Lectures on Mathematical Finance

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Contents

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All remaining errors and misprints are, of course on my sole responsibility.

Begin at the beginning, and go on till you come to the end. Then, stop.

L. Caroll, Alice's Adventures in Wonderland

Chapter 1

Pricing and Hedging

Assume that a family of underlying assets is given on a time horizon $[0, T]$. We shall first focus on the problem of pricing and hedging derivative products. A derivative security is a security whose value depends on the value of the basic underlying variables. The "price" of the derivative is the amount of money that the buyer agrees to give to the seller of the derivative at time 0 to receive the derivative at date T (the maturity time). When the derivative product is redundant in the market, we shall see that it has a unique fair price, that of a portfolio of underlying assets which gives the same cash flow. Otherwise any investor could achieve a return with no initial investment. When it is not redundant, it may be given several prices. The "hedging strategy" is the portfolio of underlying assets needed by the seller of the derivative to hedge himself against the delivery of the product.

1.1 Discrete time

1.1.1 Binomial approach

The simplest example is the so-called two dates "binomial model". There are two trading dates, 0 and 1, and two assets : a bond, with price 1 at time 0 and $(1 + r)$ at time 1 (r is the interest rate for the period). The asset price equals S at time 0 and is a random variable S_1 at date 1, equal to uS with probability p and to dS with probability $1 - p$, with $d < u$. In this simple model, a derivative product is any financial product with payoff C_1 at date 1 equal to $h(S_1)$ for some function h. Examples of derivative products are "call options" and "put options". A call option of strike K has payoff $C_1 = (S_1 - K)^+$, i.e. $h(x) = (x - K)^+$ (while a put option of strike K has payoff $P_1 = (K - S_1)^+$). Indeed one unit of a call option (bought at date 0) confers to the buyer the right (but not the obligation) to buy the asset at price K at date 1. At date 1, if $S_1 > K$, the buyer of the call buys the asset at price K and sells it right away at price S_1 making profit $S_1 - K$ while if $S_1 < K$, the buyer doesn't do anything and his profit is 0. The seller has to deliver the asset at price K if $S_1 > K$, this is why he needs to hedge himself against this potential loss.
Let (α, θ) be a portfolio of α shares of bond and θ shares of asset. There are no constraints on α and θ, these numbers can be negative. If θ is negative, the investor is "short" in the asset. The portfolio hedges the derivative if it has same value at time 1, hence if $\alpha(1 + r) + \theta S_1 = h(S_1)$ equivalently if the following two equations are fulfilled

$$\begin{cases} h(uS) & = & \alpha(1+r) + \theta\, uS \\ h(dS) & = & \alpha(1+r) + \theta\, dS\,. \end{cases}$$

In a general case, we get that the hedging portfolio is given by

$$\alpha = \frac{1}{1+r}\frac{uS\,h(dS) - dS\,h(uS)}{uS-dS}\ ,\quad \theta = \frac{h(uS)-h(dS)}{uS-dS}\ ,$$

and the time-0 value of the hedging portfolio can be written as

$$h_0 \ :=\ \alpha + \theta S \ =\ \frac{1}{1+r}\left(\pi h(uS) + (1-\pi)h(dS)\right)\ , \tag{1.1}$$

where

$$\pi \ :=\ \frac{1}{u-d}\left((1+r)-d\right)\ . \tag{1.2}$$

For an European call, the hedging portfolio is given by

$$\alpha = \frac{1}{1+r}\frac{uC_d - dC_u}{u-d}\ ,\quad \theta = \frac{C_u - C_d}{uS-dS}\ ,$$

with $C_u = (uS-K)^+, C_d = (dS-K)^+$ and the time-0 value of the hedging portfolio can be written as

$$C \ :=\ \alpha + \theta S \ =\ \frac{1}{1+r}\left(\pi C_u + (1-\pi)C_d\right)\ . \tag{1.3}$$

Note that $1 > \theta > 0$.

The right member of (1.3) can be interpreted as an expectation if π belongs to $]0,1[$, which is verified if and only if $d < 1+r < u$. The number π is then called the "risk neutral probability" since

$$S = \frac{\pi uS + (1-\pi)bS}{1+r}\ , \tag{1.4}$$

in other words, the value of the asset is equal to the expectation of its discounted payoff. Equality (1.3) proves that the value of any portfolio is equal to the expectation of its discounted payoff. If the inequality $d < 1+r < u$ is not fulfilled, then there are ways to make money with no initial investment. More precisely, an arbitrage opportunity is a portfolio (α,θ) such that the initial value is non-positive, i.e., $\alpha + \theta S \leq 0$ and the date 1 value is non-negative

$$\alpha(1+r) + \theta uS \geq 0,\ \alpha(1+r) + \theta dS \geq 0$$

and at least one of the two last inequalities is strict. For example, let us show that if $(1+r) < d$, then there is an arbitrage. Indeed, at date 0, an investor may borrow the amount of money S at interest rate r and with the money, buy the asset. At date 1, he reimburses $S(1+r)$ and sells the asset at price $S_1 \geq dS$ making the non-negative net profit $S_1 - S(1+r)$ in both states and a strictly positive profit in the up state. Hence the portfolio $(-S,1)$ is an arbitrage. Symmetrically, in the case $u < (1+r)$, the portfolio $(S,-1)$ is an arbitrage (the investor shorts the asset). Hence if there is no-arbitrage, $d < 1+r < u$ and the price of the asset is its expected discounted payoff under the risk neutral probability. Similarly the value of any portfolio is its discounted expected payoff under the risk neutral probability.

It can be proved that if the price of the derivative was different from h_0, as defined in (1.3), then there would exist arbitrage opportunities.

1.1.2 Two dates, several assets and several states of the world

We now consider a two dates financial market where uncertainty is represented by a finite set of states $\{1,\dots,k\}$. There are $d+1$ assets. At date 0, asset i, $0\le i\le d$, has value S^i and pays $d^i(j)$ at date 1, in units of accounts, in state j. Let $d^i\in I\!R^k$ be asset's i payoff vector. Assume that asset 0 is riskless (in other words that $d^0(j)=1,\ \forall\, j$) and let interest rate r be defined by $S^0=\frac{1}{1+r}$. A portfolio $\theta=(\theta^0,\theta^1,\dots,\theta^d)$ where $\theta^i\in I\!R$ is the fraction of asset i hold by an investor has market value $\sum_{i=0}^{d}\theta^iS^i$ at date 0 and payoff $\sum_{i=0}^{d}\theta^id^i(j)$ at date 1 in state j.

We shall say that $V\ge 0$ for a vector V if any component is non-negative. Let $S\in I\!R^{d+1}$ be the date 0 assets market values vector and D be the $(k\times(d+1))$ matrix of payoffs, i.e.

$$D=\begin{pmatrix} d^0(1) & d^1(1) & \dots & d^d(1)\\ d^0(2) & d^1(2) & \dots & d^d(2)\\ \dots & \dots & \dots & \dots\\ d^0(k) & d^1(k) & \dots & d^d(k)\end{pmatrix}$$

We use the notation $V\ge 0$ if any component of the vector V is non-negative. Then $\sum_{i=0}^{d}\theta^iS^i=\S\cdot\theta$ and $\sum_{i=0}^{d}\theta^id^i(j)$ is the j-th component of the vector $D\theta$. There is no-arbitrage if $D\theta=0$ implies $S\cdot\theta=0$ and $D\theta\ge 0$, $D\theta\ne 0$ implies $S\cdot\theta>0$. In other words, there is no-arbitrage if there is no portfolio offering something for nothing. It follows from a convex analysis type argument that there is no-arbitrage iff there exists a "state price" vector $\beta\in I\!R^k_{++}$ such that

$$S^i \;=\; \sum_{j=1}^{k} d^i(j)\,\beta_j \qquad i\in\{0,\dots,d\}.$$

As $S^0=\frac{1}{1+r}=\sum_{j=1}^{k}\beta_j$, define $\pi_j=(1+r)\beta_j$. Then $\sum_{j=1}^{k}\pi_j=1$. We now have a vector of probabilities (such a probability is called "risk neutral") and can write

$$S^i \;=\; \frac{1}{1+r}\sum_{j=1}^{k}\pi_jd^i(j) \qquad i\in\{0,\dots,d\},,$$

or in a concise form

$$S \;=\; \frac{1}{1+r}\,D^T\pi.$$

Hence if there is no-arbitrage, the price of an asset is its expected discounted payoff under a well chosen probability.

A contingent claim is a date 1 random payoff and is identified to an element z of $I\!R^k$. Markets are complete if span $D=I\!R^k$. In other words, any contingent claim $z\in I\!R^k$ may then be hedged (for any $z\in I\!R^k$, there exists a porfolio θ such that $z=D\theta$). It follows from elementary algebra that, in a complete market, β and (π,r) are uniquely defined. The date 0 value of a contingent claim z is the initial value of any hedging portfolio θ and is equal to

$$\theta\cdot S=\frac{1}{1+r}\sum_{j=1}^{k}\pi_j\,z_j \;=\; \sum_{j=1}^{k}\beta_j\,z_j$$

its payoff value at state price β or to its expected payoff under the risk neutral probability. One easily shows that this is the only fair price of the contingent claim: if the contingent claim was given any other price, then there would exist an arbitrage.
One can think of β_j as the cost of obtaining one unit of account in state j.

1.1.3 Multiperiod discrete time model

Let us now study the case of N trading dates.

Let us first assume that there are only two assets, a riskless and a risky asset. The riskless asset has price $(1+r)^n$ at date n (we assume here that the interest rate is constant over time and denote by $R_n = (1+r)^{-n} = (S_n^0)^{-1}$ the time n discount factor) while the risky asset has price S_n. Let us assume that the investor observes past prices and make decisions that depend only on those observations. To model that assumption, we associate with the investor's information a tree. We shall consider that at time 1, there are two states u and d; state u in term is followed by states uu and ud at date 2, the state uu is followed by uuu and uud and so on. A state of nature at time n is a sequence of length n of digits u and d; if e_n is such a sequence, the following states of nature at time $n+1$ are denoted by (e_n, u) and (e_n, d). Let $S_n(e_n)$ be the value of the asset at time n in state e_n. A portfolio $(\alpha_n(e_n), \theta_n(e_n))$ held at time n in state e_n, has value $\alpha_n(e_n)(1+r)^n + \theta_n(e_n)S_n(e_n)$ in that state and value $\alpha_n(e_n)(1+r)^{n+1} + \theta_n(e_n)S_{n+1}(e_n, u)$ or $\alpha_n(e_n)(1+r)^{n+1} + \theta_n(e_n)S_{n+1}(e_n, d)$ at date $n+1$. At date $n+1$, the investor may rebalance his portfolio under a "self-financing" constraint: $(\alpha_{n+1}, \theta_{n+1})$ has to fulfill at date $n+1$ in state e_{n+1},

$$\alpha_n(e_n)(1+r)^{n+1} + \theta_n(e_n)S_{n+1}(e_{n+1}) = \alpha_{n+1}(e_{n+1})(1+r)^{n+1} + \theta_{n+1}(e_{n+1})S_{n+1}(e_{n+1})\,.$$

In other words, the value at date $n+1$ of the portfolio bought at date n equals the value at date $n+1$ of the portfolio bought at date $n+1$. In that case, if we denote by V_n the value of the portfolio at time n

$$V_{n+1} - V_n = \alpha_n(S_{n+1}^0 - S_n^0) + \theta_n(S_{n+1} - S_n)$$

or, if $\Delta V_n = V_{n+1} - V_n$,

$$\Delta V_n = \alpha_n \Delta S_n^0 + \theta_n \Delta S_n\,.$$

When the market is arbitrage free between succeeding states of nature, one may construct, as in (1.2), node by node a probability on the tree, such that the discounted asset price process is a martingale. More precisely, for any n and any e_n, we introduce two nonnegative numbers $\pi_n(e_n; u)$ and $\pi_n(e_n; d)$ such that $\pi_n(e_n; u) + \pi_n(e_n; d) = 1$ and (cf. (1.4))

$$S_n(e_n)(1+r) = \pi_n(e_n; u)\, S_{n+1}(e_n, u) + \pi_n(e_n; d)\, S_{n+1}(e_n, d)\,.$$

In an explicit form

$$\pi_n(e_n; u) = \frac{(1+r)S(e_n) - S(e_n, d)}{S(e_n, u) - S(e_n, d)}$$

represents the risk-neutral probability between time n and $n+1$ for the branch of the tree starting at the node e_n. The discounted value of any self-financing portfolio is then also a martingale. Furthermore, one may compute the time n value of a terminal payoff C_N and a hedging strategy by a backward induction argument. Indeed, the $N-1$ time value in state e_{N-1} of payoff C_N is (cf. (1.3))

$$C_{N-1}(e_{N-1})R_{N-1} = R_N \sum_{s_N} \pi_{N-1}(e_{N-1}; s_N)\, C_N(e_{N-1}, s_N)$$

where $s_N = u$ or d. Similarly the $N-2$ time value in state e_{N-2} of payoff $C_{N-1}$is

$$\begin{aligned} C_{N-2}(e_{N-2})R_{N-2} &= R_{N-1}\sum_{s_{N-1}} \pi_{N-2}(e_{N-2};s_{N-1})\, C_{N-1}(e_{N-2},s_{N-1}) \\ &= R_N \sum_{s_{N-1},s_N} \pi_{N-2}(e_{N-2};s_{N-1})\, \pi_{N-1}(e_{N-2},s_{N-1};s_N)\, C_N(e_{N-2},s_{N-1},s_N) \end{aligned}$$

where $s_{N-1} = u$ or d and $\pi_{N-1}(e_{N-2},s_{N-1};s_N) = \pi_{N-1}(e_{N-1};s_N)$ is the risk neutral probability at node $(e_{N-2},s_{N-1}) = e_{N-1}$ between time $N-1$ and N. The time n-value of payoff C_N in the state $e_n = e$ is therefore obtained by induction

$$C_n(e)R_n = R_N \sum_{s_{n+1},s_{n+2},\ldots,s_N} \pi_n(e;s_{n+1}) \ldots \pi_{N-1}(e,s_{n+1},\ldots,s_{N-1};s_N) C_N(e,s_{n+1},s_{n+2},\ldots,s_N)$$

where $s_i = u$ or d. The discounted value of time n-value of payoff C_N is therefore the conditional expectation of the discounted terminal value, given the information up to time n, i.e., knowing which states of nature is realized.

Let us now assume that uncertainty is represented by a finite set of states $\{1,\ldots,k\}$ at each date (for simplicity, we assume that the number of states is constant over time). A state of nature at time n is a sequence of length n of states at dates $\ell \le n$; if e_n is such a sequence, then $e_{n+1} = (e_n, j)$, $j \in \{1,\ldots,k\}$. We assume that there are $d+1$ assets. At date n, the i-th asset has ex-dividend price S^i_n and pays dividend d^i_n (the cum-dividend price is $S^i_n + d^i_n$). A portfolio (α_n, θ_n) held at time n in state e_n, has value $\alpha_n(e_n)(1+r)^n + \theta_n(e_n)\cdot S_n(e_n)$ in that state at time n and value

$$\begin{aligned} \alpha_n(e_n)(1+r)^{n+1} &+ \theta_n(e_n)\cdot(S_{n+1}(e_{n+1}) + d_{n+1}(e_{n+1})) \\ &= \alpha_n(e_n)(1+r)^{n+1} + \sum_{i=1}^{d} \theta^i_n(e_n)\,(S^i_{n+1}(e_{n+1}) + d^i_{n+1}(e_{n+1})) \end{aligned}$$

at date $n+1$. We assume that strategies are "self-financing": the portfolio $(\alpha_{n+1}, \theta_{n+1})$ at date $n+1$ in state e_{n+1} has to be such that

$$\begin{aligned} \alpha_n(e_n)(1+r)^{n+1} &+ \theta_n(e_n)\cdot[S_{n+1}(e_{n+1}) + d_{n+1}(e_{n+1})] \\ &= \alpha_{n+1}(e_{n+1})(1+r)^{n+1} + \theta_{n+1}(e_{n+1})\cdot S_{n+1}(e_{n+1}) \end{aligned}$$

When the market is arbitrage free between succeeding states of nature (in other words if there doesn't exist strategies such that for a pair (e_n, e_{n+1}), the following inequalities are satisfied $\alpha_n(e_n)(1+r)^n + \theta_n(e_n)S_n(e_n) \le 0$ while $\alpha_n(e_n)(1+r)^{n+1} + \theta_n(e_n)[S_{n+1}(e_{n+1}) + d_{n+1}(e_{n+1})] \ge 0$ with a strict inequality for some state), one may construct node by node a probability Q on the tree, such that defining

$$\tilde S^i_n = \frac{S^i_n}{(1+r)^n},\ \tilde d^i_n = \frac{d^i_n}{(1+r)^n}$$

the discounted price and dividend processes of the i-th asset and $\tilde G^i_n = \sum_{\ell=1}^{n} \tilde d^i_\ell + \tilde S^i_n$ the discounted gain, one has

$$\tilde G^i_{n-1} = E_Q[\, \tilde G^i_n \mid \mathcal{F}_{n-1}] \,.$$

The discounted gain process is therefore a martingale.

1.2 Continuous time model

1.2.1 The Bachelier model

In continuous time, the first model was Bachelier's one (1900) [1]. Bachelier assumes that the prices are given by $S_t = x + \nu t + \sigma W_t$ where W is a Gaussian process, with zero mean and covariance equal to $\inf(t,s)$. Recall that a Gaussian process X is a process such that, for any $(a_i, i \leq n)$, and any $0 < t_1 < \ldots, < t_n$, the random variable $\Sigma_{i=1}^n a_i X_{t_i}$ is a Gaussian variable. A simple calculus leads to

$$E(\exp(\lambda W_t - \frac{\lambda^2 t}{2})) = 1\,.$$

In a differential form, the Bachelier model can be written $dS_t = \nu dt + \sigma dW_t$.
In his work, Bachelier gave the price of a European option. The only weakness of that model is that prices can be negative. Samuelson (1963) proposes to model prices as

$$S_t = x\exp(\nu t + \sigma W_t)\,.$$

From Itô's calculus, this can be written as

$$dS_t = S_t(\mu dt + \sigma dW_t)\,,$$

where $\nu = \mu - \frac{\sigma^2}{2}$. Let us recall that a Brownian motion is a continuous process W with independent stationary increments such that $W_{t+s} - W_s$ is a Gaussian variable, with mean 0 and variance t. It is therefore, a martingale.

1.2.2 Martingales

Martingales are processes such that the expectation of X_t given the past before time s, with $s < t$ is equal to X_s. If W is a Brownian motion, writing

$$W_t = (W_t - W_s) + W_s$$

and using that $W_t - W_s$ is independent of the past before s, we obtain easily the martingale property. In the same way,

$$\exp(\lambda W_t - \frac{\lambda^2 t}{2}) = \exp(\lambda(W_t - W_s) - \frac{\lambda^2 (t-s)}{2})\exp(\lambda W_s - \frac{\lambda^2 s}{2})\,,$$

leads easily to the martingale property : A Brownian motion is (the only) continuous process such that for any λ, the process $(\exp(\lambda W_t - \frac{\lambda^2 t}{2}), t \geq 0)$ is a martingale with expectation equal to 1.
An interesting feature of martingales is that, as soon as the terminal value is known, it is possible to construct all the process, by taking the expectation with respect to the past. If a target is given, there exists a unique process, which is a martingale and reaches the target.

1.2.3 Black and Scholes model

In the model used by Black and Scholes model, there are two assets, a bond with price

$$S_t^0 = e^{rt} = 1/R(t)$$

where r is the interest rate, supposed to be constant and a risky asset. The price of the risky asset evolves according to the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t)\,,$$

which solution from the initial condition $S_0 = x$ is

$$S_t = x\exp(\mu t)\,\exp(\sigma W_t - \frac{\sigma^2 t}{2})\,.$$

Here $(W_t, t \geq 0)$ is a Brownian motion and μ and σ are constants. Hence $S_t \geq 0$ and

$$\ln S_t = \ln x + \mu t + \sigma W_t - \frac{\sigma^2}{2}t$$

is a Gaussian process with mean $\ln x + \mu t - \frac{\sigma^2}{2}t$ and covariance $\sigma^2(t \wedge s)$. This is why the process S is also called log-normal. In particular, $E(S_t) = xe^{\mu t}$. The coefficient μ thus measures the global behavior of S while the coefficient σ, called the volatility of the price, measures the importance of the noise (indeed $\mathrm{Var}[\log(S_t)] = \sigma^2 t$). The larger is σ, the bigger is the influence of the Brownian part. The sign of σ is irrelevant since $(-W)$ is also a Brownian motion.

1.2.4 PDE approach

A portfolio (α, θ) is a pair of adapted processes (in other words, processes that at date t only depend on the past of the Brownian motion up to time t or more precisely which are $\mathcal{F}_t = \sigma(W_s, s \leq t)$ measurable) such that α_t (resp. θ_t) is the number of shares of the bond (resp. of the asset) owned by an investor. The time t value of the portfolio is $V_t = \alpha_t S_t^0 + \theta_t S_t$. The portfolio defines an *hedging strategy* for the contingent claim H if its terminal value is equal to H:

$$\alpha_T S_T^0 + \theta_T S_T = H\,.$$

The contingent claim H is of the form $H = h(S_T) = (S_T - K)^+$ for a Call.

A portfolio is *self-financing* if its changes in value are due to changes of prices, not to rebalancing of the portfolio, equivalently if one has

$$dV_t = \alpha_t dS_t^0 + \theta_t dS_t\,.$$

Black and Scholes methodology is to find an hedging strategy for the contingent claim. They *assume* that the value of the hedging portfolio is a smooth function of time and underlying, say $C(t, S_t)$. They construct a self financing portfolio, made of π_t shares of the underlying asset which hedges the derivative. Then, setting for simplicity $C(t, S_t) = V_t$

$$\begin{aligned}
dV_t &= rV_t dt + \pi_t(dS_t - rS_t dt) \\
&= [rV_t + (\mu - r)\pi_t S_t]dt + \pi_t \sigma S_t dW_t \\
&= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial x}dS_t + \frac{1}{2}\frac{\partial C^2}{\partial x^2}S_t^2\sigma^2 dt \\
&= [\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x}\mu + \frac{1}{2}\frac{\partial C^2}{\partial x^2}S_t^2\sigma^2]dt + \frac{\partial C}{\partial x}\sigma S_t dW_t
\end{aligned}$$

Therefore, by identification, they obtain

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x}\mu + \frac{1}{2}\frac{\partial C^2}{\partial x^2}S_t^2\sigma^2 = rV_t + (\mu - r)\pi_t S_t$$

and

$$\frac{\partial C}{\partial x}\sigma S_t = \pi_t \sigma S_t \,.$$

Hence

$$\begin{aligned} \pi_t &= \frac{\partial C}{\partial x} \\ \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial C^2}{\partial x^2} S_t^2 \sigma^2 &= rC - rS_t \frac{\partial C}{\partial x} \end{aligned}$$

Therefore, the price of a European option is $C(t, S_t)$ where C is the solution of

$$\frac{\partial C}{\partial t}(t,x) + \frac{1}{2}x^2\sigma^2 \frac{\partial C^2}{\partial x^2}(t,x) + rx\frac{\partial C}{\partial x}(t,x) = rC(t,x)$$

which satisfies the terminal condition

$$C(T,x) = (x-K)^+$$

One can notice that the same procedure works for any contingent claim of the form $h(S_T)$.

Using method from PDE, they solve explicitely this equation and obtain also the hedging strategy.

$$C = S_0 \mathcal{N}(d_1(S_0,T)) - Ke^{-rT}\mathcal{N}(d_2(S_0,T))$$

with

$$d_1(x,T) = \frac{1}{\sigma\sqrt{T}} \ln(\frac{x}{Ke^{-rT}}) + \frac{\sigma\sqrt{T}}{2},\ d_2(x,T) = d_1(x,T) - \sigma\sqrt{T}\,.$$

The price of the call at time t is

$$C(t,x) = x\mathcal{N}(d_1(x,T-t)) - Ke^{-r(T-t)}\mathcal{N}(d_2(x,T-t))\,.$$

The hedging strategy is $\theta_t = d_1(x, T-t)$.

1.2.5 Martingale approach

The process S is not a martingale. However, setting $\kappa = \dfrac{\mu - r}{\sigma}$, the process

$$S_t \exp(-rt - \kappa W_t - \frac{1}{2}\kappa^2 t) = S_t e^{-rt} L_t = x\exp((\sigma-\kappa)W_t - \frac{1}{2}(\sigma-\kappa)^2 t)$$

is a martingale (where $L_t = \exp(-\kappa W_t - \frac{1}{2}\kappa^2 t)$). The choice of $H_t = e^{-rt}L_t$ as a multiplier is such that as well the price of the risky asset and the price of the riskless asset multiplied by this factor are martingaales. Moreover, it can be shown from Itô's calculus that, if V is the value of a self financing portfolio, the process $V_t e^{-rt} L_t$ is a martingale. Another way to say that is to use Girsanov's theorem.

1.2.6 Discounted processes

Using some elementary algebra, if a portfolio is self-financing, then

$$dV_t = rV_t dt + \theta_t(dS_t - rS_t dt) \tag{1.5}$$

or, using integration by parts formula

$$d(e^{-rt}V_t) = e^{-rt}dV_t - re^{-rt}V_t dt = \theta_t d(e^{-rt}S_t)\,. \tag{1.6}$$

The reverse characterization holds : if x is a fixed number and if θ is an adapted process, then there exists a self financing portfolio (α, θ) with initial value x. Indeed, form (1.6), we obtain the value of the associated portfolio as

$$V_t e^{-rt} = x + \int_0^t \theta_s d(e^{-rs}S_s)$$

and the value of α is

$$\alpha_t = e^{-rt}(V_t + \theta_t S_t)$$

Hence the time t value of a self-financing portfolio only depends on its initial value and on the amount invested in the risky asset :

$$\tilde{V}_t = V_t e^{-rt} = V_0 + \int_0^t \theta_s d\tilde{S}_s = V_0 + \int_0^t \theta_s R_s (dS_s - rS_s ds) \tag{1.7}$$

where $\tilde{S}_t = e^{-rt}S_t$ is the discounted price. In this general setting, a contingent claim H is hedgeable, if there exists x and θ such that

$$He^{-rT} = V_0 + \int_0^T \theta_s R_s (dS_s - rS_s ds) = x + \int_0^t \theta_s d\tilde{S}_s$$

The quantity $\int_0^t \theta_s d\tilde{S}_s$ is interpreted as the gain of the strategy.

If (α, θ) is a strategy, the quantity

$$C_t(\alpha, \theta) = \tilde{V}_t - \int_0^t \theta_s d\tilde{S}_s = \alpha + e^{-rt}\theta_t S_t - \int_0^t \theta_s R_s d\tilde{S}_s$$

represents its cost. If the discounted cost is a constant, then the strategy is self financing, and the constant is the initial value.

It may be shown, in the Black and Sholes framework, using deep results of stochastic calculus (mainly the representation theorem) that there exists a unique hedging portfolio for any H. By definition, the price of the contingent claim is the initial value V_0 of the hedging portfolio.

1.2.7 Girsanov's theorem

It turns out that the discounted price of the risky asset is a martingale after a change of probability. A change of probability changes the law of the variable or of the process. Setting $W_t^* = W_t + \kappa t$, where $\kappa = \frac{\mu - r}{\sigma}$ the dynamics of S may therefore be written as

$$dS_t = S_t(rdt + \sigma dW_t^*)$$

or, in an equivalent form the dynamics of the discounted price $\widetilde{S}_t = S_t e^{-rt}$ are

$$d\widetilde{S}_t = \sigma \widetilde{S}_t dW_t^* \tag{1.8}$$

which solution is

$$\widetilde{S}_t = S_0 \exp(\sigma W_t^* - \frac{\sigma^2 t}{2})\,.$$

Hence, the discounted price $(\widetilde{S}_t, t \geq 0)$ is a martingale under the risk neutral probability Q as soon as W^* is a Brownian motion under the probability Q. Now Girsanov's theorem states that there exists a probability measure Q, equivalent to P, such that, under Q, the process $(W_t^*, t \geq 0)$ is a Brownian motion. The probability measure Q is defined by its Radon-Nykodym density : $dQ = L_t dP$ on the σ-algebra $\mathcal{F}_t$ with $L_t = \exp(-\kappa W_t - \frac{1}{2}\kappa^2 t)$. Furthermore the discounted value of any self-financing portfolio is a martingale, since

$$V_t e^{-rt} = V_0 + \int_0^t \sigma\theta_s \, \widetilde{S}_s \, dW_s^* \,. \tag{1.9}$$

Indeed, the right hand side is a stochastic integral with respect to a Brownian motion, hence a martingale. It follows that

$$V_t e^{-rt} = E_Q(V_T e^{-rT} | \mathcal{F}_t) = E_Q(H e^{-rT} | \mathcal{F}_t) \tag{1.10}$$

and $V_0 = E_Q(He^{-rT})$. Hence the time t discounted value of a derivative is the conditional expectation of its discounted terminal value under the risk neutral probability with respect to the information at time t. The hedging portfolio is obtained from the fact that a martingale with respect to a Brownian filtration may be written as a stochastic integral with respect to the Brownian motion, therefore

$$E_Q(He^{-rT}|\mathcal{F}_t) = V_0 + \int_0^t \psi_s dW_s^* \,,$$

and, by identification with (1.9) $\theta_s \sigma \tilde{S}_s = \psi_s$. In general, no explicit form may be given for ψ_s hence for the hedging portfolio, except in a Markovian setting. Indeed, if S is a markovian diffusion, there exists a function φ such that $E_Q(h(S_T)e^{-rT}|\mathcal{F}_t) = \varphi(t, S_t)$. Hence, Itô's formula links the hedging portfolio with the derivative of φ with respect to the underlying.

1.2.8 European options

Let us apply the preceeding method to obtain the Black and Scholes' formula which gives the price C of a call option. As the pay-off of a call option is $h(S_T)$ with $h(x) = (x - K)^+$, we get

$$C = E_Q(e^{-rT}(S_T - K)^+) = E_Q(e^{-rT} S_T \mathbb{1}_{S_T \geq K}) - Ke^{-rT} E_Q(\mathbb{1}_{S_T \geq K}) \,.$$

As $S_T e^{-rT} = S_0 \exp(\sigma W_T^* - \frac{\sigma^2 T}{2})$, some computations on Gaussian variables lead to the formula

$$C = S_0 \mathcal{N}(d_1(S_0, T)) - Ke^{-rT} \mathcal{N}(d_2(S_0, T))$$

with

$$d_1(x, T) = \frac{1}{\sigma\sqrt{T}} \ln(\frac{x}{Ke^{-rT}}) + \frac{\sigma\sqrt{T}}{2}, \; d_2(x, T) = d_1(x, T) - \sigma\sqrt{T} \,.$$

The price of the call at time t equals $C(t, S_t) = E_Q(e^{-rT}(S_T - K)^+ | \mathcal{F}_t)$ with

$$C(t, x) = x\mathcal{N}(d_1(x, T - t)) - Ke^{-r(T-t)} \mathcal{N}(d_2(x, T - t)) \,.$$

These formulas have been obtained by Bachelier at the beginning of the century. Black and Scholes' main contribution was to provide the hedging strategy. The spirit of their method is as follows. As $(S_t, t \geq 0)$ is a Markov process, the time t value of the hedging portfolio may be written in the form $H(t, S_t)$. Assuming that H is smooth enough, one may apply Itô's formula to get

$$d\widetilde{V}_t = e^{-rt}(\frac{\partial H}{\partial t} + rS_t \frac{\partial H}{\partial x} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 H}{\partial x^2} - rH)(t, S_t)dt + e^{-rt}\sigma S_t \frac{\partial H}{\partial x}(t, S_t) dW_t^* \,.$$

As $e^{-rt}H(t,S_t) = \widetilde{V}_t$ is a martingale, the drift term is equal to 0, i.e.,

$$\frac{\partial H}{\partial t} + rx\frac{\partial H}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 H}{\partial x^2} - rH = 0,\, \forall t \geq 0, \forall x > 0\,.$$

Hence the price of a call option is the solution of a partial differential equation with terminal condition $H(T,x) = h(x)$. A surprising fact is that μ does not appear in the P.D.E.. Furthermore

$$d\widetilde{V}_t = e^{-rt}\sigma S_t \frac{\partial H}{\partial x}(t,S_t)dW_t^*$$

or in an integrated form

$$\widetilde{V}_t = V_0 + \int_0^t \sigma e^{-rs} S_s \frac{\partial H}{\partial x}(s,S_s)\, dW_s^* = V_0 + \int_0^t \frac{\partial H}{\partial x}(s,S_s)\, d\tilde{S}_s\,.$$

Hence, (see (1.7)) the hedging portfolio consists of $\theta_s = \dfrac{\partial H}{\partial x}(s,S_s)$ shares of the risky asset and is therefore obtained by solving a partial differential equation. The non risky part may be computed from the relation $H(t,S_t) = \alpha_t S_t^0 + \theta_t S_t$. In the case of a call, the hedging portfolio is called the " Delta" and equals

$$\theta_t = \frac{\partial C}{\partial x}(t,S_t) = \mathcal{N}(d_1)\,.$$

1.2.9 Derivative products

Pay-offs of derivative products may depend on the asset in a fairly complicated way. The simplest derivatives have pay-offs which are function of the terminal value of the asset, S_T, hence of the form $h(S_T)$, for some function h. Derivatives' pay-offs can be path-dependent and depend on all the history of $(S_t, t \geq 0)$. For example, Asian options have a payoff equal to $(\frac{1}{T}\int_0^T S_u du - K)^+$ and thus depend on the average of S over the time interval $[0,T]$. Lookback options have a payoff equal to $(\max_{0\leq t\leq T} S_t - K)^+$ and depend on the maximum of S over the time interval.

Exotic options

Besides call options, many products are traded over the counter, especially options on exchange rates. We already mentionned Asians and lookback options. Barrier options are options which disappear when the underlying asset hits a prespecified barrier. Boost options are options which payoff depends on the time that the underlying asset spends under a barrier. In any complete market, one may use the martingale approach to price those options as the expectation under the risk neutral probability measure of their discounted payoff. The computation of the expectation may be done using specific algorithms, as Monte-Carlo methods. Unfortunately, the martingale method only gives the price of the option and not the hedging strategy, contrary to the PDE approach, which gives in a Markovian setting the hedging portfolio as the "delta" of the current price. The reader may find more details on the PDE approach to exotic products in Willmott et al. (1995).

The PDE approach may not be used for path-dependent payoffs (since the Markov property does not hold anymore). In particular, exotic options are difficult to price and to hedge even in a Black and Scholes model. Asian options prices, for example, fulfill a two dimensional PDE but they are difficult to compute.

It is worthwhile to emphasize that, in a complete market, prices can be computed, at least using a Monte-Carlo method to approximate the expectation of the discounted payoff, or a finite difference method to solve the PDE (which requires some Markov property). The hedging strategy is much more difficult to obtain.
In an incomplete market, neither prices nor hedging strategies exist.

More general models

Black and Scholes model has been generalised to stochastic volatility models, multidimensional processes, processes with jumps. Furthermore, portfolio constraints, transactions costs, may be introduced. The reader is refered to Bingham and Kiesel (1998) or Björk (1998) for more information. More advanced models can be found in Karatzas and Shreve (1998), or Shyriaev (1999).

The general idea is as follows: the fundamental theorem of asset pricing states that the market is arbitrage free if there exists at least an e.m.m. (a probability equivalent to the original probability such that discounted prices are martingales), When this probability is unique, markets are complete and the price of a derivative is the expectation of its discounted payoff under the e.m.m..

When the interest rate is stochastic, a zero-coupon is needed to complete the market.

Bibliography

[1] L. Bachelier. Thérie de la spéculation. *PhD thesis*, Paris, 1900.

[2] N. Bellamy and M. Jeanblanc. Incomplete markets with jumps. *Finance and Stochastics*, 4:209–222, 1999.

[3] B. Biais, T. Björk, J. Cvitanić, N. El Karoui, E. Jouini, and J.C. Rochet. *Financial Mathematics, Bressanone, 1996.* Ed W. Runggaldier, Lecture notes in mathematics 1656, Springer, Berlin, 1997.

[4] N El Karoui and L. Mazliak. *Backward stochastic differential equations.* Longman, Pitman research notes in Mathematics series, 364, Harlow, 1997.

[5] J. Ma and J. Yong. Forward-Backward stochastic differential equations. volume 1702 of *Lecture Notes in Maths.* Springer-Verlag, Berlin, 1999.

[6] M. Frittelli. The minimal entropy martingale measure and the valuation problem in incomplete markets. *Preprint*, 1996.

Chapter 2

Single jump and Default processes

The aim of this chapter is to provide a relatively concise overview of mathematical notions and results which underpin the valuation of defaultable claims.

Our goal is to furnish results which cover the *intensity-based* methodology. We provide a detailed analysis of the relatively simple case when the flow of informations available to an agent reduces to the observations of the random time which models the default event. The focus is on the evaluation of conditional expectations with respect to the filtration generated by a default time with the use of the intensity function. Then, we study the case when an additional information flow - formally represented by some filtration $\mathbf{F}$ - is present.

At the intuitive level, $\mathbf{F}$ is generated by prices of some assets, or by other economic factors (e.g., interest rates). Though in typical examples $\mathbf{F}$ is chosen to be the Brownian filtration, most theoretical results do not rely on such a specification of the filtration $\mathbf{F}$. Special attention is paid here to the hypothesis (H), which postulates the invariance of the martingale property with respect to the enlargement of $\mathbf{F}$ by the observations of a default time. This hypothesis prevails in the literature, and means that all the contingent claims (with or without default, in particular the $\mathbf{F}$-measurable ones) are hedgeable. We establish a representation theorem, in order to understand the meaning of complete market in a defaultable world. The main part of this chapter can be found in the surveys of Jeanblanc and Rutkowski [14, 15]. For a complete study of credit risk, see the forthcoming book of Bielecki and Rutkowski [3].

2.1 A toy model

Let us begin with an easy example. A riskless asset, with deterministic interest rate $(r(s); s \geq 0)$ is the only asset available on the default-free market. We denote as usual by $R(t) = \exp - \int_0^t r(s)ds$ the discount factor. The price of a zero-coupon with maturity T is $P(0,T) = R(T)$, whereas the t-time price $P(t,T)$ of a zero-coupon with maturity T is $P(t,T) = R_T^t \stackrel{def}{=} \exp - \int_t^T r(s)ds$.
The default appears at time τ (where τ is a non-negative random variable with density f). We denote by F the right-continuous cumulative function of τ defined as $F(t) = P(\tau \leq t)$ and we assume that $F(t) < 1$ for any $t < T$, where T is the maturity date.

We emphasize that we do not use the term “risk-neutral” probability nor “e.m.m.” in what follows because the risk is not hedgeable for the moment.

2.1.1 Payment at Maturity

A defaultable zero-coupon with maturity T, and rebate paid at maturity, consists of

- the payment of one monetary unit at time T if the default has not appeared before time T,
- A payment of δ, done at maturity, if $\tau < T$, where $0 < \delta$. The case $\delta > 1$ can be viewed as a life insurance with a payment at τ, the death time.

Value of the defaultable zero-coupon

The value of the defaultable zero-coupon is

$$\begin{aligned} P_d(0,T) &= E(R(T)\,[\mathbb{1}_{T<\tau} + \delta\mathbb{1}_{\tau\le T}]) \\ &= E(R(T)\,[1-(1-\delta)\mathbb{1}_{\tau\le T}]) \\ &= P(0,T) - (1-\delta)R(T)F(T)\,. \end{aligned}$$

The t-time value depends whether or not the default is appeared before this time. If the default has appeared before time t, the payment of δ will be done, and the value of the defaultable zero coupon is δR_T^t. If the default has not yet appeared, nobody knows when it will appear. The value of the defaultable zero-coupon is the expectation of the discounted payoff $R_T^t\,[\mathbb{1}_{T<\tau} + \delta\mathbb{1}_{\tau\le T}]$ knowing that $t < \tau$, i.e.

$$\begin{aligned} P_d(t,T) &= E(R_T^t\,[\mathbb{1}_{T<\tau} + \delta\mathbb{1}_{\tau\le T}]\,|t<\tau) \\ &= R_T^t(1-(1-\delta)P(\tau\le T|t<\tau)) \\ &= R_T^t(1-(1-\delta)\frac{P(t<\tau\le T)}{P(t<\tau)}) \\ &= P(t,T) - R_T^t(1-\delta)\frac{F(T)-F(t)}{1-F(t)}\,. \end{aligned}$$

Let us remark that the value of the defaultable zero-coupon is discontinuous at time τ, except if $F(T) = 1$. In this case, the default appears with probability one before maturity.

The payment can be a function of the default time, i.e. $\delta(\tau)$. In that case, the value of this defaultable zero-coupon is

$$\begin{aligned} P_d(0,T) &= E(R(T)\,\mathbb{1}_{T<\tau} + R(T)\delta(\tau)\mathbb{1}_{\tau\le T}) \\ &= P(T<\tau)R(T) + R(T)\int_0^T \delta(s)f(s)ds\,. \end{aligned}$$

The t-time value $P_d(t,T)$ satisfies, if the default has not occured before t

$$\begin{aligned} R(t)P_d(t,T) &= E(R(T)\,\mathbb{1}_{T<\tau} + R(T)\delta(\tau)\mathbb{1}_{\tau\le T}|(t<\tau)) \\ &= \frac{P(T<\tau)}{P(t<\tau)}R(T) + \frac{R(T)}{P(t<\tau)}\int_t^T \delta(s)f(s)ds\,. \end{aligned}$$

Particular case

In the case where F is differentiable, let us introduce the function γ defined as $\gamma(t) = \dfrac{f(t)}{1-F(t)}$ where $f(t) = F'(t)$, i.e.,

$$1 - F(t) = \exp\left(-\int_0^t \gamma(s)ds\right)\,.$$

Then,

$$P_d(t,T) = P(t,T) - (1-\delta)R_T^t \left(1 - \exp(-\int_t^T \gamma(s)ds)\right) .$$

Therefore the default appears as a spread on interest rate.

In particular, if the law of τ is an exponential law with parameter λ, i.e., $1 - F(t) = e^{-\lambda t}$, we get $\gamma = \lambda$ and

$$P_d(t,T) = P(t,T) - (1-\delta)R_T^t(1 - e^{-\lambda(T-t)}) .$$

2.1.2 Payment at hit

A defaultable zero-coupon with maturity T consists of

- The payment of one monetary unit at time T if the default has not yet appeared,
- A payment of $\delta(\tau)$, where δ is a deterministic function, done at time τ if $\tau < T$.

Value of the defaultable zero-coupon

The value of this defaultable zero-coupon is

$$\begin{aligned} P_d(0,T) &= E(R(T)\,\mathbb{1}_{T<\tau} + R(\tau)\delta(\tau)\mathbb{1}_{\tau\le T}) \\ &= P(T<\tau)R(T) + \int_0^T R(s)\delta(s)dF(s) . \end{aligned}$$

The t-time value $P_d(t,T)$ satisfies, if the default has not occured before t

$$\begin{aligned} R(t)P_d(t,T) &= E(R(T)\,\mathbb{1}_{T<\tau} + R(\tau)\delta(\tau)\mathbb{1}_{\tau\le T}|(t<\tau)) \\ &= \frac{P(T<\tau)}{P(t<\tau)}R(T) + \int_t^T R(s)\delta(s)\frac{dF(s)}{1-F(t)} \end{aligned}$$

Hence,

$$R(t)G(t)P_d(t,T) = G(T)R(T) - \int_t^T R(s)\delta(s)dG(s)$$

with $G = 1 - F$. If $\tau < t$, the payment of $\delta(\tau)$ was done, and if this payment is not re-invested in the market, the value of P_d is 0.

Let us introduce the function $\Gamma(t) = -\ln(1 - F(t))$. Then,

$$P_d(0,T) = e^{-\Gamma(T)}R(T) + \int_0^T R(s)e^{-\Gamma(s)}\delta(s)d\Gamma(s) . \tag{2.1}$$

The t-time value $P_d(t,T)$ satisfies

$$R(t)e^{-\Gamma(t)}P_d(t,T) = e^{-\Gamma(T)}R(T) + \int_t^T R(s)e^{-\Gamma(s)}\delta(s)d\Gamma(s)$$

Particular case

if F is differentiable, the function $\gamma = \Gamma'$ satisfies $f(t) = \gamma(t)e^{-\Gamma(t)}$. Then,

$$P_d(0,T) = e^{-\Gamma(T)}R(T) + \int_0^T R(s)\gamma(s)e^{-\Gamma(s)}\delta(s)ds , \tag{2.2}$$

and

$$R_d(t)P_d(t,T) = R_d(T) + \int_t^T R_d(s)\gamma(s)\delta(s)ds$$

with $R_d(t) = \exp - \int_0^t [r(s) + \gamma(s)]ds$.

Note that in the case where τ is the first jump of a Poisson process with deterministic intensity γ,

$$f(t) = P(\tau \in dt) = \gamma(t)\exp\left(-\int_0^t \gamma(s)ds\right) = \gamma(t)e^{-\Gamma(t)}$$

and $P(\tau < t) = F(t) = 1 - e^{-\Gamma(t)}$.

2.1.3 Risk neutral probability measure, martingales

It is usual to translate the absence of arbitrage opportunities as the existence of a risk-neutral probability, i.e., a probability such that the discounted assets are martingales. Here, so far, we do not have any martingale, since we do not have a reference filtration. The only source of noise is the default time. As we have noticed, the t-time value of the defaultable zero-coupon takes different form whether or not the default has appeared before time t. If the default has not appeared, the value of the zero-coupon is deterministic, if the default has appeared, the value depends on this time.

In a more mathematical language, we shall model the information by the filtration $(\mathcal{D}_t, t \geq 0)$ generated by the process $D_t = 1\!\!1_{\tau \leq t}$. Any random variable $\mathcal{D}_t$-measurable is equal to a deterministic constant on the set $\{t < \tau\}$ and to a function of τ on the set $\{t \geq \tau\}$.
The set of equivalent martingale measure is the set of probability, equivalent to the historical one, such that the discounted price of the tradeable assets are martingales.
Now, the problem is to define the tradeable assets. If the primary market consists only of the risk-free asset, there exists infinitely many e.m.m. : if the support of the density of τ is $I\!\!R^+$, any probability such that τ admits a density with support $I\!\!R^+$ is a change of law. In particular, the range of prices of a defaultable zero-coupon bond is trivial and equals $]0, R_T[$ (obtained as limit cases : the default appears at time 0^+, or never.)
If there exist tradeable defaultable zero-coupons (DZC in short) , their prices are given by the market, and the equivalent martingale measure Q is such that, on the set $t < \tau$,

$$P_d(t,T) = R_T^t E_Q\left([1\!\!1_{T<\tau} + \delta 1\!\!1_{t<\tau\leq T}]\,|(t<\tau)\right) = R_T^t E_Q\left([1\!\!1_{T<\tau} + \delta 1\!\!1_{\tau\leq T}]\,|\mathcal{D}_t\right).$$

Therefore, we can characterize the cumulative function of τ under Q from the market prices of the DZC as follows.

Fixed Payment at maturity

If the prices of defaultable zero-coupon with different maturities are known, then

$$\frac{P(0,t) - P_d(0,t)}{R(t)(1-\delta)} = F(t)$$

and $F(t) = Q(\tau \leq t)$, so that the law of τ is known under the e.m.m. However, as noticed in Hull and White,*extracting default probabilities from bond prices [is] in practice, usually more complcated. First, the recovery rate is usually non-zero. Second, most corporate bonds are not zero-coupon bonds* [12].

Payment at hit

In this case the cumulative function can be obtained using the derivative of the defaultable zero-coupon price with respect to the maturity. Indeed, we get

$$\frac{\partial}{\partial T}P_d(0,t) = g(t)R(t)(1-\delta(t)) - R(t)G(t)r(t), \forall t$$

therefore, solving this equation leads to

$$1 - Q(\tau \le t) = \int_0^t \partial_T P_d(0,s) \exp\left(-\int_0^s \frac{r(u)}{1-\delta(u)}du\right) ds\,.$$

Martingales

Processes of the form $E(X|\mathcal{D}_t)$ are obviously martingales. A question is now to find a generating martingale, such that any martingale can be written as a stochastic integral with respect to the generating one. The process

$$D_t - \Lambda(t \wedge \tau)$$

where $\Lambda(t) = -\ln(1 - F(t-))$ is this generating martingale. We shall check that this process is indeed a martingale in the following section.

Example 2.1.1 Let us study the case where τ is the first time where a Poisson process N jumps. Let $\lambda(s)$ be the deterministic intensity and $D_t = N_{t\wedge\tau}$. It is well known that $N_t - \int_0^t \lambda(s)ds$ is a martingale. Therefore, the process stopped at time τ is also a martingale, i.e., $D_t - \int_0^{t\wedge\tau} \lambda(s)ds$ is a martingale. Furthermore, we have seen that we can reduce our attention to that case.

2.2 Successive default times

The previous results can easily be generalized to the case of successive default times. We assume here that $r = 0$.

2.2.1 Two times

Let us start with the case with two random times τ_1, τ_2. We denote by $T_1 = \inf(\tau_1, \tau_2)$ and $T_2 = \sup(\tau_1, \tau_2)$, and we assume, for simplicity that $P(\tau_1 = \tau_2) = 0$. We denote by $(D^i_t, t \ge 0)$ the default process associated with $(T_i, i = 1, 2)$, and by $D_t = D^1_t + D^2_t$ the process associated with two defaults. As usual $\mathbf{D}$ is the filtration generated by the process D. The σ-algebra $\mathcal{D}_t$ is equal to $\sigma(T_1 \wedge t) \vee \sigma(T_2 \wedge t)$.

A $\mathcal{D}_t$-measurable random variable is equal to a constant on the set $t < T_1$, equal to a $\sigma(T_1)$-measurable random variable on the set $T_1 \le t < T_2$, and to a $\sigma(T_2)$-measurable random variable on the set $T_2 \le t$.

Payment at maturity

Suppose that a payment of 1 monetary unit is done at maturity if no default has appeared, δ_1 if one and only one default has appeared and δ_2 in the remaining case, where $0 \le \delta_2 < \delta_1 < 1$. Therefore, to obtain the value of this claim we have to compute

$$E(\mathbb{1}_{T<T_1} + \delta_1 \mathbb{1}_{T_1 \le T < T_2} + \delta_2 \mathbb{1}_{T_2 \le T}|\mathcal{D}_t)$$

It is then straightforward to prove, for $t < T$

$$\begin{aligned} P(T < T_1|\mathcal{D}_t) &= 1\!\!1_{t<T_1}\frac{P(T<T_1)}{P(t<T_1)} \\ P(T_1 \le T < T_2|\mathcal{D}_t) &= 1\!\!1_{t<T_1}\frac{P(t<T_1\le T<T_2)}{P(t<T_1)} + 1\!\!1_{T_1\le t<T_2}\frac{P(T<T_2|T_1)}{P(t<T_2|T_1)} \\ P(T_2 \le T|\mathcal{D}_t) &= 1\!\!1_{t<T_1}\frac{P(t\le T_1<T_2<T)}{P(t<T_1)} + 1\!\!1_{T_1\le t<T_2}\frac{P(t<T_2<T|T_1)}{P(t<T_2|T_1)} + 1\!\!1_{T_2<t} \end{aligned}$$

In the very particular case where τ_1 and τ_2 are independent with density f and g, the law of the pair T_1, T_2 is easy to compute

$$P(T_1 \in du, T_2 \in dv) = 1\!\!1_{u<v}[f(u)g(v) + g(u)f(v)]du\,dv$$

and the conditional law of T_2 with respect to T_1 follows

$$P(T_2 \in dv|T_1 = u) = 1\!\!1_{u<v}\frac{f(u)g(v)+g(u)f(v)}{\int_0^\infty dv[f(u)g(v)+g(u)f(v)]}dv$$

Payment at hit

Suppose that a payment of 1 monetary unit is done at maturity if no default has appeared, $\delta_1(T_1)$ if the first default arrives before T (and the second after T) and $\delta_2(T_2)$ in the remaining case, where $0 \le \delta_2 < \delta_1 < 1$. Therefore, we have to compute

$$E(1\!\!1_{T<T_1} + \delta_1(T_1)1\!\!1_{T_1\le T<T_2} + \delta_2(T_2)1\!\!1_{T_2\le T}|\mathcal{D}_t)$$

It is then straightforward to prove

$$\begin{aligned} E(\delta_1(T_1)1\!\!1_{T_1\le T<T_2}|\mathcal{D}_t) &= 1\!\!1_{t<T_1}\frac{E(\delta_1(T_1)1\!\!1_{t<T_1\le T<T_2})}{P(t<T_1)} + 1\!\!1_{T_1\le T<T_2}\delta(T_1)\,\frac{P(T<T_2|T_1)}{P(t<T_2|T_1)} \\ E(\delta_2(T_2)1\!\!1_{T_2\le T}|\mathcal{D}_t) &= 1\!\!1_{t<T_1}\frac{E(\delta_2(T_2)1\!\!1_{t\le T_1<T_2<T})}{P(t<T_1)} + 1\!\!1_{T_1\le t<T_2}\frac{E(\delta_2(T_2)1\!\!1_{t<T_2<T|T_1})}{P(t<T_2|\tau_1)} \\ &\quad + 1\!\!1_{T_2<t}\delta_2(T_2) \end{aligned}$$

In order to compute the different terms, we need the joint law as well as the conditional joint law.

2.2.2 Copulas

A recent approach [5] for modeling dependent credit risks is the use of copulas.

Definition 2.2.1 *A copula C is a joint cumulative distribution of n random variables uniformly distributed on* $[0, 1]$

Let F be an n-dimensional cumulative distribution with continuous margin F_i. Then there exists a copula C such that

$$H(x) = C(F_1(x_1), \cdots, F_n(x_n))$$

In the case where n companies are studied, and if F_i is the cumulative distribution for the default of the ith company, the probability of all companies defaulting is

$$P(\tau_i \le T, \forall i) = C(F_1(T), \cdots, F_n(T))$$

2.2.3 More than two times

Suppose that the default times are modeled via a Poisson process with intensity h. The terminal payoff is $\prod_{T_i \leq T}(1-\delta(T_i))$. The value of this payoff is $E(\prod_{T_i \leq T}(1-\delta(T_i)))$. In the case of constant $\delta(s) = \delta$, we get

$$E(\prod_{T_i \leq T}(1-\delta(T_i))) = E((1-\delta)^{N_T}) = \exp(\delta \int_0^T h(s)ds)\,.$$

In the general case,

$$E(\prod_{T_i \leq T}(1-\delta(T_i)) = \exp(\int_0^T \delta(s)h(s)ds)\,.$$

2.3 Elementary martingale

We now present the result of the previous section in a different form. Let us start with some well known facts. Suppose that τ is an $\overline{I\!R}^*_+$-valued random variable on some probability space $(\Omega, \mathcal{G}, P)$. Let us denote by D_t the right-continuous increasing process $D_t = 1\!\!1_{t \geq \tau}$ and by $(\mathcal{D}_t)$ the natural filtration generated by the process D. The filtration $\mathbf{D}$ is the smallest filtration (satisfying the usual hypotheses) on Ω such that τ is an $\mathbf{D} = (\mathcal{D}_t, t \geq 0)$-stopping time. The σ-algebra $\mathcal{D}_t$ is generated by the sets $\tau \leq s$ for $s \leq t$ and the atom $\tau > t$.

Any integrable r.v. H, $\mathcal{D}_t$ measurable, is of the form $h(\tau \wedge t) = h(\tau)1\!\!1_{\tau \leq t} + h(t)1\!\!1_{t<\tau}$ where h is a Boreal function.

Lemma 2.3.1 *If X is any integrable, $\mathcal{G}$-measurable r.v.* $E(X|\mathcal{D}_t)1\!\!1_{t<\tau} = 1\!\!1_{t<\tau}E(X1\!\!1_{t<\tau})/P(t<\tau)$.

PROOF: The r.v. $E(X|\mathcal{D}_t)$ is $\mathcal{D}_t$-measurable, therefore it can be written on the form $E(X|\mathcal{D}_t) = Y_t 1\!\!1_{t \geq \tau} + A 1\!\!1_{t<\tau}$ where A is constant. By multiplying both members by $1\!\!1_{t<\tau}$, and taking the expectation, we obtain

$$E[1\!\!1_{t<\tau}E(X|\mathcal{D}_t)] = E[E(1\!\!1_{t<\tau}X|\mathcal{D}_t)] = E[1\!\!1_{t<\tau}X] = AP(t<\tau)$$

□

Let $F(t) = P(\tau \leq t)$ the right-continuous cumulative function of τ. We will denote by $F(t-)$ the left limit of $F(s)$ when s goes to t .

Proposition 2.3.1 *The process*

$$M_t = D_t - \int_0^{\tau \wedge t} \frac{dF(s)}{1-F(s-)} = D_t - \int_0^t (1-D_s)\frac{dF(s)}{1-F(s-)}$$

is a $\mathbf{D}$*-martingale.*

PROOF: From the previous lemma $E(D_t - D_s|\mathcal{D}_s) = E(1\!\!1_{s<\tau\leq t}|\mathcal{D}_s) = 1\!\!1_{s<\tau}A + 1\!\!1_{s\geq\tau}B$. We have proved that the constant A is $A = \dfrac{P(s<\tau\leq t)}{P(s<\tau)} = \dfrac{F(t)-F(s)}{1-F(s)}$. Multiplying the two members of the previous equality by $1\!\!1_{s\geq\tau}$, we obtain that the r.v. B is equal to 0. On the other hand,

$$E\left[\int_{\tau\wedge s}^{\tau\wedge t} \frac{dF(u)}{1-F(u-)}|\mathcal{D}_s\right] = 1\!\!1_{\tau>s}g(s)$$

where

$$\begin{aligned} g(s) &= \frac{1}{1-F(s)} E\left[1\!\!1_{\tau>s} \int_{s\wedge\tau}^{t\wedge\tau} \frac{DFL(u)}{1-F(u-)} \right] \\ &= \frac{1-F(t)}{1-F(s)} \int_s^t \frac{DFL(u)}{1-F(u-)} - \frac{1}{1-F(s)} \int_s^t DFL(v) \int_s^v \frac{DFL(u)}{1-F(u-)} . \end{aligned}$$

Fubini's theorem proves the equality $g(s) = \dfrac{F(t)-F(s)}{1-F(s)}$. □

Successive default times

We reproduce now the result of [4], in order to obtain the martingales in the filtration $\mathbf{G}$, in case of two default time.
Let us denote by $F_1(t)$ the distribution function of T_1 and by $F_2(t;u)$ the conditional distribution function of T_2 with respect to T_1, i.e.,

$$F_2(t;u) = P(T_2 \le t | T_1 = u)$$

The process $M_t \overset{def}{=} D_t - \Lambda_{t\wedge\tau}$ is a $\mathbf{D}$-martingale, where

$$\Lambda_t = \Lambda_1(t) 1\!\!1_{t<T_1} + [\Lambda_1(T_1) + \Lambda_2(t,T_1)]\, 1\!\!1_{T_1 \le t < T_2} + [\Lambda_1(T_1) + \Lambda_2(T_2,T_1)]\, 1\!\!1_{T_2 \le t}$$

with $\Lambda_1(t) = \displaystyle\int_0^t \frac{dF_1(s)}{1-F_1(s)}$ and $\Lambda(t;u) = \displaystyle\int_u^t \frac{dF_2(s;u)}{1-F_2(s;u)}$. It is proved in [4] that any $\mathbf{D}$-martingale is a stochastic integral with respect to M. This result admits an immediate extension to the case of n successive defaults.

2.3.1 Intensity process

The function $\Lambda_t = \displaystyle\int_0^t \frac{dF(s)}{1-F(s-)}$ is continuous and increasing. Therefore, we have obtained the decomposition of the submartingale D_t as $M_t + \Lambda(t\wedge\tau)$. The process $A_t = \Lambda_{t\wedge\tau}$ is called the compensator of D.
In particular, if F is differentiable, τ admits a density $f = F'$, and the process

$$M_t = D_t - \int_0^{\tau\wedge t} \lambda(s) ds$$

is a martingale, where $\lambda(s) = \dfrac{dF(s)}{1-F(s)} = \dfrac{f(s)}{1-F(s)}$ is a deterministic positive function, called the intensity.

Proposition 2.3.2

$$E(1\!\!1_{\tau>T} | \mathcal{D}_t) = 1\!\!1_{\tau>t} \exp\left(-\int_t^T \lambda(s) ds \right)$$

PROOF: Let $T > t$. From the lemma $E(1\!\!1_{\tau>T}|\mathcal{D}_t) = 1\!\!1_{t<\tau} A$ where A is a constant equal to $\dfrac{P(T<\tau)}{P(t<\tau)} = \dfrac{1-F(T)}{1-F(t)}$ □

This result gives the value of a defaultable zero-coupon bond $E(\mathbb{1}_{\tau>T}|\mathcal{D}_t)$ when the agent knows only the information $\mathcal{D}_t$ and when the spot rate is equal to 0.

If a zero-coupon is tradeable at a price ρ_t, then, under any risk-neutral probability $\rho_t R(t) = E(R(T)\mathbb{1}_{T<\tau}|t<\tau) = \mathbb{1}_{t<\tau}\exp\left(-\int_t^T \lambda(s)ds\right)$ therefore, the value of $\int_0^u \lambda(s)ds$ is known as soon as there are zero-coupon for each maturity, and the risk-neutral intensity can be obtained from the market data.

Proposition 2.3.3 *The process* $L_t \stackrel{def}{=} \mathbb{1}_{\tau>t}\exp\left(\int_0^t \lambda(s)ds\right)$ *is a* $\mathbf{D}$*-martingale.*

PROOF: This result is obvious from the previous proposition. Since the function λ is deterministic,

$$E(L_t|\mathcal{D}_s) = \left(\exp\int_0^t \lambda(u)du\right) E(\mathbb{1}_{t<\tau}|\mathcal{H}_s) = \mathbb{1}_{\tau>s}\exp\int_0^s \lambda(u)du\,.$$

Another way is to apply Itô's formula to the process $L_t = (1-N_t)\exp\int_0^t \lambda(s)ds$

$$dL_t = -dD_t\exp\int_0^t \lambda(s)ds + \lambda(t)[\exp\int_0^t \lambda(s)ds](1-N_t)dt = -[\exp\int_0^t \lambda(s)ds]\,dM_t\,.$$

A sophisticated way is to note that L is the exponential martingale solution of the SDE $dL_t = -L_t dM_t$. □

2.3.2 Representation theorem

Proposition 2.3.4 *Let h be a function. Then*

$$E(h(\tau)|\mathcal{D}_t) = x + \int_0^{t\wedge\tau} e^{\Delta\Gamma(s)}(H_{s-} - h(s))(dD_s - d\Lambda(s))\,.$$

PROOF: Let $H_t = E(h(\tau)|\mathcal{D}_t)$. From the measurability argument

$$H_t = h(\tau)\mathbb{1}_{\tau\le t} + \mathbb{1}_{t<\tau}\frac{E(h(\tau)\mathbb{1}_{t<\tau})}{P(t<\tau)}\,.$$

An integration by parts yields to

$$\begin{aligned}
e^{\Gamma(t)}E(h(\tau)\mathbb{1}_{t<\tau}) &= e^{\Gamma(t)}\int_t^\infty h(u)dF(u)\\
&= x - \int_0^t e^{\Gamma(s-)}h(s)dF(s) + \int_0^t E(h(\tau\mathbb{1}_{s<\tau})e^{\Gamma(s)}d\Gamma(s)\\
&= x - \int_0^t e^{\Gamma(s-)}h(s)DFL(s) + \int_0^t H_s d\Gamma(s) + \int_0^t H_s\frac{DFL_s}{1-F(s)}\,.
\end{aligned}$$

Hence

$$E(h(\tau)|\mathcal{D}_t) = x + \int_0^{t\wedge\tau}(H_{s-} - h(s))\frac{dF(s)}{1-F(s)}\,.$$

2.3.3 Partial information

Duffie and Lando's result

Duffie and Lando [8] assume that $\tau = \inf\{t : V_t \leq m\}$ where V satisfies

$$dV_t = \mu(t, V_t)dt + \sigma(t, V_t)dW_t$$

Here the process W is a Brownian motion. If the information is the Brownian filtration, the time τ is a stopping time w.r.t. a Brownian filtration, therefore is predictable and admits no intensity. We will discuss this point latter on. If the agents do not know the behavior of V, but only the minimal information $\mathcal{D}_t$, i.e. he knows when the default appears, the price of a zero-coupon is, in the case where the default is not yet appeared, $\exp - \int_t^T \lambda(s)ds$ where $\lambda(s) = \dfrac{f(s)}{G(s)}$ and $G(s) = P(\tau > s), f = -G'$, as soon as the cumulative function of τ is differentiable. Duffie and Lando have obtained that the intensity is $\lambda(t) = \dfrac{1}{2}\sigma^2(t,0)\dfrac{\partial f}{\partial x}(t,0)$ where $f(t,x)$ is the conditional density of V_t when $T_0 > t$, i.e. the differential w.r.t. x of $\dfrac{P(V_t \leq x, T_0 > t)}{P(T_0 > t)}$, where $T_0 = \inf\{t\,;\, V_t = 0\}$. In the case where V is an homogenous diffusion, i.e. $dV_t = \mu(V_t)dt + \sigma(V_t)dW_t$, the equality between Duffie-Lando and our result is obvious. In fact, Duffie and Lando proved that λ_s is the limit, when h goes to 0 of

$$\frac{1}{hP(T_0 > t)} \int_0^\infty P(V_t \in dx, T_0 > t)P_x(T_0 < h)$$

This last quantity can be written as

$$\begin{aligned}
& \frac{1}{hP(T_0 > t)} \int_0^\infty P(V_t \in dx, T_0 > t)(1 - P_x(T_0 > h)) \\
= \quad & \frac{1}{hP(T_0 > t)} \left[P(T_0 > t) - \int_0^\infty P(V_t \in dx, T_0 > t + h)) \right] \\
= \quad & \frac{1}{hP(T_0 > t)} (P(T_0 > t) - P(T_0 > t + h))
\end{aligned}$$

Extensions

The previous problem admits an obvious extension to the case where the observation is $\mathcal{F}_t = \sigma(V_s, s \leq [t])$ where $[t]$ is the greatest integer smaller than t, which corresponds to observation at discrete times.

2.4 Cox Processes and Extensions

2.4.1 Construction of Cox Processes with a given stochastic intensity

Let $(\Omega, \mathcal{G}, P)$ be a probability space, and $(X_t, t \geq 0)$ a continuous diffusion process on this space. We denote by $\mathbf{F}$ its canonical filtration, satisfying the usual conditions. *A nonnegative* $\mathbf{F}$*-adapted process* λ *is given.* We assume that there exists a random variable Θ, independent of X, with an exponential law: $P(\Theta \geq t) = e^{-t}$. We define the random time τ as the first time when the process $\Lambda_t = \int_0^t \lambda_s \, ds$ is above the random level Θ, i.e.,

$$\tau = \inf \{t \geq 0 \,:\, \Lambda_t \geq \Theta\}.$$

We assume that $\Lambda_t < \infty, \Lambda_\infty = \infty$.

The mutual independence of Θ and X will avoid us to enter in the enlargement of filtration's world. This will be done in a next section, in the general case, that is, when the independence hypothesis is relaxed.

Another example is to choose $\tau = \inf\{t \geq 0 : \tilde{N}_{\Lambda_t} = 1\}$, where $\Lambda_t = \int_0^t \lambda_s \, ds$ and $\tilde{N}$ is a Poisson process with intensity 1, independent of the filtration $\mathbf{F}$. The second method is in fact equivalent to the first. Cox processes are used in a great number of studies (see, e.g., [18]). Obviously, this model extends to the case where the intensity is given as a $\mathbf{F}$-adapted increasing process.

Remark 2.4.1 In Wong [22], the time of default is given as

$$\tau = \inf\{t \,:\, \Lambda_t \geq \Sigma\}$$

where Σ a non-negative r.v. independent of $\mathcal{F}_\infty$. This model reduces to the previous one: if Φ is the cumulative function of Σ, the r.v. $\Phi(\Sigma)$ has a uniform distribution and

$$\tau = \inf\{t \,:\, \Phi(\Lambda_t) \geq \Phi(\Sigma)\} = \inf\{t \,:\, \Psi^{-1}[\Phi(\Lambda_t)] \geq \Sigma\}$$

where Ψ is the cumulative function of the exponential law, and

$$F_t = P(\tau \leq t|\mathcal{F}_t) = P(\Lambda_t \geq \Sigma|\mathcal{F}_t) = \Phi(\alpha_t)\,.$$

See also [2] for an extension of this model.

2.4.2 Conditional Expectations

Lemma 2.4.1 *The conditional distribution function of τ given the σ-field $\mathcal{F}_t$ is for $t \geq s$*

$$P(\tau > s|\mathcal{F}_t) = \exp\Big(-\Lambda_s\Big).$$

Proof: The proof follows from the equality $\{\tau > s\} = \{\Lambda_s > \Theta\}$. From the independence assumption and the $\mathcal{F}_t$-measurability of Λ_s for $s \leq t$, we obtain

$$P(\tau > s|\mathcal{F}_t) = P\Big(\Lambda_s \geq \Theta \,\Big|\, \mathcal{F}_t\Big) = \exp\Big(-\Lambda_s\Big).$$

In particular, we have

$$P(\tau \leq t|\mathcal{F}_t) = P(\tau \leq t|\mathcal{F}_\infty), \tag{2.3}$$

and, for $t \geq s$, $P(\tau > s|\mathcal{F}_t) = P(\tau > s|\mathcal{F}_s)$. Let us notice that the process $F_t = P(\tau \leq t|\mathcal{F}_t)$ is here an increasing process. □ The particular case where τ is independent of $\mathcal{F}_\infty$ is interesting. In that case, Λ is a deterministic increasing function.

Remark 2.4.2 If the process λ is not non-negative, we get, for $s < t$

$$P(\tau > s|\mathcal{F}_t) = \exp(-\sup_{u \leq s} \Lambda_u)\,.$$

We write $D_t = 1\!\!1_{\{\tau \leq t\}}$ and $\mathcal{D}_t = \sigma(D_s \,:\, s \leq t)$. We introduce the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$, that is, the enlarged filtration generated by the underlying filtration $\mathbf{F}$ and the process D. (We denote by $\mathbf{F}$ the original **F**iltration and by $\mathbf{G}$ the enlar**G**ed one.) We shall frequently write $\mathbf{G} = \mathbf{F} \vee \mathbf{D}$.

It is easy to describe the events which belong to the σ-field $\mathcal{G}_t$ on the set $\{\tau > t\}$. Indeed, if $G_t \in \mathcal{G}_t$, then $G_t \cap \{\tau > t\} = B_t \cap \{\tau > t\}$ for some event $B_t \in \mathcal{F}_t$.

Therefore any $\mathcal{G}_t$-measurable random variable Y_t satisfies $1\!\!1_{\{\tau > t\}} Y_t = 1\!\!1_{\{\tau > t\}} y_t$, where y_t is an $\mathcal{F}_t$-measurable random variable.

Proposition 2.4.1 *Let Y be an integrable r.v. Then,*

$$1\!\!1_{\{\tau>t\}}E(Y|\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}\frac{E(Y1\!\!1_{\{\tau>t\}}|\mathcal{F}_t)}{E(1\!\!1_{\{\tau>t\}}|\mathcal{F}_t)} = 1\!\!1_{\{\tau>t\}}e^{\Lambda_t}E(Y1\!\!1_{\{\tau>t\}}|\mathcal{F}_t).$$

PROOF: From the remarks on the $\mathcal{G}_t$-measurability, if $Y_t = E(Y|\mathcal{G}_t)$, then there exists y_t, which is $\mathcal{F}_t$-measurable such that

$$1\!\!1_{\{\tau>t\}}E(Y|\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}y_t$$

and multiplying both members by the indicator function, we deduce $y_t = \frac{E(Y1\!\!1_{\{\tau>t\}}|\mathcal{F}_t)}{E(1\!\!1_{\{\tau>t\}}|\mathcal{F}_t)}$. □

We shall now compute the expectation of a predictable process at time τ and we shall give the intensity of τ.

Lemma 2.4.2 (i) *If h is a* $\mathbf{F}$*-predictable process then*

$$\begin{aligned} E(h_\tau) &= E\Big(\int_0^\infty h_u\lambda_u \exp\big(-\Lambda_u\big)\,du\Big) \\ E(h_\tau|\mathcal{F}_t) &= E\Big(\int_0^\infty h_u\lambda_u \exp\big(-\Lambda_u\big)\,du\,\Big|\,\mathcal{F}_t\Big) \end{aligned}$$

and

(2.4) $$E(h_\tau|\mathcal{G}_t) = E\Big(\int_t^\infty h_u\lambda_u \exp\big(\Lambda_t-\Lambda_u\big)\,du\,\Big|\,\mathcal{F}_t\Big)1\!\!1_{\{\tau>t\}} + h_\tau 1\!\!1_{\{\tau\le t\}}.$$

(ii) *The process $(D_t - \int_0^{t\wedge\tau}\lambda_s ds, t\ge 0)$ is a* $\mathbf{G}$*-martingale.*

Definition 2.4.1 *The process λ is called the intensity of τ.*

PROOF: Let $h_t = 1\!\!1_{]v,w]}(t)B_v$ where $B_v \in \mathcal{F}_v$. Then,

$$\begin{aligned} E(h_\tau|\mathcal{F}_t) &= E\Big(E(1\!\!1_{]v,w]}(\tau)B_v|\mathcal{F}_\infty)\,\Big|\,\mathcal{F}_t\Big) = E\Big(B_v(e^{-\Lambda_v}-e^{-\Lambda_w})|\mathcal{F}_t\Big) \\ &= E\Big(B_v\int_v^w \lambda_u e^{-\Lambda_u}\,du\,\Big|\,\mathcal{F}_t\Big) = E\Big(\int_0^\infty h_u\lambda_u e^{-\Lambda_u}du\,\Big|\,\mathcal{F}_t\Big) \end{aligned}$$

and the result follows from the monotone class theorem.

The martingale property (ii) follows from integration by parts formula. Let $t < s$. Then, on the one hand

$$\begin{aligned} E(D_s - D_t|\mathcal{G}_t) &= P(t<\tau\le s|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}\frac{P(t<\tau\le s|\mathcal{F}_t)}{P(t<\tau|\mathcal{F}_t)} \\ &= 1\!\!1_{\{t<\tau\}}E(1-\exp(\Lambda_s-\Lambda_t)|\mathcal{F}_t) \end{aligned}$$

On the other hand, from part (i)

$$E\Big(\int_{t\wedge\tau}^{s\wedge\tau}\lambda_u du\,\Big|\,\mathcal{G}_t\Big) = E(\Lambda_{s\wedge\tau}-\Lambda_{t\wedge\tau}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}E\Big(\int_t^\infty h_u\lambda_u e^{-(\Lambda_u-\Lambda_t)}du\,\Big|\,\mathcal{F}_t\Big)$$

where $h_u = \Lambda(s\wedge u) - \Lambda(t\wedge u)$. Consequently,

$$\begin{aligned} \int_t^\infty h_u\lambda_u e^{-(\Lambda_u-\Lambda_t)}du &= \int_t^s(\Lambda_u-\Lambda_s)\lambda_u e^{-(\Lambda_u-\Lambda_t)}du + (\Lambda_t-\Lambda_s)\int_s^\infty \lambda_u e^{-(\Lambda_u-\Lambda_t)}du \\ &= -(\Lambda_s-\Lambda_t)e^{-(\Lambda_s-\Lambda_t)} + \int_t^s\lambda(u)e^{-(\Lambda_u-\Lambda_t)}du + (\Lambda_s-\Lambda_t)e^{-(\Lambda_s-\Lambda_t)} \\ &= 1-e^{-(\Lambda_s-\Lambda_t)}. \end{aligned}$$

This ends the proof. □ If τ is independent of $\mathcal{F}_\infty$

$$E(h_\tau \mathcal{G}_t) = 1\!\!1_{\tau \geq t} \int_t^\infty \lambda_u E(h_u|\mathcal{F}_t) e^{\Lambda_t - \Lambda_u} du + 1\!\!1_{\tau \leq t} h_\tau$$

Remark 2.4.3 We emphasize that the enlarged filtration $\mathbf{G} = \mathbf{F} \vee \mathbf{D}$ is here the filtration which should be taken into account; the filtration generated by $\mathcal{F}_t$ and $\sigma(\Theta)$ is too large. In the latter filtration, in the case where $\mathbf{F}$ is a Brownian filtration, τ would be a predictable stopping time.

2.4.3 Conditional Expectation of $\mathcal{F}_\infty$-Measurable Random Variables

Lemma 2.4.3 *Let X be an $\mathcal{F}_\infty$-measurable r.v.. Then*

$$E(X|\mathcal{G}_t) = E(X|\mathcal{F}_t), \tag{2.5}$$

$$E(X 1\!\!1_{\{\tau > T\}}|\mathcal{G}_t) = 1\!\!1_{\tau > t} e^{\Lambda_t} E(X e^{-\Lambda_T}|\mathcal{F}_t) \tag{2.6}$$

Proof: Let X be an $\mathcal{F}_\infty$-measurable r.v. Then,

$$E(X 1\!\!1_{\{\tau > t\}}|\mathcal{F}_t) = E(E(X 1\!\!1_{\{\tau > t\}}|\mathcal{F}_\infty)|\mathcal{F}_t) = P(\tau > t|\mathcal{F}_t) E(X|\mathcal{F}_t).$$

Formula (2.6) follows from prop. 2.4.1. Indeed, $E(X 1\!\!1_{\{\tau > T\}}|\mathcal{G}_t)$ is equal to 0 on the $\mathcal{G}_t$-measurable set $\tau < t$, whereas

$$E(X 1\!\!1_{\{\tau > T\}}|\mathcal{F}_t) = E(X 1\!\!1_{\{\tau > T\}}|\mathcal{F}_T|\mathcal{F}_t) = E(X e^{\Lambda_T}|\mathcal{F}_t).$$

To prove that $E(X|\mathcal{G}_t) = E(X|\mathcal{F}_t)$, it suffices to check that

$$E(B_t h(\tau \wedge t) X) = E(B_t h(\tau \wedge t) E(X|\mathcal{F}_t))$$

for any $B_t \in \mathcal{F}_t$ and any $h = 1\!\!1_{[0,a]}$. For $t \leq a$, the equality is obvious. For $t > a$, we have

$$\begin{aligned} E(B_t 1\!\!1_{\{\tau \leq a\}} E(X|\mathcal{F}_t)) &= E(B_t E(X|\mathcal{F}_t) E(1\!\!1_{\{\tau \leq a\}}|\mathcal{F}_\infty)) = E(E(B_t X|\mathcal{F}_t) E(1\!\!1_{\{\tau \leq a\}}|\mathcal{F}_t)) \\ &= E(X B_t E(1\!\!1_{\{\tau \leq a\}}|\mathcal{F}_t)) = E(B_t X 1\!\!1_{\{\tau \leq a\}}) \end{aligned}$$

as expected. □

Remark 2.4.4 Let us remark that (2.5) implies that every $\mathbf{F}$-square integrable martingale is a $\mathbf{G}$-martingale. However, equality (2.5) does not apply to any $\mathcal{G}$-measurable random variable; in particular $P(\tau \leq t|\mathcal{G}_t) = 1\!\!1_{\{\tau \leq t\}}$ is not equal to $F_t = P(\tau \leq t|\mathcal{F}_t)$.

2.4.4 Defaultable Zero-Coupon Bond

Therefore, for $t < T$

$$E(1\!\!1_{\{T < \tau\}}|\mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} E\Big(\exp\big(-\int_t^T \lambda_s \, ds\big) \,\Big|\, \mathcal{F}_t\Big).$$

Suppose that the price at time t of a default-free bond paying 1 at maturity t is

$$B(t,T) = E\Big(\exp\big(-\int_t^T r_s \, ds\big) \,\Big|\, \mathcal{F}_t\Big).$$

The value of a defaultable zero-coupon bond is

$$E\Big(1\!\!1_{\{T < \tau\}} \exp\big(-\int_t^T r_s \, ds\big) \,\Big|\, \mathcal{G}_t\Big) = 1\!\!1_{\{\tau > t\}} E\Big(\exp\big(-\int_t^T [r_s + \lambda_s] \, ds\big) \,\Big|\, \mathcal{F}_t\Big).$$

The t-time value of a corporate bond, which pays δ at time T in case of default and 1 otherwise, is given by

$$E\Big(e^{-\int_t^T r_s\,ds}\,(\delta 1\!\!1_{\{\tau\le T\}} + 1\!\!1_{\{\tau>T\}})\,\Big|\,\mathcal{F}_t\Big).$$

The last quantity is equal to

$$\delta B(t,T) + 1\!\!1_{\{\tau>t\}}(1-\delta)E\Big(\exp\big(-\int_t^T [r_s+\lambda_s]\,ds\big)\,\Big|\,\mathcal{F}_t\Big).$$

It can be easily proved that, if h is some $\mathbf{F}$-predictable process,

$$E(h_\tau 1\!\!1_{\{\tau\le T\}}|\mathcal{G}_t) = h_\tau 1\!\!1_{\{\tau\le t\}} + 1\!\!1_{\{\tau>t\}}E\Big(\int_t^T h_u e^{\Lambda_t-\Lambda_u}\lambda_u du|\mathcal{F}_t\Big).$$

The credit risk model of this kind was studied extensively by Lando [18].

2.4.5 Stochastic boundary

We have seen in the previous subsection that, under the equality (2.3), we have an invariance property of the martingales. This invariance property, i.e. the $\mathbf{F}$ martingales are $\mathbf{G}$ martingales, is called (H) hypothesis.

In this section (taken from [10]) we show that under ((2.3)), the default time can be viewed as an hitting time of a stochastic barrier. Therefore, it is a generalization of the constant boundary model, also called structural approach.

Suppose that

$$P(\tau\le t|\mathcal{F}_\infty) = e^{-M_t}$$

where M is an arbitrary continuous strictly increasing $\mathbf{F}$-adapted process. Our goal is to show that there exists a random variable Θ, independent of $\mathcal{F}_\infty$, with exponential law of parameter 1, such that $\tau \overset{law}{=} \inf\{t\ge 0 : M_t > \Theta\}$. Let us set $\Theta \overset{def}{=} M_\tau$. Then

$$\{t<\Theta\} = \{t<M_\tau\} = \{C_t<\tau\},$$

where C is the right inverse of M, so that $M_{C_t}=t$. Therefore

$$P(\Theta>u|\mathcal{F}_\infty) = e^{-M_{C_u}} = e^{-u}.$$

We have thus established the required properties, namely, the probability law of Θ and its independence of the σ-field $\mathcal{F}_\infty$. Furthermore, $\tau = \inf\{t : M_t > M_\tau\} = \inf\{t : M_t > \Theta\}$.

2.4.6 Representation theorem

Kusuoka [16] establishes the following representation theorem.

Theorem 2.4.1 *Under (H), any $\mathbf{G}$-square integrable martingale admits a representation as a sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale M.*

We assume for simplicity that F is continuous and $F_t<1, \forall t\in I\!\!R^+$. Since (H) hypothesis holds, F is an increasing process. Then,

$$dF_t = e^{-\Gamma_t}d\Gamma_t$$

and

$$d(e^{\Gamma_t}) = e^{\Gamma_t}d\Gamma_t = e^{\Gamma_t}\frac{dF_t}{1-F_t}\,. \tag{2.7}$$

Proposition 2.4.2 *Suppose that hypothesis* (H) *holds under* P *and that any* **F***-martingale is continuous. Then, the martingale* $H_t = E_P(h_\tau | \mathcal{G}_t)$ *, where* h *is a* **F***-predictable process, admits a decomposition in a continuous martingale and a discontinuous martingale as follows*

$$(2.8) \qquad H_t = m_0^h + \int_0^{t\wedge\tau} e^{\Gamma_u} dm_u^h + \int_{]0,t\wedge\tau]} (h_u - J_{u-})\, dM_u,$$

where m *is the continuous* **F***-martingale*

$$m_t^h = E_P\Big(\int_0^\infty h_u dF_u \,|\, \mathcal{F}_t\Big),$$

$J_t = e^{\Gamma_t}(m_t^h - \int_0^t h_u dF_u)$ *and* M *is the discontinuous* **G***-martingale* $M_t = D_t - \Lambda_{t\wedge\tau}$ *where* $d\Lambda_u = \dfrac{dF_u}{1-F_u}$.

PROOF: From (2.4) we know that

$$(2.9) \qquad H_t = E(h_\tau | \mathcal{G}_t) = \mathbb{1}_{\{\tau\le t\}} h_\tau + \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} E\Big(\int_t^\infty h_u dF_u \,\Big|\, \mathcal{F}_t\Big) = \mathbb{1}_{\{\tau\le t\}} h_\tau + \mathbb{1}_{\{\tau>t\}} J_t\,.$$

From integration by part formula, using that Γ is an increasing process and m^h a continuous martingale, we deduce that

$$dJ_t = e^{\Gamma_t} dm_t^h + J_{t-} e^{-\Gamma_t}\, d(e^{\Gamma_t}) - h_t(e^{\Gamma_t} dF_t)\,.$$

Therefore, from (2.7)

$$dJ_t = e^{\Gamma_t} dm_t^h + (J_{t-} - h_t)\frac{dF_t}{1-F_t}$$

or, in an integrated form,

$$J_t = m_0 + \int_0^t e^{\Gamma_u} dm_u^h + \int_0^t (J_{u-} - h_u) d\Lambda_u.$$

Note that $J_u = H_u$ for $u < \tau$. Therefore, on $\{t < \tau\}$

$$H_t = m_0^h + \int_0^{t\wedge\tau} e^{\Gamma_u} dm_u^h + \int_0^{t\wedge\tau} (J_{u-} - h_u) d\Lambda_u$$

From (2.9), the jump of H at time τ is $h_\tau - J_\tau = h_\tau - J_{\tau-} = h_\tau - H_{\tau-}$. Then, (2.8) follows. □

Remark 2.4.5 Since hypothesis (H) holds, the processes $(m_t, t \ge 0)$ and $(\int_0^{t\wedge\tau} e^{\Gamma_u} dm_u,\, t \ge 0)$ are also **G**-martingales.

2.4.7 Hedging contingent claims

Let $(m_t^X - \mu_t^X S_t, \mu_t^X)$ be the hedging portfolio for the default free contingent claim Xe^{Γ_T}.
Here

$$\begin{aligned} m_t^X &= E(Xe^{-\Gamma_T} | \mathcal{F}_t) \\ m_t^X &= m_0^X + \int_0^t \mu_s^X dS_s \end{aligned}$$

If a defaultable zero-coupon is traded at price ρ_t then under the e.m.m. CHOSEN BY THE MARKET

$$\rho_t = E(\mathbb{1}_{T<\tau} | \mathcal{G}_t) = L_t m_t$$

where $m_t = E(e^{-\Gamma_T} | \mathcal{F}_t)$.

The hedging portfolio for the defaultable contingent claim $X 1\!\!1_{T<\tau}$ is based on the riskless bond, the asset and the defaultable zero-coupon. The characterization of this hedging portfolio is

(i) a long position of $\frac{m_t^X}{m_t}$ defaultable zero-coupon,

(ii) the number of asset'shares to be hold is $e^{\Gamma_t}(\mu_t^X - \frac{m_t^X}{m_t}\mu_t)$

(iii) an amount of

$$-e^{\Gamma_t}(\mu_t^X - \frac{m_t^X}{m_t}\mu_t)S_t$$

is invested in the riskless bond.

2.5 General Case

Suppose that in addition to τ, a Brownian motion B lives on the space $(\Omega, \mathcal{G}, P)$. We denote by $\mathcal{F}_t = \sigma(B_s, s \leq t)$ its canonical filtration and by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ the enlarged filtration generated by the pair (B, D). In this general setting, the Brownian motion B is no longer a Brownian motion in the **G** filtration and is not always a semi-martingale. However, B remains a semi-martingale if we restrict our attention to the behavior of the processes before time τ.
Nevertheless, the conditional expectation with respect to $\mathcal{G}_t$ is easy to compute from the expectation w.r.t. $\mathcal{F}_t$.
In this section, we use results proven in Dellacherie [6].

2.5.1 Conditional expectation

It is well known that if (Z_t) is any **G** predictable process, then there exists a unique **F** predictable process (z_t) such that $Z_t 1\!\!1_{t<\tau} = z_t 1\!\!1_{t<\tau}$. The uniqueness of z follows from the assumption that $F_t < 1$ (See Dellacherie et al. [7] for comments)

We recall some well known results (See Dellacherie, [6])

Lemma 2.5.1 *Let X be an integrable $\mathcal{G}_T$-measurable, r.v. Then*

$$E(X|\mathcal{G}_t) 1\!\!1_{t<\tau} = \frac{E(X 1\!\!1_{t<\tau}|\mathcal{F}_t)}{E(1\!\!1_{t<\tau}|\mathcal{F}_t)} 1\!\!1_{t<\tau}\,.$$

Proof: This result is obvious, as soon as we work with the left continuous version of the conditional expectation, from the remarks on the measurability of the restriction of a **G**-predictable process to the set $t < \tau$. Indeed, on the set $\{t < \tau\}$, the conditional expectation $E(X|\mathcal{G}_t)$ equals a $\mathcal{F}_t$-measurable random variable z_t. Therefore, taking conditional expectation with respect to $\mathcal{F}_t$ of both members of the equality

$$E(X|\mathcal{G}_t) 1\!\!1_{t<\tau} = z_t 1\!\!1_{t<\tau}$$

leads to

$$E(X 1\!\!1_{t<\tau}|\mathcal{F}_t) = z_t E(1\!\!1_{t<\tau}|\mathcal{F}_t)$$

□

In particular, we obtain $E(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{t<\tau}\frac{E(X 1\!\!1_{T<\tau}|\mathcal{F}_t)}{E(1\!\!1_{t<\tau}|\mathcal{F}_t)}$. It is easy to check that $E(X 1\!\!1_{T<\tau}|\mathcal{G}_t)$ is equal to 0 on the set $\{t \geq \tau\}$. Indeed,

$$E(X 1\!\!1_{T<\tau}|\mathcal{G}_t) 1\!\!1_{t\geq\tau} = E(X 1\!\!1_{T<\tau} 1\!\!1_{t\geq\tau}|\mathcal{G}_t) = 0$$

This result gives an interesting interpretation of the intensity. From the definition, $E(D_t - D_s|\mathcal{G}_s) = 1\!\!1_{s<\tau} E(\int_s^{t\wedge\tau} \lambda_u du|\mathcal{G}_s) = \frac{E(s<\tau\leq t|\mathcal{F}_s)}{E(s<\tau|\mathcal{F}_s)}$ and $\lambda_s 1\!\!1_{s<\tau} = \lim_{t\to s} \frac{P(s<\tau\leq t)}{P(s<\tau)}$.

Definition 2.5.1 *The hazard process is the process* $\Gamma_t = -\ln(1-F_t)$ *where* $F_t = P(\tau \leq t|\mathcal{F}_t)$

We assume that Γ is continuous.

Lemma 2.5.2 *The process*

$$L_t := 1\!\!1_{\tau<t} e^{\Gamma_t} = (1-D_t)e^{\Gamma_t} = \frac{1-D_t}{1-F_t}$$

is a **G**-*martingale. Moreover, for any* **F**-*martingale* m, *the product* Lm *is a* **G**-*martingale.*

PROOF: It is enough to check that for any $t \leq s$

$$E(1\!\!1_{\{\tau>s\}} e^{\Gamma_s}|\mathcal{G}_t) = 1\!\!1_{t<\tau} e^{\Gamma_t}.$$

This can be rewritten as follows

$$1\!\!1_{t<\tau} e^{\Gamma_t} E(1\!\!1_{\{\tau>s\}} e^{\Gamma_s}|\mathcal{F}_t) = 1\!\!1_{t<\tau} e^{\Gamma_t}.$$

To complete the proof of the first statement, it is enough to observe that

$$E(1\!\!1_{\{\tau>s\}} e^{\Gamma_s}|\mathcal{F}_t) = E(e^{\Gamma_s} E(1\!\!1_{\{\tau>s\}}|\mathcal{F}_s)|\mathcal{F}_t) = 1.$$

For the second part of the lemma, notice that for $t \leq s$ in view of (2.6) we have

$$E(L_s m_s|\mathcal{G}_t) = E(1\!\!1_{\{\tau>t\}} L_s m_s|\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} e^{\Gamma_t} E(m_s|\mathcal{F}_t) = (1-D_t)e^{\Gamma_t} m_t = L_t m_t \quad (2.10)$$

so that Lm is a **G**-martingale. □

Proposition 2.5.1 *The martingale* $(L_t = (1-D_t)\exp\int_0^t \Lambda_u du\,,\ t \geq 0)$ *satisfies* $dL_t = L_{t-} dM_t$.

PROOF: From Itô's lemma (see the following chapter)

$$dL_t = \exp\int_0^t \Lambda_u du[-dD_t + (1-D_t)\lambda_t dt] = \exp\int_0^t \Lambda_u du(-dM_t)$$

Again, the process Z is the exponential martingale solution of $dZ_t = -Z_t dM_t$.

2.5.2 Ordered Random Times

This section is taken from [14]. Consider two **F**-adapted increasing continuous processes, Λ^1 and Λ^2, which satisfy $\Lambda^2_0 = \Lambda^1_0 = 0$ and $\Lambda^2_t < \Lambda^1_t$ for every $t \in I\!\!R_+$. Let Θ be a random variable which is exponentially distributed with mean 1 and is independent of the processes $\Lambda^i, i = 1,2$. For $i = 1,2$ we set

$$\tau_i = \inf\{\, t \in I\!\!R_+ \,:\, \Lambda^i_t \geq \Theta\,\}. \quad (2.11)$$

so that obviously $\tau_1 < \tau_2$ with probability 1.

We shall write $\mathbf{G}^i = \mathbf{D}^i \vee \mathbf{F}$, for $i = 1,2$, and $\mathbf{H} = \mathbf{D}^1 \vee \mathbf{D}^2 \vee \mathbf{F}$. An analysis of each random time τ_i with respect to its 'natural' enlarged filtration $\mathbf{G}^i$ can be done along the same lines as in the previous section. cess. It is clear that for each i the process Λ^i represents the $(\mathbf{F}, \mathbf{G}^i)$-martingale hazard process of the random time τ_i. We shall call the process Γ defined as $\Gamma_t = -\ln(1-F_t)$ where $F_t = P(\tau \le t|\mathcal{F}_t)$ the $\mathbf{F}$-hazard process of τ. In particular, Γ_1 is the $\mathbf{F}$-hazard process of τ_1.

We shall focus on the study of hazard processes with respect to the enlarged filtration. We find it convenient to introduce the following auxiliary notation:[1] $\mathbf{F}^i = \mathbf{D}^i \vee \mathbf{F}$, so that $\mathbf{H} = \mathbf{D}^1 \vee \mathbf{F}^2 = \mathbf{D}^2 \vee \mathbf{F}^1$.

Let us start by an analysis of τ_1. We are looking for the $\mathbf{F}^2$-hazard process $\tilde{\Gamma}^1$ of τ_1, as well as for the $(\mathbf{F}^2, \mathbf{H})$-martingale hazard process $\tilde{\Lambda}^1$ of τ_1. We shall first check that $\tilde{\Gamma}^1 \neq \Gamma^1$. Indeed, by virtue of the definition of a hazard process we have, for $t \in I\!\!R_+$,

$$e^{-\Gamma^1_t} = P(\tau_1 > t|\mathcal{F}_t) = e^{-\Lambda^1_t}.$$

and

$$e^{-\tilde{\Gamma}^1_t} = P(\tau_1 > t|\mathcal{F}^2_t) = P(\tau_1 > t|\mathcal{F}_t \vee \mathcal{D}^2_t).$$

Equality $\tilde{\Gamma}^1 = \Gamma^1$ would thus imply the following relationship, for every $t \in I\!\!R_+$,

$$P(\tau_1 > t|\mathcal{F}_t \vee \mathcal{D}^2_t) = P(\tau_1 > t|\mathcal{F}_t). \tag{2.12}$$

The relationship above is manifestly not valid, however. In effect, the inequality $\tau_2 \le t$ implies $\tau_1 \le t$, therefore on the set $\{\tau_2 \le t\}$, which clearly belongs to the σ-field $\mathcal{D}^2_t$, we have $P(\tau_1 > t|\mathcal{F}_t \vee \mathcal{D}^2_t) = 0$, and this contradicts (2.12). This shows also that the $\mathbf{F}^2$-hazard process $\tilde{\Gamma}^1$ is well defined only strictly before τ_2.

As one might easily guess, the properties of τ_2 with respect to the filtration $\mathbf{F}^1$ are slightly different. First, we have

$$e^{-\tilde{\Gamma}^2_t} = P(\tau_2 > t|\mathcal{F}^1_t) = P(\tau_2 > t|\mathcal{F}_t \vee \mathcal{D}^1_t).$$

We claim that $\tilde{\Gamma}^2 \neq \Gamma^2$, that is, the equality

$$P(\tau_2 > t|\mathcal{F}_t \vee \mathcal{D}^1_t) = P(\tau_2 > t|\mathcal{F}_t) \tag{2.13}$$

is not valid, in general. Indeed, the inequality $\tau_1 > t$ implies $\tau_2 > t$, and thus on set $\{\tau_1 > t\}$, which belongs to $\mathcal{D}^1_t$, we have $P(\tau_2 > t|\mathcal{F}_t \vee \mathcal{D}^1_t) = 1$, in contradiction with (2.13). Notice that the process $\tilde{\Gamma}^2$ is not well defined after time τ_1.

On the other hand, it can be checked that the process $D^1_t - \Lambda^1_{t\wedge\tau_1}$, which is of course stopped at τ_1, is not only a $\mathbf{G}^1$-martingale, but also a $\mathbf{H}$-martingale. We conclude that Λ^1 coincides with the $(\mathbf{F}^2, \mathbf{H})$-martingale hazard process of τ_1. Furthermore, the process $D^2_t - \Lambda^2_{t\wedge\tau_2}$ is a $\mathbf{G}^2$-martingale; it does not follow a $\mathbf{H}$-martingale, however (otherwise, the equality $\tilde{\Gamma}^2 = \Gamma^2 = \Lambda^2$ would hold up to time τ_2, but this is clearly not true). The exact evaluation of the $(\mathbf{F}^1, \mathbf{H})$-martingale hazard process $\tilde{\Lambda}^2$ of τ_2 seems to be rather difficult. Let us only mention that it is reasonable to expect that $\tilde{\Lambda}^2$ it is discontinuous at τ_1. See the following section.

Let us finally notice that τ_1 is a totally inaccessible stopping time not only with respect to $\mathbf{G}^1$, but also with respect to the filtration $\mathbf{H}$. On the other hand, τ_2 is a totally inaccessible stopping time with respect to $\mathbf{G}^1$, but it is a predictable stopping time with respect to $\mathbf{H}$. Indeed, we may easily find an announcing sequence τ^n_2 of $\mathbf{H}$-stopping times, for instance,

$$\tau^n_2 = \inf\{\, t \ge \tau_1 \,:\, \Lambda^2_t \ge \Theta - \tfrac{1}{n}\,\}.$$

[1] Though in the present setup $\mathbf{F}^i = \mathbf{G}^i$, this double notation will appear useful in what follows.

Therefore the **H**-martingale hazard process $\hat{\Lambda}^2$ of τ_2 coincides with the **H**-predictable process $D_t^2 = 1\!\!1_{\{\tau_2 \le t\}}$. Let us set $\tau = \tau_1 \wedge \tau_2$. In the present setup, it is evident that $\tau = \tau_1$, and thus the **H**-martingale hazard process $\hat{\Lambda}$ of τ is equal to Λ^1. It is also equal to the sum of **H**-martingale hazard processes of τ_i, $i = 1, 2$, stopped at τ. Indeed, we have

$$\hat{\Lambda}_{t\wedge\tau} = \Lambda^1_{t\wedge\tau} = \Lambda^1_{t\wedge\tau} + D^2_{t\wedge\tau}.$$

This property is universal (though not always very useful).

2.6 Infimum and supremum, general case

In this section we investigate an elementary example, which proves than the dependence of the intensity with respect to the filtration is a delicate tool. We study the case where two random times τ_i are given on probability space $(\Omega, \mathcal{G}, P)$, endowed with a reference filtration $(\mathcal{F}_t, t \ge 0)$. As usual, we denote by D^i the process $D^i_t = 1\!\!1_{\tau_i \le t}$. We denote by $\mathcal{D}^i_t$ the σ-algebra generated by $(D^i_s, s \le t)$, and $\mathcal{F}^i_t = \mathcal{F}_t \vee \mathcal{D}^i_t$. The random time τ is the infimum of the random times τ_i, i.e., $\tau = \tau_1 \wedge \tau_2$, D is the associated default process. We introduce the filtrations $\mathcal{D}_t = \sigma(D_s, s \le t)$ $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$, $\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{D}^1_t \vee \mathcal{D}^2_t$. We shall also use, as before, the notation $\mathbf{G}^i = \mathbf{F}^i$. Obviously, $\mathcal{F}_t \subset \mathcal{F}^i_t \subset \mathcal{H}_t$, $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{H}_t$. Now, our aim is to make precise the links between the different intensities, i.e., the processes $\lambda^i, \lambda, \psi^i, \varphi_i$ defined as below (in case of existence)
The first group consists of **F**-adapted intensities
(1) For any $i = 1, 2$, the process λ^i is the $\mathbf{F} - \mathbf{G}^i$-intensity of τ_i, i.e., a **F**-adapted process such that $D^i_t - \int_0^{t\wedge\tau_i} \lambda^i_s ds$ is a $\mathbf{F}^i$-martingale, $\Lambda^i_t = \int_0^t \lambda^i_s ds$.
(2) The process λ is the $\mathbf{F} - \mathbf{G}$-intensity of τ, i.e., a **F**-adapted process such that $D_t - \int_0^{t\wedge\tau} \lambda_s ds$ is a **G**-martingale, $\Lambda_t = \int_0^t \lambda_s ds$.
The second group consists of $\mathbf{F}^i$-adapted intensities
(3) For any pair (i, j) with $i \neq j$ the process φ^i is the $\mathbf{F}^i - \mathbf{H}$-intensity of τ_j, i.e., a $\mathbf{F}^i$-adapted process such that $D^j_t - \int_0^{t\wedge\tau_j} \varphi^i_s ds$ is a **H**-martingale for $j \neq i$, $\Phi^i_t = \int_0^t \varphi^i_s ds$.
(4) The process ψ^i is the $\mathbf{F}^i - \mathbf{H}$-intensity of τ, i.e., a $\mathbf{F}^i$-adapted process such that $D_t - \int_0^{t\wedge\tau} \psi^i_s ds$ is a **H**-martingale, $\Psi^i_t = \int_0^t \psi^i_s ds$.

We know that the three first processes are easy to characterize with hazard processes.

Computation of Λ^i

Let $F^i_t = P(\tau_i \le t | \mathcal{F}_t)$, and $\widetilde{F}^i$ its compensator (i.e. the increasing process such that $F - \widetilde{F}$ is a **F**-martingale). We assume for simplicity that F is a continuous process. Then $\Lambda^i_t = \int_0^t \frac{d\widetilde{F}^i_s}{1 - F^i_s}$. In the case where F^i is increasing, $\Lambda^i_t = -\ln(1 - F^i_t)$.

Computation of Λ

In the same way, denoting $F_t = P(\tau \le t | \mathcal{F}_t)$, we have $\Lambda_t = \int_0^t \frac{d\widetilde{F}_s}{1 - F_s}$.

In the case where $\mathcal{D}_t^1$ and $\mathcal{D}_t^2$ are independent conditionally with respect to $\mathcal{F}_t$, then $(1-F^1)(1-F^2) = 1 - F$. Moreover, if F^i are increasing, $\Lambda_t = \Lambda_t^1 + \Lambda_t^2$.

Computation of Φ^i

Finally, if $H_t^i = P(\tau_j \leq t | \mathcal{F}_t^i)$, then $\Phi_t^i = \int_0^t \frac{d\widetilde{H}_s^i}{1 - H_s^i}$. The computation of H^1 leads to

$$1 - H_t^1 = 1\!\!1_{\tau_1 \leq t} P(\tau_2 > t | \mathcal{F}_t \vee \sigma(\tau_1)) + 1\!\!1_{t < \tau_1} \frac{P(\tau_1 > t, \tau_2 > t | \mathcal{F}_t)}{P(\tau_1 > t | \mathcal{F}_t)}$$

In particular, on the set $\{t < \tau\}$

$$(1 - H_t^1)(1 - H_t^2) = \frac{[P(\tau_1 > t, \tau_2 > t | \mathcal{F}_t)]^2}{P(\tau_1 > t | \mathcal{F}_t) P(\tau^2 > t | \mathcal{F}_t)} \,.$$

In the case where $\mathcal{D}_t^1$ and $\mathcal{D}_t^2$ are independent conditionally with respect to $\mathcal{F}_t$, then $(1 - H_t^1)(1 - H_t^2) = (1 - F_t^1)(1 - F_t^2)$.

Computation of Ψ^i

The computation of ψ^i can be done using other tools. On the set $\{t < \tau\}$, using that martingales are stable by additivity, we get

$$\psi_t^1 = \psi_t^2 = \varphi_t^1 + \varphi_t^2$$

It is important to notice that the knowledge of the processes ψ^i requires the computation of processes φ, therefore the knowledge of information after time τ. This is the approach used be Duffie [9].

2.7 Correlated default time

Let us study the case

$$\tau^i = \inf\{t \,:\, \Lambda_t^1 = \int_0^t \lambda_s^i ds > \Theta^i\}$$

where the joint distribution function of Θ^1, Θ^2 is

$$\Phi(x, y) = G^1(x) G^2(y)[1 + \alpha(1 - G^1(x))(1 - G^2(y))]$$

In that case, if $\tau = \tau_1 \wedge \tau_2$, the $\mathbf{F}$-hazard process of τ is

$$F_t = P(\tau \leq t | \mathcal{F}_t) = \Phi(\Lambda_t^1, \Lambda_t^2) \,.$$

In the case $G^1(x) = G^2(x) = 1 - e^{-x}$, $\lambda^1 = \lambda^2$

$$F_t = (1 - e^{-\Lambda_t})^2 \, (1 + \alpha e^{-2\Lambda_t}) \,.$$

Bibliography

[1] Black, F. and Scholes, M.: The pricing of options and corporate liabilities, Journal of Political Economy, 81, 637-654, 1973.

[2] Bérlanger, A. , Shreve, S.E. and Wong, D. : A unified model for credit derivatives, Preprint, 2001.

[3] Bielecki, T. and Rutkowski, M.: Modeling of credit risk: intensity based approach. 1999. To appear Springer-Verlag.

[4] Chou, C.S. and Meyer, P.A.: Sur la représentation des martingales comme intégrales stochastiques dans les processus ponctuels, Séminaire de Probabilités IX, Lect. Notes in Math., 226-236, Springer-Verlag, Berlin, 1975.

[5] Copulas : see papers of Frey on web page of Risklab, Zurich and Riboulet and Roncalli, Web page of CreditLyonnais, France. http://gro.creditlyonnais.com

[6] Dellacherie, C.: Un exemple de la théorie générale des processus, *Séminaire de Probabilités IV*, p. 60-70, Lecture Notes in Math. 124, Springer-Verlag, Berlin, 1970.

[7] Dellacherie, C. and Maisonneuve, B. and Meyer, P.A., *Probabilités et potentiel, chapitres XVII-XXIV, Processus de Markov (fin). Compléments de calcul stochastique*, Hermann, Paris, 1992.

[8] Duffie, D. and Lando, D. , Term structure of credit spreads with incomplete accounting information, To appear Econometrica, 2000.

[9] Duffie, D.: First to default valuation, preprint, 1998.

[10] El Karoui, N.: Modélisation de l'information, CEA-EDF-INRIA, Ecole d'été, 1999. Journal of Banking and Finance, 19, 299-322, 1995.

[11] Jarrow, R.A. and Turnbull, S.M.: Pricing derivatives on financial securities subject to credit risk, Journal of Finance, 50, 53-85, 1995.

[12] Hull, J. and White, A. : Valuing credit default swaps I: no counterparty default risk,*Preprint, Rotman School Toronto* 2000.

[13] Jarrow, R.A. and Turnbull, S.M. *Derivative securities,* Southwestern college publish., Cincinnati, 1996.

[14] Jeanblanc, M. and Rutkowski, M. : Modeling default risk: an overview. *Mathematical Finance: theory and practice, Fudan University. Modern Mathematics Series, High Education press.* Beijin, 2000.

[15] Jeanblanc, M. and Rutkowski, M. : Modeling default risk: Mathematical tools. *Fixed Income and Credit risk modeling and Management, New York University, Stern scoll of business, Statistics and operations research department, Workshop, May 5, 2000.*

[16] Kusuoka, S.: A remark on default risk models, *Advances in Mathematical Economics*, 1, 69-82, 1999.

[17] Lando, D.: Modelling bonds and derivatives with default risk, preprint, 1996.

[18] Lando, D.: On Cox processes and credit risky securities, Review of Derivatives Research, 2, 99-120, 1998.

[19] Merton, R.: On the pricing of corporate debt: the risk structure of interest rates, Journal of Finance, 3, 449-470, 1974.

[20] Schönbucher, P.J. and Schlögl, E.: Credit risk derivatives and competition in the loan market: simplified version, preprint, 1997.

[21] Schönbucher, P.J. The pricing of credit risk and credit risk derivatives, preprint, 1997.

[22] Wong, D.: A unifying credit model, ScScotia Capital Markets Group,1998.

Chapter 3

Optimal portfolio

This chapter deals on optimal choice of investment strategies and is mainly an introduction to the portfolio insurance. The main part of this survey is taken from a survey Dana -Jeanblanc.

Optimal portfolio-consumption problems have been studied in finance for a long time. We shall show that not only there is a strong relation between absence of arbitrage and existence of an optimal solution but solving a portfolio-consumption problem provides a way to price any asset. Indeed assume that there is a family of underlying assets and that an extra asset is introduced in the market and that the investor's demand for that asset is zero while his consumption and portfolio demand for the other assets is as before, then we shall show the asset may be given a viable price. Furthermore the study of investors demands is the first step to the analysis of how prices depend on the fundamentals of an economy that we shall develop in next section. We consider a financial market with two assets, a riskless one with interest rate r and a risky asset with price S at time 0 and S_1 with

$$P(S_1 = uS) = p, P(S_1 = dS) = 1 - p\,.$$

A utility function U is given. Let us recall that a utility function is a non decreasing function, i.e., $U'(x) > 0)$ -preference for more than less- and concave, i.e. $U''(x) < 0$ -risk aversion. The aim of the agent is to choose a portfolio, in order to maximize its expected utility of terminal wealth under the budget constraint of his initial wealth x, i.e.,

$$\max E[U(X_T^{x,\theta})]$$

3.1 Discrete time

3.1.1 Two dates, 2 assets, complete case

First approach

The budget constraints writes

$$x = \theta_0 + \theta_1 S\,.$$

The terminal wealth is

$$X^{x,\theta} = \theta_0(1+r) + \theta_1 S_1 = \begin{cases} \theta_0(1+r) + \theta_1 uS &= x(1+r) + \theta_1 S(u-1-r) \quad \text{Upper state} \\ \theta_0(1+r) + \theta_1 dS &= x(1+r) + \theta_1 S(d-1-r) \quad \text{Low state} \end{cases}$$

Hence

$$E[U(X^{x,\theta})] = pU(x(1+r) + \theta_1 S(u-1-r)) + (1-p)U[x(1+r) + \theta_1 S(d-1-r)]$$

and the maximum is reached for θ_1 such that the derivative with respect to θ_1 equals 0 :

$$pS(u-1-r)U'[x(1+r)+\theta_1 S(u-1-r)] + (1-p)S(d-1-r)U'[x(1+r)+\theta_1 S(d-1-r)] \,.$$

This can be written as

$$pS_1(u)\,U'(x_u) + (1-p)S_1(d)\,U'(x_d) = S(1+r)[p\,U'(x_u) + (1-p)\,U'(x_d)$$

$$q_u\,S_1(u) + q_d\,S_1(d) = S(1+r)$$

with

$$\begin{aligned} q_u &= \frac{pU'(x_u)}{p\,U'(x_u) + (1-p)\,U'(x_d)} \\ q_d &= \frac{(1-p)U'(x_d)}{p\,U'(x_u) + (1-p)\,U'(x_d)} \end{aligned}$$

Let us remark that

$$q_u, q_d \in [0,1], q_u + q_d = 1$$

so that

$$q_u\,S_1(u) + q_d\,S_1(d) = E_q(S_1)$$

and. from

$$E_q(S_1) = S(1+r)$$

we deduce that q is the risk neutral probability that we have defined in the first lecture

$$q_u := \frac{1}{u-d}\left((1+r)-d\right) . \tag{3.1}$$

Note that we can write

$$E_q(S_1) = E_p(LS_1)$$

where L is equal to p/q_u in the up state and to $(1-p)/q_d = (1-p)/(1-q_u)$ in the low case.

Why we need utility functions? The problem

$$\max E(X^{x,\theta})$$

is equivalent to

$$Max\,\{x(1+r) + \theta_1[E(S_1) - (1+r)]\}$$

and has no solution, except if there are constraints on the portfolio.

Second approach

In a first step, one determines the optimal terminal wealth in each state of nature. This will give us a target to reach, or a payoff to hedge, which will be possible due to the completeness of the market. Let x_u and x_d the value of the terminal wealth. The problem is to maximize

$$pU(x_u) + (1-p)U(x_d)$$

under the budget constraint. We know (first chapter) that there is a strong relation between the terminal value of a portfolio and the initial value

$$x = \frac{1}{1+r}[\pi x_u + (1-\pi)x_d]$$

where π is the riskneutral probability. Hence our problem is to find a pair x_u, x_d such that

$$\max pU(x_u) + (1-p)U(x_d) \quad \text{under the constraint } \; x = \frac{1}{1+r}[\pi x_u + (1-\pi)x_d]$$

We obtain using Lagrange multiplier

$$\begin{aligned} pU'(x_u) &= \nu\frac{1}{1+r}\pi \\ (1-p)U'(x_d) &= \nu\frac{1}{1+r}(1-\pi) \end{aligned}$$

or

$$\begin{aligned} x_u &= I(\nu\frac{1}{1+r}\frac{\pi}{p}) \\ x_d &= I(\nu\frac{1}{1+r}\frac{1-\pi}{1-p}) \end{aligned}$$

where I is the inverse of U,i.e.

$$X^* = I(\nu\frac{1}{1+r}L)$$

where L is a random variable equal to $\frac{\pi}{p}$ in the up state and to $\frac{1-\pi}{1-p}$ in the lower state and ν is adjusted in order to satisfy the budget constraint

$$x = \frac{1}{1+r}[\pi x_u + (1-\pi)x_d] = \frac{1}{1+r}E_\pi[I(\nu\frac{1}{1+r}L)$$

In order to find the portfolio, we have to solve

$$\begin{cases} x_u &= \alpha(1+r) + \theta\, uS \\ x_d &= \alpha(1+r) + \theta\, dS\,. \end{cases}$$

hence, $\theta = \dfrac{x_u - x_b}{uS - dS}$.

3.1.2 Two dates Model, $d+1$ assets

We consider a two dates financial market where uncertainty is represented by a finite set of states $\{1, \ldots, k\}$. There are $d+1$ assets. The notations are those of the pricing and hedging chapter. We assume here that assets pay in units of a consumption good. Let us consider an investor with endowment e_0 at date 0 and $e_1(j)$ at date 1 in state j. At date 0, the investor buys a portfolio θ and consumes c_0 without running into debt and consumes $c_1(j)$ at date 1 in state j. The set of feasible consumptions-portfolios for the investor is

$$\begin{cases} e_0 \;\geq\; c_0 + \sum_{i=0}^d \theta^i\, S^i \\ e_1(j) \;\geq\; c_1(j) - \sum_{i=0}^d \theta^i\, d^i(j) \quad , \quad j \in \{1, \ldots, k\}. \end{cases}$$

Define the set of feasible consumptions by

$$B(S) \;:=\; \{c \in I\!R_+^{k+1} \;;\; \exists\, \theta \in I\!R^{d+1},\ \text{fulfilling (i) et (ii)}\}\,.$$

Let us further assume that the investor has a probability μ over states and that he is a von-Neumann Morgenstern maximizer, with utility function over feasible consumptions

$$u(c_0, c_{11}, \ldots, c_{1k}) \ = \ v_0(c_0) + \alpha \sum_{j=1}^{k} \mu_j \, v_1(c_{1j}) \ = v_0(c_0) + \alpha E_\mu(v_1(c_1))$$

with v_0 et v_1 strictly concave, strictly increasing, C^2 and α the discount factor. It may easily be proven that there exists an optimal portfolio-consumption solution iff there is no-arbitrage in the financial market. It follows from the first order conditions that

$$S^i \ = \ \sum_{j=1}^{k} \alpha \, \mu_j \, \frac{v_1'(c_{1j}^*)}{v_0'(c_0^*)} \ d^i(j) \ .$$

In other words, the market value of an asset is its payoff value at state price β with $\beta_j = \alpha \, \mu_j \dfrac{v_1'(c_{1j}^*)}{v_0'(c_0^*)}$. The j-th state price is therefore proportional to the probability of the state, to the discount factor and is higher in states where the optimal consumption is scarcer. Equivalently, one also has:

$$\frac{1}{1+r} = \alpha \frac{E_\mu(v_1'(c_1^*))}{v_0'(c_0^*)}$$

$$S^i \ = \frac{1}{1+r} \frac{E_\mu(v_1'(c_1^*) d^i)}{E_\mu(v_1'(c_1^*))} \ .$$

The risk-neutral probability in state j is therefore proportional to the marginal utility of the optimal consumption in state j. If the investor was risk-neutral ($v_1' = $ cste), he would pay $\dfrac{\mu_j}{1+r}$ at date 0 to get one unit of good in state j. As he is risk averse, he is willing to pay, $\dfrac{\mu_j}{1+r} \dfrac{v_1'(c_{1j})}{E_\mu(v_1'(c_1^*))}$.

3.1.3 Incomplete markets

Assume that a contingent claim with payoff z is introduced in the market. Assume that the investor's demand for that contingent claim is zero and that his consumption and portfolio demand for the other assets is as before. It follows from the first order conditions that the price $S(z)$ of the contingent claim is

$$S(z) \ = \ \sum_{j=1}^{k} \alpha \, \mu_j \, \frac{v_1'(c_{1j}^*)}{v_0'(c_0^*)} \ z(j) \ .$$

We may further write

$$S(z) \ = \ \frac{E_\mu(z)}{1+r} + \frac{1}{(1+r) E_\mu(v_1'(c_1^*))} \operatorname{cov}_\mu (v_1'(c_1^*), z).$$

Let $R(z) = \dfrac{z}{S(z)}$ be the contingent's claim return and r be the riskless rate, we then have

$$E_\mu(R(z)) - (1+r) = - \operatorname{cov}_\mu (R(z), \frac{v'(c_1^*)}{E_\mu(v_1'(c_1^*))})$$

Contingent's claim risk premium is therefore positive (resp. negative) if its payoff is negatively correlated (resp. positively) with $v'(c_1^*)$.

3.1.4 Complete case

For further use, let us lastly compute the optimal consumption-portfolio in the case of complete markets. In that case, there exists a unique risk-neutral probability π, the feasible set is simply

$$B(S) \ = \{c \in I\!R_+^{k+1} \ \mid \ c_0 + \frac{1}{1+r}\sum_{j=1}^{k} \pi_j c_{1j} \leq e_0 + \frac{1}{1+r}\sum_{j=1}^{k} \pi_j e_1(j)\} \ .$$

The investor's problem is therefore a "one constraint" demand problem

$$\begin{cases} \max \quad v_0(c_0) + \alpha \displaystyle\sum_{j=1}^{k} \mu_j v_1(c_{1j}) \quad \text{under the constraint} \\ \qquad c_0 + \dfrac{1}{1+r}\displaystyle\sum_{j=1}^{k} \pi_j c_{1j} \leq e_0 + \dfrac{1}{1+r}\displaystyle\sum_{j=1}^{k} \pi_j e_1(j) \end{cases}$$

Let $I_0 : I\!R_+ \to I\!R_+$ (resp. $I_1 : I\!R_+ \to I\!R_+$) be the inverse of v_0' (resp. v_1'). Then, using the Lagrangian

$$c_0^* = I_0(\lambda)\,; \qquad c_{1j}^* = I_1(\frac{\lambda \pi_j}{\mu_j \alpha(1+r)}), \quad \forall\, j = 1, \ldots, k$$

where λ is the unique solution of the equation

$$I_0(\lambda) + \frac{1}{1+r}\sum_{j=1}^{k} I_1\left(\frac{\lambda \pi_j}{\mu_j \alpha(1+r)}\right) = e_0 + \frac{1}{1+r}\sum_{j=1}^{k} \pi_j e_1(j)\,.$$

The optimal consumption is therefore a decreasing function of the risk-neutral density. The optimal hedging portfolio θ^* is obtained by solving $D\theta^* = c_1^* - e_1$.

3.1.5 Multiperiod Discrete time model

Let us now study the case of N trading dates. Let us consider an investor with endowment e_n at date n. At date n, the investor buys a portfolio (α_n, θ_n) and consumes c_n under the self-financing constraints:

$$e_n + \alpha_{n-1}(1+r)^n + \theta_{n-1} \cdot (S_n + d_n) = \alpha_n(1+r)^n + \theta_n \cdot S_n + c_n, \quad 0 \leq n \leq N-1$$

$$e_N + \alpha_{N-1}(1+r)^N + \theta_{N-1} \cdot d_N = c_N$$

(We have assumed that $S_N = 0$). Let us further assume that the investor has a probability μ over states and that he is a von Neumann Morgenstern maximizer, with utility function over feasible consumption processes $u(c_0, \ldots, c_N) \ = E_\mu \left[\sum_{n=0}^{N} \alpha^n v(c_n)\right]$ with v strictly concave, strictly increasing, C^2 and α the discount factor. Eliminating consumptions, the investor maximizes the indirect utility of a strategy

$$E_\mu \left[\sum_{n=0}^{N-1} \alpha^n \, v(e_n + \alpha_{n-1}(1+r)^n + \theta_{n-1} \cdot (S_n + d_n) - \alpha_n(1+r)^n - \theta_n \cdot S_n)\right] + \\ E_\mu \left[\alpha^N \, v(e_N + \alpha_{N-1}(1+r)^N + \theta_{N-1} \cdot d_N)\right]$$

It may be proven that there exists an optimal portfolio-consumption solution $(\alpha_n^*, \theta_n^*, c_n^*)_{n=0}^N$ iff there is no-arbitrage in the financial market. It follows from the first order conditions that

$$S_{n-1}^i = \frac{\alpha E_\mu[v'(c_n^*)(S_n^i + d_n^i) \mid \mathcal{F}_{n-1}]}{v'(c_{n-1}^*)} \quad \text{and} \quad S_{n-1}^0 = \frac{\alpha E_\mu[v'(c_n^*)(1+r)^n \mid \mathcal{F}_{n-1}]}{v'(c_{n-1}^*)}$$

Let $\tilde{S}_n^i = \frac{S_n^i}{(1+r)^n}$, $\tilde{d}_n^i = \frac{d_n^i}{(1+r)^n}$ be the discounted price and dividend processes of the i-th asset and $\tilde{G}_n^i = \sum_{\ell=1}^n \tilde{d}_\ell^i + \tilde{S}_n^i$ be the discounted gain. Equivalently we have:

$$\tilde{G}_{n-1}^i = \frac{E_\mu[v'(c_n^*)\tilde{G}_n^i \mid \mathcal{F}_{n-1}]}{E_\mu[v'(c_n^*) \mid \mathcal{F}_{n-1}]} .$$

The gain process is therefore a martingale. It easily follows that

$$S_n^i = \frac{E_\mu[\sum_{N \geq \ell > n} \alpha^{\ell-n} d_\ell \, v'(c_\ell^*) \mid \mathcal{F}_n]}{v'(c_n^*)}, \quad 0 \leq n \leq N .$$

3.1.6 Markovitz efficient portfolio

Markovitz assumes that investors are risk-adverse and that the measure of the risk is the variance of returns. More precisely, assume that the underling assets are random variables $S_i(1), i \leq d$ at time 1. The return of the i-th asset is the r.v.

$$R_i = \frac{S_i(1) - S_i(0)}{S_i(0)} .$$

A portfolio ($\theta = (\theta_i, i \leq d)$) has value $\sum_{i=0}^d \theta_i S_i(0)$ at time 0 and $\sum_{i=0}^d \theta_i S_i(1)$ at time 1. The return of the portfolio is

$$\frac{\sum_{i=0}^d \theta_i S_i(1) - \sum_{i=0}^d \theta_i S_i(0)}{\sum_{i=0}^d \theta_i S_i(0)} = \frac{1}{x} \sum_{i=0}^d \theta_i S_i(0) R_i = \sum_{i=0}^d \pi_i R_i ,$$

where x is the initial wealth and $\pi_i = \dfrac{\theta_i S_i(0)}{x}$ is the proportion of wealth invested in the i-th asset.(Note that $\sum_{i=0}^d \pi_i = 1$). Hence, denoting by ρ_i the expected return of the i-th asset and $\sigma_{i,j} = \text{Cov}(R_i, R_j)$ (these quantities are supposed to be known) the expectation of the return is

$$E(R(\pi)) = \sum_{i=0}^d \pi_i E(R_i) = \sum_{i=0}^d \pi_i \rho_i$$

while its variance is

$$\text{Var} R(\pi) = \sum_{i,j} \pi_i \pi_j \sigma_{i,j} .$$

In the Markovitz approach, the aim of the agent is to find the portfolio with the smallest variance between portfolio with the same expected return, or the portfolio with the largest expected return between all portfolio with a given variance. It appears that there exists a relation between these two quantities.

Plotting the curve (variance, expectation) leads to a hyperbolic curve when there are no riskless asset and to a line when there are a riskless asset. Moreover, this line is tangent to the hyperbole associated with the risky assets. It can be shown that this approach is linked with a utility function of the form $u(x) = x - ax^2$.

3.2 Continuous time models. Maximization of terminal wealth in a complete market.

3.2.1 A continuous time two assets model

As in Black and Scholes' model, there are two assets, a bond with price $S_t^0 = e^{rt}$ and a risky asset with price

$$dS_t = S_t(\mu dt + \sigma dW_t)\,,\; S_0 = s\,.$$

Then, the market is complete. If an investor holds portfolio (α_t, θ_t) and consumes at rate c_t at time t, his wealth V, with $V_t = \alpha_t S_t^0 + \theta_t S_t$ evolves according to the stochastic differential equation

$$dV_t = \alpha_t S_t^0 r dt + \theta_t dS_t - c_t dt = rV_t dt + \theta_t(dS_t - rS_t dt) - c_t dt\,, \tag{3.2}$$

due to the self financing condition $dV_t = \alpha_t dS_t^0 + \theta_t dS_t - c_t dt$.

Assume that his objective is to maximize

$$E_P\left(U(V_T) + \int_0^T u(c_s)ds\right)\,,$$

where u and U are strictly increasing, strictly concave and sufficiently differentiable functions. We shall solve this problem by two methods. The first one, called the martingale method is very useful to compute the optimal consumption and terminal bequest in a complete market, while the second called the dynamic programming method gives the hedging portfolio.

3.2.2 Historical probability

Let $L_t = \exp[-\kappa W_t - \frac{1}{2}\kappa^2 t]$. Using Itô's formula, the process

$$e^{-rt}V_t L_t$$

can be shown to be a martingale. Therefore, its expectation is constant. In particular,

$$E\left(e^{-rT}L_T V_T\right) = x \tag{3.3}$$

This is the budget constraint.

Reciprocally, if a positive random variable (the terminal wealth that the agent would like to obtain, or the payoff he would receive) V_T is given such that (3.3) holds, thanks to the martingale property the current wealth is given by via

$$V_t L_t e^{-rt} = E\left(e^{-rT}L_T V_T | \mathcal{F}_t\right)$$

and the portfolio which hedges this terminal wealth is given via a representation theorem. I other words, the market being complete, it is possible to hedge he contingent claim $H = V_T$.

Solving the problem

The Lagrangian of the constrained problem is

$$E_P\left[U(V_T) - \lambda\left(V_T e^{-rT} L_T - x\right)\right].$$

The first order condition is (the derivative with respect to the terminal wealth equals 0)

$$U'(V_T^*) = \lambda e^{-rT} L_T$$

where λ is such that the budget constraint holds, i.e.,

$$E\left[e^{-rT} L_T (U')^{-1}(\lambda e^{-rT} L_T)\right] = x.$$

Hence V_T^* may more or less easily be computed from the above formula. The hedging portfolio is much harder to be computed. It is obtained from a representation theorem.

Examples

If $U(x) = \ln(x)$, then $I(y) = y^{-1}$. The optimal terminal wealth is $V_T^* = (\lambda e^{-rT} L_T)^{-1}$, where the parameter λ is adjusted so that the budget constraint holds

$$E(e^{-rT} L_T (\lambda e^{-rT} L_T)^{-1}) = x$$

or

$$\lambda = 1/x$$

The optimal wealth is obtained via

$$V_t L_t e^{-rt} = E\left(e^{-rT} L_T (\lambda e^{-rT} L_T)^{-1} | \mathcal{F}_t\right) = 1/\lambda = x$$

i.e.

$$V_t = x e^{rt} L_t^{-1}$$

Hedging portfolio associated with the optimal utility for the log case. The optimal wealth is

$$V_t = x e^{rt} L_t^{-1} = x \exp[rt + \kappa W_t + \frac{1}{2}\kappa^2 t] = x \exp[rt + \kappa^2 t] \exp[\kappa W_t - \frac{1}{2}\kappa^2 t]$$

so that

$$\begin{aligned} dV_t &= [r + \kappa^2] V_t dt + V_t \kappa dW_t \\ &= rV_t dt + V_t \kappa [dW_t + \kappa dt] = rV_t dt + V_t \frac{\kappa}{\sigma}[\sigma dW_t + (\mu - r)dt] \\ &= rV_t dt + V_t \frac{\kappa}{\sigma S_t} S_t [\sigma dW_t + (\mu - r)dt] = rV_t dt + V_t \frac{\kappa}{\sigma S_t}[dS_t - rS_t dt] \end{aligned}$$

so that the hedging portfolio is $\theta = V_t \frac{\kappa}{\sigma S_t}$.It is interesting to remark that we obtain that $e^{rt} L_t^{-1}$ is the value of a portfolio. This is the so-called numeraire portfolio [12], [1].

3.2.3 The Dynamic programming method

Let us now present the Dynamic programming method. The optimal portfolio is obtained in terms of the value function that we shall next define. Assume that the investor's wealth equals x at time t and that he invests the proportion π of his wealth in the risky asset (this number is related with the portfolio by $\pi X = \theta S$), then his wealth fulfills the stochastic differential equation

$$dX_s^{t,x,\pi} = rX_s^{t,x,\pi}ds + \pi_s X_s^{t,x,\pi}[\sigma dW_s + (\mu - r)ds]\,,\ s \geq t \tag{3.4}$$

with initial condition $X_t^{t,x,\pi} = x$. Let v the value function be defined by

$$v(t,x) = \sup_{\pi} E\left\{U(X_T^{t,x,\pi})\right\}$$

where the supremum runs over the portfolio and where $X_T^{t,x,\pi}$ is the terminal wealth of the investor. The value function satisfies the dynamic programming equation

$$v(t,x) = \sup_{\pi} E\left\{v(\tau, X_\tau^{t,x,\pi})\right\}$$

for any time τ. When it is smooth enough, it fulfills the Hamilton Jacobi Bellman equation

$$\begin{cases} \dfrac{\partial v}{\partial t} + \sup\limits_{\pi}\left\{(rx + (\mu - r)\pi)\dfrac{\partial v}{\partial x} + \dfrac{1}{2}\sigma^2\pi^2\dfrac{\partial^2 v}{\partial x^2}\right\} = 0 \\ v(T,x) = U(x) \end{cases}$$

Therefore, the optimal portfolio π^* is the value of π for which the supremum is attained and is the solution of a quadratic problem. Hence π^* is the function of wealth defined by the relation

$$\pi^*(t,x) = -\frac{\mu - r}{\sigma^2}\frac{\partial v}{\partial x}(t,x)\left(\frac{\partial^2 v}{\partial x^2}(t,x)\right)^{-1},.$$

In the particular case where $U(x) = \ln x$, it may be proven that $v(t,x) = p(t)\ln x$ where $p(t)$ fulfills a differential equation, hence π^* is constant. The optimal portfolio is therefore to invest a fixed multiple of wealth in the risky asset at all dates.

3.3 Consumption and terminal wealth

3.3.1 The martingale method

The aim of thia approach is to characterize the optimal terminal wealth and, in a second step to reach this target. We have seen the power of the martingale approach to reach target.

Risk neutral probability

In the martingale method, one works under the risk neutral probability Q introduced before. Denoting by $\tilde{V}_t = e^{-rt}V_t$ the discounted wealth, the integration by part formula leads to $d\tilde{V}_t = \theta_t S_t e^{-rt}[\sigma dW_t + (\mu - r)dt] - e^{-rt}c_t dt$ and $\tilde{V}_t + \int_0^t e^{-rs}c_s ds$ is a Q-martingale. Indeed, as $W_t^* = W_t + \kappa t$ is a Q-martingale,

$$e^{-rt}V_t + \int_0^t e^{-rs}c_s ds = x + \int_0^t \theta_s e^{-rs}\sigma[dW_s + \kappa ds] = x + \int_0^t \theta_s e^{-rs}\sigma dW_s^* \tag{3.5}$$

and the right member (hence the left member) is a Q-martingale. Therefore, its expectation is constant

$$E_Q\left(e^{-rT}V_T+\int_0^T e^{-rs}c_s ds\right)=x \tag{3.6}$$

The left-hand side is $E_P\left(H_TV_T+\int_0^T H_sc_sds\right)$ with $H_t=R_tL_t$ where L_T is the Radon-Nikodym density dQ/dP and $H_t=e^{-rt}L_t=e^{-rt}E_P(L_T|\mathcal{F}_t)$. From Itô's formula, we obtain

$$dH_t=-H_t(rdt+\kappa dW_t) \tag{3.7}$$

Reciprocally, if a positive terminal wealth V_T and a consumption process are given such that (3.6) holds, thanks to the martingale property the current wealth is given by

$$V_te^{-rt}=E_Q\left(e^{-rT}V_T+\int_t^T e^{-rs}c_sds|\mathcal{F}_t\right)$$

and the portfolio which hedges this terminal wealth and consumption is given via (3.5). Let us remark that this method allows us to avoid the positive wealth constraint : as soon as the terminal wealth is positive, the value of a self-financing portfolio which finances some consumption is non-negative. The investor thus faces a unique budget constraint on consumption and terminal wealth (V_T,c)

$$x=E_P\left(H_TV_T+\int_0^T H_sc_sds\right),$$

Historical probability

One can avoid the risk neutral probability as follows : Let $L_t=\exp\theta W_t-\frac{1}{2}\theta^2t$. Using Itô's formula, the process

$$e^{-rt}V_tL_t+\int_0^t e^{-rs}L_sc_sds$$

can be shown to be a martingale. Therefore, its expectation is constant

$$E\left(e^{-rT}L_TV_T+\int_0^T e^{-rs}L_sc_sds\right)=x \tag{3.8}$$

This is the budget constraint. Reciprocally, if a positive terminal wealth V_T and a consumption process are given such that (3.8) holds, thanks to the martingale property the current wealth is given by via

$$V_tL_te^{-rt}=E\left(e^{-rT}L_TV_T+\int_t^T e^{-rs}L_sc_sds|\mathcal{F}_t\right)$$

and the portfolio which hedges this terminal wealth and consumption is given via a representation theorem.

Solving the problem

The Lagrangian of the constrained problem is

$$E_P\left[U(V_T)+\int_0^T u(c_s)ds-\lambda\left(V_TH_T+\int_0^T H_sc_sds-x\right)\right].$$

The first order conditions are

$$\begin{cases} U'(V_T^*) &= \lambda H_T \\ u'(c_t^*) &= \lambda H_t \,, \ \forall\ t \end{cases}$$

where λ satisfies

$$E\left[H_T(U')^{-1}(\lambda H_T) + \int_0^T H_s(u')^{-1}(\lambda H_s)ds\right] = x.$$

Hence c_t^* and V_T^* may easily be computed from the above formulas. The hedging portfolio is much harder to be computed. It is obtained from a representation theorem.

Examples

If $u(x) = U(x) = x^\delta$, then $I(y) = y^{-1/\delta}$. The optimal terminal wealth is $X_T^* = (\lambda H_T)^{-1/\delta}$, the optimal consumption is $c_t^* = (\lambda H_t)^{-1/\delta}$ where the parameter λ is adjusted so that the budget constraint holds

$$E(H_T(\lambda H_T)^{-1/\delta} + \int_0^T H_t(\lambda H_t)^{-1/\delta} dt) = x$$

or

$$E(H_T^{1-1/\delta} + \int_0^T H_t^{1-1/\delta} dt) = x\lambda^{1/\delta}$$

This condition reduces to the computation of $E(H_t^\alpha$, with $\alpha = 1 - (1/\delta)$. This can easily be done. Indeed, from Itô's formula and (3.7)

$$\begin{aligned} dH_t^\alpha &= \alpha H_t^{\alpha-1} dH_t + \frac{1}{2}\alpha(\alpha-1)H_t^{\alpha-2} dH_t \cdot dH_t \\ &= -\alpha H_t^\alpha(rdt + \kappa dW_t) + \frac{1}{2}\alpha(\alpha-1)H_t^\alpha \kappa^2 dt \\ &= H_t^\alpha[\alpha(rdt + \kappa dW_t) + \frac{1}{2}\alpha(\alpha-1)\kappa^2 dt] \\ &= H_t^\alpha[\nu dt + \beta dW_t] \end{aligned}$$

where $\nu = \alpha(r + \frac{1}{2}(\alpha-1)\kappa^2)$; therefore, H_t^α is a geometric Brownian motion,

$$H_t^\alpha = \exp[\nu t + \beta W_t - \frac{1}{2}\beta^2 t]$$

hence $E(H_t^\alpha) = e^{\nu t}$.

3.3.2 The Dynamic programming method

Let us now present the Dynamic programming method. The optimal portfolio is obtained in terms of the value function that we shall next define. Assume that the investor's wealth equals x at time t and that he invest the proportion π of his wealth in the risky asset (this number is related with the portfolio by $\pi X = \theta S$) and that he consumes at rate c, then his wealth fulfills the stochastic differential equation

$$dX_s^{t,x,\pi,c} = rX_s^{t,x,\pi,c} ds + \pi_s X_s^{t,x,\pi,c}[\sigma dW_s + (\mu - r)ds] - c_s ds \,, \ s \geq t \tag{3.9}$$

with initial condition $X_t^{t,x,\pi,c} = x$. Let the value function be defined by

$$v(t,x) = \sup_{c,\pi} E\left\{\int_t^T u(c_s)ds + U(X_T^{t,x,\pi,c})\right\}$$

where the supremum runs over the consumption-portfolio pairs and where $X_T^{t,x,\pi,c}$ is the terminal wealth of the investor. The value function satisfies the dynamic programming equation

$$v(t,x) = \sup_{c,\pi} E\left\{\int_t^\tau u(c_s)ds + v(\tau, X_\tau^{t,x,\pi,c})\right\}$$

for any time τ. When it is smooth enough, it fulfills the Hamilton Jacobi Bellman equation

$$\begin{cases} \dfrac{\partial v}{\partial t} + \sup_{c,\pi}\left\{(rx - c + (\mu - r)\pi)\dfrac{\partial v}{\partial x} + \dfrac{1}{2}\sigma^2\pi^2\dfrac{\partial^2 v}{\partial x^2} + u(c)\right\} = 0 \\ v(T,x) = U(x) \end{cases}$$

Therefore, the optimal portfolio π^* is the value of π for which the supremum is attained and is the solution of a quadratic problem. Hence π^* is the function of wealth defined by the relation

$$\pi^*(t,x) = -\frac{\mu - r}{\sigma^2}\frac{\partial v}{\partial x}(t,x)\left(\frac{\partial^2 v}{\partial x^2}(t,x)\right)^{-1},$$

while the optimal consumption c^* satisfies $u'(c^*(t,x)) = \dfrac{\partial v}{\partial x}(t,x)$.
We emphasize that the optimal consumption is obtained via the optimal wealth.

In the particular case where $U(x) = u(x) = x^\delta$, $0 < \delta < 1$, it may be proven that $v(t,x) = p(t)x^\delta$ where $p(t)$ fulfills a differential equation, hence π^* is constant. The optimal portfolio is therefore to invest a fixed multiple of wealth in the risky asset at all dates. The optimal consumption is also a fixed multiple of the wealth.

Generalization

Suppose that there are d risky assets in the market which prices fulfill the following stochastic differential equation

$$dS_t^i = S_t^i(b_i dt + \sum_{j=1}^d \sigma_{i,j} dW_t^j), \quad 0 \le i \le d$$

The methods mentioned above may still be used. The dynamic programming method leads to an optimal portfolio defined in terms of the value function

$$\pi_t^* = -(\sigma\sigma^T)^{-1}(b - r\mathbf{1})\frac{\partial v}{\partial x}(t,x)\left(\frac{\partial^2 v}{\partial x^2}(t,x)\right)^{-1}$$

where b (resp. $\mathbf{1}$) is the vector with coordinate b_i (resp. 1), σ is the matrix $\sigma_{i,j}$ and $\pi \in \mathbb{R}^d$ is the vector of fraction of wealth invested in the risky assets. The optimal portfolio π_t^* is therefore proportional to the vector $(\sigma\sigma^T)^{-1}(b - r\mathbf{1})$. Hence we obtain a mutual fund result: if the investor is a von-Neumann Morgenstern maximizer, then he will invest in only two assets, the bond and the risky fund $(\sigma\sigma^T)^{-1}(b - r\mathbf{1})$. This result which doesn't require mean variance utilities, was originally obtained by Merton at the end of the sixties and was one of the first example of the use of continuous time models in finance.

3.3.3 Income

If the agent gets an income $(e_t, t \geq 0)$ in continuous time, its wealth evolves according to

$$dV_t = rV_t dt + \theta_t[\sigma dW_t + (\mu - r)dt] - c_t dt + e_t dt$$

hence

$$V_t e^{-rt} = E_Q\left(e^{-rT} V_T + \int_t^T e^{-rs}(c_s - e_s)ds|\mathcal{F}_t\right)$$

The constraint of positive wealth at any time is now binding.

Bibliography

[1] Bajeux-Besnainou, I. and Portait, R. (1998), Dynamic asset allocation in a mean-variance framework, *Management Science*, vol. 44, No. 11, S79-S95.

[2] Biais, B. and Björk, T. and Cvitanić, J. and El Karoui, N. and Jouini, E. and Rochet, J.C., *Financial Mathematics, Bressanone, 1996*, Runggaldier, W. edt, volume 1656, Lecture Notes in Maths. Springer-Verlag, Berlin, 1997. [Contains a series of paper on Risk sharing, adverse selection and market structure; Interest rate theory; Optimal trading under constraints; Non-linear pricing theory; Market imperfections, equilibrium and arbitrage.]

[3] Bingham, N.H. and Kiesel, R. *Risk-neutral valuation,* Springer, Berlin, 1998. [Contains an introduction to incomplete markets, as well as exotic options.]

[4] Björk, T., *Arbitrage theory in continous time,* Oxford University Press, Oxford, 1998. [Deals with change of numeraire, incomplete markets, term structure]

[5] Chung, K. L. and Williams, R. J., *Introduction to Stochastic Integration,* 2nd edition, Birkhäuser-Verlag, 1990. [Is a good introduction to stochastic calculus.]

[6] Duffie, D., *Dynamic asset pricing theory* , Princeton University Press, Princeton, 1992. [Essential]

[7] Elliott, R. and Kopp,P. E., *Mathematics of Financial markets,* Springer-Verlag, Berlin, 1998. [Deals with discrete and continuous time models. Contains a chapter on American options are studied.]

[8] Karatzas, I. and Shreve, S. *Methods of Mathematical Finance,* Springer-Verlag, Berlin, 1998.

[9] Korn, R., *Optimal portfolio,* World, Singapour, 1998.[Contains a detailed presentation of consumption/investment optimization.]

[10] Kwok, Y.K., *Mathematical models of financial derivatives,* Springer Finance, Berlin, 1998. [Various derivative products are presented.]

[11] Lamberton, D. and Lapeyre, B., *Introduction to stochastic calculus applied to finance,* Chapman and Hall, 1998. [A good introduction to stochastic calculus, as well as an introduction to option pricing. Contains chapters on American options and processes with jumps.]

[12] Long, J. B. (1990), The numeraire portfolio, *Journal of Financial Economics*, **26**, 29-69.

[13] Mikosch, T., *Elementary Stochastic calculus with finance in view,* World Scientific, Singapore, 1999. [A concise and excellent introduction to stochastic calculus.]

[14] Musiela, M. and Rutkowski, M., *Martingale Methods in Financial Modelling,* Springer- Verlag, Heidelberg, 1997. [A reference book for Financial mathematics, in particular this book contains a study of the term structure and derivative products on interest rates.]

[15] Oksendal, B., *Stochastic Differential Equations,* Fith edition Springer-Verlag, Berlin, 1998. [A complete presentation of stochastic techniques for Brownian motion.]

[16] Pliska, S.R., *Introduction to mathematical finance,* Blackwell, Oxford, 1997. [Discrete time finance, pricing and optimization techniques.]

[17] Shiryaev, A. *Essential of stochastic Finance,* World Scientific, Singapore, 1999. [A reference book for stochastic finance, in particular for discontinuous models in continuous time.]

[18] Wilmott, P. and Howison, S. and Dewynne, J., *The Mathematics of Financial derivatives. A student introduction.* , Cambridge University press, Cambrigde. 1995. [PDE approach to exotic options pricing.]

Chapter 4

Portfolio Insurance

This chapter is a detailled version of the paper " Optimal portfolio management with American capital guarantee" by Nicole El Karoui, Monique Jeanblanc, Vincent Lacoste.

The aim of the paper is to investigate finite horizon portfolio strategies which maximize a utility criterion when a constraint is imposed to a terminal date (European guarantee) or for every intermediary date (American Guarantee). Classical automatic strategies such as the Cushion method - also known as Constant Proportional Portfolio Insurance - as well as the Option Based Portfolio Insurance are studied. In the case of a European guarantee, we prove the optimality of the OBPI method for CRRA utility functions. We then focus on the extension of the OBPI method to the American case, and we prove the strategy based on American puts to be optimal for the maximization of an expected CRRA utility function criterion with American constraint. The solutions are fully described in a Black-Scholes environment as well as in the more general case of complete markets. Finally, all the results are extended to general utility functions.

4.1 Introduction

A large choice of strategies are offered to fund managers. The most celebrated and the simplest one is the Buy and Hold strategy, where in reference to an investment horizon, a well-diversified portfolio (for example an Index portfolio) is tailored, without readjustment before the end. Using dynamical strategies readjusted according to market evolution can improve the performance, but in both cases, the portfolio might support large losses, as in October 1987, or 1998. In order to avoid large losses the manager may decide to "insure" a specified-in-advance minimum value for the portfolio, which implies to give up some potential gains. Legal constraints may also impose to institutional investors that the liquidative value of specific funds never drops below a threshold at any time up to the horizon. For these reasons and many others, such as the increasing number of pension funds, practioners as well as academics have recently paid a particular attention to the problem of portfolio protection.

Along these lines, it is noticeable that Leland and Rubinstein (1976) [10], by referring to the option replicating strategies, introduced the OBPI (Option Based Portfolio Insurance) strategies, using traded or synthetic options. Later on, Perold and Sharpe (1986) [15] and then Black and Jones (1987) [3] developed automated strategies among which the Cushion method (also known as the Constant Proportion Portfolio Insurance method) has become very popular among practioners. Both methods guarantee that the portfolio current value dominates the discounted value of a pre-specified final floor.

More generally, practioners have well understood the separation principle which was introduced by Markowitz (1959) [12] and extended by Merton (1970) [13] : the asset allocation is made optimal through the use of two separate funds, the first one being a combination of basic securities, the second one being the money market account. Merton assumed only that the terminal liquidative value of the fund is non-negative. More recent papers (see for example Cox and Huang (1989) [4]) have extended this result to an often called three fund separation principle, when an extra insurance is required by the investor. A third fund is added to Merton's funds, which pays a derivative written on the fund depending on the basic securities. The OBPI method is an application of such a separation principle. More recently, various authors have proposed dynamic fund strategies in the case of American protection (see Gerber and Pafumi (2000) [6] and Boyle and Imai (2000) [2]). The main results of the present paper are firstly to extend the three fund separation principle to an American constraint, secondly to fully describe the optimal dynamic portfolio in a most general framework.

The paper is organized as follows: in the second section, we recall the classical CPPI and OBPI portfolio insurance methods, which are intensively used by practionners and compare their respective terminal performances. The third section focuses on the maximization of an expected utility criterion, over all self-financing strategies which value satisfies a European constraint. The Put Based Strategy written on the optimal portfolio solving the unconstrained problem is proven to be optimal for CRRA utility functions. In such a case, the three fund separation principle appears to be strictly valid. The fourth section the extension of the OBPI method for the fund to satisfy an American rather than European constraint : the strategy is based on American put options. In order to remain self-financing, we introduce a path dependent gearing parameter. As a result the amount invested in the risky assets increases when the value drops below a given exercize boundary. The description is first done within a Black-Scholes environment. The fifth section extends this result to the more general case of complete markets using the properties of American options developed by El Karoui and Karatzas (1995) [5]. The optimality of the strategy is then proven for a CRRA utility function criterion. In the sixth section, we extend our optimality results to a general class of utility functions. It is shown that the non linearity of the unconstrained optimal portfolio with respect to the initial wealth makes the OBPI still optimal, but the strategic allocation is changed through the initial cost of the protection.

Importantly enough, all through the paper, we set the problem in the general framework of *complete*, *arbitrage free* and *frictionless* markets.

4.2 Classical insurance strategies

4.2.1 Strategic allocation and general framework

The first step in the management of investment funds is to define a *strategic allocation* related to a finite horizon. According to the investor's risk aversion, the manager decides the proportion of indexes, securities, coupon bonds, to be hold in a well-diversified portfolio with positive values. An example would be the efficient portfolio in the Markovitz setting, or the portfolio constructed using the mutual fund result (See Merton [13, 14]), or the optimal portfolio introduced in section 4.3 of this paper.

We denote by S_t the t-time value of *one unit of the* **S***trategic allocation*. Without loss of generality, we assume that $S_0 = 1$. An initial amount $\lambda > 0$ invested at date 0 in the strategic allocation evolves in the future following $(\lambda S_t)_{t\geq 0}$.

At this step, we do not need to specify the dynamics of $(S_t)_{t\geq 0}$. Only recall that we have assumed the market to be *complete*, *arbitrage free* and *frictionless*. The following assumptions give the restrictions we impose on the strategic allocation :

Assumptions
A1. $(S_t)_{t\geq 0}$ follows a continuous diffusion process, $\mathbb{R}^+$-valued.
A2. All dividends and coupons are assumed to be reinvested, in such a way that the strategic allocation is self-financing.
A3. At any date $t \geq 0$ we can find in the market zero-coupon bonds for all maturities $T \geq t$.

We shall use the following general characterization for self-financing strategies :
(SF) $(X_t, t \geq 0)$ is the value process of a self-financing strategy if and only if the process RX is a Q-martingale, where R is the discounted factor, i.e., $R_t = \exp\left(-\int_0^t r(s)ds\right)$ and Q the risk-neutral probability measure.

4.2.2 European versus American guarantee

We now focus on the second step for the manager which is to define the *tactic allocation*, that is to manage dynamically the strategic allocation to fulfil the guarantee.

More precisely, we assume that the manager requires his portfolio to be protected against downfalls of the *strategic allocation*, with horizon date $T \geq 0$ and current floor value we denote by $(K_t)_{0\leq t\leq T}$. One example is to define a minimal final value for the fund, denoted by $K_T = K$, being a percentage of the initial fund value. In such a case, K defines the level of the *capital guarantee* proposed by the manager (for e.g. $K = 90\%$ of the initial capital).
When the guarantee holds for the only terminal date T (for e.g. $T = 8$ years for life insurance contracts ; $T = 5$ years for French tax-free equity funds), the protection is said to be European.
Another type of contracts propose such a guarantee for any intermediary date between 0 and T (this might be a legal requirement as for life insurance contracts). The guarantee is then said to be American. In such a case the floor value is defined as a time dependent function (or process) $(K_t)_{0\leq t\leq T}$.

Remark 4.2.1 In practice, K_t can either be pre-determined or related to a market benchmark. When K_t does not depend on a benchmark, the protection is said to be a capital guarantee. When K_t is marked on a benchmark, the fund proposes a *performance guarantee* : for e.g. $K_t = \alpha\frac{I_t}{I_0}$, where I_t is the current value of an index, and α is a fixed proportion between 0 and 100%.

The present paper is more concerned with the first type of *capital guaranteed* funds.

Let us now consider the case where the current floor value K_t equals the discounted value of a final strike price K :

$$K_t = KB_{t,T},$$

where $B_{t,T}$ denotes the t-time value of a zero-coupon bond paying \$1 at time T. A simple arbitrage argument implies that the American guarantee then reduces to a European one. Let us remark the same situation holds for the performance guarantee if the index does not deliver dividends.

The following subsections present the two most classical insurance methods valid in such a European case.

4.2.3 Stop loss strategy

This strategy is also said to be the "All or Nothing" strategy. It follows the following argument : The investor takes a long position in the strategic allocation whose initial price S_0 is, w.l.g., supposed to be greater than $KB(0,T)$.
At the first hitting time when $S_t < KB(t,T)$, he sells the totality of the strategic allocation to buy K zero-coupon bonds. When the situation is reversed, the orders are inverted. Hence, at maturity, V_T is greater than $\max(S_T, K)$.
The well known drawback of the method is that it cannot be used in practice when the price of the risky asset fluctuates around the floor $G_t = KB(t,T)$, because of transaction costs. Moreover, even in the case of constant interest rate, the strategy is not a self-financing since $e^{-rt}\sup(S_t, KB(t,T))$ is not a martingale. Indeed, the value of this strategy is greater than $KB(t,T)$. If such a strategy is self financing, and if there exists τ such that its value is equal to $KB(\tau,T)$, then it would remain equal to $KB(t,T)$ after time τ, and this is obviously not the case. (See Lakner for details) In other terms,

$$e^{-rt}\sup(S_t, KB(t,T)) = x + \text{martingale} + L_t$$

where L is the local time of $(S_t e^{-rt}, t \geq 0)$ at the level Ke^{-rT}.
Sometimes, practitioners introduce a corridor around the floor and inverse the strategy only when the asset price is outside this corridor. More precisely, the tactic allocation is

$$S_t \mathbb{1}_{t<T_1} + (K-\epsilon)\mathbb{1}_{T_1 \leq t < T_2} + S_t \mathbb{1}_{T_2 \leq t < T_3} + \dots$$

where

$$\begin{aligned} T_1 &= \inf\{t \,:\, S_t \leq K - \epsilon\},\, T_2 = \inf\{t \,:\, t > T_1, S_t \geq K + \epsilon\}, \\ T_3 &= \inf\{t \,:\, t > T_2, S_t \leq K - \epsilon\} \dots \end{aligned}$$

The terminal value of the portfolio when the length of the corridor tightens to 0 can be proved to converge a.s. to : $V_T = \max(S_T, K) - L_T^K$, where L_T^K represents the local time of $(S_t, t \in [0,T])$ around K. We do not provide a proof of this standard result

4.2.4 CPPI strategy

The Constant Proportional Portfolio Insurance was introduced by Perold and Sharpe (1986)[15] and Black and Jones (1987) [3]. The manager finances the protection by taking a long position on K zero-coupon bonds, and dynamically manages the *cushion* $C_t = V_t - KB_{t,T}$, where V_t denotes the current liquidative value of the protected fund.

The cushion is managed in such a way that the proportion of the wealth currently invested in the underlying strategic allocation $(S_t, t \geq 0)$ is a constant m proportion of the liquidative value. The parameter m is usually called the *leverage* of the fund and it is often chosen in practise close to 4.
More precisely, let us denote by $(r_t, t \geq 0)$ the short rate and by $\sigma_t, t \geq 0)$ the volatility of the strategic allocation (Obviously, r and σ are suppossed to be adapted processes with reasonable

integrability properties, i.e. $\int_0^T r_s ds < \infty, \int_0^T \sigma_s^2 ds < \infty$, a.s.). A self-financing strategy with leverage effect m evolves as

$$C_t = m\frac{C_t}{S_t}S_t + (1-m)C_t$$

and $dC_t = m\frac{C_t}{S_t}dS_t + (1-m)C_t r_t dt$, or equivalently,

$$dC_t = C_t[r_t dt + m(\frac{dS_t}{S_t} - r_t dt)]. \quad (4.1)$$

From (4.1), the volatility of C is $m\sigma_t^2$, therefore, using the positivity of C,

$$\begin{aligned} d\ln C_t &= (r_t - \frac{m^2\sigma_t^2}{2})dt + m\left(d\ln S_t + (\frac{1}{2}\sigma_t^2 - r_t)dt\right) \\ &= -(m-1)(r_t + \frac{m}{2}\sigma_t^2)dt + m\, d\ln S_t\,. \end{aligned}$$

Hence, knowing that $S_0 = 1$,

$$C_T = C_0\left(S_T e^{-\frac{m-1}{m}\int_0^T (r_t + \frac{m}{2}\sigma_t^2)dt}\right)^m, \; C_0 = 1 - KB(0,T)\,.$$

4.2.5 OBPI Strategy

The Option Based Portfolio Insurance, pioneered by Leland and Rubinstein (1976) [10], similarly with the CPPI method, has both actions : firstly to protect the portfolio value at maturity; and secondly to take advantage of rises in the underlying strategic allocation.

The Put Based Strategy is to buy (or duplicate) a *Put option* to insure a long position on the underlying strategic allocation $(S_t)_{t\geq 0}$.

The initial capital invested in the fund, supposed to be normalized at 1 and strictly larger than $KB(0,T)$ is then split into two parts, say λ and $1-\lambda$, where λ lies between 0 and 1. With the first part, the manager buys at date 0 a fraction λ of the strategic allocation, and with the remaining part, he insures his position with a put written on his long position which current value is $(\lambda S_t)_{t\geq 0}$. The strike price of the put option is K, being the final floor value for the fund.

Note that we have to require that $1 \geq KB_{0,T}$ in order to obtain the existence of a portfolio satisfying the terminal contraint.

Let us denote by $P^e(t,K,T)$ the t-time price of an European Put with maturity T and strike K written on one unit of the strategic allocation, where the superscript e stands for European. Suppose that such options may be traded in the market for every strike K. In our case, the market is complete and this condition holds since any bounded derivative may be replicated with self-financing portfolio.

Remark 4.2.2 The put option can also be written as a put on λS_T with strike price K following : $(K - \lambda S_T)^+ = \lambda(\frac{K}{\lambda} - S_T)^+$. It is noticeable that the strategic allocation has to perform above K/λ (note that λ depends on K) for the investor to get more than its guaranteed capital. For simultaneously small values of λ and high values of K, K/λ might become large, and therefore the final payoff might look poorly attractive (see figure 1 for more insight).

Hence, at maturity, the manager obtains a protected value for the fund :

$$V_T(\lambda) = \lambda S_T + (K - \lambda S_T)^+ = \sup(\lambda S_T, K).$$

The parameter λ is usually called the *gearing* of the fund.

The value of λ has to be determined at date 0, and it is to be adjusted by the *budget constraint :*

$$V_0(\lambda) = \lambda + \lambda P^e(0, T, K/\lambda) = 1.$$

Therefore λ highly depends on the volatility market, through the price of the put option.

Proposition 4.2.1 *There exists a unique e constant λ, with $0 < \lambda < 1$ such that*

$$\lambda + \lambda P^e(0, T, K/\lambda) = 1\,.$$

The strategy defined by $V_t(\lambda) = \lambda S_t + \lambda P^e(t, T, K/\lambda)$ is the protected fund with terminal value

$$V_T(\lambda) = \lambda S_T + (K - \lambda S_T)^+ = \sup(\lambda S_T, K)$$

satisfies the budget constraint.

PROOF: The liquidative value of the fund at maturity T : $V_T(\lambda) = \sup(\lambda S_T, K)$, is an non-decreasing function with respect to λ valued in $[K, +\infty[$ and satisfies

$$\text{for } \lambda > \lambda',\ 0 \leq V_T(\lambda) - V_T(\lambda') \leq (\lambda - \lambda')S_T\,.$$

From the no-arbitrage assumption, knowing that $S_t > 0$ for all $t \geq 0$, the 0 time value of this protected fund $V_0(\lambda)$ is a non-decreasing function of λ valued in $]KB_{0,T}, +\infty[$, and lipschitzian with respect to λ with a Lipschitz contant equal to 1. Indeed, if $V_0(\lambda) = V_0(\lambda')$ for $\lambda > \lambda'$, then $\sup(\lambda S_T, K) = \sup(\lambda' S_T, K)$, which implies that $\lambda' \leq K$. In that case, the terminal value of the protected fundis equal to the guarantee in all states of the world, and its 0-time value is given ky $KB(0, T)$ which is impossible since this quantity is assumed to be strictly smaller than 1.
It is obvious to check that $\lambda \in [0, 1]$. □

Remark 4.2.3 When the dynamics of the underlying $(\lambda S_t)_{t \geq 0}$ are assumed to be Markovian, the value of an European put on S, $P^e_S(t, T, K)$ turns out to be a deterministic function of time t and current value of the underlying S_t, which for the sake of simplicity we shall denote by $P^e(t, S_t, T, K)$. The function $P^e(\cdot, T, \cdot)$ then solves the classical valuation Partial Differential Equation. The function $\lambda P^e(t, S_t, T, K/\lambda)$ whick represents the price of a Put option written on λS with strike K differs in general of the function $P^e(t, \lambda S_t, K, T)$, except in the Black and Scholes framework where the dependence with respect to the initial condition is linear.

The OBPI strategy can also be written using calls : the Call Based Strategy is to buy a Call, and K zero-coupon bonds. Due to the Put-Call parity, for both Put Based and Call Based Strategies the terminal wealth is :

$$V_T = K + (\lambda S_T - K)^+ = \sup(\lambda S_T, K).$$

The residual initial wealth invested in the call is as for the CPPI strategy $1 - KB_{0,T}$. By a long position on a Call, the manager takes advantage of rises of the underlying. Using similar notations as previously for the put, the parameter λ is adjusted such that :

$$C^e_{\lambda S}(0) = 1 - KB_{0,T}. \tag{4.2}$$

4.2.6 Comparison of performances

For those two different strategies, we can compare the performances $(V_T - V_0)/V_0$ or, equivalently their terminal values V_T. We know from the no-arbitrage condition that there does not exist a dominating strategy in all states of the world.

Figure 1 charts the respective payoffs with horizon date 5 years of the CPPI method with leverage 2 and the OBPI method calculated with a floor value of 90% of the initial capital. The strategic allocation is supposed to follow a Black-Scholes dynamics, with interest rate $r = 5\%$, and annual volatility $\sigma = 20\%$. The gearing of the OBPI is calculated such that the budget constraint (4.2) is verified. It equals 94.60%.

Insert Figure 1 here.

Figure 1 shows that the CPPI method outperforms the OBPI for small or negative performances of the strategic allocation. The OBPI becomes most profitable for medium performances. The exponential form of the cushion implies that CPPI outperforms OBPI for much larger upper changes. Figure 2 gives two other examples with lower and larger strike values, and a higher leverage $m = 4$ for CPPI.

Insert Figure 2 here.

Figure 2 shows that for small floor values, the OBPI method tends to a Buy and Hold strategy, whereas the exponential form of the CPPI method used with a high leverage makes the fund outperform for very high performances of the strategic allocation.
When the protection goes to 100% of the capital, the OBPI drives away from the Buy and Hold strategy, and looses part of the performance.

Next Figure 3 charts the dependence of the gearing parameter with respect to the strike level. It is a decreasing function which tends to 0 when the strike tends to $\frac{1}{B_{0,T}}$. In this latter case, the fund in both methods reduces a zero coupon. The *cushion* is nul, and therefore no investment is done in the risky assets.

Insert Figure 3 here.

The following sections study the optimal policy to follow which maximizes an expected utility criterion, and proves the OBPI to be most generally optimal.

4.3 OBPI Optimality for a European Guarantee

In this section, we study the optimal portfolio policy in the case of a European Guarantee, for which the constraint holds only at the terminal date T.

We now consider the risk aversion of the investor by the mean of an expected utility criterion to be maximized under the subjective probability P. Given a *utility function* u (concave, strictly increasing, defined on $\mathbb{R}^+$) we compare $E[u(V_T)]$ for the different strategies where the manager allocates his wealth in financial assets.
More precisely, we are looking for an optimal solution of the program :

$$\max E[u(V_T)]; \qquad \text{under the constraints } V_T \geq K, \text{and } V_0 = 1\,, \tag{4.3}$$

over all self-financing portfolios.

We prove the optimality of the Put Based Strategy written on an appropriate strategic allocation in the case of a constant relative risk aversion (CRRA) utility function u defined as $u(x) = \dfrac{x^{1-\gamma}}{1-\gamma}$, for all $x \in I\!R^+$, with $\gamma \in]0,1[$. The general case is studied in subsection 4.6.1.

4.3.1 Choice of the strategic allocation and properties

We choose the strategic allocation S as the solution of the free problem :

(4.4) $$\max E[u(X_T^\lambda)]; \qquad \text{under the budget constraint } X_0^\lambda = \lambda\,,$$

where λ denotes the initial wealth. It is well known (see for example Karatzas and Shreve (1998) [9]) that the terminal value $\hat{X}_T^\lambda$ of the optimal strategy with initial wealth λ satisfies the first order condition[1] :

(4.5) $$E(u'(\hat{X}_T^\lambda)(X_T^\lambda - \hat{X}_T^\lambda)) = 0,$$

for any X_T^λ, terminal value of a self-financing portfolio with initial value λ.

Moreover, in the CRRA case, the optimal terminal wealth $\hat{X}_T^\lambda$ is proportional to the initial wealth, i.e., $\hat{X}_T^\lambda = \lambda S_T$, where $S_T = \hat{X}_T^1$ is the optimal unconstrained strategy with initial value 1.

Remark 4.3.1 *The linear property of the solution with respect to the initial wealth for a CRRA utility function criterion allows to describe the optimal solution as a proportion of one unit of the optimal strategic allocation. This justifies the fact the initial fund value has been previously chosen equal to* 1*.*

Consequently, we have : $u'(\hat{X}_T^\lambda) = \lambda^{-\gamma} u'(S_T)$. Hence, the first order condition (4.5) can be re-written :

(4.6) $$E(u'(S_T)(X_T^\lambda - \hat{X}_T^\lambda)) = 0\,.$$

4.3.2 Choice of the tactic allocation

As in subsection 4.2.5, we assume that the initial wealth 1 invested in the fund is split into two parts, say λ and $1-\lambda$, where λ is the amount the manager invests in the optimal unconstrained strategy $(S_t)_{t\geq 0}$. With the remaining part, the manager buys a European put option on his long position $(\lambda S_t)_{t\geq 0}$ with strike K and terminal date T[2]. Therefore the insured portfolio, whose current value is denoted by $\hat{V}_t$ and which combines the two positions, satisfies at date T :

(4.7) $$\hat{V}_T = \lambda S_T + (K - \lambda S_T)^+ = \max(\lambda S_T, K) \geq K.$$

The parameter λ is to be adjusted by means of the budget constraint :

$$\lambda + P^e_{\lambda S}(0) = 1\,,$$

where $P^e_{\lambda S}(0)$ is the price at date 0 of the European put on the underlying $(\lambda S_t)_{t\geq 0}$.

Remark 4.3.2 *The value of* $1-\lambda$ *is the initial cost of the insurance. This cost depends on the anticipation of the agent only via the choice of the strategic allocation. In a bullish market, the final payoff is* λS_T*, to be compared with* S_T*.*

---

[1] Consider the portfolio with terminal value $V_T(\epsilon) = \epsilon \hat{X}_T^\lambda + (1-\epsilon) X_T^\lambda$ and write that $\epsilon = 1$ is the maximum of $E[u(V_T(\epsilon))]$: $\left(\frac{\partial}{\partial \epsilon} E[u(V_T(\epsilon))]\right)_{\epsilon=1} = 0.$

[2] Due to the assumption of completeness of the market, it is possible to find or duplicate put options with any strike.

4.3.3 Optimality of the tactic allocation

Proposition 4.3.1 *The Put Based Strategy written on the optimal portfolio with no constraint solves the optimization problem with European constraint for CRRA utility functions. More precisely, if V_T is the terminal value of a portfolio with initial value 1 such that $V_T \geq K$ and $\hat{V}_T$ is the terminal value of the Put Based Strategy defined in 4.7, then :*

$$E[u(\hat{V}_T)] \geq E[u(V_T)] .$$

Proof: The concavity of u yields to :

$$u(V_T) - u(\hat{V}_T) \leq u'(\hat{V}_T)(V_T - \hat{V}_T) .$$

From equality (4.7) and the CRRA property of u we get :

$$u'(\hat{V}_T) = u'(\lambda S_T) \wedge u'(K) = [\lambda^{-\gamma} u'(S_T)] \wedge u'(K) .$$

Since $u'(\hat{V}_T) \geq u'(K)$ is equivalent to $\hat{V}_T = K$ due to the constraint $\hat{V}_T \geq K$ and the decreasing property of u', we obtain :

$$\begin{aligned} [[\lambda^{-\gamma} u'(S_T)] \wedge u'(K)\,]\,(V_T - \hat{V}_T) \quad &= \quad \lambda^{-\gamma}\, u'(S_T)\,(V_T - \hat{V}_T) \\ &\quad - [\lambda^{-\gamma} u'(S_T) - u'(K)]^+\,(V_T - K) . \end{aligned}$$

On one hand, from the first order condition (4.6) written for $\lambda = 1$, we have :

$$E[\,u'(S_T)(V_T - \hat{V}_T)] = E[\,u'(S_T)(V_T - \hat{X}_T)] + E[\,u'(S_T)(\hat{X}_T - \hat{V}_T)] = 0,$$

and on the other hand, from the terminal constraint on V_T, the following inequality holds :

$$-E\left([\lambda^{-\gamma} u'(S_T) - u'(K)]^+ \,(V_T - K) \right) \leq 0.$$

Hence, $E(u(\hat{V}_T)) \geq E(u(V_T))$. □

4.4 American case in the Black and Scholes framework

We now adress the problem of an American guarantee.

We first exhibit how to build *self-financing* Put-based strategies when $(S_t)_{t\geq 0}$ is known to follow a Black-Scholes dynamics. The proof of the optimality of our strategy, written on the unconstrained optimal portfolio is given for CRRA utility function in subsection 5.3. The result is extended to a general class of utility functions in subsection 4.6.2.
In this section, the dynamics of the strategic allocation $(S_t)_{t\geq 0}$ are given by :

$$dS_t = S_t(r\,dt + \sigma dW_t)\,,\; S_0 = 1, \tag{4.8}$$

where $(W_{t\geq 0}$ is a Brownian motion under the risk-neutral probability Q. The interest rate r is assumed to be constant, as well as the volatility σ.

4.4.1 American Put Based strategy

By analogy with the European case, we introduce an American put on the position $(\lambda S_t)_{t\geq 0}$, where λ is now to be adjusted such that :

$$1 = \lambda + P^a_{\lambda S}(0), \tag{4.9}$$

where $P^a_{\lambda S}(0)$ is the price at time 0 of the American put on the underlying $(\lambda S_t\,; t \geq 0)$, with strike K and maturity T. Note that by definition of an American contract $P^a_{\lambda S}(t) \geq (K - \lambda S_t)^+$, hence, defining X_t as the t-time value of a portfolio consisting in both positions on the strategic allocation and the American derivative :

$$X_t \stackrel{def}{=} \lambda S_t + P^a_{\lambda S}(t) \geq K.$$

However, the part of the portfolio which consists in the American put is not self-financing. Some cash is needed to hedge the anticipated exercize as soon as the stopping region is attained, that is after time $\sigma(\lambda)$ where :

$$\sigma(\lambda) = \inf\{t \,:\, P^a_{\lambda S}(t) = K - \lambda S_t\}\,.$$

4.4.2 Properties of the American put price

We recall some well known properties of the American put price in the Black-Scholes framework. The price $P^a_S(t)$ of an American option on the underlying S is a deterministic function of time t and current value of the underlying S_t. We denote by $P^a(t,x)$ such a function, hence the price of a put on the underlying $(\lambda S_t,\, t \geq 0)$ is $P^a(t, \lambda S_t)$. By definition :

$$P^a(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} E_Q(K - X^{t,x}_\tau)^+ e^{-r(\tau - t)}\,)$$

where $\mathcal{T}_{t,T}$ is the set of stopping times taking values in the interval $[t,T]$ and $X^{t,x}_T$ denotes the T-time value of the solution of the Black-Scholes equation (4.8) which equals x at time t. Let us denote by $\mathcal{C}$ the continuity region defined as $\mathcal{C} = \{(t,x) | P^a(t,x) > (K-x)^+\}$. In the Black and Scholes framework, this continuity region is also described via the increasing exercise boundary $(b(t), t \geq 0)$ where b is the deterministic function defined as :

$$b(t) = \sup\{x \,:\, P^a(t,x) = (K-x)^+\}.$$

Therefore :

$$\mathcal{C} = \{(t,x) \,:\, x > b(t)\}\,.$$

The function $P^a(t,x)$ satisfies :

$$\begin{cases} \partial_t P^a(t,x) + \dfrac{1}{2}\sigma^2 x^2 \partial_{xx} P^a(t,x) + rx\partial_x P^a(t,x) - rP^a(t,x) = 0, \quad \forall (t,x) \in \mathcal{C}\,, \\ P^a(t,x) = K - x\,,\ \forall (t,x)\ \notin \mathcal{C}\,, \\ P^a(t,x) \geq K - x\,,\ \forall (t,x)\,, \end{cases} \tag{4.10}$$

and (smoothfit principle) $\partial_x P^a(t, b(t)) = -1$.
We introduce

$$A(t,x) \stackrel{def}{=} x + P^a(t,x)\,.$$

The function $A(t,x)$ is C^2 with respect to x and the second derivative admits only one discontinuity. Moreover, introducing the operator $\mathcal{L}$ defined as :

$$\mathcal{L} = \partial_t + \frac{1}{2}\sigma^2 x^2 \partial_{xx} + rx\partial_x ,$$

we have :

$$\left\{ \begin{array}{rcl} A(t,x) & = & K, \text{ for } (t,x) \notin \mathcal{C}\,, \\ \mathcal{L}A(t,x) & = & \mathcal{L}P^a(t,x) + rx = rA(t,x) \text{ for } (t,x) \in \mathcal{C}\,, \\ \mathcal{L}A(t,x) & = & 0, \text{ for } (t,x) \notin \mathcal{C}\,. \end{array} \right. \tag{4.11}$$

and $\partial_x A(t,b(t)) = 0$.
Let us consider $A(t, \lambda S_t)$, which is the value of a long position on $(\lambda S_t)_{t\geq 0}$ and an American Put on the underlying λS with strike K :

$$A(t, \lambda S_t) = \lambda S_t + P^a(t, \lambda S_t)\,.$$

This process follows :

$$\begin{aligned} dA(t,\lambda S_t) &= \lambda\Delta(t,\lambda S_t)(dS_t - rS_t dt) + \mathcal{L}A(t,\lambda S_t)dt \\ &= \lambda\Delta(t,\lambda S_t)(dS_t - rS_t dt) + rA(t,\lambda S_t)1\!\!1_{\{(t,\lambda S_t)\in\mathcal{C}\}}dt \\ &= \lambda\Delta(t,\lambda S_t)(dS_t - rS_t dt) + rA(t,\lambda S_t)1\!\!1_{\{\lambda S_t \geq b(t)\}}dt \\ &= rA(t,\lambda S_t)dt + \lambda\Delta(t,\lambda S_t)(dS_t - rS_t dt) - rK1\!\!1_{\{\lambda S_t \leq b(t)\}}dt\,, \end{aligned} \tag{4.12}$$

where Δ is the derivative of A with respect to the underlying value, i.e., $\Delta(t,x) = \partial_x A(t,x)$.
Therefore, from the condition **(SF)**, the process $(A(t, \lambda S_t), t \geq 0)$ is the value of a self-financing portfolio **up to the hitting time of the boundary**. If the exercize boundary is reached before maturity, the portfolio generates a continuous dividend rate rK, which has to be re-invested in order to remain self-financing.

4.4.3 An adapted self-financing strategy

We are now looking for a continuous and adapted non-negative process $(\lambda_t, t \geq 0)$ such that the portfolio we denote by $(V_t)_{t\geq 0}$:

$$V_t \stackrel{def}{=} \lambda_t S_t + P^a(t, \lambda_t S_t) = A(t, \lambda_t S_t)$$

is self-financing. We have recalled that the self-financing property holds in the continuity region $\mathcal{C}$. Therefore, we choose $(\lambda_t, t \geq 0)$ such that λ is constant as long as $\lambda_t S_t \in \mathcal{C}$ and such that $\lambda_t S_t \geq b(t)$ in order to remain within the continuity region or at the boundary. Hence, the choice of an increasing process[3] for λ leads to :

$$\lambda_t = \sup_{u\leq t}(\lambda_0, \frac{b(u)}{S_u}) = \lambda_0 \vee \sup_{u\leq t}\left(\frac{b(u)}{S_u}\right)\,,$$

where λ_0 is to be adjusted to satisfy the budget constraint.

Proposition 4.4.1 *Let $(S_t)_{t\geq 0}$ follow a Black-Scholes dynamics (4.8).*
(i) The strategy

$$V_t = \lambda_t S_t + P^a(t, \lambda_t S_t),$$

[3]The choice of λ as an increasing process is justified while dealing with optimality.Intuitively, λ is increasing because outside the continuity region the dividend rate rK can be reinvested in buying more stocks.

is self-financing and satisfies $V_t \geq K, \forall t$, where :

$$\lambda_t = \sup_{u \leq t}(\lambda_0, \frac{b(u)}{S_u}),$$

and λ_0 is adjusted by the budget constraint

$$V_0 = \lambda_0 + P^a(0, \lambda_0) = 1\,.$$

When the constraint is active ($V_t = K$), we have a null position in the strategic allocation.
(ii) The terminal value of this strategy is

$$V_T = \lambda_T S_T + (K - \lambda_T S_T)^+ = \sup\left(K, \sup_{u \leq T}(\lambda_0, \frac{b(u)}{S_u})S_T\right)\,. \tag{4.13}$$

The final payoff V_T therefore has a path dependent lookback feature.

Proof: Since $(\lambda_t)_{t \geq 0}$ is a continuous bounded variation process Itô's formula implies that :

$$dV_t = [dA(t, \lambda S_t)]_{\lambda = \lambda_t} + S_t \partial_x A(t, \lambda_t S_t) d\lambda_t\,.$$

Now, using that λ increases only at the boundary and that the smoothfit principle implies that $\partial_x A(t, b(t)) = 0$, we get, as in (4.12) :

$$\begin{aligned} dV_t &= rA(t, \lambda_t S_t)dt + \lambda_t \Delta(t, \lambda_t S_t)(dS_t - rS_t dt)] - rK \mathbb{1}_{\lambda_t S_t \leq b(t)} dt \\ &\quad + S_t \Delta(t, \lambda_t S_t) \mathbb{1}_{\lambda_t S_t = b(t)} d\lambda_t\,. \end{aligned}$$

We have noticed that $\Delta A(t, \lambda_t S_t) = 0$ on the set $\{\lambda_t S_t = b(t)\}$. Therefore :

$$dV_t = rA(t, \lambda_t S_t)dt + \lambda_t \Delta(t, \lambda_t S_t)\left[dS_t - rS_t dt\right] - Kr \mathbb{1}_{\{\lambda_t S_t \leq b(t)\}} dt\,.$$

The set $\{(t, \omega) : \lambda_t S_t \leq b(t)\}$ is equal to the set $\{(t, \omega) : S_t = \frac{b(t)}{\lambda_t}\}$ and has a zero $dP \otimes dt$ measure, since the process $\frac{b(t)}{\lambda_t}$ has bounded variation. Hence, from **(SF)**, the portfolio $(V_t, t \geq 0)$ is self-financing. □

The Put Based strategy described in Proposition 4.4.1 which is now proven to be self-financing appears to be a good candidate for optimality. This will be established in Proposition 4.5.3.

Remark 4.4.1 The function $A(t, \lambda_t x)$ is solution of a Neuman problem (with the smoothfit condition at the boundary). The probabilistic representation of this solution is the reflected process at the boundary,i.e., $S_t \sup_{u \leq t} \frac{b(u)}{S_u}$.

4.4.4 Description of the American Put Based Strategy

The particularity of the strategy defined in proposition 4.4.1 resides in the adapted bounded variation process $(\lambda_t)_{t \geq 0}$. In practice, the manager of a fund who follows such a policy should increase the gearing of the fund any time the strategic allocation drops below a given exercize boundary. This implies in particular, refering to the first remark in subsection 4.2.5, that he has to adapt continuously the strike level of the American put he currently holds. Indeed, the strike value of the put written on S_t is K/λ_t. The same type of management is already known by academics and practioners who deal with lookback options (see Gatto, Goldman and Sosin (1989) for full details), for which the hedging portfolio based on straddles has to be rebalanced continuously in order to track the minimum of the underlying process (see equation (4.13) for a description of the lookback feature).

Another practical implication of our result is that, even though fund managers have a global position which is nul on the strategic allocation when the boundary is reached, they should not get off the market when the market drops drastically. Considering that they should track the minimum value of their strategic portfolio, they should firstly keep buying more stocks, and secondly, rebalance their position on the put market.

Following figure 4 plots a bearish simulated path of the strategic allocation, simultaneously with the modified position of the fund, namely $(\lambda_t S_t)_{t\geq 0}$. The unmodified position is also plotted, with an initial gearing of 91.69% supposed to be fixed for the whole life of the fund. 91.69% is calculated so that the budget constraint (4.9) is verified, with the same market conditions as for Figures 1 to 4 ($\sigma = 20\%$, $r = 5\%$, $T = 5$ years, $K = 90\%$). We also plot the exercize boundary of the American Put, in order to make visible the control made on the fund position by means of the gearing parameter.

Insert Figure 4 here.

Because of the downfall of the strategic allocation below the exercize boundary, the final gearing parameter is 135.88%, much greater than the initial 91.69%.

Figure 5 compares the liquidative values of three optimal strategies : the Buy and Hold strategy, the European Put Based strategy (which gearing was calculated in the previous section equal to 94.6%), and our American Put Based strategy. The path followed by the strategic allocation is the same as in Figure 4.

Insert Figure 5 here.

In the present case of a bearish market, the American strategy outperforms the other ones as soon as the market drops below the strike price. Noticeably enough, the fund terminates with a positive final performance of +17.16%, when the European strategy gets a negative performance limited to −10% due to the capital guarantee, and the Buy and Hold strategy gets a negative performance of −13.77%.

It is important to note that the sample has been chosen to illustrate the efficiency of the American protection. In most cases, and in particular when the market is performing well, the order of performances is inverted, the Buy and Hold strategy being the first one above the European guarantee, and lastly the American one. In such a case, the cost of the insurance in terms of performance is well quantified by the initial gearing parameters of 94.6% for the European case and 91.69% for the American case.

Remark 4.4.2 *Our strategy is close to the one of Gerber and Pafumi (2000)[6]. In their seminal paper, the authors propose a protected level given by :*

$$\lambda_t^G = \sup_{0\leq u\leq t} (\lambda_0^G, \frac{K}{S_u}).$$

They derive a closed formed formula for the price of this guarantee in the Black and Scholes framework. The main difference with our approach is that we use the American boundary instead of the strike level. We also prove our strategy to maximize an expected utility criterion as soon as the strategic allocation in chosen in an optimal way.

4.5 American case for general complete markets

In the first subsection, we describe Put-Based self-financing strategies in the general case of complete markets. In the second subsection, we prove, for CRRA utility functions, the optimality of the strategy when the strategic allocation solves the unconstrained maximization problem (4.4). The general case follows using the same tools and will be solved in the last section of his paper.

In order to build Put-Based self-financing strategies in the general setting of complete markets, we use the Gittins index methodology, and we follow the ideas of El Karoui and Karatzas (1995)[5]. The starting point is that, if a family of processes, indexed by a parameter are martingales, then the family of the derivatives with respect to the parameter are also martingales. The remarkable result is that we obtain the same representation for the strategy as in the Markovian case (compare equations (4.13) and (4.19)).

4.5.1 Price of an American put

We now suppose that S is an arbitrary continuous, strictly positive process, which represents the value of a self-financing strategy. We assume, without loss of generality that $S_0 = 1$. We denote by Q the risk-neutral probability measure.

We introduce $P_t^a(\lambda)$, the American Put price on the underlying $(\lambda S_t, t \geq 0)$ and strike K, defined as :

$$P_t^a(\lambda) = \text{esssup}_{\tau \in \mathcal{T}_{t,T}} E_Q(R_\tau^t (K - \lambda S_\tau)^+ | \mathcal{F}_t),$$

where $R_s^t = R_s / R_t$ and $\mathcal{T}_{t,T}$ is the set of stopping times taking values in $(t, T]$. Let us remark that $P_t^a(\lambda)$ is decreasing with respect to λ, $P_t^a(0) = K$, $P_T^a(\lambda) = (K - \lambda S_T)^+$ and that $P_t^a(\lambda) \geq (K - \lambda S_t)^+$. We denote by $\sigma(\lambda)$ the associated optimal stopping time :

$$\sigma(\lambda) \stackrel{def}{=} \inf\{u \geq 0 \,:\, P_u^a(\lambda) = (K - \lambda S_u)^+\}\,.$$

The map $\lambda \to \sigma(\lambda)$ is non-decreasing and right-continuous. From the value of $P_T^a(\lambda)$, we observe that $\sigma(\lambda) \leq T$. We define, for $t < T$, the stochastic critical price b_t by :

$$\frac{b_t}{S_t} \stackrel{def}{=} \sup\{\lambda\,,\, P_t^a(\lambda) = (K - \lambda S_t)^+\},$$

and we note $\gamma_t \stackrel{def}{=} \dfrac{b_t}{S_t}$. We set $b_T^+ = K$ (in this general setting, it may happen that $\lim_{t \to T} b_t \neq K$.) We define the so-called Gittins index as the right-continuous inverse of σ, i.e. :

$$G_t = \sup_{0 \leq u < t} \gamma_u, \text{ for } t < T,$$

and we set :

$$G_T^+ = \left(\sup_{0 \leq u < T} \gamma_u \right) \vee \frac{K}{S_T}\,. \tag{4.14}$$

Let us remark that, for $t < T$:

$$\{G_t < \lambda\} = \{\sigma(\lambda) > t\},$$

and

$$\{G_T^+ < \lambda\} \subseteq \{\{\sigma(\lambda) = T\} \cap \{K < \lambda S_T\}\} \subseteq \{G_T^+ \leq \lambda\}. \tag{4.15}$$

Proposition 4.5.1 *The price of the American put is :*

$$P_0^a(\lambda) = E_Q(R_T S_T (G_T^+ - \lambda)^+)\,. \tag{4.16}$$

PROOF: From the envelop theorem[4] the supremum and the differentiation can be inverted (see [5] for details), hence :

$$\frac{\partial P^a}{\partial \lambda}(\lambda) = -E_Q(R_{\sigma(\lambda)} S_{\sigma(\lambda)} 1\!\!1_{\{K \geq \lambda S_{\sigma(\lambda)}\}})\,.$$

From the Q-martingale property of RS, we prove that the right-hand side equals :

$$-E_Q(R_T S_T) + E_Q(R_{\sigma(\lambda)} S_{\sigma(\lambda)} 1\!\!1_{K < \lambda S_{\sigma(\lambda)}})\,.$$

On the set $\{K < \lambda S_{\sigma(\lambda)}\} \cap \{T > \sigma(\lambda)\}$, we would get $P^a_{\sigma(\lambda)}(\lambda) = 0$, which is absurd, therefore $\sigma(\lambda)$ is equal to T on $\{K < \lambda S_{\sigma(\lambda)}\}$ and :

$$E_Q(R_{\sigma(\lambda)} S_{\sigma(\lambda)} 1\!\!1_{K < \lambda S_{\sigma(\lambda)}}) = E_Q(R_T S_T 1\!\!1_{K < \lambda S_T} 1\!\!1_{\sigma(\lambda) = T})\,.$$

From (4.15), we deduce :

$$E_Q(R_T S_T 1\!\!1_{\{G_T^+ > \lambda\}}) \leq -\frac{\partial P^a}{\partial \lambda}(\lambda) \leq E_Q(R_T S_T 1\!\!1_{\{G_T^+ \geq \lambda\}})\,, \tag{4.17}$$

and by integration with respect to λ of this inequality, it follows that the price of the American put can be written as (4.16). □

Remark 4.5.1 The value of the American Put at any time t can be obtained with the same ideas, with the help of

$$\sigma_t(\lambda) \stackrel{def}{=} \inf\{u \geq t \,:\, P_u^a(\lambda) = (K - \lambda S_u)^+\}\,,$$

and the Gittins index

$$G_{t,u} = \sup_{t \leq \theta < u} \gamma_\theta, \text{ for } u < T, \quad G_{t,T}^+ = \left(\sup_{t \leq \theta < T} \gamma_\theta\right) \vee \frac{K}{S_T}\,.$$

With this notation,

$$P_t^a(\lambda) = E_Q(R_T^t S_T (G_{t,T}^+ - \lambda)^+ | \mathcal{F}_t) = E_Q(R_T^t (S_T G_{t,T}^+ - \lambda S_T)^+ | \mathcal{F}_t)\,. \tag{4.18}$$

[4]The envelop theorem states that, if $a^*(\lambda) = \text{argmax} f(a,\lambda)$, then $\partial_\lambda f(a^*(\lambda),\lambda) = \sup \partial_\lambda f(a,\lambda)$.

4.5.2 Self-financing strategy

Proposition 4.5.2 *The strategy*

$$V_t = S_t\lambda_t + P_t^a(\lambda_t)$$

is self-financing with terminal value

$$V_T = K \vee S_T\lambda_T = S_T(G_T^+ \vee \lambda_0)\,.$$

and satisfies $V_t \geq K, \forall t$ *when choosing* λ_t *such that :*

$$\lambda_t = G_t \vee \lambda_0 = \left(\sup_{u\leq t} \frac{b_u}{S_u}\right) vee\lambda_0\,, \tag{4.19}$$

where λ_0 *is to be adjusted by means of the budget constraint* $\lambda_0 + P_0^a(\lambda_0) = 1$*. When the constraint is active, i.e.,* $V_t = K$*, we have a null position on the strategic allocation.*

Proof: From the $\mathcal{F}_t$-measurability of λ_t and equality (4.18),

$$V_t = S_t\lambda_t + P_t^a(\lambda_t) = E_Q(S_t\lambda_t + R_T^t S_T(G_{t,T}^+ - \lambda_t)^+|\mathcal{F}_t).$$

Using the martingale property of RS,

$$\begin{aligned} V_t &= E_Q(R_T^t S_T[\lambda_t + (G_{t,T}^+ - \lambda_t)\mathbb{1}_{G_{t,T}^+ > \lambda_t}]|\mathcal{F}_t) \\ &= E_Q(R_T^t S_T[G_{t,T}^+ \vee \lambda_t]|\mathcal{F}_t)\,. \end{aligned} \tag{4.20}$$

From

$$\sup_{0\leq u<t}(\gamma_u) \vee \sup_{t\leq u<T}(\gamma_u) = \sup_{0\leq u<T}(\gamma_u),$$

we obtain $G_{t,T}^+ \vee \lambda_t = G_{0,T}^+ \vee \lambda_0$. Therefore :

$$V_t = E_Q(R_T^t S_T[G_{0,T}^+ \vee \lambda_0]|\mathcal{F}_t),$$

and the process RV is a Q-martingale ; hence V_t is the value of a self-financing strategy. In particular,

$$V_T = S_T[G_{0,T}^+ \vee \lambda_0] = S_T\lambda_T \vee K\,.$$

From the definition of $G_{0,T}^+ = \sup_{t\in[0,T[}\left(\frac{b_t}{S_t}\right) \vee \frac{K}{S_T}$. We find back the lookback feature noticed in equation (4.13). □

At the boundary, i.e., when $P_t^a(\lambda_t) = (K - \lambda_t S_t)^+$, we obtain $V_t = K$. Moreover, the Gittins index is increasing only at the boundary. Therefore, as in the Black and Scholes framework, the process λ is increasing with support included in the set $\{V_t = K\}$.

4.5.3 Optimality

Let u be a CRRA utility function, and S the optimal strategy for the free problem (4.4) with initial wealth 1 as defined in subsection 3.1.
In this subsection we prove that the process $(V_t, t \geq 0)$ defined in Proposition 4.5.2 is the optimal portfolio for the problem with American guarantee.

In order to give a precise proof, let us introduce the state price process $(H_t, t \geq 0)$, such that for all self-financing portfolios with value $(X_t, t \geq 0)$, the budget constraint can be written :

$$X_0 = E[H_T X_T]\,. \tag{4.21}$$

We recall that H_t is the product of the discount factor $\exp(-\int_0^t r_s ds)$ and the Radon-Nikodym density of the equivalent risk-neutral martingale measure Q with respect to the subjective probability P. We also know that the Market Numeraire $M_t = H_t^{-1}$ is a portfolio, namely M_t is the optimal portfolio to hold for an unconstrained log-utility agent, as deduced from following equation (4.22) (see for example Long [11] (1990) or Bajeux and Portait [1](1998) for more details).
From optimization theory (see Karatzas and Shreve (1998) [9]), we know that the solution for the free problem (4.25) with initial wealth λx, which we denote by $\hat{X}_T^{\lambda x}$, satisfies the marginal utility condition :

$$u'(\hat{X}_T^{\lambda x}) = (yM_T)^{-1}, \tag{4.22}$$

where y is a parameter (the inverse of the Lagrange multiplier) to be adjusted as a function of the initial wealth λx by means of the budget constraint $E(H_T \hat{X}_T^{\lambda x}) = \lambda x$.
In case of CRRA utility function, it is well known that $\hat{X}^z = z\hat{X}^1$.

Proposition 4.5.3 *Let u be a CRRA utility function and $S = \hat{X}^1$ be the optimal strategy for the free problem with initial wealth equal to 1 and λ_t the gearing parameter decribed in Proposition 4.5.2. The strategy*

$$\hat{V}_t = \lambda_t S_t + P_t^a(\lambda_t)$$

is the optimal strategy for the problem with American guarantee.

Proof: Let $(V_t, t \geq 0)$ be any self-financing portfolio such that $V_t \geq K, \forall t$. From the concavity of u :

$$u(V_T) - u(\hat{V}_T) \leq u'(\hat{V}_T)(V_T - \hat{V}_T)\,.$$

The same arguments as in the European case lead to :

$$u'(\hat{V}_T)(V_T - \hat{V}_T) = u'(S_T\lambda_T)(V_T - \hat{V}_T) - [u'(S_T\lambda_T) - u'(\hat{V}_T)]^+ (V_T - K)\,.$$

Then using that for a CRRA function u, $u'(xy) = u'(x)u'(y)$, and that $u'(S_T) = \nu H_T$, we obtain :

$$u'(S_T\lambda_T)(V_T - \hat{V}_T) = \nu H_T u'(\lambda_T)(V_T - \hat{V}_T)\,.$$

An integration by parts formula, and the fact that the process $(u'(\lambda_t), t \geq 0)$ is a decreasing process provide :

$$E(H_T u'(\lambda_T)(V_T - \hat{V}_T)) = E\left(\int_0^T u'(\lambda_s)\, d(H_s(V_s - \hat{V}_s)) + \int_0^T H_s(V_s - \hat{V}_s)\, du'(\lambda_s)\right). \tag{4.23}$$

From the martingale property of $H\hat{V}$ and HV, the first term in the right-hand side of (4.23) is equal to 0. The process $(u'(\lambda_t), t \geq 0)$ is decreasing with support $\{\omega, t : \hat{V}_t(\omega) = K\}$, therefore :

$$E(\int_0^T H_s(K - \hat{V}_s) du'(\lambda_s)) = 0\,.$$

It follows that :

$$E\left(\int_0^T H_s(V_s - \hat{V}_s)\, du'(\lambda_s)\right) = E\left(\int_0^T H_s(K - \hat{V}_s) du'(\lambda_s)\right).$$

The process λ is increasing, hence $E\left(\int_0^T H_s(V_s - K)du'(\lambda_s)\right)$ is non-positive. Putting all the inequalities together, we establish that

$$E(u(\hat{V}_T)) \geq E(u(V_T))\,.$$

□

4.6 Optimality results for general utility functions case

4.6.1 European guarantee

For general utility functions, the linear property of the solution of problem (4.4) fails to be true : the optimal solution depends on the value of the initial wealth. The concept of one *unit* of strategic allocation therefore does not stand any longer, nor the concept of one unit of fund. We therefore introduce a new parameter $x > 0$ being the initial fund value. Consequently we now consider the maximization problem :

$$\max E[u(V_T)]; \qquad \text{under the contraints } V_T \geq Kx, \text{and} : V_0 = x\,, \tag{4.24}$$

and the free problem :

$$\max E[u(X_T^{\lambda x})]; \qquad \text{under the budget constraint } X_0^{\lambda x} = \lambda x\,. \tag{4.25}$$

We prove in this section that, nonetheless the strategic allocation cannot be defined independently, a separation principle similar to Proposition 3.1 still applies : the optimal policy consists in investing an initial amount λx in an optimal unconstrained portfolio, and protecting the fund by buying a put with strike Kx on that position. Let us remark that λ still represents the fraction of initial capital invested in risky assets at date 0.

Recall the the solution of the free problem is (see 4.22) $u'(\hat{X}_T^{\lambda x}) = (yM_T)^{-1}$,. As we show in our proofs, it is better to parametrize S with y rather than λ and to refer to the strategic allocation as :

$$S_T(y) = (u')^{-1}(H_T/y)\,. \tag{4.26}$$

The *modified strategic allocation* is given by :

$$S_t(y) = E_Q(R_T^t\,[S_T(y)]\,|\mathcal{F}_t) = E_Q(R_T^t\,(u')^{-1}\,[H_T/y]\,|\mathcal{F}_t)\,, \tag{4.27}$$

where Q is the risk-neutral probability and y is adjusted by means of the budget constraint :

$$E_Q(R_T S_T(y)) + P_0^e(S(y)) = x,$$

which is not linear with respect to y. Here $P^e(S(y))$ is the price of the European put with strike Kx written on the underlying $(S_t(y), t \geq 0)$. Therefore, the first order condition (4.5) takes the form :

$$E[H_T(X_T - S_T(y))] = 0\,, \tag{4.28}$$

for any X_T, terminal value of a self-financing portfolio such that $X_0 = S_0(y)$. Let us now consider the Put Based strategy described in section 4.3 associated with the strategic allocation $S_T(y)$ with terminal wealth :

$$\hat{V}_T^x = \max(S_T(y), Kx).$$

Proposition 4.6.1 *The Put Based Strategy written on the optimal portfolio with no constraint solves the optimization problem with a European constraint for any utility function.*

PROOF: The proof is similar to the CRRA case. Indeed, for any V_T terminal value of an admissible strategy with initial wealth x satisfying the constraint $V_T > Kx$, we get from the concavity of u and the definition of $S_T(y)$:

$$u(V_T) - u(\hat{V}_T^x) \leq u'(\hat{V}_T^x)(V_T - \hat{V}_T^x) = \left[y^{-1}H_T \wedge u'(Kx)\right] (V_T - \hat{V}_T^x)$$

The right-hand side of this last equation is equal to :

$$y^{-1}H_T(V_T - \hat{V}_T^x) - \left[u'(S_T(y)) - u'(Kx)\right]^+ (V_T - Kx) .$$

As previously, from the first order condition[5] (4.28) :

$$E[H_T(V_T - \hat{V}_T^x)] = 0 ,$$

and using the terminal constraint on V_T, we deduce :

$$E[u(V_T) - u(\hat{V}_T^x)] = -E\left(\left[u'(S_T(y)) - u'(Kx)\right]^+ (V_T - Kx)\right) \leq 0 .$$

□

4.6.2 American guarantee

We now deal with the maximization problem with American constraint. Again we follow closely the method exposed in the CRRA case. The difference is the choice of the parametrization : we now refer to the parametrized strategic allocation (see equation (4.27)) of the form :

$$S_t(y) = E_Q\left(R_T^t (u')^{-1}[H_T/y]|\mathcal{F}_t\right) .$$

Let us remark that $S_t(y)$ is increasing with respect to y.

The process $(R_tS_t(y), t \geq 0)$ being a martingale, the process $(R_t\partial_y S_t(y), t \geq 0)$ is also a martingale. The price of an American put on $(S_t(y), t \geq 0)$ is $P^a(y) = \sup_\tau E_Q(R_\tau(K - S_\tau(y))^+)$ and is decreasing with respect to y. As before, we have also $P^a_u(y) \geq (K - S_u(y))^+$ and $P^a_u(0) = K$.

Let $\sigma(y)$ be the optimal stopping time :

$$\sigma(y) = \inf\{u, P^a_u(y) = (K - S_u(y))^+\},$$

and note that σ is increasing with respect to y and that $\sigma(y) \leq T$. Then :

$$P^a_0(y) = E_Q(R_{\sigma(y)}(K - S_{\sigma(y)}(y))^+) .$$

For $t < T$ let us note $\gamma_t = \sup\{y : P_t(y) = (K - S_t(y))^+\}$ and G the right continuous inverse of $\sigma(\lambda)$ such that :

$$\{\sigma(y) > t\} = \{G_t < y\} .$$

Note that we obtain as before : $G_t = \sup_{u<t} \gamma_u$.

[5]This can be viewed as a budget constraint.

Proposition 4.6.2 *The value of the American put is*

$$P^a(y) = E_Q\left[R_T\left(S_T(G_T^+) - S_T(y)\right)^+\right],$$

where G_T^+ is defined in the proof below by (4.29).

PROOF: The derivative with respect to y of the American put price is :

$$\begin{aligned}\frac{\partial P^a(y)}{\partial y} &= -E_Q\left(R_{\sigma(y)}\frac{\partial S_{\sigma(y)}(y)}{\partial y}\mathbb{1}_{K>S_{\sigma(y)}(y)}\right)\\ &= -E_Q(R_T\frac{\partial S_T(y)}{\partial y}) + E_Q(R_{\sigma(y)}\frac{\partial S_{\sigma(y)}(y)}{\partial y}\mathbb{1}_{S_{\sigma(y)}>K}).\end{aligned}$$

We remark that, on the set $\{S_{\sigma(y)} > K\}$ the stopping time $\sigma(y)$ equals T, therefore :

$$\frac{\partial P^a(y)}{\partial y} = -E(R_T\frac{\partial S_T(y)}{\partial y}) + E(R_T\frac{\partial S_T(y)}{\partial y}\mathbb{1}_{K<S_T(y)}\mathbb{1}_{\sigma(y)=T}).$$

Setting

(4.29) $$G_T^+ = \sup_{t<T}\gamma_t \vee \kappa(K),$$

where $\kappa(K)$ is defined via the increasing property of $S_T(\cdot)$ as :

$$\kappa(K) = \sup_y\{K \geq S_T(y)\},$$

(i.e., $y < \kappa(K)$ if and only if $K \geq S_T(y)$), we get :

$$-E(R_T\frac{\partial S_T(y)}{\partial y}) + E(R_T\frac{\partial S_T(y)}{\partial y}\mathbb{1}_{G_T^+<y} \leq \frac{\partial P^a(y)}{\partial y} \leq -E(R_T\frac{\partial S_T(y)}{\partial y}) + E(R_T\frac{\partial S_T(y)}{\partial y}\mathbb{1}_{G_T^+\leq y}).$$

Therefore, by integration with respect to y :

$$P^a(y) = E_Q[R_T(S_T(G_T^+) - S_T(y))^+].$$

Starting at time t, and working with $G_{t,u} = \sup_{t\leq\theta<u} G_\theta$ and $G_{t,T}^+ = \sup_{u<T} G_{t,u} \vee \kappa(K)$ leads to :

$$P_t^a = E_Q(R_T^t(S_T(G_{t,T}^+) - S_T(G_t))^+|\mathcal{F}_t)$$

□

Let us now define $\widetilde{G}_t = G_t \vee \lambda_0$, where λ_0 will be adjusted by mean of constraint budget.

Proposition 4.6.3 *(i) The strategy*

$$V_t = S_t(\widetilde{G}_t) + P_t^a(\widetilde{G}_t)$$

is self-financing, with terminal value $V_T = S_T(\widetilde{G}_T) \vee K$.
(ii) This strategy, based on the optimal strategy for the free problem is optimal for the constrained problem.

PROOF: We follow the proof of Proposition 4.4.1. Using the previous Proposition 4.6.2, and using that, for any t, the process $(R^t_u S_u(\widetilde{G}_t), t \leq u)$ is a Q-martingale, we can write, as in (4.20) :

$$V_t = E_Q(S_t(\widetilde{G}_t) + R^t_T[S_T(G^+_{t,T}) - S_T(\widetilde{G}_t)]^+)|\mathcal{F}_t) = E(R^t_T[S_T(G^+_{t,T}) \vee S_T(\widetilde{G}_t)]|\mathcal{F}_t)\,.$$

On the set $S_T(G^+_{t,T}) > S_T(\widetilde{G}_t)$, the increasing property of $S_T(\cdot)$ implies that $G^+_{t,T} \geq \widetilde{G}_t$, hence $G^+_{t,T} = G^+_T \vee \lambda_0$. On the complementary set $S_T(G^+_{t,T}) \leq S_T(\widetilde{G}_t)$, the equality $G^+_{t,T} = G^+_T \vee \lambda_0$ still holds. Finally,

$$V_t = E(R^t_T S_T(G^+_T \vee \lambda_0)|\mathcal{F}_t),$$

and result (i) follows.

The proof of optimality is the same as in Proposition 4.5.3. The only change to make is to replace $u'(S_T\lambda_T)$ by

$$u'(S_T(\widetilde{G}_T)) = H_T/\widetilde{G}_T$$

and to work with the decreasing process $(\widetilde{G}_t)^{-1}$ rather than $u'(\lambda_t)$. This process decreases only at the boundary, i.e. when $V_t = K$, and we are done. □

Bibliography

[1] Bajeux-Besnainou, I. and Portait, R. (1998), Dynamic asset allocation in a mean-variance framework, *Management Science*, vol. 44, No. 11, S79-S95.

[2] Boyle, P. P. and Imai, J. (2000) Dynamic fund protection, Preprint, Université de Waterloo.

[3] Black, F. and Jones, R. (1987) Simplifying portfolio insurance, *Journal of portfolio management*, 48-51.

[4] Cox, J. C. and Huang, C.-F. (1989) Optimal consumption and portfolio policies when asset prices follow a diffusion process, *Journal of Economic Theory*, 49, 33-83.

[5] El-Karoui, N. and Karatzas, I. (1995) The optimal stopping problem for a general American put-option, *Mathematical Finance, M.H.Davis and al..ed.* IMA vol in mathematics n.65.

[6] Gerber, H. and Pafumi, G. (2000) Pricing dynamic investment fund protection, *North American Actuarial Journal,* **4**(2) 28-41.

[7] Gatto, M.-A., Goldman, M.-B. and Sosin, H.-B. (1979) Path-dependent options: "Buy at low, sell at the high", *Journal of Finance,* **34**, 1111-1127.

[8] Jensen, B.A. and Sørensen, C. (2000) Paying for minimum interest rate guarantees: Who should compensate who?. Working paper.

[9] Karatzas, I. and Shreve, S. (1998), *Methods of Mathematical Finance*, Springer-Verlag, Berlin.

[10] Leland, H.E. and Rubinstein, M. (1988) The evolution of portfolio insurance, in D.L. Luskin (ed.) *Portfolio insurance: a guide to dynamic hedging.* Wiley.

[11] Long, J. B. (1990), The numeraire portfolio, *Journal of Financial Economics*, **26**, 29-69.

[12] Markowitz, H. M. (1959), *Portfolio selection*, Yale University Press, New Haven, CT.

[13] Merton, R. (1971), Optimum consumption and portfolio rules in a continuous time model, *Journal of Economic Theory*, **3**, 373-413.

[14] Merton, R. (1973), An intertemporal capital asset pricing model, *Econometrica*, **41**, 867-888.

[15] Perold, A. and Sharpe, W. (1988) Dynamic strategies for asset allocation, *Financial analysts journal. January-February* 16-27.

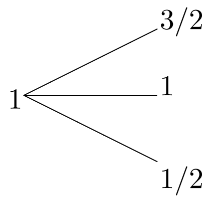
Chapter 5

Incomplete markets

5.1 Discrete time. Example

We use the notation of the chapter on hedging.

Let us study the following case in a two dates and 3 states of nature model: one riskless asset with value 1 at time 0 and time 1 (i.e. we have chosen $r = 0$) and a risky asset with value 1 at time 0 and value 3/2, 1 or 1/2 at date 1 (there are three states of nature : up, middle and down).



This market is incomplete: for example, it is not possible to hedge the contingent claim $H = (2, 1, 1)$. Indeed, an hedging strategy would be a pair (α, θ) such that

$$\begin{cases} \alpha + \frac{3}{2}\theta & = & 2 \\ \alpha + \theta & = & 1 \\ \alpha + \frac{1}{2}\theta & = & 1 \end{cases}$$

which has no solution. The set of risk neutral probability is the set of (p_1, p_2, p_3) such that

$$\begin{cases} p_i & \geq & 0, \\ p_1 + p_2 + p_3 & = & 1 \\ \frac{3}{2}p_1 + p_2 + \frac{1}{2}p_3 & = & 1\,. \end{cases}$$

Therefore this set $\mathcal{Q}$ contains an infinite number of solutions

$$\mathcal{Q} = \{p_1; p_1 = p_3, p_2 = 1 - 2p_1, 0 \leq p_1 \leq \frac{1}{2}\}.$$

The fact that $\mathcal{Q}$ is not empty guarantee that there are no arbitrages.

5.1.1 Case of a contingent claim

Range of prices and super-replication

For $H = (2,1,1)$,

$$\sup_Q E_Q(H) = \sup(2p_1 + p_2 + p_3) = \sup(p_1 + 1) = \frac{3}{2}$$

whereas

$$\inf_Q E_Q(H) = \inf(2p_1 + p_2 + p_3) = \inf(p_1 + 1) = 1$$

The range of prices is the interval $]1, \frac{3}{2}[$.

Let us check that if the contingent claim H is traded at price p, with $p \in]1, \frac{3}{2}[$, it does not induce arbitrage opportunity. Indeed, in that case, there exists a (unique) risk neutral probability[1]. Indeed, the system

$$\begin{cases} p_i & \geq & 0, \\ p_1 + p_2 + p_3 & = & 1 \\ 3p_1/2 + p_2 + p_3/2 & = & 1 \\ 2p_1 + p_2 + p_3 & = & p\,. \end{cases}$$

has a unique solution

$$\begin{aligned} p_1 &= p_3 = p - 1 \in [0,1] \\ p_2 &= 3 - 2p \in [0,1] \end{aligned}$$

Let us check on that example that the upper bound of interval of prices is the superreplication price. The superreplication price is defined as

$$\begin{aligned} p &= \inf\{x : \exists(\alpha,\theta)\ x = \alpha + \theta, \alpha + \theta S_1 \geq H\} \\ &= \inf\{x : \exists(\alpha,\theta)\ x = \alpha + \theta, \begin{cases} \alpha + \theta/2 & \geq & 1 \\ \alpha + 3\theta/2 & \geq & 2 \\ \alpha + \theta & \geq & 1 \end{cases} \} \end{aligned}$$

We can solve this program by geometrical approach, leading to the solution

$$x = 3/2, \alpha = 1/2, \theta = 1$$

We also check that any price outside the range interval yields to an arbitrage opportunity. Let $p \geq 3/2$ and assume that the contingent claim H is traded at price p.Take a short position on H,

[1] Let us recall the fundamental theorem of pricing : the market is arbitrage free if and only if there exists at least a risk neutral probability, the market is complete if this risk neutral probability is unique

buy one share of the risky asset and $p-1$ shares of the bond. The needed initial wealth is 0, the terminal wealth $p-1+S_1-H$ is non negative in all states of world

$$\begin{aligned} p-1+\frac{3}{2}-2=p-3/2 &\geq 0 \\ p-1+1-1=p-1 &\geq 0 \\ p-1+1/2-1=p-\frac{3}{2} &\geq 0\,. \end{aligned}$$

The lower bound of range of prices is the buyer price, i.e.

$$\sup\{x\,:\,\exists\alpha,\theta,x=\alpha+\theta,\,\alpha+\theta S_1\leq H\}$$

or

$$\sup\left\{x\,:\,\exists\alpha,\theta,x=\alpha+\theta\,\left\{\begin{array}{rcl}\alpha+3\theta/2 & \leq & 2\\ \alpha+\theta & \leq & 1\\ \alpha+\theta/2 & \leq & 1\end{array}\right.\right\}$$

which leads to

$$x=1,\alpha+\theta=1\,.$$

If $p\leq 1$, a long position on H, a short position on S and holding $1-p$ shares of the riskless assets is an arbitrage opportunity.

Completion of the market

Moreover, if the contingent claim H is traded, the market is complete. Indeed, for any contingent claim $G=(a,b,c)$, there exists a portfolio (α,θ,θ_2), where $theta_2$ is the number of shares of the contingent claim, such that

$$\left\{\begin{array}{lcl}\alpha+3\theta/2\,+2\theta_2 & = & a\\ \alpha+\theta+\theta_2 & = & b\\ \alpha+\theta/2+\theta_2 & = & c\end{array}\right.$$

It is worthwhile to mention that hedgeable contingent claims do not complete the market, e.g. $G=(3,2,1)$.

Variance hedging

Here we set the probabilities of going up, or down equal to 1/3, i.e.,

$$P(S_1=3/2)=P(S_1=1)=P(S_1=1/2)=1/3\,.$$

. A portfolio with initial value $h=\alpha+\theta$ has terminal value $\alpha+\theta S_1=h+\theta(S_1-1)$. Our aim is to find a pair h,θ such that the difference between the contingent claim and the terminal value of a portfolio is small in the mean square sense, i.e. to find (h,θ) to minimize $E([H-h-\theta(S_1-1))]^2)$. The quantity $E([H-h-\theta(S_1-1)]^2)$ equals

$$1/3[(2-h-\theta/2)^2+(1-h)^2+(1-h+\theta/2)^2]$$

The derivative w.r.t. θ is $2/3[-1/2(2-h-\theta/2)+1/2(1-h+\theta/2)]$ and is equal to 0 for $\theta=1$ The derivative w.r.t. h is $-2/3[(2-h-\theta/2)+(1-h)+(1-h+\theta/2)]=-2/3[4-3h]$ and is equal to 0 if $4-3h=0$. Hence, this strategy corresponds to an initial value equal to 4/3. (this value is in the range of prices)

Reservation price

The Hodges-Neuberger approach is based on an optimization problem. Suppose that an agent has a utility function u. In the primary market (the riskless asset and the risky one), with an initial wealth x, he solves its investment problem $V(x) = \sup_\theta E(u(X_1^{x,\theta}))$ where $X_1^{x,\theta} = \alpha + \theta S_1$ is its terminal wealth associated with the portfolio (α, θ) with initial value $x = \alpha + \theta$. If the contingent claim H is traded at price p, the agent can take a short position on it, selling the claim at price p. Then, its initial wealth is $p + x = \alpha + \theta$. He manages this wealth, and has to deliver H at time 1. Therefore, its terminal wealth is $X^{x+p,\theta} - H = \alpha + \theta S_1 - H$ and he solves the optimization program

$$V(x + p, H) = \sup_\theta E[u(X^{x+p} - H)]\,.$$

The reservation price for seller is defined as the value of p such that

$$V(x) = V(x + p, H)\,.$$

We now choice as an exemple $u(x) = x^\alpha$. In our case, $X^{x,\theta} = x + \theta(S_1 - 1)$, hence

$$E(u(X^x)) = \frac{1}{3}\left[(x + \frac{1}{2}\theta)^\alpha + x^\alpha + (x - \frac{1}{2}\theta)^\alpha\right]$$

and the maximum is reached for $\theta = 0$, hence $V(x) = x^\alpha$. In the same way,

$$E[u(X^{x+p} - H)] = \frac{1}{3}\left[(x + p + \theta/2 - 2)^\alpha + (x + p - 1)^\alpha + (x + p - \theta/2 - 1)^\alpha\right]$$

and the maximum is reached for $\theta = 1$, therefore $V(x + p, H) = \frac{1}{3}\left[2(x + p - \frac{3}{2})^\alpha + (x + p - 1)^\alpha\right]$. The reservation price for an agent with initial wealth x and utility function x^α is the value p such that

$$x^\alpha = \frac{1}{3}\left[2(x + p - \frac{3}{2})^\alpha + (x + p - 1)^\alpha\right]$$

It can be checked that p belongs to the range of prices. In general, p depends on the initial wealth.

In a symmetric way, it is possible to define the reservation price for buyer as the value of p such that

$$V(x) = V(x - p, -H)\,.$$

Davis'price

Davis (1997) defines the price of the contingent claim H using a marginal rate : suppose that H is traded at price p. An investor invests an amount of δ in H and keeps this position till maturity. His final wealth is $X_T^{x-\delta,\theta} + \frac{\delta}{p}H$. His investment program is

$$V(\delta, x, p) = \sup_\theta E[U(X^{x-\delta,\theta} + \frac{\delta}{p}H)]$$

The fair price of H is defined as p^* solution of

$$\frac{\partial V}{\partial \delta}(0, p, x) = 0\,.$$

In our example,

$$E[u(X_T^{x-\delta,\theta}+\frac{\delta}{p}H)]=\frac{1}{3}\left[(x-\delta+\frac{1}{2}\theta+2\frac{\delta}{p})^\alpha+(x-\delta+\frac{\delta}{p})^\alpha+(x-\delta+\frac{1}{2}\theta+\frac{\delta}{p})^\alpha\right]$$

The maximum, w.r.t. θ is obtained for $\theta=-\frac{\delta}{p}$. Then,

$$V(\delta,x,p)=\frac{1}{3}\left[(x-\delta+\frac{3}{2}\frac{\delta}{p})^\alpha+(x-\delta+\frac{\delta}{p})^\alpha+(x-\delta+\frac{3}{2}\frac{\delta}{p})^\alpha\right]$$

The derviative w.r.t. δ, for $\delta=0$, is up to a constant factor

$$x^{\alpha-1}[3-\frac{8}{2p}]$$

hence the Davis' price is $p=3/4$. It can be checked that p belongs to the range of prices.

Shortfall

The superreplication price is too large for investors who can prefer to have some risk. They can choose to trade the contingent claim at price

$$\inf\{x\,:\,\exists\alpha,\theta,P(X^{x,\theta}\geq H)\geq 1-\epsilon\}$$

In our case, only three values of ϵ are relevant. Let us choose $\epsilon=1/3$. In that case, we chose to superhedge in two states of nature, mainly

(1)two upper states $\left\{\begin{array}{rcl}\alpha+3\theta/2&\geq&2\\ \alpha+\theta&\geq&1\end{array}\right.$ $x=1,\alpha=-1,\theta=2$

(2) two lower states $\left\{\begin{array}{rcl}\alpha+\theta&\geq&1\\ \alpha+\theta/2&\geq&1\end{array}\right.$ $x=1,\alpha=1,\theta=0$

(3) two extreme states $\left\{\begin{array}{rcl}\alpha+3\theta/2&\geq&2\\ \alpha+\theta/2&\geq&1\end{array}\right.$ $x=\frac{3}{2}$

The infimum is reached for $x=1$. The loss is the non-hedged state is equal to $x+\theta/2-2=-1$ in the first case, and $x-\theta/2-1-1=-1$ in the second case. Therefore, the loss is large (equal to the initial capital)

5.1.2 Range of price for a European call

Let us study the case of a European call with strike 1. Let us first compute the range of viable prices

$$\begin{aligned}\sup_Q E_Q((S_1-1)^+)&=&\sup\frac{1}{2}p_3=\frac{1}{4}\\ \inf_Q E_Q((S_1-1)^+)&=&\inf\frac{1}{2}p_3=0\end{aligned}$$

The supereplication price is

$$\inf\{x|\exists\alpha,\theta:\alpha+\theta=x,\alpha+\frac{3}{2}\theta\geq\frac{1}{2},\alpha+\frac{3}{2}\theta\geq 0,\alpha+\frac{1}{2}\theta\geq 0\}$$

A geometric study shows that this quantity is equal to 1/4, i.e. the upper bound of the range of prices.

5.1.3 Two dates, continuum prices

Let S be the price of the risky asset at time 0. We assume that there exists two numbers S_b and S_h such that the price at time 1 is a random variable S_1 which takes values in the interval $[S_b, S_h]$. More precisely, we assume that the random variable S_1 admits a density f such that $P(S_1 \in A) > 0$, for any $A \subset [S_b, S_h]$. This technical hypothesis will help us to characterize equivalent probabilities. Indeed, under this hypothesis, an equivalent probability measure is characterized by a density h, i.e. a function strictly positive on $]S_b, S_h[$ such that $\int_{S_b}^{S_h} h(x)dx = 1$. We also assume (for no-arbitrage purpose) that $S_b \leq (1+r)S \leq S_h$. Let $\mathcal{P}$ be the set of risk neutral probabilities, i.e. the set of probabilities Q which admit a strictly positive density h on $]S_b, S_h[$ such that $E_Q(\frac{S_1}{1+r}) = \frac{1}{1+r}\int_{S_b}^{S_h} xh(x)dx = S$.

Proposition 5.1.1 *Let g be a convex function (for example $g(x) = (x-K)^+$), then*

$$\sup_{P \in \mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right) = \frac{g(S_h)}{1+r}\,\frac{S(1+r) - S_b}{S_h - S_b} + \frac{g(S_b)}{1+r}\,\frac{S_h - S(1+r)}{S_h - S_b}.$$

If g is a C^1 function, then

$$\inf_{P \in \mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right) = \frac{g((1+r)S)}{1+r}.$$

Remark 5.1.1 The supremum is reached for a probability Q such that $Q(S_1 = S_h) + Q(S_1 = S_b) = 1$. This probability is not equivalent to P (except if, under P, the r.v. S_1 has a Bernoulli law). The infimum is reached for a probability such that $Q(S_1 = S(1+r)) = 1$.

PROOF:
Let g be a convex function, and $y = \mu x + \nu$ the line which contains the two points $(S_b, g(S_b))$ and $(S_h, g(S_h))$. Then,

$$\forall x \in [S_b, S_h]\,,\ g(x) \leq \mu x + \nu, \qquad g(S_b) = \mu S_b + \nu, \qquad g(S_h) = \mu S_h + \nu\,,$$

hence, for any $Q \in \mathcal{P}$

$$E_Q(g(S_1)) \leq \mu E_Q(S_1) + \nu = \mu S(1+r) + \nu\,.$$

Let P^* be the probability measure such that

$$P^*(S_1 = S_h) = p, \qquad P^*(S_1 = S_b) = 1 - p, \qquad E_{P^*}(S_1) = S(1+r)\,.$$

This last condition determines p :

$$p = \frac{S(1+r) - S_b}{S_h - S_b},\ 1 - p = \frac{S_h - S(1+r)}{S_h - S_b}.$$

Then $E_{P^*}(g(S_1)) = \mu S(1+r) + \nu$. The supremum is reached for P^*. However, P^* does not belong to $\mathcal{P}$. However, let us remark that for $g(x) = ax + b$, the quantity $E_Q(g(S_1)) = aE_Q(S_1) + b = a(1+r)S + b$ does not depend on Q. This will remain true in a general setting: in an incomplete market, if the contingent claim is hedgeable, the expectation of its discounted value does not depend on the choice of the risk-neutral probability measure.

The lower bound is obtained easily :

$$\inf_{Q\in\mathcal{P}} E_Q(\frac{g(S_1)}{1+r}) = \frac{g(S(1+r))}{1+r}.$$

Indeed, let $y = \gamma x + \delta$ be the equation of the tangent to $y = g(x)$ at point $(S(1+r), g(S(1+r))\,)$. Then

$$g(x) \geq \gamma x + \delta, \quad \gamma S(1+r) + \delta = g(S(1+r))\,,$$

therefore $E_Q(g(S_1)) \geq E_Q(\gamma S_1 + \delta) = g(S(1+r))$ and the minimum is reached for a Dirac measure at point $S(1+r)$.

5.1.4 Bid-ask price

The selling price (or the bid price) is the smallest value x such that the seller can hedge against the delivery. In other words, x is the smallest value such that it i spossible to construct a portfolio (α, β) such that the terminal value of the portfolio is greater than $g(S_1)$. Hence, the bid price is

$$\inf_{(\alpha,\beta)\in\mathcal{A}} (\alpha + \beta S)$$

with $\mathcal{A} = \{(\alpha, \beta)|\ \alpha(1+r) + \beta x \geq g(x),\ \forall x \in [S_b, S_h]\}$. A main result is

$$\inf_{(\alpha,\beta)\in\mathcal{A}} (\alpha + \beta S) = \sup_{P\in\mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right).$$

Indeed, from the definition of $\mathcal{A}$, we obtain $\alpha(1+r) + \beta S_1 \geq g(S_1)$, hence

$$\inf_{(\alpha,\beta)\in\mathcal{A}} (\alpha + \beta S) \geq \sup_{P\in\mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right).$$

Now, using the pair μ, ν of the previous subsection, we check that $(\frac{\nu}{1+r}, \mu)$ EST Dans $\mathcal{A}$,

$$\inf_{(\alpha,\beta)\in\mathcal{A}} (\alpha + \beta S) \leq \mu S + \frac{\nu}{1+r} = \sup_{P\in\mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right).$$

The problems

$$\sup_{P\in\mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right) \quad \text{and} \quad \inf_{(\alpha,\beta)\in\mathcal{A}} (\alpha + \beta S)$$

are dual problems.

In the same way, we define the buyer price, i.e. the maximal amount which can be borrowed against the contingent claim. This price is defined as

$$\sup_{(\alpha,\beta)\in\mathcal{C}} (\alpha + \beta S)$$

with $\mathcal{C} = \{(\alpha, \beta)|\ \alpha(1+r) + \beta x \leq g(x),\ \forall x \in [S_b, S_h]\}$. This is obviously equal to

$$\inf_{(\alpha,\beta)\in\mathcal{A}(-g)} (\alpha + \beta S)$$

with $\mathcal{A}(-g) = \{(\alpha, \beta) | \ \alpha(1+r) + \beta x \geq -g(x), \ \forall x \in [S_b, S_h]\}$. The equality

$$\sup_{(\alpha,\beta)\in\mathcal{C}} (\alpha + \beta S) \ = \inf_{P\in\mathcal{P}} E_P\left(\frac{g(S_1)}{1+r}\right)$$

holds.

Any price outside the price interval creates arbitrage.

5.2 Discrete time, general setting : Bid-ask spread

The notation of this section are the notation introduced in the first chapter. They are different from those of the previous section. If markets are incomplete, i.e., not complete, span D is different of $\mathbb{R}^k$, one can similarly price any contingent claim z in span D by the value of any hedging portfolio. If $z \notin$ span D, one cannot price z neither by arbitrage, nor by hedging, one can only define a "bid-ask" spread. Let

$$\bar{S}(z) := \inf\{\theta \cdot S \mid D\theta \geq z \ \}$$

be the minimum expenditure of the seller of the contingent claim and

$$\underline{S}(z) := \sup\{\theta \cdot S \mid D\theta \leq z \ \} = -\inf\{\theta \cdot S \mid D\theta \geq -z \ \},$$

be the maximal amount of money that the buyer of z can borrow against z. Any price in $]\underline{S}(z), \bar{S}(z)[$ is a no-arbitrage price. Furthermore

$$\bar{S}(z) = \sup\{ \frac{\pi^T z}{1+r} \mid \pi \gg 0, \ \frac{D^T\pi}{1+r} = S\} \quad \text{and} \quad \underline{S}(z) = \inf\{ \frac{\pi^T z}{1+r} \mid \pi \gg 0, \ \frac{D^T\pi}{1+r} = S\}\,.$$

Similarly if there are portfolio constraints, one may only define a "bid-ask" spread. For example, assume that investors bear the constraints $\theta^\ell \geq 0, \ 0 \leq \ell \leq d_0$. Then the definition of no-arbitrage has to be changed: there is no-arbitrage in the market if there is no feasible portfolio that gives something for nothing (in other words, there is no-arbitrage if $\theta^\ell \geq 0, \ 0 \leq \ell \leq d_0, \ D\theta \geq 0 \ , \ D\theta \neq 0$ implies $S \cdot \theta > 0$). If one defines $\bar{S}(z) := \inf\{\theta \cdot S \mid \theta^\ell \geq 0, \ 0 \leq \ell \leq d_0, \ D\theta \geq z \ \}$ and $\underline{S}(z) := \sup\{\theta \cdot S \mid \theta^\ell \geq 0, \ 0 \leq \ell \leq d_0, \ D\theta \leq z \ \}$, any price in $]\underline{S}(z), \bar{S}(z)[$ is a no-arbitrage price.

5.3 Continuous time

When the set of e.m.m. is infinite, as in discrete time, it is not possible to hedge all the contingent claim. Recall some definitions. Let $(S_t, t \geq 0)$ be the asset's price process, and $S_t^0 = \exp \int_0^t r(s)ds$ the value of the riskless asset. We denote by $R_t = \exp - \int_0^t r(s)ds$ the discounted factor, and $R_s^t = \exp - \int_t^s r(u)du$. We do not make precise the dynamics of the risky asset for the moment. A pair (α_t, θ_t) is a portfolio if the processes are adapted with respect to the filtration generated by S, the value of this portfolio is

$$V_t = \alpha_t S_t^0 + \theta_t S_t$$

The portfolio is said to be self-financing if

$$dV_t = \alpha_t dS_t^0 + \theta_t dS_t = r_t V_t dt + \theta_t (dS_t - r_t S_t dt)\,,$$

A contingent claim H is hedgeable if there exists a self financing strategy such that $X_T = H$.
In particular, if $r = 0$, a contingent claim is hedgeable if there exists (h, θ) such that

$$H = h + \int_0^T \theta_s dS_s$$

This can be view, from a mathematical point of view as a representation theorem.
We emphasize that here, the process S can be d-dimensional.

Remark : At that point, we have to add some integrability conditions. Indeed, Dudley proved that, if S is a martingale, for any contingent claim H, there exists a process η such that $H = \int_0^T \eta_s dS_s$. This would lead to a self-financing portfolio with zero initial value, i.e., an arbitrage opportunity. In order to avoid that, either we can restrict our attention to positive contingent claim and portfolio with value bounded above by a constant, or to square integrable value.

Note that, if there exists an e.m.m., the value of an hedgeable contingent claim is $V_t = E_Q(HR_T^t|\mathcal{F}_t)$, for any Q. Even in an incomplete market, some claims are hedgeable, as for example, in the case of constant interest rate, the contingent claim $a + bS_T$, or $\displaystyle\int_0^T (a + bS_s)ds$.
Let us check that, in the case of zero interest rate (to be simple) the contingent claim $\displaystyle\int_0^T (a+bS_s)ds$ is hedgeable. From integration by parts formula

$$d(tS_t) = tdS_t + S_tdt$$

hence

$$\int_0^T S_sds = TS_T - 0 - \int_0^T tdS_t$$

It follows that

$$\int_0^T (a + bS_s)ds = aT + b\int_0^T S_sds = aT + bTS_T - \int_0^T sdS_s = aT + bTS_0 + \int_0^T \theta_sdS_s$$

with $\theta_s = bT - s$.

5.3.1 Superhedging price

When markets are incomplete, there do not exists an hedging strategy, and there exists several e.m.m. A price is "viable" for a contingent claim H if it is equal to the expectation of the discounted payoff under an e.m.m. Viable prices do not induce arbitrage opportunities., however, they do not give an hedging strategy. Due to the convexity of the set $\mathcal{Q}$ of e.m.m., the set $\{E_Q(R_TH), Q \in \mathcal{Q}\}$ is an interval, and any choice of initial price outside the interval would provide an arbitrage. Results of El Karoui and Quenez [4][3](1991-95) imply that when the dynamics of the stock is driven by a Wiener process, the supremum of the possible prices is equal to the minimum initial value of an admissible self-financing strategy that replicates the contingent claim, this result was generalized by Kramkov [7] (1996) and Hugonnier [6] (2000).
Eberlein and Jacod [8](1997) showed the absence of non-trivial bounds on European option prices in a model where prices are driven by a purely discontinuous Lévy process with unbounded jumps, this result is generalized using different methods by Bellamy and Jeanblanc [2] and Jakubenas [10]. However, this interval is too large , as we shall show in what follows.
A superhedging portfolio is a portfolio that allows for some consumption and whose terminal value

is larger than the contingent claim. It is characterized by a triple (x, θ, C) where x is the initial value, θ the process of number of shares of the risky asset and C the increasing process of cumulated consumption. Its value $V^{x,\theta,C}$ follows

$$dV_t = rV_t dt + \theta_t(dS_t - rS_t dt) - dC_t\,, \quad V_0 = x\,, \tag{5.1}$$

and the terminal value has to be greater than the contingent claim B. The superhedging price is defined as

$$\inf\{x \,:\, \exists(\theta, C)\,\text{such that} V_T^{x,\theta,C} \geq B\}\,.$$

The smallest superhedging price of a contingent claim B is equal to the supremum of the viable prices. As prices range is large, many authors choose a specific e.m.m.

5.3.2 Choice of the model

Let us study a model where the dynamics of the prices are modeled as

$$dS_t = S_t(\mu dt + \sigma_1 dW_t^{(1)}) + \sigma_2 dW_t^{(2)}$$

This model seems incomplete. Nevertheless, we can write the dynamics as

$$dS_t = S_t(\mu dt + \sigma_3 dW_t)$$

where

$$W_t = \frac{1}{\sigma_3}(\sigma_1 dW_t^{(1)} + \sigma_2 dW_t^{(2)})$$

is a Brownian motion. We use here that W is a martingale with bracket equal to 1, or that $W_t^2 - t$ is a martingale. It is also possible to use that the associated exponential is a martingale.

The gap between both models is that in the first one, contingent claims are measurable with respect to the pair of Brownian motions $(W^{(1)}, W^{(2)})$, whereas in the second model they are adapted with respect to the asset prices.

5.3.3 Bounds for stochastic volatility

Let

$$dS_t = S_t(\mu dt + \sigma_t dW_t)$$

where σ_t is a random process, such that

$$0 \leq \sigma_1 \leq \sigma_t \leq \sigma_2$$

If σ is a process which depends of a second Brownian motion, the market is incomplete. However, we can prove that any viable price of the contingent claim H is in the interval

$$]C(t, \sigma_1, S_t), C(t, \sigma_2, S_t)[$$

, where

$$C(t, \sigma, x) = x\mathcal{N}(d_1(\sigma, x, T-t)) - Ke^{-r(T-t)}\mathcal{N}(d_2(\sigma, x, T-t))\,.$$

$$d_1(\sigma, x, T) = \frac{1}{\sigma\sqrt{T}}\ln(\frac{x}{Ke^{-rT}}) + \frac{\sigma\sqrt{T}}{2},\ d_2(\sigma, x, T) = d_1(\sigma, x, T) - \sigma\sqrt{T}\,.$$

Indeed, under any risk neutral probability Q,

$$dS_t = S_t(rdt + \sigma_t d\tilde{W}_t)\,,$$

where $\tilde{W}$ is a Q Brownian motion. From ITô's formula

$$\begin{aligned} e^{-rT}C(\sigma_2,T,S_T) = e^{-rt}C(\sigma_2,t,S_t) &+ \int_t^T e^{-rs}[\partial C_t + rS_s\partial C_x + \frac{1}{2}\sigma_s^2 S_s^2 \partial_{xx}C - rC](\sigma_2,s,S_s)ds \\ &+ \int_t^T e^{-rs}\sigma_s S_s \partial_x C(\sigma_2,s,S_s)d\tilde{W}_s \end{aligned}$$

The quantity $\partial_x C(\sigma_2,t,S_t)$ is bounded hence the stochastic integral is a martingale. The left hand side is equal to $e^{-rT}(S_T-K)^+$. If $C_t^{e,Q} = e^{rt}E_Q[e^{-rT}(S_T-K)^+|\mathcal{F}_t]$ is the viable price computed under Q, taking the conditional expectation with respect to $\mathcal{F}_t$ leads to

$$e^{-rt}C_t^{e,Q} = e^{-rt}C(\sigma_2,t,S_t) + E_Q[\int_t^T [\partial C_t + rS_s\partial C_x + \frac{1}{2}\sigma_s^2 S_s^2\partial_{xx}C - rC](\sigma_2,s,S_s)ds|\mathcal{F}_t]$$

Now, recall that

$$\partial C_t + rx\partial C_x + \frac{1}{2}\sigma_2 x^2 \partial_{xx}C - rC](\sigma_2,t,x) = 0$$

Hence

$$\partial C_t + rS_s\partial C_x + \frac{1}{2}\sigma_s^2 S_s^2\partial_{xx}C - rC = \frac{1}{2}[\sigma_s^2 - \sigma_2^2]S_s^2\partial_{xx}C$$

and, from the assumption, this quantity is negative. It follows that

$$e^{-rt}C_t^{e,Q} \le e^{-rt}C(\sigma_2,t,S_t)$$

In a general framework, the stochastic volatility model are on the form

$$\begin{aligned} dS_t &= S_t[\mu(t,S_t,Y_t)dt + \sigma(t,S_t,Y_t)dW_t] \\ dY_t &= \eta(t,S_t,Y_t)dt + \gamma(t,S_t,Y_t)dB_t \end{aligned}$$

The case where

$$0 \le \sigma_1 \le \sigma(t,s,y) \le \sigma_2$$

and $r=0$ is the Avellaneda, Lévy, Paras model. The super-replication price is the solution of the so-called Black-Scholes-Barrenblatt equation

$$-\partial_t C + \inf_y[-\frac{1}{2}\sigma^2(t,s,y)s^2\partial_{ss}C] = 0$$

and the associated strategy is $\theta = \partial_s C(t,S_t)$.

5.3.4 Jump diffusion processes

We consider a financial market with a riskless asset with deterministic return rate r and a risky asset with dynamics under the historical probability

(5.2) $$dS_t = S_{t-}(b(t)dt + \sigma(t)dW_t + \phi(t)dM_t)$$

where b, σ and ϕ are deterministic bounded functions with $|\sigma(t)| > c, -1 < \phi(t), \frac{1}{c} < |\phi(t)| < c$ where c is a strictly positive constant. Here, W is a Brownian motion and M the compensated martingale associated with a Poisson process with deterministic intensity λ, i.e., $M_t = N_t - \lambda t$. Let us remark that the filtration generated by prices is the filtration generated by the pair (W,N). The condition $-1 < \phi(t)$ ensures that prices remain positive. Indeed, at a jump time

$$\Delta S_t = S_t - S_{t-} = S_{t-}\phi(t)\Delta M_t = S_{t-}\phi(t)\Delta N_t = S_{t-}\phi(t)$$

Hence, $S_t = S_{t-}(1+\phi(t))$.

The market being incomplete, it is not possible to give a hedging price for each contingent claim $B \in \mathcal{F}_T$. We define a t-time viable price $V^\gamma(t)$ for the contingent claim B as the conditional expectation (with respect to the information $\mathcal{F}_t$) of the discounted contingent claim under the martingale-measure P^γ, i.e. $R(t)V_t^\gamma \stackrel{def}{=} E^\gamma(R(T)\,B|\mathcal{F}_t)$.
(it can be proved that the set of e.m.m. is parameterized by mean of a process γ valued in $]-1,\infty[$.)
We study here the range of viable prices, i.e., the interval $]\inf_{\gamma\in\Gamma} V_t^\gamma\,,\,\sup_{\gamma\in\Gamma} V_t^\gamma[$.

We restrict our attention to the European case, i.e. when $B = (S_T - K)^+$.
We denote by C the Black-Scholes function, i.e., the function $C(t,x)$ such that

$$R(t)C(t,X_t) = E(R(T)(X_T-K)^+\,|X_t)\,,\;\; C(T,x) = (x-K)^+$$

when

(5.3) $$dX_t = X_t(r(t)dt + \sigma(t)\,dW_t)\,, X_0 = x.$$

In other words, $C(t,x) = R_T^t E[h(x(R_T^t)^{-1}\exp[\Sigma(t)G - \frac{1}{2}\Sigma^2(t)]\,)]$ where $h(x) = (x-K)^+$, G is a standard normal random variable, $R_T^t = \dfrac{R(T)}{R(t)}$ and $\Sigma^2(t) = \int_t^T \sigma^2(s)ds$. We recall that C is a convex function of x which satisfies

(5.4) $$\mathcal{L}(C)(t,x) = rC(t,x)$$

where

$$\mathcal{L}(f)(t,x) = \frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x)\,.$$

Furthermore, $|\partial_x C(t,x)| \le 1$.

Theorem 5.3.1 *Let $P^\gamma \in \mathcal{Q}$. Then, the associated viable price is bounded below by the Black-Scholes function, evaluated at the underlying asset value, and bounded above by the underlying asset value, i.e.,*

$$R(t)C(t,S_t) \le E^\gamma(R(T)\,(S_T-K)^+|\mathcal{F}_t) \le R(t)\,S_t$$

The range of viable prices $V_t^\gamma = \dfrac{R(T)}{R(t)}E^\gamma((S_T-K)^+|\mathcal{F}_t)$ is exactly the interval $]C(t,S_t),\,S_t[$.

Before giving the proof, we give Itô's formula for jumping processes. Let W be a BM and M the compensated martingale associated with a PP with constant intensity λ. Let F be a $C^{1,2}$ function and

$$dX_t = f_t dW_t + g_t dM_t + h_t dt\,.$$

Then,

$$\begin{aligned} F(t,X_t) &= F(0,X_0) + \int_0^t \partial_s F(s,X_s)\,ds + \int_0^t F'(s,X_{s-})dX_s \\ &+ \frac{1}{2}\int_0^t F''(s,X_s)f_s^2\,ds + \sum_{s\le t}[F(s,X_s) - F(s,X_{s^-}) - F'(s,X_{s-})\Delta X_s]\,. \end{aligned}$$

Here the sum is taken over almost surely finite number of jump times that occur prior to t and is equal to $\int_0^t [F(s,X_s) - F(s,X_{s^-}) - F'(s,X_{s-})g_s]\,dN_s$. Note also that $X_s = X_{s^-}(1+g_t)$.

PROOF: We give the proof of the theorem in the case $t = 0$. Itô's formula for mixed processes leads to

$$R(T)C(T,S_T) = C(0,S_0) + \int_0^T [\mathcal{L}(RC)(s,S_s) + R(s)\lambda(s)(\gamma_s+1)\Lambda C(s,S_s)]\,ds$$
$$+ \int_0^T R(s)\frac{\partial C}{\partial x}(s,S_{s_-})\,S_{s_-}\,(\sigma(s)dW_s^\gamma + \phi(s)dM_s^\gamma) + \int_0^T R(s)\Lambda C(s,S_{s_-})\,dM_s^\gamma$$

where

$$\Lambda f(t,x) = f(t,(1+\phi(t))\,x) - f(t,x) - \phi(t)x\,\frac{\partial f}{\partial x}(t,x)\,.$$

The convexity of $C(t,\cdot)$ implies that $\Lambda C(t,x) \geq 0$ and the Black-Scholes equation (5.4) provides $\mathcal{L}[RC](s,x) = 0$. The stochastic integrals are martingales; indeed $\left|\frac{\partial C}{\partial x}(t,x)\right| \leq 1$ imply that $|\Lambda C(t,x)| \leq 2xc$ where c is the bound for the size of the jumps ϕ. Taking expectation with respect to P^γ leads to

$$\begin{aligned} E^\gamma(R(T)C(T,S_T)) &= E^\gamma(R(T)(S_T-K)^+) \\ &= C(0,S_0) + E^\gamma\left(\int_0^T R(s)\lambda(s)(\gamma_s+1)\Lambda C(s,S_s)\,ds\right) \end{aligned}$$

The lower bound follows. The upper bound is a trivial one. □

5.3.5 Transaction costs

The transaction costs are supposed to be proportional to the amount of the transaction. In that case, it can be shown that the superreplication price of a European call option is the price of the underlying.

5.3.6 Variance hedging

Let us assume that $r = 0$. A *self-financing* portfolio is such that its value V satisfies

$$dV_t = rV_t dt + \theta_t(dS_t - rS_t dt) = \theta_t dS_t \qquad (5.5)$$

or

$$V_t = V_0 + \int_0^t \theta_s dS_s$$

A strategy hedges H if $V_T = H$.

Under the hypothesis that P is a risk neutral probability, Föllmer and Sondermann succeed to minimize the variance

$$E((V_T - H))^2)$$

The proof is based on the orthogonal decomposition ; if H is any contingent claim, it can be written as

$$H = h + \int_0^T \theta_s dS_s + M_T^\top$$

where $M_T^\top$ is the terminal value of a martingale orthogonal to S (i.e. such that $E(M_t^\top S_t) = 0$)

This method was extended by Rheinlander and Schweizer and by Laurent, Pham and Gourieroux to the general case. It can be proved that this lead to the choice of a particular e.m.m., called the minimal martingale measure.

5.3.7 Remaining risk

Let us assume that $r = 0$. Let (α, θ) be any strategy and $V_t = \alpha_t + \theta_t S_t$ its value. We do not assume here that the strategy is self-financing. The gain process associated to this strategy is defined as $G_t = \int_0^t \theta_s dS_s$ the cost process up to time t is

$$C_t = V_t - \int_0^t \theta_s dS_s$$

A strategy hedges H if $V_T = H$.
If the market is complete, there exists a strategy for which

$$H = V_T = V_0 + G_T$$

so that the corresponding cost process satisfies

$$C_T = V_0 = E_Q(H)\,.$$

Following Schweizer, we define the remaining risk as

$$R_t = E[(C_T - C_t)^2 | \mathcal{F}_t)]$$

A strategy is risk-minimizing if it minimizes the remaining risk at any time.

5.3.8 Reservation price

A different approach is initiated by Hodges and Neuberger [5], and studied in El Karoui and Rouge [11], Hugonnier [6] and Bouchard-Denize [1].

Let x be the initial endowment of an agent and U a utility function. The reservation price of the contingent claim H is defined as the infimum of h such that

$$\sup E[U(X_T^{x+h} - H)] > \sup E[U(X_T^x)]$$

The agent selling the option starts with an initial endowment $x + h$, he gets an optimal portfolio with terminal value X_T^{x+h} and he has to deliver the contingent claim H.

5.3.9 Davis approach

Another way, studied by Davis [2] (1997), is to value options for an agent endowed with a particular utility function. Related results have been obtained by a number of authors in various contexts. Davis defines the price of the contingent claim ζ using a marginal rate : suppose that ζ is traded at price p. An investor invests an amount of δ in ζ and kep this position till maturity. Her final wealth is $X_T^{x-\delta,\pi} + \frac{\delta}{p(\zeta)}\zeta$. Her investment program is

$$W(\delta, x, p) = \sup_{pi} E(U(X^{x-\delta,\pi} + \frac{\delta}{p(\zeta)}\zeta))$$

Definition 5.3.1 *Assume that the equation*

$$\frac{\partial W}{\partial \delta}(0, p, x) = 0$$

has a unique solution p^*. The fair price of ζ is defined as p^*.

Theorem 5.3.2 *Let* $V(x) = \sup_\pi E(U(X_T^{\pi,x})) = E(U(X_T^{\pi^*,x}))$*. Assume that* V *is differentiable and that* $V'(x) > 0$*. Then* p^* *satisfies* $p^* = \frac{E(U'(X_T^{\pi^*,x})\zeta)}{V'(x)}$

It can be proved that this lead to the choice of a particular e.m.m., called the Davis martingale measure.

5.3.10 Minimal entropy

Another way to price options is to choose a particular equivalent martingale measure, e.g. the minimal entropy measure as in Frittelli [6] (1996).
Let S be the dynamics of the prices and $\mathcal{M}(P)$ the set of e.m.m., i.e., the set of probability equivalent to P such that S is a Q martingale. We assume that this set is not empty. For any $Q \in \mathcal{M}(P)$, the entropy of Q with respect to P is defined as

$$H(Q|P) = E_P(\frac{dQ}{dP} \ln \frac{dQ}{dP})$$

Theorem 5.3.3 *Let* $\mathcal{M}_{L \ln L} = \{Q \in \mathcal{M} : H(Q|P) < \infty$*. There exists a unique probability* $Q^e \in \mathcal{M}_{L \ln L}$ *such that*

$$\forall Q \in \mathcal{M}(P)\,, H(Q|P) \geq H(Q^e|P)$$

Bibliography

[1] Bouchard-Denize. B. *Contrôle*. Thèse, Paris 1, 2000.

[2] M.H.A. Davis. Option pricing in Incomplete markets. In M.H.A. Demtser and S.R. Pliska, editors, *Mathematics of Derivative Securities*, Publication of the Newton Institute, pages 216–227. Cambridge University Press, 1997.

[3] N. El Karoui and M-C. Quenez. Programmation dynamique et évaluation des actifs contingents en marchés incomplets. *CRAS, Paris*, 331:851–854, 1991.

[4] N. El Karoui and M-C. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. control and Optim.*, 33:29–66, 1995.

[5] S.D. Hodges and A. Neuberger. Optimal replication of contingent claims under transaction costs. *Rev. Future Markets*, 8:222–239, 1989.

[6] J.N. Hugonnier. Utility based pricing of contingent claims. *Preprint*, 2000.

[7] D. Kramkov. Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Prob. Theo. and related fields*, 105:459–479, 1996.

[8] E. Eberlein and J. Jacod. On the range of option pricing. *Finance and Stochastic*, 1:131–140, 1997.

[9] H. Föllmer and M. Schweizer. Hedging of contingent claims under incomplete information. *Applied Stochastic Analysis eds. M.H.A. Davis and R.J. Elliott, Gordon and Breach, London*, 1990.

[10] P. Jakubenas. Range of prices. *Preprint*, 1999.

[11] N. El Karoui and R. Rouge. Pricing via utility maximization and entropy. *Mathematical Finance*, 10:259–276, 2000.