{t-} \tilde{Z}^2_t)(\tilde{Z}^1_t)^{-1}\, dZ^1_t + \phi^2_{t-}\, dZ^2_t.$$

Proposition 3.7. *The wealth process V satisfies for $t \in [0,T]$*

$$V_t = Z^1_t\Big(\tilde{V}_0 + \int_0^t \phi^2_u\, dZ^{2*}_u\Big)$$

where $\tilde{V}_0 = V_0/\tilde{Z}^1_0 = V_0/Z^1_0$ and $Z^{2*}_t = \tilde{Z}^2_t/\tilde{Z}^1_t$.

Proof. It suffices to note that, setting $\tilde{V}_t = \phi^1_t \tilde{Z}^1_t + \phi^2_t \tilde{Z}^2_t$, we have $d\tilde{V}_t = \phi^1_t\, d\tilde{Z}^1_t + \phi^2_t\, d\tilde{Z}^2_t$. □

3.4.3. *Case C. Single defaultable tradeable asset and a single default-free asset.* Let us finally consider the case of two tradeable assets, with prices $Z^1_t = 1\!\!1_{\{\tau>t\}}\tilde{Z}^1_t$ and Z^2_t, where $\tilde{Z}^1, Z^2$ are strictly positive, continuous, $\mathbb{F}$-semimartingales. We now have

$$V_t = \phi^1_t Z^1_t + \phi^2_t Z^2_t = \phi^1_t 1\!\!1_{\{\tau>t\}}\tilde{Z}^1_t + \phi^2_t Z^2_t$$

and

$$dV_t = (V_{t-} - \phi^2_t Z^2_t)(\tilde{Z}^1_t)^{-1} dZ^1_t + \phi^2_t\, dZ^2_t.$$

It is clear that equality $V_t = 0$ on $\{\tau \le t\}$ implies that $\phi^2_t = 0$ for every $t \in [0,T]$. Therefore, $dV_t = V_{t-}(\tilde{Z}^1_t)^{-1} dZ^1_t$ and the possibility of replication of a defaultable claim with zero-recovery is unlikely within this setup (except for some trivial cases).

3.5. Replicating strategies for defaultable claims. Our goal is to examine the possibility of the exact replication of a generic defaultable claim. By a *replicating strategy* we mean here a self-financing trading strategy with the wealth process which coincides with the pre-default value of the claim at any time prior to default (and prior to the maturity date), and with the claims payoff at the time of default or at the claim's maturity, whichever comes first. Suppose that V stands for the wealth process of this strategy. We require that $V = \tilde{U}$ on the stochastic interval $[\![0, \tau \wedge T[\![$ and: $V_\tau = Z_\tau$ on $\{\tau \le T\}$, $V_T = Y$ on $\{\tau > T\}$.

3.5.1. *Case A. Single defaultable tradeable asset and two default-free assets.*

Replication of the promised payoff. We shall first examine the possibility of an exact replication of a defaultable contingent claim with zero recovery. To this end, we shall make use of Proposition 3.5 (notice that we replace Z^1 with $\tilde{Z}^1$, however).

Corollary 3.1. *Suppose that there exists a process $\hat{\phi}^2$ such that*

$$\tilde{Z}^1_T\Big(\tilde{U}_0(Y) + \int_0^T \hat{\phi}^2_t\, d\hat{Z}^2_t\Big) = Y$$

where $\hat{Z}^2$ is defined in Proposition 3.5. Then the trading strategy (ϕ^1, ϕ^2, ϕ^3) given by

$$\phi^1_t = \tilde{U}_t(Y)/\tilde{Z}^1_t, \qquad \phi^2_t = \hat{\phi}^2_t \bar{Z}^1_t e^{-\bar{\alpha}_t}, \qquad \phi^3_t = -\phi^2_t Z^2_t / Z^3_t,$$

replicates the defaultable claim $(Y, 0, \tau)$.

PROOF. We shall make use of Proposition 3.5. Suppose that the process $\hat{\phi}^2$ exists. Then we define

$$V_t = \tilde{Z}^1_t \Big(\tilde{U}_0(Y) + \int_0^t \hat{\phi}^2_u \, d\hat{Z}^2_u \Big)$$

and we set

$$\phi^1_t = V_t/\tilde{Z}^1_t, \quad \phi^2_t = \hat{\phi}^2_t \bar{Z}^1_t e^{-\bar{\alpha}_t}, \quad \phi^3_t = -\phi^2_t Z^2_t / Z^3_t.$$

In view of Proposition 3.5 the strategy (ϕ^1, ϕ^2, ϕ^3) is self-financing on $[\![0, \tau \wedge T[\![$. At the default time τ the wealth necessarily jumps to zero (recall that Z^1 is subject to zero recovery), and at the maturity of the claim it equals Y on the set $\{\tau > T\}$. □

As usual, we say that a defaultable claim is *attainable* if it admits at least one replicating strategy.

COROLLARY 3.2. *Suppose that a defaultable claim represented by the random variable X is attainable. Let $\hat{\mathbf{Q}}$ be a probability measure such that $\hat{Z}^2$ is an $\mathbb{F}$-martingale under $\hat{\mathbf{Q}}$. Then the value at time 0 of the promised payoff (i.e., of the claim $(Y, 0, \tau)$) equals*

$$U_0(Y) = Z^1_0 \, \mathbf{E}_{\hat{\mathbf{Q}}}(Y/\tilde{Z}^1_T).$$

It is useful to notice that within the present setup the replication of all defaultable claims with zero recovery is possible, provided that the underlying default-free market is complete (this assumption was done in Section 2).

Replication of the recovery payoff. Let us now focus on the recovery payoff Z at time of default. Let $\tilde{U}_t(Z)$ be the pre-default value of this payoff. In order to examine the replicating strategy, we shall make use of Proposition 3.6 (in particular, we assume now that $\phi^2_t Z^2_t + \phi^3_t Z^3_t = Z_t$ for every $t \in [0, T]$).

COROLLARY 3.3. *Suppose that there exists a process $\hat{\phi}^2$ such that*

$$Z^1_T \Big(\tilde{V}_0 + \int_0^T \hat{\phi}^2_u \, d\hat{Z}^2_u + \int_0^T \bar{Z}_u \, d(\bar{Z}^1_u)^{-1} \Big) = 0.$$

Then the replicating strategy for the recovery payoff Z (i.e., for the defaultable claim $(0, Z, \tau)$) equals

$$\phi^1_t = \frac{\tilde{U}_t(Z) - Z_t}{\tilde{Z}^1_t}, \quad \phi^2_t = e^{-\bar{\alpha}_t} \bar{Z}^1_t \hat{\phi}^2_t, \quad \phi^3_t = \frac{Z_t - \phi^2_t Z^2_t}{Z^3_t},$$

where the pre-default value at time t of the recovery payoff equals

$$\tilde{U}_t(Z) = V_t = \tilde{Z}^1_t \Big(\tilde{V}_0 + \int_0^t \hat{\phi}^2_u \, d\hat{Z}^2_u + \int_0^t \bar{Z}_u \, d(\bar{Z}^1_u)^{-1} \Big).$$

PROOF. It suffices to observe that the terminal value of the recovery process is null, and to apply Proposition 3.6. □

COROLLARY 3.4. *Suppose there exists a probability measure $\hat{\mathbf{Q}}$ such that $\hat{Z}^2$ is an $\mathbb{F}$-martingale under $\hat{\mathbf{Q}}$. Then*

$$V_t = \tilde{Z}^1_t \mathbf{E}_{\hat{\mathbf{Q}}} \Big(\int_t^T \bar{Z}_u \, d(\bar{Z}^1_u)^{-1} \, \Big| \, \mathcal{F}_t \Big).$$

REMARK 3.9. The case of a defaultable asset Z^1 with non-zero recovery can be dealt with in a similar way. Let us notice that the existence of a closed-form expression for the process $\hat{\phi}^2$ depends on additional assumptions.

3.5.2. *Case B. Two defaultable tradeable assets.* We shall now make use of Proposition 3.7 and we shall focus on the replication of the promised payoff. Let us consider two defaultable tradeable assets Z^1 and Z^2 with zero recovery, and an associated trading strategy (ϕ^1, ϕ^2). Clearly, it replicates a defaultable claim $X = \mathbb{1}_{\{\tau > T\}} Y$ whenever the following equality holds

$$\tilde{Z}^1_T \Big(\tilde{V}_0 + \int_0^T \phi^2_t \, dZ^{2*}_t \Big) = Y.$$

Corollary 3.5. *Suppose that a defaultable claim represented by X is attainable. Let $\tilde{\mathbf{Q}}$ be a probability measure such that Z^{2*} is an $\mathbb{F}$-martingale under $\tilde{\mathbf{Q}}$. Then the value at time* 0 *of the claim* $(Y, 0, \tau)$ *equals*

$$U_0(Y) = Z^1_0 \, \mathbf{E}_{\tilde{\mathbf{Q}}}(Y / \tilde{Z}^1_T).$$

From the viewpoint of market completeness, the situation in different than in the previous case. Indeed, a defaultable claim X is attainable if and only if the associated promised payoff Y can be replicated with the use of pre-default value processes $\tilde{Z}^1$ and $\tilde{Z}^2$. In addition, even if a default-free asset is introduced, a replicating strategy for an arbitrary defaultable claim will always involve a null position in this asset. Therefore, the introduction of a tradeable default-free asset is not relevant if we restrict our attention to defaultable claims.

3.6. Equity derivatives. We postulate that the dynamics of the stock price S are given by (2.1). As the first example, we shall examine how to value and hedge a *vulnerable* European call option with the terminal payoff

$$\hat{C}_T = \mathbb{1}_{\{\tau > T\}} (S_T - K)^+.$$

Notice that $\hat{C}_T = \tilde{C}_T$, where we denote

$$\tilde{C}_T = \mathbb{1}_{\{\tau > T\}} (S_T \mathbb{1}_{\{\tau > T\}} - K)^+ = (S_T \mathbb{1}_{\{\tau > T\}} - K)^+.$$

Hence, the considered contract can also be seen as either a vulnerable or a non-vulnerable option on a defaultable stock. We argue that the financial interpretation of a particular real-life derivative contract is of great importance here. To support this point, we shall consider several possible models, with different choices of tradeable assets that are used for hedging purposes, and we shall show that both the claim's price and its hedging strategy depends on the model's choice.

3.6.1. *Case A. Single defaultable tradeable asset and two default-free assets.* We first consider the case of a vulnerable option written on a non-defaultable stock. Specifically, the stock price process is assumed to be tradeable and default-free. In addition, we postulate that the default-free and defaultable discount bonds, maturing at time T, are also tradeable.

Valuation. To value a vulnerable call option, it suffices to apply Corollary 3.2. Let us denote by $F^S_t = S_t / B(t,T)$ the forward price of the stock, and let us define $\Gamma(t,T) = \tilde{D}^0(t,T)/B(t,T)$. The next result gives the basic properties of the process $\Gamma(t,T)$ (for a fixed T).

Lemma 3.10. *For a fixed $T > 0$, the process $\Gamma(t,T)$, $t \in [0,T]$, is a continuous $\mathbb{F}$-submartingale. It is a process of finite variation* (*in fact, an increasing function*) *if and only if the hazard process Γ is deterministic. Indeed, in this case we have $\Gamma(t,T) = e^{\Gamma_t - \Gamma_T}$.*

PROOF. Recall that $\hat{B}_t = B_t e^{\Gamma_t}$ and notice that

$$\Gamma(t,T) = \frac{\tilde{D}^0(t,T)}{B(t,T)} = \frac{\hat{B}_t \, \mathbf{E}_{\mathbf{Q}^*}(\hat{B}_T^{-1} \,|\, \mathcal{F}_t)}{B_t \, \mathbf{E}_{\mathbf{Q}^*}(B_T^{-1} \,|\, \mathcal{F}_t)} = \mathbf{E}_{\mathbf{Q}_T}(e^{\Gamma_t - \Gamma_T} \,|\, \mathcal{F}_t) = e^{\Gamma_t} M_t$$

where $M_t = \mathbf{E}_{\mathbf{Q}_T}(e^{-\Gamma_T} \,|\, \mathcal{F}_t)$ and $\mathbf{Q}_T$ stands for the so-called forward martingale measure, given on $(\Omega, \mathcal{G}_T)$ (as well as on $(\Omega, \mathcal{F}_T)$) through the formula

$$\frac{d\mathbf{Q}_T}{d\mathbf{Q}^*} = \frac{1}{B_T B(0,T)}, \qquad \mathbf{Q}^* - \text{a.s.}$$

We conclude that $\Gamma(t,T)$ is the product of a strictly positive, increasing, right-continuous, $\mathbb{F}$-adapted process e^{Γ_t} and a strictly positive, continuous[3] $\mathbb{F}$-martingale M. It is well known that a continuous martingale is never of finite variation, unless it is a constant process. □

REMARK 3.11. It is easy to check that

$$\mathbf{Q}_T(t < \tau \leq T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \mathbf{E}_{\mathbf{Q}_T}(e^{\Gamma_t - \Gamma_T} \,|\, \mathcal{F}_t)$$

so that Γ is also the hazard process of τ with respect to $\mathbb{F}$ under $\mathbf{Q}_T$. Notice that Lemma 3.10 is valid no matter whether interest rates are random or deterministic.

Corollary 3.2 yields

$$\hat{C}_0 = D^0(0,T) \, \mathbf{E}_{\hat{\mathbf{Q}}} Y = \Gamma(0,T) B(0,T) \, \mathbf{E}_{\hat{\mathbf{Q}}} Y$$

where $\hat{\mathbf{Q}}$ is the martingale measure for the process $\hat{S}$ given by $\hat{S}_t = F_t^S e^{-\bar{\alpha}_t}$, where $\bar{\alpha}_t = \langle \ln \Gamma(t,T), \ln F^S \rangle_t$. If $\Gamma(t,T)$ is increasing, we have $\hat{S}_t = F_t^S$ so that $\hat{S}$ is simply the forward price of the stock. If the interest rate r is deterministic then we have (cf. (2.1))

$$d\hat{S}_t = \hat{S}_t \sigma \, dW_t^*, \qquad \hat{S}_0 = S_0 / B(0,T).$$

The price $\hat{C}_0$ thus equals $\Gamma(0,T) C_0$, where C_0 denotes the Black-Scholes price of a (non-vulnerable) European call. This result can be easily generalized to the case of random interest rates (e.g., within the Gaussian HJM framework).

Hedging. Let us now examine hedging of a vulnerable option. In general, the replicating strategy for X satisfies on the set $\{\tau > t\}$

$$\phi_t^1 \tilde{D}^0(t,T) + \phi_t^2 S_t + \phi_t^3 B(t,T) = \tilde{U}_t(Y)$$

and

$$\phi_t^1 \, d\tilde{D}^0(t,T) + \phi_t^2 \, dS_t + \phi_t^3 \, dB(t,T) = d\tilde{U}_t(Y).$$

To hedge perfectly the *jump risk* we need to take

$$\phi_t^1 = \tilde{U}_t(Y) / \tilde{D}^0(t,T).$$

Consequently, we necessarily have $\phi_t^2 S_t + \phi_t^3 B(t,T) = 0$.

REMARK 3.12. The remaining risk, referred to as the *spread risk* (or, the *volatility risk*) is hedged by matching the diffusion terms (recall that we postulated the completeness of the default-free market model). It is thus clear that the component ϕ^1 is chosen to perfectly hedge the jump risk, and the components ϕ^2, ϕ^3 are tailored to hedge the spread risk. Notice also that the trading strategy introduced

[3] Recall that the filtration $\mathbb{F}$ is generated by a process W^*, which is a Wiener process with respect to $\mathbf{P}^*$, $\mathbf{Q}^*$ as well as with respect to the forward martingale measure $\mathbf{Q}_T$. All martingales with respect to a Brownian filtration are known to be continuous processes.

above replicates the defaultable claim not only prior to the default time, but also after τ.

Formally, we shall consider a self-financing cash strategy (ϕ^1, ϕ^2, ϕ^3) such that at any date there is zero net investment in stock and default-free bond, so that

$$\phi_t^2 S_t + \phi_t^3 B(t,T) = 0, \qquad \forall\, t \in [0,T].$$

The following result is a straightforward consequence of Proposition 3.5. In the remaining part of Section 3.6.1, it is assumed that the default intensity is deterministic.

Corollary 3.6. *Assume that the default intensity γ is deterministic. Then the wealth process V satisfies for $t \in [0,T]$*

$$V_t = D^0(t,T)\Big(\frac{V_0}{D^0(0,T)} + \int_0^t \hat{\phi}_u^2 \, d\hat{S}_u\Big)$$

where $\hat{\phi}_t^2 = \phi_t^2/\Gamma(t,T)$ and $\hat{S}_t = F_t^S$.

Proof. Since $\Gamma(t,T)$ is of finite variation, we have $\bar{\alpha}_t = 0$, $\hat{\phi}_t^2 = \phi_t^2/\Gamma(t,T)$ and $\hat{S}_t = F_t^S$. □

Consider the defaultable claim $\hat{C}_T = 1\!\!1_{\{\tau > T\}}(S_T - K)^+$. In view of Corollary 3.6, we need to find a constant c and a process $\hat{\phi}^2$ such that

$$c + \int_0^T \hat{\phi}_t^2 \, d\hat{S}_t = c + \int_0^T \hat{\phi}_t^2 \, dF_t^S = (S_T - K)^+.$$

It is clear that $\hat{\phi}^2$ coincides with the Black-Scholes replicating strategy and $V_0/D^0(0,T) = C_0/B(0,T)$, where C_0 is the Black-Scholes price of a European call option. Thus the price at time 0 of $\hat{C}_T$ equals $\Gamma(0,T)C_0$.

Corollary 3.7. *We have $\hat{\phi}^2 = \psi$, where ψ is the Black-Scholes hedge ratio for the call option. The pre-default value at time t of $\hat{C}_T$ satisfies $V_t(X) = \Gamma(t,T)C_t$.*

The component ϕ^1 of the replicating strategy for the vulnerable call option satisfies on the set $\{\tau > t\}$

$$\phi_t^1 = V_t(X)/\tilde{D}(t,T) = C_t/B(t,T).$$

Moreover $\phi_t^2 = \psi_t \Gamma(t,T)$ and $\phi_t^3 = -\psi_t \Gamma(t,T) F_t^S$.

Example 3.13. Assume that r and γ are constant. Then

$$\phi_t^1 = C_t e^{r(T-t)}, \ \phi_t^2 = \psi_t e^{-\gamma(T-t)}, \ \phi_t^3 = -\psi_t e^{(r-\gamma)(T-t)} S_t$$

where $\psi_t = N(d_1(S_t, T-t))$ is the classic Black-Scholes hedge ratio of a European call option.

3.6.2. *Case B. Two defaultable tradeable assets.* We shall now consider the payoff

$$\tilde{C}_T = 1\!\!1_{\{\tau > T\}}(\tilde{S}_T 1\!\!1_{\{\tau > T\}} - K)^+ = (\tilde{S}_T 1\!\!1_{\{\tau > T\}} - K)^+$$

representing a (vulnerable or non-vulnerable) option written on a defaultable stock. To replicate this claim, we postulate that the stock price process is a tradeable, but defaultable, asset. Thus the price process S of the stock admits the following generic representation $S_t = \mathbb{1}_{\{\tau>t\}}\tilde{S}_t$, where, by assumption, the pre-default value is governed by

$$d\tilde{S}_t = \tilde{S}_t\big(\mu_t\,dt + \sigma\,dW_t^*\big).$$

In addition, we postulate that the defaultable discount bond with maturity T is tradeable, with the price $D^0(t,T)$ and the pre-default value $\tilde{D}^0(t,T)$ given by (3.2).

Valuation. The valuation procedure is based on Corollary 3.5. In the case of deterministic default intensity $\gamma(t)$ and deterministic short-term interest rate $r(t)$, the martingale property of the process $S_t^* = \tilde{S}_t/\tilde{D}^0(t,T)$ under $\tilde{\mathbf{Q}}$ is equivalent to

$$d\tilde{S}_t = \tilde{S}_t\big(\hat{r}(t)\,dt + \sigma\,dW_t^*\big)$$

where $\hat{r}(t) = r(t) + \gamma(t)$ is the credit-risk-adjusted interest rate. The price at time 0 of the contract equals

$$\tilde{C}_0 = D^0(0,T)\,\mathbf{E}_{\tilde{\mathbf{Q}}}Y = \Gamma(0,T)B(0,T)\,\mathbf{E}_{\tilde{\mathbf{Q}}}(\tilde{S}_T - K)^+$$

where $\tilde{\mathbf{Q}}$ is the martingale measure for the process S^*.

Example 3.14. In the case of constant r and γ, the result is exactly the same as the Black-Scholes price of a (default-free) European call under the assumption that the risk-free interest rate equals $\hat{r} = r + \gamma$.

Hedging. Using Proposition 3.7, we arrive at the following equality for the wealth process V of a self-financing strategy:

$$V_t = D^0(t,T)\Big(\tilde{V}_0 + \int_0^t \phi_u^2\,dS_u^*\Big)$$

where $\tilde{V}_0 = V_0/\tilde{S}_0 = V_0/S_0$. Replication of the claim $\tilde{C}_T$ is thus equivalent to the following equality

$$c + \int_0^T \phi_t^2\,dS_t^* = (\tilde{S}_T - K)^+.$$

Example 3.15. It is rather clear that in the special case of constant r and γ, the hedging strategy will be exactly the same as in the (default-free) Black-Scholes model, but with the default-free interest rate substituted with the credit-risk-adjusted interest rate $\hat{r} = r + \gamma$.

4. Defaultable claims: pricing and hedging à la Markowitz

For the case study presented below we simplify the model of the primary default-free market as follows: we fix $T > 0$ and we assume that the spot rate process r is zero, which means that $B_t = 1$ for every $t \in [0,T]$. Furthermore, we postulate that under the actuarial probability, say $\mathbf{P}$, the stock price process S evolves according to

$$dS_t = S_t\,\big(\nu\,dt + \sigma\,dW_t\big), \qquad S_0 > 0,$$

where ν and $\sigma \neq 0$ are constants, and W is a Wiener process on a probability space $(\Omega, \mathcal{G}, \mathbf{P})$. We denote by $\mathbb{F}$ the filtration generated by the Wiener process W (it is also generated by the stock price S). Observe that in this case we have that

- the unique martingale measure for S on $(\Omega, \mathcal{F}_T)$ is given by

$$\frac{d\mathbf{P}^*}{d\mathbf{P}}\Big|\mathcal{F}_t = \Lambda(t), \qquad \forall\, t \in [0,T],$$

where we denote by Λ the *deflator*, i.e., the process satisfying

$$d\Lambda(t) = -\Lambda(t)\theta\, dW_t, \qquad \Lambda(0) = 1,$$

where $\theta = \nu/\sigma$, or, explicitly

$$\Lambda(t) = \exp\Big(-\theta W_t - \frac{1}{2}\theta^2 t\Big),$$

- $W_t^* = W_t + \theta t$, $t \in [0, T]$, is a Wiener process under $\mathbf{P}^*$,
- the natural filtrations generated by W and W^* coincide.

We assume that the two assets, the savings account B and the stock S, are liquid and available for investment by some economic agent. Now, imagine that a new investment opportunity becomes available for the agent. Namely, the agent may purchase at time $t = 0$ a contingent claim X, whose corresponding cash flow of X units occurs at time T. The random variable X is supposed to be $\tilde{\mathcal{F}}_T$-measurable, where $\tilde{\mathbb{F}}$ is some filtration in $\mathcal{G}$, which is not a subfiltration of $\mathbb{F}$. Thus, we are now dealing with a full market consisting of the primary market and of the claim X.

We assume that B and S are the *only* assets available for trading, and that the full market is incomplete; in particular, we assume that $X \ / \ \in_T \mathcal{F}$ But, this requirement alone may not suffice for the non-attainability of X in the full market. In the present context, the *incompleteness* of the full market means that the claim X is non-attainable, that is, it can not be represented as[4] $x + \int_0^T \phi_t^1\, dS_t$, where the integrand ϕ^1 is an $\mathbb{F} \vee \tilde{\mathbb{F}}$-adapted process and x is a constant. To summarize, we only allow for the primary assets B and S to be used in the agent's portfolio, but we allow a trading strategy to be based on the full information, formally represented by the joint filtration $\mathbb{G} = \mathbb{F} \vee \tilde{\mathbb{F}}$.

The question that we want to study is: how much would the agent be willing to pay at time $t = 0$ for the claim X, and how the agent should hedge her investment? [A symmetric study can be conducted for an agent creating such an investment opportunity by selling the claim.]

The question under consideration is a non-trivial one in our set-up. Of course, in practice the claim X may be available for purchase at some *ask price*, but this may not be the price that the agent would be willing to pay for it. Since we consider the case when the claim X can not be replicated by a self-financing portfolio consisting of only the two underlying instruments (i.e., a portfolio involving the savings account B and the risky asset S), the standard arbitrage argument for the determination of a (unique) price for the claim X can not be applied here. In this section, we propose the *mean-variance paradigm* in order to determine both the price and the hedging strategy for the non-attainable claim X.

Special case. For the sake of concreteness, in Section 4.2 we shall further specify our model of the full market. Namely, we shall assume in this section that the claim X represents a cash flow of a defaultable (corporate) discount bond, maturing at time T, which is subject to the so-called *fractional recovery of Treasury value* scheme. Specifically, we shall assume that the bond's cash flow X at time T is given as

$$X = L\left(\mathbb{1}_{\{\tau > T\}} + \delta \mathbb{1}_{\{\tau \le T\}}\right) = \delta L + L(1-\delta)\mathbb{1}_{\{\tau > T\}} \tag{4.1}$$

[4]Recall that $B_t = 1$ for all t, so that $dB_t = 0$.

for some constants L and δ. That is, if the bond defaults prior or at the maturity (that is, if $\tau \leq T$) then the fraction δ of the notional amount is paid at maturity; otherwise, the bond pays at maturity the full promised notional amount L. Observe that in this set-up we may, and we do take $\tilde{\mathbb{F}} = \mathbb{H}$ where, as before, $\mathbb{H}$ is the natural filtration generated by the process $H_t = \mathbb{1}_{\{\tau \leq t\}}$. Consequently, the full filtration $\mathbb{G}$ satisfies $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.

4.1. Mean-variance paradigm for pricing and hedging of non-attainable claims. We consider a class of *admissible trading strategies* $\phi = (\phi^0, \phi^1)$, defined as follows:

- ϕ^0_t is the amount of money held in the savings account, and ϕ^1_t is the number of shares of the risky asset S held in the agent's portfolio at time t,
- ϕ^0 and ϕ^1 are predictable processes with respect to the full filtration $\mathbb{G}$,
- ϕ is self-financing: $\phi^0_t B_t + \phi^1_t S_t = \phi^0_0 + \phi^1_0 S_0 + \int_0^t \phi^1_u \, dS_u$ for every $t \in [0,T]$,
- $\mathbf{E_P}\big(\int_0^T \|\phi_t\|^2 \, dt\big) < \infty$.

Suppose now that the agent has initially (that is, at time $t = 0$) the amount $v > 0$ available for investment. Thus, in the absence of outside endowments and in the absence of consumption, the *wealth process* of the agent, which is defined as follows

$$V^{\phi,v}_t = \phi^0_t + \phi^1_t S_t, \qquad \forall\, t \in [0,T],$$

satisfies

$$dV^{\phi,v}_t = \phi^1_t \, dS_t, \qquad V^{\phi,v}_0 = v.$$

For the sake of notational simplicity, we shall usually write V instead of $V^{\phi,v}$.

We postulate that the agent's objective for investment is the classical mean-variance portfolio selection objective. That is, for any given initial wealth v the agent is interested in solving the following problem over the horizon $[0,T]$[5]:

Problem MV(d,v): Minimize $\mathbf{V_P}(V_T)$ over the set of all admissible trading strategies, subject to

$$\mathbf{E_P} V_T \geq d, \qquad V_0 = v.$$

Remark 4.1. Observe that the above problem is non-trivial only if $d > v$, as otherwise investing in the savings account B alone generates the wealth process $V_t = v$, that obviously satisfies the terminal condition $\mathbf{E_P} V_T \geq d$, and for which the variance of V_T is zero. Thus, we shall assume from now on that $d > v$.

Remark 4.2. It is not difficult to show (cf. Bielecki et al. (2003) or Bielecki and Jeanblanc (2003)) that the above problem MV(d,v) admits a solution, so that there exists an optimal trading strategy, say $\phi^{d,v,*}$ and the optimal wealth process, say $V^{d,v,*}_t$. Let us denote by $\sigma^{2,v,d,*}$ the value of $\mathbf{V_P}(V^{d,v,*}_T)$.

When the claim X is available for purchase at time $t = 0$ for price p, the agent will decide whether to purchase it or not on the basis of the following reasoning: First, the agent solves the associated *mean-variance problem* MV(d,v,p).

Problem MV(d,v,p): Minimize $\mathbf{V_P}(V_T + X)$ over the set of all admissible trading strategies, subject to

$$\mathbf{E_P} V_T \geq d - m, \qquad V_0 = v - p \geq 0,$$

[5]By $\mathbf{V_P}$ we denote the variance operator.

where we denote $m = \mathbf{E_P} X$.

Next, the agent will be ready to pay for the claim X the amount that is no more then

$$p^{MV;d,v}(X) := \sup \{p \in [0,v] : \text{the problem MV}(d,v,p) \text{ admits a solution,} \\ \text{and } \sigma^{2,d,v,p,X,*} \leq \sigma^{2,v,d,*}\},$$

where, as usually, we set $\sup \emptyset = -\infty$.

Remark 4.3. We shall state below that if the parameters d, v and p as well as the random variable X satisfy certain additional sufficient conditions, then there exists an optimal strategy, say $\phi^{d,v,p,X,*}$, for the problem MV(d,v,p). We denote by $V_T^{d,v,p,X,*}$ the value of the terminal wealth corresponding to an optimal strategy $\phi^{d,v,p,X,*}$, and we set $\sigma^{2,d,v,p,X,*} = \mathbf{V_P}(V_T^{d,v,p,X,*} + X)$.

Definition 4.4. The value $p^{MV;d,v}(X)$ is called the *buyer's mean-variance price* of the claim X. The corresponding optimal strategy is called the *buyer's mean-variance hedging strategy.*

Remark 4.5. Observe that we require that $p \leq v$ in the formulation of problem MV(d,v,p).

Remark 4.6. Observe that, unlike as in the case of the problem MV(d,v), the problem MV(d,v,p) may be non-trivial even if $d - m \leq v - p$; although investing in the savings account alone will produce in this case a wealth process for which the condition $\mathbf{E_P} V_T \geq d - m$ is satisfied, the variance $\mathbf{V_P}(V_T + X) = \mathbf{V_P}(X)$ may not be minimal.

Remark 4.7. For simplicity of presentation of our mean-variance hedging idea we did not postulate above that agent's wealth is non-negative for all times. Problem MV(d,v) with such a postulate added has been recently studied in Bielecki et al. (2003). Observe that non-negativity of the wealth process V is not implicit in our formulation of the problem, as we are using investment strategies representing numbers of shares of the underlying primary assets. If we were using so called proportion strategies, then the wealth process would be a positive semi-martingale by definition. We refer to Bielecki et al. (2003) for discussion regarding the relationship between the two classes of strategies. We shall study in a separate work the more general problem of mean-variance pricing and hedging under the assumption of non-negativity of the wealth process. In particular, we shall compare optimal solution to our problem under the non-negativity constraint, to the optimal solution of a problem MV(d,v) where only proportion strategies are allowed.

Comments on the pricing rule. Let us denote by $\mathcal{N}(X)$ the *no-arbitrage price interval* for the claim X, that is, $\mathcal{N}(X) = [MBP(X), MAP(X)]$ where $MBP(X)$ ($MAP(X)$, resp.) stands for the maximal bid price (minimal ask price, resp.) It may be so that our mean-variance price $p^{MV;d,v}(X)$ of this claim does not fall into the no-arbitrage interval. Since this possibility may appear as an unwanted feature of our approach to pricing and hedging, we comment below on this aspect of our pricing method.

When we consider the valuation issue from the perspective of the entire market, then it is natural to apply the no-arbitrage paradigm. According to this paradigm, the market as a whole will "preserve" only those prices of a financial asset, which fall into the no-arbitrage interval. Prices from outside this interval can't be sustained in a longer term due to market forces, which will tend to eliminate any persistent arbitrage opportunity.

Now, suppose that an individual investor is interested in putting some of her initial capital $v > 0$ into an investment opportunity provided by some asset X. Thus, the investor needs to decide whether to acquire the investment opportunity, and if so then how much to pay for it, based on her overall attitude towards risk and reward. Let us denote this attitude by $\mathcal{RR}$.[6] Suppose that the investment opportunity is not spanned by the underlying primary securities (the primary market) that are available for liquid trading. Let us denote the primary market by $\mathcal{PM}$. Finally, let us suppose that an investor makes investment (valuation) decisions regarding assets such as X using some pricing rule, say Π. That is, the investor decides about the price, say p, that she is willing to pay for the asset X, by evaluating her pricing rule, that is

$$p = \Pi(v, X, \mathcal{RR}, \mathcal{PM}).$$

The number p is the price that investor is willing to pay for the investment opportunity given her initial capital v, given her attitude towards risk and reward, and given the primary market model. The investor then "submits" her price to the market. Now, suppose that the market recognized no-arbitrage interval for the asset X is $\mathcal{N}(X)$. If it happens that $p \in \mathcal{N}(X)$ then the investor's bid price for X can be accepted by the market.[7] But, there is no fundamental reason why the investor's pricing rule should produce a price p from the no-arbitrage interval. Neither our mean-variance price, nor the indifference price based on some utility function, necessarily fall into the no-arbitrage interval. In the case when $p \notin \mathcal{N}(X)$, the investor's bid price may not be accepted by the market, and the investor may not enter into the investment opportunity.[8]

In view of the above remarks, we are now willing to suggest the following principle for determining the investor's price of the investment opportunity X, say $p^I(X)$,

$$p^I(X) = \Pi(v, X, \mathcal{RR}, \mathcal{PM}) \quad \text{if} \quad \Pi(v, X, \mathcal{R}R, \mathcal{PM}) \in \mathcal{N}(X)$$

and $p^I(X) = -\infty$ otherwise. This principle simply "marks to market" the investor's pricing rule Π. In our case the investor's pricing rule is given by Definition 4.4, and this rule needs to be marked-to-market in the way described above.

Case of an attainable claim. We recall again, that if the claim X were measurable with respect to $\mathcal{F}_T$ then we would be dealing with a complete full market, and a unique arbitrage price would be computed using for example the risk-neutral valuation approach. In this case, we shall verify that our mean-variance price coincides with this unique arbitrage price. In fact, we have the following result.

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[6]In our case this attitude is rooted in the *mean-variance* paradigm. But, it could be *mean-VaR* paradigm, or any other paradigm that connects risk (measured in some way) with reward (measured in some way). Of course, use of utility functionals is included here as a possibility.

[7]Most likely some bargaining process will be involved starting from the investor's bid price and some seller's ask price. This bargaining process will ultimately decide the price at which the claim X will change hands.

[8]Of course, in such case, the investor may want revise her attitude towards risk and reward, so that such revision may lead to a new price p' that will belong to the no-arbitrage interval.

PROPOSITION 4.1. *Assume that the claim X is attainable in the primary market model, and assume that its (no-)arbitrage price, say x, satisfies $x \leq v$. Then $p^{MV;d,v}(X) = x$.*

PROOF. Since X is attainable in the primary market model, we have that

$$X = x + \phi_T^0 + \phi_T^1 S_T = x + \int_0^T \phi_t^1 \, dS_t$$

for some admissible trading strategy ϕ. Note that for the claim X it holds that $V_T^{v,d,p,*} = V_T^{v-p+x,d,*}$ and that $\sigma^{2,v,d,p,*} = \sigma^{2,v-p+x,d,*}$. It can be shown (see Bielecki et al. (2004) or Bielecki and Jeanblanc (2003)) that for $v \leq d$ the function $v \mapsto \sigma^{2,v,d,*}$ is strictly decreasing. Consequently, the function $p \mapsto \sigma^{2,v,d,p,*}$ is strictly decreasing for p such that $v - p + x \leq d$, that is $p \geq v - d + x$. In view of our underlying assumption that $v \leq d$ we have that $v - d + x \leq x$. Finally, note that for $p = x$ we have that $\sigma^{2,v,d,x,*} = \sigma^{2,v,d,*}$. This completes the proof. $\square$

4.2. Solution to problem MV(d, v, p): the mean-variance price and hedging strategy. For the future reference, we denote by $\mathcal{P}^v$ and $\mathcal{P}^0$ the set of all random variables $V_T^v = v + \int_0^T \phi_t^1 \, dS_t$ and $V_T^0 = \int_0^T \phi_t^1 \, dS_t$, respectively, where ϕ^1 is the second component of any admissible strategy ϕ. Thus, $\mathcal{P}^v$ is simply a translation of $\mathcal{P}^0$ by a deterministic constant v. In addition, we denote by $\Pi_{\mathbf{P}}$ the orthogonal projection operator (in the norm of the space $L^2(\Omega, \mathcal{G}, \mathbf{P})$) from $L^2(\Omega, \mathcal{G}, \mathbf{P})$ on (the closed, linear subspace) $\mathcal{P}^0$.

We shall now focus on the defaultable claim X given by (4.1). Notice that $X = \delta L + Y(T, \delta)$ where we denote $Y(T, \delta) = L(1 - \delta)\mathbb{1}_{\{\tau > T\}}$. The first step in solving the problem MV(d, v, p) for X given by (4.1) is to examine the following problem:

Problem MV$(d, v, p; \delta)$: Minimize $\mathbf{V_P}[V_T + Y(T, \delta)]$ over the set of all admissible trading strategies,[9] subject to

$$\mathbf{E_P} V_T \geq d(\delta, T) := d - \mathbf{E_P} X, \qquad V_0 = v - p \geq 0.$$

We first solve the auxiliary problem:

Problem MVA$(d, v, p; \delta)$: Minimize $\mathbf{V_P}[V_T + Y(T, \delta)]$ over the set of all admissible trading strategies, subject to

$$\mathbf{E_P} V_T = d(\delta, T), \qquad V_0 = v - p \geq 0.$$

The latter problem is in turn equivalent to the following one:

Problem MVB$(d, v, p; \delta)$: Minimize $\mathbf{E_P}[V_T + Y(T, \delta)]^2$ over the set of all admissible trading strategies, subject to

$$\mathbf{E_P} V_T = d(\delta, T), \qquad V_0 = v - p \geq 0.$$

We split the solution to the above problem into two phases. In phase one, we solve:

$$\min_{\hat{V} \in \mathcal{P}^{v-p}} \mathbf{E_P}[\hat{V} + Y(T, \delta)]^2, \tag{4.2}$$

[9] Recall that admissible trading strategies are adapted to the full filtration $\mathbb{G}$.

subject to

$$\mathbf{E_P}\hat{V} = d(\delta, T), \quad \mathbf{E_P}(\Lambda(T)\hat{V}) = v - p \geq 0. \tag{4.3}$$

We rewrite the above problem as

$$\min_{\tilde{V}\in\mathcal{P}^0} \mathbf{E_P}[\tilde{V} + v - p + Y(T,\delta)]^2, \tag{4.4}$$

subject to

$$\mathbf{E_P}\tilde{V} = d(\delta, T, v, p) := d(\delta, T) - (v - p), \quad \mathbf{E_P}(\Lambda(T)\tilde{V}) = 0. \tag{4.5}$$

It is clear that if a random variable $\tilde{V}^*$ is an optimal solution for the problem (4.4)-(4.5), then $\hat{V}^* = \tilde{V}^* + (v-p)$ is an optimal solution of the problem (4.2)-(4.3). The Lagrangian corresponding to the problem (4.4)-(4.5) is

$$\mathbf{E_P}\big([v - p + \tilde{V} + Y(T,\delta)]^2 - \lambda_1\tilde{V} - \lambda_2\Lambda(T)\tilde{V}\big) - (d(\delta,T,v,p))^2 + \lambda_1 d(\delta,T,v,p).$$

Let $Y(T,\delta) = I_1 + I_2$ be the orthogonal decomposition of $Y(T,\delta)$, with $I_1 = \Pi_{\mathbf{P}}\left(Y(T,\delta)\right)$. Thus, the optimal solution for problem (4.4)-(4.5) is given by

$$2\tilde{V}^* = \lambda_1 + \lambda_2\Lambda(T) - 2I_1 - 2(v-p),$$

and, consequently, the optimal solution for problem (4.2)-(4.3) is given by

$$2\hat{V}^* = \lambda_1 + \lambda_2\Lambda(T) - 2I_1$$

where the Lagrange multipliers λ_1, λ_2 satisfy

$$\lambda_1 + \lambda_2 - 2\alpha_1 = 2d(\delta,T), \quad \lambda_1 + \lambda_2\exp(\theta^2 T) - 2\beta_1 = 2(v-p),$$

and where $\alpha_1 = \mathbf{E_P}I_1$, $\beta_1 = \mathbf{E_P}(\Lambda(T)I_1) = \mathbf{E_{P^*}}I_1 = 0$. Hence, setting $\hat{v} := v - p$, we get

$$\begin{aligned}\hat{V}^* &= \frac{1}{e^{\theta^2 T} - 1}\Big(\big(d(\delta,T) + \alpha_1\big)e^{\theta^2 T} - \hat{v} - \big(\alpha_1 - \hat{v} + d(\delta,T)\big)\Lambda(T)\Big) - I_1 \\ &=: \bar{V}^* - I_1.\end{aligned}$$

The quantities I_1 and α_1 can be computed using Proposition 6.3, Lemma 6.1 and Lemma 6.2 in Bielecki and Jeanblanc (2003). The corresponding optimal variance is

$$\mathbf{V_P}\big[\hat{V}^* + Y(T,\delta)] = \frac{\big(\hat{\alpha} - \hat{\beta} - \hat{v} + d(\delta,T)\big)^2 - \big(\hat{\beta} - \mathbf{E_P}I_2\big)^2}{e^{\theta^2 T} - 1} + \mathbf{V_P}(I_2)$$

where $\hat{\alpha} = \alpha(L,\delta,T) = \mathbf{E_P}[Y(T,\delta)]$ and $\hat{\beta} = \beta(L,\delta,T) = \mathbf{E_{P^*}}I_2$.

It is apparent that the optimal random variable $\hat{V}^*$ for the problem (4.2)-(4.3) also determines an optimal terminal wealth, say V_T^*, for the auxiliary problem MVA$(d,v,p;\delta)$. Furthermore, the following proposition was established in Bielecki and Jeanblanc (2003).

Proposition 4.2. *Suppose that* $\hat{\alpha} - \hat{\beta} > -v + p + d - m > 0$. *Then, the optimal terminal wealth* V_T^* *for the auxiliary problem* $MVA(d,v,p;\delta)$ *is also the optimal terminal wealth for the problem* $MV(d,v,p;\delta)$.

In the following section, we shall complete solving problem MV$(d,v,p;\delta)$ by determining an optimal strategy. This optimal strategy is nothing else but the mean-variance hedging strategy corresponding to a particular value p.

The mean-variance hedging strategy for a given p. We now turn to phase two of solving the auxiliary problem MVA$(d, v, p; \delta)$. In view of Proposition 4.2, it is also phase two of solving our original problem MV$(d, v, p; \delta)$. Here, we derive the mean-variance hedging strategy for a given value of p. In the next step, we shall find the mean-variance hedging strategy.

Similarly as in Bielecki and Jeanblanc (2003), we can use an appropriate BSDE in order to construct an $\mathbb{F}$-adapted process, say $\hat{\phi}^{1;p}$ so that

$$\bar{V}^* = \mathbf{E}_{\mathbf{P}^*}\bar{V}^* + \int_0^T \hat{\phi}_t^{1;p}\, dS_t.$$

Using Proposition 6.3, Lemma 6.1 and Lemma 6.2 in Bielecki and Jeanblanc (2003), we can determine a process $\widetilde{\psi}^{Y(\delta,T),\mathbf{P},p}$ so that

$$I_1 = \Pi_{\mathbf{P}}(Y(\delta,T)) = \int_0^T \widetilde{\psi}_t^{Y(\delta,T),\mathbf{P},p}\, dS_t.$$

It appears then that the optimal wealth for the problem MV$(d, v, p; \delta)$ can be represented as

$$\begin{aligned} V_T^* &= \hat{V}_1^* - I_1 = \mathbf{E}_{\mathbf{P}^*} V_1^* + \int_0^T \left(\hat{\phi}_t^{1;p} - \widetilde{\psi}_t^{Y(T,\delta),\mathbf{P},p}\right) dS_t \\ &= v - p + \int_0^T \phi_t^{1,v,\delta,p;*}\, dS_t \end{aligned}$$

where

$$\phi_t^{1,v,\delta,p;*} = \frac{dV_t^*}{dS_t} = \hat{\phi}_t^{1;p} - \widetilde{\psi}_t^{Y(T,\delta),\mathbf{P},p}.$$

Observe that the delta hedging strategy $\phi^{1,v,\delta,p;*}$ is composed of two components: the part $\hat{\phi}_t^{1;p}$ that hedges against the risk of the primary market, and the part $-\widetilde{\psi}_t^{Y(T,\delta),\mathbf{P},p}$ that hedges against the default risk of the claim X.

The buyer's mean-variance price and hedging strategy. We are now in the position to determine the mean-variance price and the mean-variance hedging strategy for the claim X. In view of our definition of the buyer's mean-variance price (cf. Definition 4.4) we are looking for a maximum p in the interval $[0, v]$ so that

$$\frac{\left(\hat{\alpha} - \hat{\beta} - v + p + d(\delta,T)\right)^2 - \left(\hat{\beta} - \mathbf{E}_{\mathbf{P}} I_2\right)^2}{e^{\theta^2 T} - 1} + \mathbf{V}_{\mathbf{P}}(I_2) \leq \frac{(d-v)^2}{e^{\theta^2 T} - 1}.$$

Let us denote $\gamma_1 = \hat{\alpha} - \hat{\beta} - v + d(\delta,T)$, $\gamma_2 = (d-v)^2$ and

$$\gamma_3 = \left(e^{\theta^2 T} - 1\right)\mathbf{V}_{\mathbf{P}}(I_2) - \left(\hat{\beta} - \mathbf{E}_{\mathbf{P}} I_2\right)^2.$$

Then we have the following result,

PROPOSITION 4.3. *If $\gamma_3 \leq \gamma_2$ then the buyer's mean-variance price is*

$$p^{MV;d,v}(X) = \min\{-\gamma_1 + \sqrt{\gamma_2 - \gamma_3}, v\} \vee 0.$$

Otherwise, $p^{MV;d,v}(X) = -\infty$. In the former case, the mean-variance hedging strategy is $\phi^{1,v,\delta,p^{MV;d,v}(X);*}$.

EXAMPLE 4.8. We shall illustrate the above results by considering a particular case of defaultable bond X given by (4.1). We shall assume here that the default time τ is provided with some simple structure. Specifically, we assume that it is the first jump of a Poisson process N, defined on $(\Omega, \mathcal{G}, \mathbf{P})$, with constant intensity

$\lambda > 0$. Thus, τ is independent of the filtration $\mathbb{F}$, and it is exponentially distributed with parameter λ. Consequently, we have that

$$m := \mathbf{E}_{\mathbf{P}} X = L\Big(e^{-\lambda T}(1-\delta) + \delta\Big)$$

and

$$\sigma^2 := \mathbf{V}_{\mathbf{P}}(X) = L^2\Big(e^{-\lambda T}(1-e^{-\lambda T})(1-\delta)^2\Big).$$

We additionally restrict ourselves to stratgies adapted only to filtration $\mathbb{F}$. Then, it follows from our previous results that in this setting the buyer's mean-variance price of the claim X is

$$p^{MV;d,v}(X) = \Big(\min\big\{m-(d-v)+\sqrt{(d-v)^2-\sigma^2(e^{\theta^2 T}-1)}, v\big\}\Big) \vee 0,$$

if $(d-v)^2 - \sigma^2(e^{\theta^2 T}-1) \geq 0$, and $-\infty$ otherwise. Moreover, in the former case, the mean-variance hedging strategy is given as

$$\phi^{1,v,\delta,p^{MV;d,v}(X);*} = \frac{d-m-v+p^{MV;d,v}(X)}{e^{\theta^2 T}-1}\,\frac{\nu}{\sigma^2}\,\frac{\Lambda(t)}{S_t}\,e^{\theta^2(T-t)}.$$

Similarly as in Collin-Dufresne and Hugonnier (1999), it is possible to show that $MBP(X) = \delta L$ and $MAP(X) = L$. Consequently, the no-arbitrage interval for the claim X is $[\delta L, L]$. It is clear then that our mean variance price $p^{MV;\delta,v}(X)$ is no more than L. However, it is not true in general that $p^{MV;\delta,v}(X)$ is greater than δL. Thus, the price $p^{MV;\delta,v}(X)$ may not belong into the no-arbitrage interval. We refer to our comments on the pricing rule with regard to the latter possibility.

References

[BSW] A. Bélanger, S. Shreve, D. Wong, *A unified model for credit derivatives*, to appear in Math. Finance, 2001.

[BJ] T.R. Bielecki, M. Jeanblanc, *Mean-variance hedging of credit risk: a case study*, working paper, 2003.

[BJPZ] T.R. Bielecki, H. Jin, S.R. Pliska, X.Y. Zhou, *Continuous-time mean-variance portfolio selection with bankruptcy prohibition*, preprint, 2003.

[BR1] T.R. Bielecki, M. Rutkowski, *Credit risk: Modelling, valuation and hedging*, Springer-Verlag, Berlin Heidelberg New York, 2002.

[BR2] T.R. Bielecki, M. Rutkowski, *Dependent defaults and credit migrations*, Appl. Math. **30** (2003), 121–145.

[BS] F. Black, M. Scholes, *The pricing of options and corporate liabilities*, J. Political Econ. **81** (1973), 637–654.

[BSJ] C. Blanchet-Scalliet, M. Jeanblanc, *Hazard rate for credit risk and hedging defaultable contingent claims*, to appear in Finance and Stochastics, 2000.

[CH] P. Collin-Dufresne, J.-N. Hugonnier, *On the pricing and hedging of contingent claims in the presence of extraneous risks*, working paper, Carnegie Mellon University, 1999.

[EJY] R.J. Elliott, M. Jeanblanc, M. Yor, *On models of default risk*, Math. Finance **10** (2000), 179–195.

[G] Y. Greenfield, *Hedging of credit risk embedded in derivative transactions*, Ph.D. thesis, Carnegie Mellon University, 2000.

[J] F. Jamshidian, *Valuation of credit default swap and swaptions*, working paper, 2002.

[JR1] M. Jeanblanc, M. Rutkowski, *Default risk and hazard processes*, Mathematical Finance – Bachelier Congress 2000, H. Geman, D. Madan, S.R. Pliska and T. Vorst, eds., Springer-Verlag, Berlin Heidelberg New York, 2002, pp. 281–312.

[JR2] M. Jeanblanc, M. Rutkowski, *Modelling and hedging of default risk*, Credit Derivatives: A Definitive Guide, Risk Books, 2003a, pp. ??–??.

[JR3] M. Jeanblanc, M. Rutkowski, *Hedging of credit derivatives within the reduced-form framework*, working paper, 2003b.

[La] D. Lando, *On Cox processes and credit-risky securities*, Rev. Derivatives Res. **2** (1998), 99–120.

[S] P.J. Schönbucher, *Pricing credit risk derivatives*, working paper, University of Bonn, 1998.

[V] N. Vaillant, *A beginner's guide to credit derivatives*, working paper, Nomura International, 2001.

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