

Invariance Times Transfer Properties

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Abstract

Invariance times are stopping times θ such that local martingales with respect to some reduced filtration and changed probability measure, stopped before θ , are local martingales with respect to the original model filtration and probability measure. They arise naturally for modeling the default time of a bank in the mathematical finance context of XVA analysis. Assuming an invariance time θ with an intensity and a positive Azéma supermartingale, this work establishes a dictionary of transfer properties between the original and changed stochastic bases, regarding in particular: expectations, martingales, stochastic integrals, random measure stochastic integrals, martingale representations, semimartingale characteristics, and solution of BSDEs. Thanks to the resulting calculus, companion papers establish the well posedness of the XVA equations.

Keywords: Progressive enlargement of filtration, invariance time, semimartingale calculus, BSDE, counterparty risk, credit risk.

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1 Introduction

Invariance times were introduced in Crépey and Song (2017b) as stopping times θ such that local martingales with respect to a reduced filtration \mathbb{F} and a possibly changed probability measure \mathbb{P} , stopped before θ , are local martingales with respect to the original model filtration \mathbb{G} and probability measure \mathbb{Q} . Seen from the smaller filtration \mathbb{F} , these are the random times θ for which the enlargement of filtration Jeulin-Yor formula can be compensated by the Girsanov formula of an equivalent change of probability measure. This topic is also related to various approaches that were introduced in the mathematical finance literature for coping with defaultable cash flows based on default intensities: cf., in particular, Duffie, Schroder, and Skiadas (1996) and Collin-Dufresne, Goldstein, and Hugonnier (2004), which are discussed in Section A.

Crépey and Song (2017b) focused on a characterization of invariance times in terms of the integrability of a tentative \mathbb{Q} to \mathbb{P} measure change density, recalled in the preliminary Section 2. The present paper establishes a dictionary between the semimartingale calculi in the original and changed stochastic bases, assuming an invariance time θ with an intensity and a positive Azéma supermartingale. In particular:

- Theorem 3.1 extends the classical reduced-form credit risk pricing formulas beyond the basic progressive enlargement of filtration setup where the Azéma supermartingale of θ has no martingale component;
- Theorem 4.1 establishes a bijection between the (\mathbb{G}, \mathbb{Q}) local martingales stopped before θ and the (\mathbb{F}, \mathbb{P}) local martingales;
- Theorem 8.1 yields the relationship between the (\mathbb{G}, \mathbb{Q}) local characteristics of a \mathbb{G} semimartingale X stopped before θ and the (\mathbb{F}, \mathbb{P}) local characteristics of the \mathbb{F} semimartingale X' , called reduction of X , that coincides with X before θ ;
- Theorems 5.1 establishes the connection between stochastic integrals in the sense of local martingales in (\mathbb{G}, \mathbb{Q}) and in (\mathbb{F}, \mathbb{P}) ;
- Theorem 6.1 establishes likewise the connection between the corresponding random measures stochastic integrals;
- Theorem 7.1 establishes the correspondence between suitable (\mathbb{G}, \mathbb{Q}) and (\mathbb{F}, \mathbb{P}) martingale representation properties;
- Theorems 9.1 and 9.2 show the equivalences, within two different spaces of solutions, between a nonstandard (\mathbb{G}, \mathbb{Q}) backward SDE (BSDE) stopped before time θ and a reduced (\mathbb{F}, \mathbb{P}) BSDE with null terminal condition; Proposition 9.1 shows that the latter is well posed among square integrable solutions, under standard assumptions on its coefficient.

Theoretical interest apart, a concrete motivation for this work is the study of the so-called XVA equations, where VA stands for valuation adjustment and X is a catch-all letter to be replaced by C for credit, F for funding, M for margin, or K for capital (see Albanese and Crépey (2017)). These are the equations related to counterparty risk and its capital and funding implications for a bank. Given a misalignment of interest between the shareholders and creditors of a bank, devising financial derivative entry prices from a shareholder indifference point of view leads to XVA BSDEs stopped before the random time θ such as the one mentioned in the last bullet point above. Based on the results of the present paper, these equations (in varying degree of generality) are shown to be well posed in Albanese and Crépey (2017, Section 7.3–7.4) and Crépey, Élie, Sabbagh, and Song (2017, Section 6).

1.1 Standing Notation and Terminology

The real line and half-line are denoted by \mathbb{R} and \mathbb{R}_+ ; $|\cdot|$ denotes any Euclidean norm (in the dimension of its argument); $\mathcal{B}(E)$ denotes the Borel σ algebra on a metrizable space E ; λ is the Lebesgue measure on \mathbb{R}_+ ; δ_a denotes a Dirac measure at a point a .

Unless otherwise stated, a function (or process) is real valued; order relationships between random variables (respectively processes) are meant almost surely (respectively in the indistinguishable sense); a time interval is random (in particular, the graph of a random time τ is simply written $[\tau]$). We do not explicitly mention the domain of definition of a function when it is implied by the measurability, e.g. we write “a $\mathcal{B}(\mathbb{R})$ measurable function h (or $h(x)$)” rather than “a $\mathcal{B}(\mathbb{R})$ measurable function h defined on \mathbb{R} ”. For a function $h(\omega, x)$ defined on a product space $\Omega \times E$, we write $h(x)$ (or h_t in the case of a stochastic process), without ω .

We use the terminology of the general theory of processes and of filtrations as given in the books by Dellacherie and Meyer (1975) and He, Wang, and Yan (1992). For any semimartingale X and for any predictable X integrable process L , the corresponding stochastic integral is denoted by $\int_0^\cdot L_t dX_t = \int_{(0, \cdot]} L_t dX_t = L \cdot X$, with the precedence convention $KL \cdot X = (KL) \cdot X$ if K is another predictable process such that KL is X integrable. The stochastic exponential of a semimartingale X is denoted by $\mathcal{E}(X)$.

We work with semimartingales on a predictable set of interval type \mathcal{I} as defined in He et al. (1992, Sect. VIII.3). In particular, X is a local martingale on \mathcal{I} (respectively $Y = L \cdot X$ on \mathcal{I}) means that

$$X^{\tau_n} \text{ is a local martingale (respectively } Y^{\tau_n} = L \cdot (X^{\tau_n})) \quad (1.1)$$

for at least one, or equivalently any, nondecreasing sequence of stopping times $(\tau_n)_{n \geq 0}$ such that $\cup[0, \tau_n] = \mathcal{I}$. The default case where $\mathcal{I} = \mathbb{R}_+$ corresponds to the standard notions of local martingale and stochastic integral.

For any càdlàg process X , for any random time τ , $\Delta_\tau X$ represents the jump of X at τ . We use the convention that $X_{0-} = X_0$ (hence $\Delta_0 X = 0$) and we write X^τ and $X^{\tau-}$ for the processes X stopped at τ and before τ , i.e., respectively,

$$X^\tau = X \mathbf{1}_{[0, \tau)} + X_\tau \mathbf{1}_{[\tau, +\infty)}, \quad X^{\tau-} = X \mathbf{1}_{[0, \tau)} + X_{\tau-} \mathbf{1}_{[\tau, +\infty)}. \quad (1.2)$$

The process X is said to be stopped at τ , respectively before τ , if $X = X^\tau$, respectively $X = X^{\tau-}$. We call compensator of a stopping time τ the compensator of $\mathbf{1}_{[\tau, \infty)}$. We say that τ has an intensity γ if τ is positive and if its compensator is given as $\gamma \cdot \lambda$, for some predictable process γ (vanishing beyond time τ). For any event A , we denote by τ_A the stopping time $\mathbf{1}_A \tau + \mathbf{1}_{A^c} \infty$.

Stochastic integrals of random functions with respect to jump measures and their compensations are meant in the sense of Jacod (1979), to which we also borrow the usage of including the optionality with respect to a reference filtration in the definition of an integer valued random measure. Random measure stochastic integrals and transform of measures by densities are respectively denoted by “ \ast ” and “ \cdot ”.

We denote by $\mathcal{P}(\mathbb{H})$ and $\mathcal{O}(\mathbb{H})$ the predictable and optional σ fields with respect to any filtration \mathbb{H} .

2 Invariance Times Revisited

In this section we recall the results of Crépey and Song (2017b) regarding their conditions (B) and (A).

We work on a space Ω equipped with a σ field \mathcal{A} , a probability measure \mathbb{Q} on \mathcal{A} , and a filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$ of sub- σ fields of \mathcal{A} satisfying the usual conditions.

2.1 Condition (B)

Let there be given a \mathbb{G} stopping time θ and a subfiltration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ of \mathbb{G} satisfying the usual conditions and the following:

Condition (B) For any \mathbb{G} predictable process L , there exists an \mathbb{F} predictable process L' , called the \mathbb{F} predictable reduction¹ of L , such that $\mathbf{1}_{(0, \theta]} L = \mathbf{1}_{(0, \theta]} L'$. ■

This holds in particular (but not only) in the classical progressive enlargement of filtration setup, where

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\theta \wedge t) \vee \sigma(\{\theta > t\}), \quad t \in \mathbb{R}_+,$$

i.e. \mathbb{G} is the smallest filtration larger than \mathbb{F} making θ a stopping time.

Let o and p denote the \mathbb{F} optional and predictable projections. In particular, $S = {}^o(\mathbf{1}_{[0, \theta)})$ is the \mathbb{F} Azéma supermartingale of θ , with canonical Doob–Meyer decomposition $S = Q - D$, where Q (with $Q_0 = S_0$) and D (with $D_0 = 0$) are the \mathbb{F} martingale component and the \mathbb{F} drift of S . We recall that

$$S_{\theta-} > 0 \text{ on } \{0 < \theta < \infty\} \tag{2.1}$$

(cf. Yor (1978, page 63) and see also Song (2016, Lemma 3.7)).

Lemma 2.2 in Crépey and Song (2017b) *Under the condition (B):*

- 1) For any \mathbb{G} stopping time τ , there exists an \mathbb{F} stopping time τ' , which we call the \mathbb{F} reduction of τ , such that $\{\tau < \theta\} = \{\tau' < \theta\} \subseteq \{\tau = \tau'\}$.
- 2) Given a measurable space (E, \mathcal{B}) , any $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}$ measurable function $\Psi_t(\omega, x)$ admits a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}$ measurable function $\Psi'_t(\omega, x)$, called predictable reduction of Ψ , such that $\mathbf{1}_{(0, \theta]} \Psi = \mathbf{1}_{(0, \theta]} \Psi'$ everywhere; Any $\mathcal{O}(\mathbb{G}) \otimes \mathcal{B}$ measurable function $\Psi_t(\omega, x)$ admits an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}$ measurable function $\Psi'_t(\omega, x)$, called optional reduction of Ψ , such that $\mathbf{1}_{[0, \theta]} \Psi = \mathbf{1}_{[0, \theta]} \Psi'$ everywhere.
- 3) Let M be a \mathbb{G} local martingale stopped before θ . For any \mathbb{F} optional reduction M' of M , M' is an \mathbb{F} semimartingale on $\{S_- > 0\}$ and

$$S_- \cdot M' + [S, M'] \text{ is an } \mathbb{F} \text{ local martingale on } \{S_- > 0\}. \tag{2.2}$$

¹Also known as pre-default process in the credit risk literature (cf. Duffie et al. (1996) and Bielecki and Rutkowski (2001b)).

Conversely, for any \mathbb{F} semimartingale K on $\{S_- > 0\}$ such that $S_- \cdot K + [S, K]$ is an \mathbb{F} local martingale on $\{S_- > 0\}$, $K^{\theta-}$ is a \mathbb{G} local martingale on \mathbb{R}_+ .

4) The Azéma supermartingale S of θ admits the multiplicative decomposition

$$S = S_0 \mathcal{Q} \mathcal{D} \text{ on } \{^P S > 0\}, \quad (2.3)$$

where $\mathcal{Q} = \mathcal{E}(\frac{1}{S_-} \cdot \mathcal{Q})$ is an \mathbb{F} local martingale on $\{^P S > 0\}$ and $\mathcal{D} = \mathcal{E}(-\frac{1}{S_-} \cdot \mathcal{D})$ is an \mathbb{F} predictable nonincreasing process on $\{^P S > 0\}$. ■

Lemma 2.3 in Crépey and Song (2017b) Under the condition (B), assuming $S_T > 0$ for some positive constant T ,

$$\text{two } \mathbb{F} \text{ optional processes that coincide before } \theta \text{ coincide on } [0, T]. \quad (2.4)$$

In particular, \mathbb{F} optional (and predictable) reductions are uniquely defined on $[0, T]$. ■

Lemma A.1 in Crépey and Song (2017b) If θ has a (\mathbb{G}, \mathbb{Q}) intensity γ , then \mathcal{D} is continuous and

$$\mathcal{E}(\pm \frac{1}{S_-} \cdot \mathcal{D}) = e^{\pm \frac{1}{S_-} \cdot \mathcal{D}}, \quad \frac{1}{S_-} \cdot \mathcal{D} = \gamma' \cdot \lambda \quad (2.5)$$

hold on $\{S_- > 0\}$. ■

Moreover, as it follows from the additional results in Song (2016) (supposing $S_T > 0$ so that reductions are uniquely defined), the \mathbb{F} optional reduction of a \mathbb{G} semimartingale is an \mathbb{F} semimartingale on $[0, T]$; the \mathbb{F} optional reduction of a \mathbb{G} optional nondecreasing process is an \mathbb{F} optional nondecreasing process on $[0, T]$.

2.2 Condition (A)

In addition to θ , \mathbb{F} , and \mathbb{G} satisfying the condition (B) as above, let there be given a positive constant T . Letters of the families “q” and “p” are used for (\mathbb{F}, \mathbb{Q}) and (\mathbb{F}, \mathbb{P}) local martingales, respectively, where \mathbb{P} refers to the following:

Condition (A) There exists a probability measure \mathbb{P} equivalent to \mathbb{Q} on \mathcal{F}_T , called invariance probability measure, such that, for any (\mathbb{F}, \mathbb{P}) local martingale P , $P^{\theta-}$ is a (\mathbb{G}, \mathbb{Q}) local martingale on $[0, T]$. ■

If so, then we call θ an invariance time and \mathbb{P} an invariance probability measure.

The most standard circumstance ensuring the condition (A) is a basic immersion setup where (\mathbb{F}, \mathbb{Q}) local martingales are (\mathbb{G}, \mathbb{Q}) local martingales without jump at θ , in which case θ is an invariance time with $\mathbb{P} = \mathbb{Q}$ (for every positive constant T). More

generally, the results of Crépey and Song (2017b) relate the condition (A) to the following integrability condition.

Theorem 3.2 in Crépey and Song (2017b) *The condition (A) holds if and only if $\mathcal{E}(\mathbf{1}_{\{\tau_S > 0\}} \frac{1}{\tau_S} \cdot \mathbb{Q})$ is a positive (\mathbb{F}, \mathbb{Q}) true martingale on $[0, T]$. In this case, a probability measure \mathbb{P} on \mathcal{A} is an invariance probability measure if and only if the \mathbb{F} density process of \mathbb{P} coincides with*

$$\mathcal{E}(\mathbf{1}_{\{\tau_S > 0\}} \frac{1}{\tau_S} \cdot \mathbb{Q})_{\cdot \wedge T} \quad (2.6)$$

on $\{\tau_S > 0\} \cap [0, T]$. In particular, \mathbb{P} defined by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \mathcal{E}(\mathbf{1}_{\{\tau_S > 0\}} \frac{1}{\tau_S} \cdot \mathbb{Q})_T \text{ on } \mathcal{A}$$

is an invariance probability measure. ■

Moreover:

Theorem 3.7 in Crépey and Song (2017b) *If θ has a (\mathbb{G}, \mathbb{Q}) intensity, then, under the condition (A),*

$$\{\mathbb{S}_- > 0\} = \{\tau_S > 0\} = \{\mathbb{S} > 0\}. \quad (2.7)$$

In addition, for any invariance probability measure \mathbb{P} ,

$$\begin{aligned} & \text{A process } P \text{ is an } (\mathbb{F}, \mathbb{P}) \text{ local martingale on } \{\mathbb{S}_- > 0\} \cap [0, T] \\ & \text{if and only if} \end{aligned} \quad (2.8)$$

$$\mathbb{S}_- \cdot P + [\mathbb{S}, P] \text{ is an } (\mathbb{F}, \mathbb{Q}) \text{ local martingale on } \{\mathbb{S}_- > 0\} \cap [0, T]. \blacksquare$$

In order to enjoy all of the above properties, we work henceforth under the following **standing assumption** (given a positive constant T):

Condition (C). The condition (A) is satisfied, $\mathbb{S}_T > 0$, and θ has a (\mathbb{G}, \mathbb{Q}) intensity. ■

In particular, we have $\{\tau_S > 0\} \supseteq [0, T]$, by (2.1). By virtue of the statement including (2.6), invariance probability measures \mathbb{P} are then uniquely determined on \mathcal{F}_T , on which they only matter anyway (because, in practice, \mathbb{P} is only used for computations in \mathbb{F} on $[0, T]$). As a consequence, we can talk of “the invariance probability measure \mathbb{P} .”

By reduction in our setup, we may and do assume the (\mathbb{G}, \mathbb{Q}) intensity of θ of the form $\gamma \mathbf{1}_{(0, \theta]}$, for an \mathbb{F} predictable process γ uniquely defined on $[0, T]$, and we write $\Gamma = \int_0^\cdot \gamma_s ds$.

Given the focus on $[0, T]$ that is implicit in the condition (A), we may and do assume that predictable reductions vanish on (T, ∞) , without loss of generality.

The \mathbb{Q} and \mathbb{P} expectations are denoted by \mathbb{E} and \mathbb{E}' .

3 Expectation Transfer Formulas

What follows provides an extension of classical results (see e.g. Bielecki, Jeanblanc, and Rutkowski (2009, Chapter 3)) beyond the basic immersion setup where $(\mathbb{F}, \mathbb{P} = \mathbb{Q})$ local martingales are (\mathbb{G}, \mathbb{Q}) local martingales without jump at θ .

Theorem 3.1 *For any \mathbb{F} stopping time $\sigma \leq T$ and \mathcal{F}_σ measurable nonnegative random variable χ , for any \mathbb{F} predictable nonnegative process K , for any \mathbb{F} optional nondecreasing process A starting from 0, we have, respectively,*

$$\mathbb{E}[\chi \mathbf{1}_{\{\sigma < \theta\}}] = \mathbb{E}'[\chi e^{-\Gamma_\sigma}], \quad (3.1)$$

$$\mathbb{E}[K_\theta \mathbf{1}_{\{\theta \leq T\}}] = \mathbb{E}'\left[\int_0^T K_s e^{-\Gamma_s} \gamma_s ds\right], \quad (3.2)$$

$$\mathbb{E}[A_T^{\theta-}] = \mathbb{E}'\left[\int_0^T e^{-\Gamma_s} dA_s\right]. \quad (3.3)$$

Proof. Since χ is \mathcal{F}_σ measurable, $\mathbb{E}[\chi \mathbf{1}_{\{\sigma < \theta\}}] = \mathbb{E}[\chi S_\sigma]$. Then (2.3), under the assumption $S_T > 0$ which is part of the condition (C), yields

$$\mathbb{E}[\chi S_\sigma] = \mathbb{E}[\chi S_0 \mathcal{E}\left(-\frac{1}{S_-} \cdot \mathbf{D}\right)_\sigma \mathcal{E}\left(\frac{1}{S} \cdot \mathbf{Q}\right)_\sigma] = \mathbb{E}'[\chi S_0 \mathcal{E}\left(-\frac{1}{S_-} \cdot \mathbf{D}\right)_\sigma], \quad (3.4)$$

by (2.6). In view of (2.5), we obtain (3.1).

For (3.2), we compute

$$\begin{aligned} \mathbb{E}[K_\theta \mathbf{1}_{\{\theta \leq T\}}] &= \mathbb{E}\left[\int_0^T K_s \mathbf{1}_{\{s \leq \theta\}} \gamma_s ds\right] = \int_0^T \mathbb{E}[K_s \mathbf{1}_{\{s < \theta\}} \gamma_s] ds \\ &= \int_0^T \mathbb{E}'[K_s e^{-\Gamma_s} \gamma_s] ds = \mathbb{E}'\left[\int_0^T K_s e^{-\Gamma_s} \gamma_s ds\right], \end{aligned}$$

where (3.1) was used for passing to the second line.

Regarding (3.3), an application of (3.2) yields

$$\mathbb{E}[A_{\theta-} \mathbf{1}_{\{\theta \leq T\}}] = \mathbb{E}'\left[\int_0^T A_s e^{-\Gamma_s} \gamma_s ds\right] = -\mathbb{E}'[A_T e^{-\Gamma_T}] + \mathbb{E}'\left[\int_0^T e^{-\Gamma_s} dA_s\right].$$

Using (3.1), we deduce

$$\begin{aligned} \mathbb{E}[A_T^{\theta-}] &= \mathbb{E}[A_T \mathbf{1}_{\{T < \theta\}}] + \mathbb{E}[A_{\theta-} \mathbf{1}_{\{\theta \leq T\}}] \\ &= \mathbb{E}'[A_T e^{-\Gamma_T}] - \mathbb{E}'[A_T e^{-\Gamma_T}] + \mathbb{E}'\left[\int_0^T e^{-\Gamma_s} dA_s\right] \\ &= \mathbb{E}'\left[\int_0^T e^{-\Gamma_s} dA_s\right]. \blacksquare \end{aligned}$$

See Section A for the discussion of alternatives to (3.2) that are known from the mathematical finance literature.

4 Martingale Transfer Formulas

Conditional expectation transfer formulas can be written in the line of Theorem 3.1. However, as explained in Section 1, our concrete motivation is the solution of BSDEs stopped before their terminal time. In order to deal with these, conditional expectations formulas are not enough: an entire semimartingale calculus is required. We denote by

- $\mathcal{M}_T(\mathbb{F}, \mathbb{P})$, the set of (\mathbb{F}, \mathbb{P}) local martingales stopped at T ,
- $\mathcal{M}_{\theta-\wedge T}(\mathbb{G}, \mathbb{Q})$, the set of (\mathbb{G}, \mathbb{Q}) local martingales stopped at $\theta - \wedge T$, i.e. before θ and at T ,
- $\mathcal{M}_T^c(\mathbb{F}, \mathbb{P})$ and $\mathcal{M}_T^d(\mathbb{F}, \mathbb{P})$, respectively $\mathcal{M}_{\theta-\wedge T}^c(\mathbb{G}, \mathbb{Q})$, and $\mathcal{M}_{\theta-\wedge T}^d(\mathbb{G}, \mathbb{Q})$, their respective subsets of continuous local martingales and purely discontinuous local martingales.

Theorem 4.1 *The following bijections hold:*

$$\begin{aligned} \mathcal{M}_T(\mathbb{F}, \mathbb{P}) &\xleftrightarrow[\cdot']{\cdot^{\theta-}} \mathcal{M}_{\theta-\wedge T}(\mathbb{G}, \mathbb{Q}), \\ \mathcal{M}_T^c(\mathbb{F}, \mathbb{P}) &\xleftrightarrow[\cdot']{\cdot^{\theta}} \mathcal{M}_{\theta-\wedge T}^c(\mathbb{G}, \mathbb{Q}), \\ \mathcal{M}_T^d(\mathbb{F}, \mathbb{P}) &\xleftrightarrow[\cdot']{\cdot^{\theta-}} \mathcal{M}_{\theta-\wedge T}^d(\mathbb{G}, \mathbb{Q}), \end{aligned} \tag{4.1}$$

where \cdot' denotes the \mathbb{F} optional reduction operator.

Proof. On $\mathcal{M}_{\theta-\wedge T}(\mathbb{G}, \mathbb{Q})$ the map \cdot' takes its values in the space $\mathcal{M}_T(\mathbb{F}, \mathbb{P})$ because, for any $M \in \mathcal{M}_{\theta-\wedge T}(\mathbb{G}, \mathbb{Q})$, the process $S_{-} \cdot M' + [S, M']$ is an (\mathbb{F}, \mathbb{Q}) local martingale on $\{S_{-} > 0\}$, by (2.2), so that $M' \in \mathcal{M}_T(\mathbb{F}, \mathbb{P})$, by (2.8). Conversely, on $\mathcal{M}_T(\mathbb{F}, \mathbb{P})$ the map $\cdot^{\theta-}$ takes its values in the space $\mathcal{M}_{\theta-\wedge T}(\mathbb{G}, \mathbb{Q})$ because, for any $P \in \mathcal{M}_T(\mathbb{F}, \mathbb{P})$, $P^{\theta-} \in \mathcal{M}_{\theta-\wedge T}(\mathbb{G}, \mathbb{Q})$, by the condition (A) which is contained in (C).

To establish the first bijection in (4.1) it remains to show that $(M')^{\theta-} = M$ and $(P^{\theta-})' = P$. As M is stopped before θ , the first identity is trivially true. Regarding the second one, $(P^{\theta-})' = P$ holds before θ , hence on $[0, T]$, by (2.4), hence on \mathbb{R}_+ as both processes $(P^{\theta-})'$ and P are stopped at T .

The second bijection in (4.1) follows by the same steps, noting that the reduction of a continuous process X is continuous on $[0, T]$, by (2.4) applied to the jump process of X .

To prove the third bijection, following He et al. (1992, Theorem 7.34), assuming $M \in \mathcal{M}_{\theta-\wedge T}(\mathbb{G}, \mathbb{Q})$, we take a (\mathbb{G}, \mathbb{Q}) continuous local martingale X and we consider the bracket $[M, X]$. Computing the quadratic variations, we obtain

$$[M, X] = [M, X^{\theta-}] = [M', X']^{\theta-}$$

on $[0, T]$, which shows that $[M', X']$ is the \mathbb{F} optional reduction of $[M, X]$. Consequently, according to (2.4), $[M, X] = 0$ on $[0, \theta \wedge T]$ if and only if $[M', X'] = 0$ on $[0, T]$. The lemma then follows from the first and second bijections in (4.1). ■

5 Transfer of Stochastic Integrals in the Sense of Local Martingales

We need the following complements to Theorem 3.1.

Lemma 5.1 *Let $(\tau_n)_{n \geq 0}$ be a nondecreasing sequence of \mathbb{G} stopping times tending to infinity. There exists a nondecreasing sequence $(\sigma_n)_{n \geq 0}$ of \mathbb{F} stopping times such that σ_n tends to infinity and*

$$\tau_n \wedge T \wedge \theta = \sigma_n \wedge T \wedge \theta.$$

Proof. We compute, using (3.1) for passing to the second line,

$$\begin{aligned} \mathbb{E}'[\mathbb{1}_{\{\tau'_n < T\}} e^{-\Gamma T}] &\leq \mathbb{E}'[\mathbb{1}_{\{\tau'_n < T\}} e^{-\Gamma \tau'_n}] \\ &= \mathbb{E}[\mathbb{1}_{\{\tau'_n < T\}} \mathbb{1}_{\{\tau'_n < \theta\}}] = \mathbb{E}[\mathbb{1}_{\{\tau_n < T\}} \mathbb{1}_{\{\tau_n < \theta\}}] \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This implies that $\mathbb{P}[\tau'_n < T] \rightarrow 0$. Hence $\mathbb{Q}[\tau'_n < T] \rightarrow 0$, as \mathbb{P} is equivalent to \mathbb{Q} on \mathcal{F}_T . The sequence $\sigma_n = (\tau'_n)_{\{\tau'_n < T\}}$, $n \geq 0$, satisfies all the desired properties. ■

Lemma 5.2 *Let A be a \mathbb{G} adapted nondecreasing process. The process $A^{\theta-}$ is (\mathbb{G}, \mathbb{Q}) locally integrable on $[0, T]$ if and only if A' is (\mathbb{F}, \mathbb{P}) locally integrable on $[0, T]$.*

Proof. Recall that A' is a nondecreasing process (cf. the last paragraph in Section 2.1). Let $(\tau_n)_{n \geq 0}$ be a nondecreasing sequence of \mathbb{G} stopping times tending to infinity. Let $(\sigma_n)_{n \geq 0}$ be associated with $(\tau_n)_{n \geq 0}$ as in Lemma 5.1. We compute

$$\begin{aligned} \mathbb{E}\left[\int_0^{\tau_n \wedge T} \mathbb{1}_{\{s < \theta\}} e^{\Gamma s} dA_s\right] &= \mathbb{E}\left[\int_0^{\tau_n \wedge T \wedge \theta} \mathbb{1}_{\{s < \theta\}} e^{\Gamma s} dA_s\right] \\ &= \mathbb{E}\left[\int_0^{\sigma_n \wedge T \wedge \theta} \mathbb{1}_{\{s < \theta\}} e^{\Gamma s} dA'_s\right] = \mathbb{E}\left[\int_0^{\sigma_n \wedge T} \mathbb{1}_{\{s < \theta\}} e^{\Gamma s} dA'_s\right] = \mathbb{E}'[A'_{\sigma_n \wedge T}], \end{aligned}$$

by (3.3). This implies the result. ■

Theorem 5.1 *Let W be a (\mathbb{G}, \mathbb{Q}) local martingale stopped before θ and let L be a \mathbb{G} predictable process. The process L is W integrable in the sense of (\mathbb{G}, \mathbb{Q}) local martingales² if and only if L' is W' integrable on $[0, T]$ in the sense of (\mathbb{F}, \mathbb{P}) local martingales (recall that we assume $L' = 0$ on (T, ∞)). If so, then*

$$(L' \cdot W' \text{ in } (\mathbb{F}, \mathbb{P}))^{\theta-} = L \cdot W \text{ in } (\mathbb{G}, \mathbb{Q}) \text{ on } [0, T].$$

Proof. For the integrability issue, we only need to check the local integrability of the processes $\sqrt{\int_0^t L_s^2 d[W, W]_s}$ and $\sqrt{\int_0^t (L'_s)^2 d[W', W']_s}$ under respectively (\mathbb{G}, \mathbb{Q}) and (\mathbb{F}, \mathbb{P}) . But these local integrabilities are equivalent because of Lemma 5.2.

²cf. He et al. (1992, Section 9.1).

To prove the identity between the stochastic integrals when they exist, we first note that the identity holds for any L in the class of \mathbb{G} predictable bounded step processes. By monotone class theorem, this is then extended to the class of \mathbb{G} predictable bounded processes L . By stochastic dominated convergence Theorem 9.30 in He et al. (1992), this is extended further to all \mathbb{G} predictable processes L which are W integrable under (\mathbb{G}, \mathbb{Q}) . ■

6 Transfer of Random Measures Stochastic Integrals

Given a Polish space E endowed with its Borel σ algebra $\mathcal{B}(E)$, we recall from He et al. (1992, Theorem 11.13) that, for any (optional) integer valued random measure π , there exists an E valued optional process β and an optional thin set, of the form $\cup_{n \in \mathbb{N}} [\tau_n]$ for some sequence of stopping times $(\tau_n)_{n \geq 0}$, such that

$$\pi = \sum_s \delta_{(s, \beta_s)} \mathbb{1}_{\{s \in \cup_{n \in \mathbb{N}} [\tau_n]\}}. \quad (6.1)$$

Hence, for any nonnegative $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$ measurable function Ψ ,

$$\Psi * \pi = \sum_s \Psi(s, \beta_s) \mathbb{1}_{\{s \in \cup_{n \in \mathbb{N}} [\tau_n]\}} = \sum_n \Psi(\tau_n, \beta_{\tau_n}) \mathbb{1}_{\{\tau_n < \infty\}}. \quad (6.2)$$

Lemma 6.1 *The \mathbb{G} optional integer valued random measure π on $\mathbb{R}_+ \times E$ admits an \mathbb{F} optional reduction, i.e. an \mathbb{F} optional integer valued random measure π' on $\mathbb{R}_+ \times E$ such that $\mathbb{1}_{[0, \theta]} \cdot \pi = \mathbb{1}_{[0, \theta]} \cdot \pi'$.*

Proof. We have, for any nonnegative $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$ measurable function Ψ ,

$$\begin{aligned} \Psi * (\mathbb{1}_{[0, \theta]} \cdot \pi) &= \sum_s \mathbb{1}_{\{s < \theta\}} \Psi(s, \beta'_s) \mathbb{1}_{\{s \in \cup_{n \in \mathbb{N}} [\tau_n]\}} \\ &= \sum_s \mathbb{1}_{\{s < \theta\}} \Psi(s, \beta'_s) \mathbb{1}_{\{s \in \cup_{n \in \mathbb{N}} [\tau'_n]\}} = \Psi * (\mathbb{1}_{[0, \theta]} \cdot \pi'), \end{aligned}$$

where $\pi' = \sum_s \delta_{(s, \beta'_s)} \mathbb{1}_{\{s \in \cup_{n \in \mathbb{N}} [\tau'_n]\}}$ defines an \mathbb{F} optional integer valued random measure, by He et al. (1992, Theorem 11.13). ■

In the remainder of the paper, we fix the space E , a \mathbb{G} optional integer valued random measure π , and the related notation as in the above. We introduce the space of random functions $\widehat{\mathcal{P}}(\mathbb{F}) = \mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(E)$ and $\widehat{\mathcal{P}}(\mathbb{G}) = \mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(E)$. We denote the (\mathbb{F}, \mathbb{P}) compensator of $\mu = \pi'$ by ν .

Lemma 6.2 *The (\mathbb{G}, \mathbb{Q}) compensator of $\mathbb{1}_{[0, \theta]} \cdot \mu$ is $\mathbb{1}_{[0, \theta]} \cdot \nu$ on $[0, T]$.*

Proof. By Lemma 5.2, for any $\Psi \in \widehat{\mathcal{P}}(\mathbb{G})$ such that the process $|\Psi| * \pi$ is (\mathbb{G}, \mathbb{Q}) integrable, the processes $|\Psi'| * \mu$ and $|\Psi'| * \nu$ are (\mathbb{F}, \mathbb{P}) locally integrable (recalling that $\Psi' = 0$ on (T, ∞) by assumption). It follows that the process

$$P = \Psi' * \mu - \Psi' * \nu$$

is an (\mathbb{F}, \mathbb{P}) local martingale (cf. He et al. (1992, p. 301)). By the condition (A), the stopped process

$$P^{\theta-} = \mathbb{1}_{[0, \theta)} \Psi' * \mu - \mathbb{1}_{[0, \theta)} \Psi' * \nu = \mathbb{1}_{[0, \theta)} \Psi * \mu - \mathbb{1}_{[0, \theta)} \Psi * \nu$$

is a (\mathbb{G}, \mathbb{Q}) local martingale, where $\mathbb{1}_{[0, \theta)} \Psi * \nu = \mathbb{1}_{[0, \theta]} \Psi * \nu$ because θ avoids the predictable stopping times. As $\mathbb{1}_{[0, \theta]} \nu$ is a \mathbb{G} predictable random measure, this proves the lemma. ■

Theorem 6.1 *For any $\Psi \in \widehat{\mathcal{P}}(\mathbb{G})$, Ψ is $(\mathbb{1}_{[0, \theta)} \cdot \mu - \mathbb{1}_{[0, \theta]} \cdot \nu)$ stochastically integrable³ in (\mathbb{G}, \mathbb{Q}) on $[0, T]$ if and only if Ψ' is $(\mu - \nu)$ stochastically integrable in (\mathbb{F}, \mathbb{P}) on $[0, T]$. If so, then*

$$(\Psi' * (\mu - \nu) \text{ in } (\mathbb{F}, \mathbb{P}))^{\theta-} = \Psi * (\mathbb{1}_{[0, \theta)} \cdot \mu - \mathbb{1}_{[0, \theta]} \cdot \nu) \text{ in } (\mathbb{G}, \mathbb{Q}) \text{ on } [0, T].$$

Proof. In view of He, Wang, and Yan (1992, Definition 11.16), the integrability relationship between Ψ and Ψ' is the consequence of Lemma 5.2.

To prove the identity between the corresponding integrals when they exist, we note that

$$(\Psi' * (\mu - \nu))^{\theta-} \text{ and } \Psi * (\mathbb{1}_{[0, \theta)} \cdot \mu - \mathbb{1}_{[0, \theta]} \cdot \nu)$$

are (\mathbb{G}, \mathbb{Q}) purely discontinuous local martingales. By virtue of He, Wang, and Yan (1992, Theorem 7.42 and Definition 11.16), they are then equal because they have the same jumps, namely

$$\begin{aligned} \Delta_t(\Psi' * (\mu - \nu))^{\theta-} &= (\Psi'(t, \beta'_t) \mathbb{1}_{\{t \in \cup_{n \in \mathbb{N}} [\tau'_n]\}} - \int_{\{t\} \times E} \Psi'(s, x) \nu(ds, dx)) \mathbb{1}_{\{t < \theta\}} \\ &= (\Psi(t, \beta_t) \mathbb{1}_{\{t \in \cup_{n \in \mathbb{N}} [\tau_n]\}} - \int_{\{t\} \times E} \Psi(s, x) \nu(ds, dx)) \mathbb{1}_{\{t < \theta\}} \\ &= \Delta_t(\Psi * (\mathbb{1}_{[0, \theta)} \cdot \mu - \mathbb{1}_{[0, \theta]} \cdot \nu)), \end{aligned}$$

as $\mathbb{1}_{[0, \theta)} \cdot \nu = \mathbb{1}_{[0, \theta]} \cdot \nu$ (because θ avoids the predictable stopping times). ■

7 Transfer of Martingale Representations Properties

We consider martingale representations with respect to martingales and compensated jump measures as in Jacod (1979), which corresponds to the notion of weak representation in He et al. (1992). As in He et al. (1992), when no jump measure is involved, we talk of strong representation.

Let W be a d variate (\mathbb{G}, \mathbb{Q}) local martingale stopped before θ . We assume the random measure π stopped before θ , in the sense that $\cup_{n \in \mathbb{N}} [\tau_n] \subseteq (0, \theta)$. We write $B = W'$, $\mu = \pi'$. Let ρ and ν denote the (\mathbb{G}, \mathbb{Q}) compensator of π and the (\mathbb{F}, \mathbb{P}) compensator of μ , so that $\rho = \mathbb{1}_{[0, \theta]} \cdot \nu$, by Lemma 6.2.

³cf. He et al. (1992, Section 11.16).

Lemma 7.1 Given $(\mathcal{P}(\mathbb{G}))^{\otimes d}$ and $\widehat{\mathcal{P}}(\mathbb{G})$ measurable integrands L and Ψ , if

$$M = L \cdot W + \Psi * (\pi - \rho) \quad (7.1)$$

holds in (\mathbb{G}, \mathbb{Q}) on $[0, T]$, then $M' = L' \cdot B + \Psi' * (\mu - \nu)$ holds in (\mathbb{F}, \mathbb{P}) on $[0, T]$.

Conversely, given $(\mathcal{P}(\mathbb{F}))^{\otimes d}$ and $\widehat{\mathcal{P}}(\mathbb{F})$ measurable integrands K and Φ , if

$$P = K \cdot B + \Phi * (\mu - \nu) \quad (7.2)$$

holds in (\mathbb{F}, \mathbb{P}) on $[0, T]$, then $P^{\theta-} = K \cdot B^{\theta-} + \Phi * (\mathbb{1}_{[0, \theta]} \cdot \mu - \mathbb{1}_{[0, \theta]} \cdot \nu)$ holds in (\mathbb{G}, \mathbb{Q}) on $[0, T]$.

Proof. This is the consequence of Theorems 5.1 and 6.1. ■

Remark 7.1 In the representation (7.1), the integrands L and Ψ corresponding to a given process M are unique modulo $d[W, W]$ and ρ negligible sets, respectively. Likewise, in the representation (7.2), the integrands K and Φ corresponding to a given process P are unique modulo $d[B, B]$ and ν negligible sets. ■

As an immediate consequence of Lemma 7.1:

Theorem 7.1 The space $\mathcal{M}_{\theta-\wedge T}(\mathbb{G}, \mathbb{Q})$ admits a (weak) representation by W and μ if and only if the space $\mathcal{M}_T(\mathbb{F}, \mathbb{P})$ admits a (weak) representation by $B = W'$ and $\mu = \pi'$. ■

Applying Theorem 7.1 with $\mu = 0$, one obtains the strong martingale representation transfer property.

8 Semimartingale Characteristic Triplets Transfer Formula

Let there be given a semimartingale X stopped before θ in some filtration \mathbb{H} under a probability measure \mathbb{M} , with jump measure π^X . The characteristic triplet of X is composed of:

$$b^{X, \mathbb{H}, \mathbb{M}}, \quad \text{the drift part of the truncated semimartingale } X - (x \mathbb{1}_{\{|x| > 1\}}) * \pi^X;$$

$$a^{X, \mathbb{H}, \mathbb{M}} = \langle X^c, X^c \rangle, \text{ the bracket of the continuous martingale part of } X \\ \text{(the diffusion part of } X \text{);}$$

$$c^{X, \mathbb{H}, \mathbb{M}} = (\pi^X)^{p, \mathbb{H}, \mathbb{M}}, \text{ the predictable dual projection of } \pi^X \\ \text{(called in He et al. (1992) the Lévy system of } X \text{, i.e. the extension to a} \\ \text{semimartingale setup of the notion of a Lévy measure).}$$

The following results show that the (\mathbb{F}, \mathbb{P}) characteristic triplet of the optional reduction X' of a (\mathbb{G}, \mathbb{Q}) semimartingale $X = X^{\theta-}$ is the predictable reduction of the (\mathbb{G}, \mathbb{Q}) characteristic triplet of X . Moreover, if the (\mathbb{G}, \mathbb{Q}) semimartingale X is special, then so is X' and the (\mathbb{F}, \mathbb{P}) drift of X' is the predictable reduction of the (\mathbb{G}, \mathbb{Q}) drift of X .

Theorem 8.1 *Let X be a (\mathbb{G}, \mathbb{Q}) semimartingale stopped before θ . Let X' be the optional reduction of X (an (\mathbb{F}, \mathbb{P}) semimartingale as recalled in the last paragraph of Section 2.1). We have*

$$\left(b^{X, \mathbb{G}, \mathbb{Q}}, a^{X, \mathbb{G}, \mathbb{Q}}, c^{X, \mathbb{G}, \mathbb{Q}}\right) = \left((b^{X', \mathbb{F}, \mathbb{P}})^\theta, (a^{X', \mathbb{F}, \mathbb{P}})^\theta, \mathbb{1}_{[0, \theta]} \cdot c^{X', \mathbb{F}, \mathbb{P}}\right) \text{ on } [0, T]. \quad (8.1)$$

Proof. We have the identity $\mathbb{1}_{[0, \theta]} \cdot \pi^X = \mathbb{1}_{[0, \theta]} \cdot \pi^{X'}$ on $[0, T]$. So,

$$X - (x \mathbb{1}_{\{|x| > 1\}})_* \pi^X = \left(X' - (x \mathbb{1}_{\{|x| > 1\}})_* \pi^{X'}\right)^{\theta-} = (P^c)^{\theta-} + (P^d)^{\theta-} + (b^{X', \mathbb{F}, \mathbb{P}})^{\theta-} \quad (8.2)$$

on $[0, T]$, where P is the (\mathbb{F}, \mathbb{P}) canonical Doob–Meyer martingale component of the (\mathbb{F}, \mathbb{P}) special semimartingale $X' - (x \mathbb{1}_{\{|x| > 1\}})_* \pi^{X'}$ on $[0, T]$, with continuous and purely discontinuous parts P^c and P^d . By the condition (A), $P^{\theta-}$ is a (\mathbb{G}, \mathbb{Q}) local martingale on $[0, T]$. Therefore, from the formula (8.2), we conclude that

$$b^{X, \mathbb{G}, \mathbb{Q}} = (b^{X', \mathbb{F}, \mathbb{P}})^{\theta-} = (b^{X', \mathbb{F}, \mathbb{P}})^\theta \text{ on } [0, T]$$

(as $\Delta_\theta b^{X', \mathbb{F}, \mathbb{P}} = 0$, because θ is totally inaccessible.) Now, applying Lemma 6.2 with $E = \mathbb{R}$, we also conclude

$$c^{X, \mathbb{G}, \mathbb{Q}} = (\pi^X)^{p, \mathbb{G}, \mathbb{Q}} = (\mathbb{1}_{[0, \theta]} \cdot \pi^X)^{p, \mathbb{G}, \mathbb{Q}} = \mathbb{1}_{[0, \theta]} \cdot (\pi^{X'})^{p, \mathbb{F}, \mathbb{P}} = \mathbb{1}_{[0, \theta]} \cdot c^{X', \mathbb{F}, \mathbb{P}}.$$

Finally, according to the second and third bijections in (4.1), we have

$$(P^c)^{\theta-} \in \mathcal{M}_{\theta \wedge T}^c(\mathbb{G}, \mathbb{Q}), \quad (P^d)^{\theta-} \in \mathcal{M}_{\theta - \wedge T}^d(\mathbb{G}, \mathbb{Q}).$$

Hence we conclude from (8.2) that $X^c = (P^c)^{\theta-}$ is the continuous martingale part of X in (\mathbb{G}, \mathbb{Q}) and therefore

$$a^{X, \mathbb{G}, \mathbb{Q}} = [X^c, X^c] = [(P^c)^{\theta-}, (P^c)^{\theta-}] = [P^c, P^c]^\theta = (a^{X', \mathbb{F}, \mathbb{P}})^\theta. \blacksquare$$

Corollary 8.1 *Suppose that the (\mathbb{G}, \mathbb{Q}) semimartingale $X = X^{\theta-}$ is special on $[0, T]$. Then X' is an (\mathbb{F}, \mathbb{P}) special semimartingale on $[0, T]$. Denoting by $\beta^{X, \mathbb{G}, \mathbb{Q}}$ and $\beta^{X', \mathbb{F}, \mathbb{P}}$ the (\mathbb{G}, \mathbb{Q}) drift of X and the (\mathbb{F}, \mathbb{P}) drift of X' , we have*

$$\beta^{X, \mathbb{G}, \mathbb{Q}} = (\beta^{X', \mathbb{F}, \mathbb{P}})^\theta \text{ on } [0, T]. \quad (8.3)$$

Proof. As X' is already known to be an (\mathbb{F}, \mathbb{P}) semimartingale and because special semimartingale means one with locally integrable jumps, the special feature of X' follows from Lemma 5.2. Note that, by He et al. (1992, Lemma 7.16 and Theorem 11.24), the function $|x| \mathbb{1}_{\{|x| > 1\}}$ is $c^{X', \mathbb{F}, \mathbb{P}}$ integrable on $[0, T]$. Consequently

$$\beta^{X', \mathbb{F}, \mathbb{P}} = b^{X', \mathbb{F}, \mathbb{P}} + (x \mathbb{1}_{\{|x| > 1\}})_* c^{X', \mathbb{F}, \mathbb{P}} \text{ on } [0, T].$$

The analogous (\mathbb{G}, \mathbb{Q}) relationship holds for X . Hence (8.3) follows from (8.1). \blacksquare

9 BSDEs Transfer Properties

In this section, θ satisfying the condition (C) on $[0, T]$ as before, we reduce a (\mathbb{G}, \mathbb{Q}) BSDE stopped before θ and at T to a simpler (\mathbb{F}, \mathbb{P}) BSDE stopped at T .

We suppose E Euclidean and $(E, \mathcal{B}(E))$ endowed with a σ finite measure m integrating $(1 \wedge |e|^2)$ on E . We consider the space \mathcal{L}_0 of the $\mathcal{B}(E)$ measurable functions u endowed with the topology of convergence in measure induced by m .

Given a $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathcal{L}_0)$ measurable function $g = g_t(z, l, \psi)$, we can define, by class monotone, a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathcal{L}_0)$ reduction $g' = g'_t(z, l, \psi)$ of g such that $\mathbb{1}_{(0, \theta]} g = \mathbb{1}_{(0, \theta]} g'$. Let A be a \mathbb{G} finite variation (càdlàg) process.

Adopting the setup of Section 7, we consider the (\mathbb{G}, \mathbb{Q}) BSDE for a \mathbb{G} adapted process Z , a $(\mathcal{P}(\mathbb{G}))^{\otimes d}$ measurable process L integrable against $B^{\theta-}$ in (\mathbb{G}, \mathbb{Q}) on $[0, T]$, and a $\widehat{\mathcal{P}}(\mathbb{G})$ measurable function Ψ stochastically integrable against $(\mathbb{1}_{[0, \theta]} \cdot \mu - \mathbb{1}_{[0, \theta]} \cdot \nu)$ in (\mathbb{G}, \mathbb{Q}) on $[0, T]$, such that, in (\mathbb{G}, \mathbb{Q}) ,

$$\left\{ \begin{array}{l} \int_0^{\theta \wedge T} |g_s(Z_{s-}, L_s, \Psi_s)| ds < \infty \text{ and} \\ \int_0^{\cdot} \mathbb{1}_{\{s < \theta\}} |dA_s| \text{ is } (\mathbb{G}, \mathbb{Q}) \text{ locally integrable on } [0, T], \\ Z_t^{\theta - \wedge T} + \int_0^{t \wedge \theta \wedge T} (g_s(Z_{s-}, L_s, \Psi_s) ds + dA_s^{\theta-}) \\ = L \cdot B_t^{\theta-} + \Psi * (\mathbb{1}_{[0, \theta]} \cdot \mu - \mathbb{1}_{[0, \theta]} \cdot \nu)_t, \quad t \in \mathbb{R}_+, \\ Z_T \mathbb{1}_{\{T < \theta\}} = 0. \end{array} \right. \quad (9.1)$$

We also consider the (\mathbb{F}, \mathbb{P}) BSDE for an \mathbb{F} adapted process U , a $(\mathcal{P}(\mathbb{F}))^{\otimes d}$ measurable process K integrable against B in (\mathbb{F}, \mathbb{P}) on $[0, T]$, and a $\widehat{\mathcal{P}}(\mathbb{F})$ measurable function Φ stochastically integrable against $(\mu - \nu)$ in (\mathbb{F}, \mathbb{P}) on $[0, T]$, such that, in (\mathbb{F}, \mathbb{P}) ,

$$\left\{ \begin{array}{l} \int_0^T |g'_s(U_{s-}, K_s, \Phi_s)| ds < \infty \text{ and} \\ \int_0^{\cdot} |dA'_s| \text{ is } (\mathbb{F}, \mathbb{P}) \text{ locally integrable on } [0, T], \\ U_t^T + \int_0^{t \wedge T} (g'_s(U_{s-}, K_s, \Phi_s) ds + dA'_s) = K \cdot B_t + \Phi * (\mu - \nu)_t, \quad t \in \mathbb{R}_+, \\ U_T = 0. \end{array} \right. \quad (9.2)$$

Note that the (\mathbb{G}, \mathbb{Q}) BSDE (9.1) is stopped at $\theta - \wedge T$, whereas the (\mathbb{F}, \mathbb{P}) BSDE (9.2) is stopped at T .

Example 9.1 Given a bank with default time θ , a \mathbb{G} stopping time τ representing the default time of a client of the bank, and a nonnegative \mathbb{G} optional process G representing the liability of the client to the bank, then the process $A = \int_0^{\cdot} G_s \delta_{\tau}(ds)$ represents the counterparty credit exposure of the bank to its client. In this case

$$|dA_s| = G_s \delta_{\tau}(ds), \quad A^{\theta-} = \int_0^{\cdot} \mathbb{1}_{\{s < \theta\}} G_s \delta_{\tau}(ds), \quad A' = \int_0^{\cdot} G'_s \delta_{\tau'}(ds).$$

The coefficient g represents the nonlinear funding costs entailed by the credit riskiness of the bank. From the point of view of the bank shareholders, financial derivative entry prices should only take into account the cash flows occurring before the default θ of the bank, because the cash flows occurring from θ onward only affect the bank creditors. Hence, in XVA equations (cf. Section 1), all cash flows need be stopped before the bank default time θ . This results in a BSDE of the form (9.1) for the valuation of counterparty risk (CVA) and of its funding implications to the bank (FVA). The cost of capital (KVA) also obeys an equation of the form (9.1). See Albanese and Crépey (2017, Sections 7.3–7.4) and Crépey, Élie, Sabbagh, and Song (2017, Section 6).

9.1 Transfer of Local Martingale Solutions

The result that follows states the equivalence between the (\mathbb{G}, \mathbb{Q}) BSDE (9.1) and the (\mathbb{F}, \mathbb{P}) BSDE (9.2) considered within the above-introduced spaces of solutions for the triples (Z, L, Ψ) and (U, K, Φ) , called local martingale solutions henceforth (as the right-hand sides in the second lines of (9.1) and (9.2) are then respectively (\mathbb{G}, \mathbb{Q}) and (\mathbb{F}, \mathbb{P}) local martingales).

Theorem 9.1 *The (\mathbb{G}, \mathbb{Q}) BSDE (9.1) and the (\mathbb{F}, \mathbb{P}) BSDE (9.2) are equivalent in the respective spaces of local martingale solutions. Specifically, if (Z, L, Ψ) solves (9.1), then $(U, K, \Phi) = (Z, L, \Psi)'$ solves (9.2). Conversely, if (U, K, Φ) solves (9.2), then $(Z, L, \Psi) = (U^{\theta-}, \mathbf{1}_{[0, \theta]}K, \mathbf{1}_{[0, \theta]}\Phi)$ solves (9.1).*

Proof. Through the correspondence stated in the theorem between the involved processes:

- The equivalence between the integrability conditions (first lines) in (9.1) and (9.2) follows from Lemma 5.2;
- The equivalence between the martingale conditions (second lines) in (9.1) and (9.2) follows from Theorems 5.1 and 6.1;
- The terminal condition in (9.2) obviously implies the one in (9.1), whereas taking the \mathcal{F}_T conditional expectation of the terminal condition in (9.1) yields

$$0 = \mathbb{E}[Z_T \mathbf{1}_{\{T < \theta\}} | \mathcal{F}_T] = \mathbb{E}[Z'_T \mathbf{1}_{\{T < \theta\}} | \mathcal{F}_T] = Z'_T S_T,$$

hence $U_T = Z'_T = 0$ (as S_T is positive under the condition (C)). ■

9.2 Transfer of Square Integrable Solutions

We now consider the (\mathbb{G}, \mathbb{Q}) BSDE (9.1) and the (\mathbb{F}, \mathbb{P}) BSDE (9.2) within suitable spaces of square integrable solutions.

We assume that the compensator ν of $\mu = \pi'$ is given as $\zeta_t(e)m(de)dt$, where ζ is a nonnegative and bounded integrand in $\mathcal{P}(\mathbb{F})$. We write, for any $t \geq 0$ and $\mathcal{B}(E)$ measurable function u ,

$$|u|_t^2 = \int_E u(e)^2 \zeta_t(e) m(de).$$

Considering the (\mathbb{G}, \mathbb{Q}) BSDE (9.1) for (Z, L, Ψ) and the reduced (\mathbb{F}, \mathbb{P}) BSDE (9.2) for (U, K, Φ) , with local martingale solutions (if any) such that

$$(U, K, \Phi) = (Z, L, \Psi)', \quad (Z, L, \Psi) = (U^{\theta-}, \mathbf{1}_{[0, \theta]} K, \mathbf{1}_{[0, \theta]} \Phi) \quad (9.3)$$

(cf. Theorem 9.1), we define,

$$\begin{aligned} \|(Z, L, \Psi)\|_2^2 &= \mathbb{E}[|Z_0|^2] + \mathbb{E}\left[\int_0^T e^{\int_0^s \gamma_u du} \mathbf{1}_{\{s < \theta\}} d(|Z|^*_s)^2\right] \\ &\quad + \mathbb{E}\left[\int_0^T e^{\int_0^s \gamma_u du} \mathbf{1}_{\{s < \theta\}} (|L_s|^2 + |\Psi_s|^2) ds\right], \\ (\|(U, K, \Phi)\|_2')^2 &= \mathbb{E}'[(|U|^*_T)^2] + \mathbb{E}'\left[\int_0^T (|K_s|^2 + |\Phi_s|^2) ds\right], \end{aligned}$$

where \cdot^* means running supremum.

Theorem 9.2 *Given local martingale solutions (Z, L, Ψ) to the (\mathbb{G}, \mathbb{Q}) BSDE (9.1) and (U, K, Φ) to the reduced (\mathbb{F}, \mathbb{P}) BSDE (9.2), we have*

$$\|(Z, L, \Psi)\|_2 = \|(U, K, \Phi)\|_2'. \quad (9.4)$$

Considered in the respective subspaces of square integrable solutions corresponding to finite norms $\|\cdot\|_2$ and $\|\cdot\|_2'$, the (\mathbb{G}, \mathbb{Q}) BSDE (9.1) and the (\mathbb{F}, \mathbb{P}) BSDE (9.2) are equivalent through the correspondence (9.3).

Proof. Given respective local martingale solutions (Z, L, Ψ) and (U, K, Φ) to (9.1) and (9.2), then related through (9.3) as seen in Theorem 9.1, the expectation transfer formula (3.3) yields:

- For $A = \int_0^\cdot e^{\int_0^s \gamma_u du} d(|Z|^*_s)^2$,
- $$\mathbb{E}\left[\int_0^T e^{\int_0^s \gamma_u du} \mathbf{1}_{\{s < \theta\}} d(|Z|^*_s)^2\right] = \mathbb{E}'[(|U|^*_T)^2] - \mathbb{E}'[|U_0|^2] = \mathbb{E}'[(|U|^*_T)^2] - \mathbb{E}[|Z_0|^2];$$
- For $A = \int_0^\cdot e^{\int_0^s \gamma_u du} (|L_s|^2 + |\Psi_s|^2) ds$,
- $$\mathbb{E}\left[\int_0^T e^{\int_0^s \gamma_u du} \mathbf{1}_{\{s < \theta\}} (|L_s|^2 + |\Psi_s|^2) ds\right] = \mathbb{E}'\left[\int_0^T (|K_s|^2 + |\Phi_s|^2) ds\right].$$

This proves (9.4).

Given the equivalence of Theorem 9.1 between (9.1) and (9.2) in the sense of local martingale solutions, their equivalence in the sense of square integrable solutions immediately follows from the transfer of norms formula (9.4). ■

9.3 Application

Assuming $\int_0^T |dA'_s|$ integrable under \mathbb{P} and a (weak) martingale representation of the form studied in Theorem 7.1, we define the process R and its (\mathbb{F}, \mathbb{P}) martingale part P given as

$$R_t = \mathbb{E}' \left[\int_t^T dA'_s \mid \mathcal{F}_t \right], \quad P_t = \mathbb{E}' \left[\int_0^T dA'_s \mid \mathcal{F}_t \right].$$

Let $f_s(u, k, \phi) = g'_s(R_{s-} + u, K_s^P + k, \Phi_s^P + \phi)$, where K^P and Φ^P are the integrands in the representation (7.2) of P (cf. Remark 7.1).

Proposition 9.1 *Suppose that $\|R\|_2' < \infty$ and*

(i) *the functions $u \mapsto f_t(u, k, \phi)$ are continuous. Moreover, f is monotonic with respect to u , i.e.*

$$(f_t(u_1, k, \phi) - f_t(u_2, k, \phi))(u_1 - u_2) \leq C(u_1 - u_2)^2;$$

(ii) $\mathbb{E}' \int_0^T \sup_{|u| \leq c} |f_t(u, 0, 0) - f_t(0, 0, 0)| dt < \infty$, for every positive c ;

(iii) f is Lipschitz continuous with respect to k and ϕ , i.e.

$$|f_t(u, k_1, \phi_1) - f_t(u, k_2, \phi_2)| \leq C(|k_1 - k_2| + |\phi_1 - \phi_2|_t);$$

(iv) $\mathbb{E}' \int_0^T |f_t(0, 0, 0)|^2 dt < +\infty$.

Then the (\mathbb{G}, \mathbb{Q}) BSDE (9.1) and the (\mathbb{F}, \mathbb{P}) BSDE (9.2) are well posed in their respective spaces of square integrable solutions, with solutions related through (9.3).

Proof. Through the correspondence

$$U = R + V, \quad K^U = K^P + K^V, \quad \Phi^U = \Phi^P + \Phi^V,$$

the (\mathbb{F}, \mathbb{P}) BSDE (9.2) for (U, K^U, Φ^U) is equivalent (in both senses of local martingale and square integrable solutions) to the following (\mathbb{F}, \mathbb{P}) BSDE for (V, K^V, Φ^V) :

$$\begin{cases} \int_0^T |f_s(V_{s-}, K_s^V, \Phi_s^V)| ds < \infty, \\ V_t + \int_0^t f_s(V_{s-}, K_s^V, \Phi_s^V) ds = K^V \cdot B_t + \Phi^V * (\mu - \nu)_t, \\ V_T = 0. \end{cases} \quad (9.5)$$

Under the assumptions of the proposition, the (\mathbb{F}, \mathbb{P}) BSDE (9.5) for (V, K^V, Φ^V) satisfies the assumptions of Kruse and Popier (2016, Theorem 1).⁴ Hence it is well posed in

⁴Derived for a Poisson measure π there, but one can readily check that all their computations performed under square integrable assumptions are still valid in our more general integer valued random measure setup.

the (\mathbb{F}, \mathbb{P}) space of square integrable solutions. So is in turn the BSDE (9.2). Therefore, through Theorem 9.2, the (\mathbb{G}, \mathbb{Q}) BSDE (9.1) is well posed in the (\mathbb{G}, \mathbb{Q}) space of square integrable solutions. ■

Beyond the illustrative case of Proposition 9.1, the reader is referred to Albanese and Crépey (2017, Sections 7.3–7.4) for more developed applications, establishing, in the XVA context of Example 9.1, the well-posedness of a variety of XVA BSDEs stopped before the bank default θ . See further Crépey, Élie, Sabbagh, and Song (2017) for applications to the XVA anticipated BSDEs (ABSDEs in the line of Peng and Yang (2009)) that occur when the possibility for the bank to use capital as a funding source is accounted for.

It is also possible to establish Markov and infinitesimal generator transfer properties between (\mathbb{G}, \mathbb{Q}) and (\mathbb{F}, \mathbb{P}) , giving a general perspective on the observations made in specific setups in Crépey, Bielecki, and Brigo (2014, Chapter 12). But this is a topic of its own which we leave for further research.

A Intensity Based Pricing Formulas, Survival Measure and Invariance Times: Discussion in the DGC Setup

In this section, we put (3.2) in perspective with other similar formulas that appear in the literature. This is done in a Markov setup where the issues at stake can be understood based on Feynman-Kac representations.

Specifically, we consider the following single-name version of the dynamic Gaussian copula (DGC) model of Crépey et al. (2014, Chapter 7)⁵ and Crépey and Song (2017a). Let

$$\theta = h^{-1}\left(\int_0^{+\infty} \varsigma(u)dB_u\right), \quad (\text{A.1})$$

where h is a continuously differentiable increasing function from $(0, +\infty)$ to \mathbb{R} , ς is a Borel function on \mathbb{R}_+ such that $\int_0^{+\infty} \varsigma^2(u)du = 1$, and B is a standard Brownian motion. The reference filtration \mathbb{F} is taken as the augmented filtration of the natural filtration \mathbb{B} of B . The full model filtration \mathbb{G} is given as the augmented filtration of the progressive enlargement of $\mathbb{F} = \mathbb{B}$ by θ . Note that the \mathbb{G} stopping time θ is \mathcal{F}_∞ measurable.

By Theorem 2.2, Lemma 3.2, and Remark 4.1 in Crépey and Song (2017a), the condition (C) is satisfied in this setup. Let $h_t = \mathbb{1}_{\{\theta \leq t\}}$ and

$$m_t = \int_0^t \varsigma(u)dB_u, \quad k_t = (h_t, \theta \wedge t), \quad \nu^2(t) = \int_t^{+\infty} \varsigma^2(v)dv, \quad (\text{A.2})$$

with ν assumed positive for all t . By application of the results of Ethier and Kurtz (1986), one can show that the process (m, k) is (\mathbb{G}, \mathbb{Q}) Markov.

⁵or Crépey, Jeanblanc, and Wu (2013) in journal version.

Remark A.1 The reason why we introduce $(\theta \wedge t)$ on top of the indicator process h_t in k_t is because of a dependence of the post θ behavior of the model on the value of θ itself. The state augmentation by $(\theta \wedge t)$ takes care of this path-dependence. ■

By definition (A.1) of θ , we have

$$\mathbb{Q}(\theta > t | \mathcal{F}_t) = \Phi\left(\frac{h(t) - m_t}{\nu(t)}\right), \quad t \in \mathbb{R}_+, \quad (\text{A.3})$$

where Φ denotes the standard normal cdf. The process on the right hand side of (A.3) has infinite variation. This shows that the reference filtration $\mathbb{F} = \mathbb{B}$ is not immersed into the full model filtration \mathbb{G} . This lack of immersion makes it more interesting from the point of view of the different approaches that we want to compare. This is our motivation for working in this particular model in this section.

Theorems 2.2 and 2.4 in Crépey and Song (2017a) show the existence of processes of the form

$$\beta_t = \beta(t, m_t, k_t) \text{ and } \gamma_t = \gamma(t, m_t, k_t) = \gamma_t \mathbb{1}_{(0, \theta]}, \quad t \in \mathbb{R}_+, \quad (\text{A.4})$$

for continuous functions β and γ with linear growth in m , such that

$$\begin{aligned} dW_t = dB_t - \beta_t dt \text{ is a } (\mathbb{G}, \mathbb{Q}) \text{ Brownian motion and} \\ \gamma \text{ is the } (\mathbb{G}, \mathbb{Q}) \text{ intensity of } \theta. \end{aligned} \quad (\text{A.5})$$

Our discussion of (3.2) relies on the formula (A.7) below.

Proposition A.1 *Let a process m^\star satisfy*

$$dm_t^\star = \varsigma(t)(dW_t^\star + \beta(t, m_t^\star, (0, t))dt), \quad 0 \leq t \leq T, \quad (\text{A.6})$$

starting from $m_0^\star = 0$, for some Brownian motion W^\star with respect to some stochastic basis $(\mathbb{G}^\star, \mathbb{Q}^\star)$. Denoting the \mathbb{Q}^\star expectation by \mathbb{E}^\star , we have, for any positive constant T and bounded Borel function $G(t, m)$,

$$\mathbb{E}[\mathbb{1}_{\theta < T} G(\theta, m_\theta)] = \mathbb{E}^\star \left[\int_0^T e^{-\int_0^t \gamma(s, m_s^\star, (0, s)) ds} \gamma(t, m_t^\star, (0, t)) G(t, m_t^\star) dt \right]. \quad (\text{A.7})$$

Proof. By the (\mathbb{G}, \mathbb{Q}) Markov property of the process (m, k) , we have

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{\theta < T} G(\theta, m_\theta) | \mathcal{G}_t] &= \mathbb{E}[\mathbb{1}_{\theta < T} G(\theta, m_\theta) | (m_t, k_t)] = \\ &v(t, m_t, k_t) = u_{h_t}(t, m_t), \quad t \in [0, \theta \wedge T], \end{aligned} \quad (\text{A.8})$$

for suitable bounded Borel functions $v(t, m, k)$ and $u_h(t, m) = v(t, m, k = (h, t))$. As a martingale, the process $u_{h_t}(t, m_t)$, $t \in [0, \theta \wedge T]$, has a vanishing drift. Hence, by an application of the Itô formula to this process (assumed regular enough)⁶, using (A.5),

⁶Given the discussion format of this section, we content ourselves with a formal argument here, without introducing viscosity solutions.

the pair function $u = (u_0(t, m), u_1(t, m))$ formally solves

$$\begin{cases} u_0(T, m) = u_1(T, m) = 0, & m \in \mathbb{R} \\ u_1(t, m) = G(t, m), & t < T, m \in \mathbb{R} \\ \partial_t u_0(t, m) + \varsigma(t)\beta(t, m, (0, t))\partial_m u_0(t, m) + \frac{\varsigma(t)^2}{2}\partial_{m^2}^2 u_0(t, m) \\ \quad + \gamma(t, m, 0)[u_1(t, m) - u_0(t, m)] = 0, & t < T, m \in \mathbb{R}, \end{cases} \quad (\text{A.9})$$

which reduces to $u_1 = \mathbb{1}_{(0, T]}G$ and the following equation for u_0 :

$$\begin{cases} u_0(T, m) = 0, & m \in \mathbb{R}, \\ \partial_t u_0(t, m) + \varsigma(t)\beta(t, m, 0)\partial_m u_0(t, m) + \frac{\varsigma(t)^2}{2}\partial_{m^2}^2 u_0(t, m) \\ \quad + \gamma(t, m, 0)G(t, m) - \gamma(t, m, 0)u_0(t, m) = 0, & t < T, m \in \mathbb{R}. \end{cases} \quad (\text{A.10})$$

Putting together (A.8) and the Feynman-Kac representation of the solution u_0 of (A.10) at the origin yields

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{\theta < T}G(\theta, m_\theta)] &= u_0(0, 0) \\ &= \mathbb{E}^* \left[\int_0^T e^{-\int_0^t \gamma(s, m_s^*, (0, s)) ds} \gamma(t, m_t^*, (0, t)) G(t, m_t^*) dt \right], \end{aligned}$$

for any process m^* satisfying (A.6). ■

A contrario, we expect that

$$\mathbb{E}[\mathbb{1}_{\theta < T}G(\theta, m_\theta)] \neq \mathbb{E} \left[\int_0^T e^{-\int_0^t \gamma(s, m_s, (0, s)) ds} \gamma(t, m_t, (0, t)) G(t, m_t) dt \right] \quad (\text{A.11})$$

(except in special cases such as $G = 0$), because, from (A.2) and (A.4)–(A.5), it holds

$$dm_t = \varsigma(t)(dW_t + \beta(t, m_t, k_t)dt), \quad t \in \mathbb{R}_+, \quad (\text{A.12})$$

which, for $t \geq \theta$ so that $k_t = (1, \theta)$, diverges from the specification (A.6). In fact, let

$$V_t = \mathbb{E} \left[\int_t^T e^{-\int_t^s \gamma(u, m_u, (0, u)) du} \gamma(s, m_s, (0, s)) G(s, m_s) ds \mid \mathcal{G}_t \right], \quad t \in \mathbb{R}_+,$$

so that V_0 is equal to the right hand side in (A.11). By an application of Duffie et al. (1996, Proposition 1) with $X = r = 0$ and $h = \gamma(\cdot, m_\cdot, (0, \cdot))$ on $[0, T]$ there (noting that any process coinciding with the (\mathbb{G}, \mathbb{Q}) intensity of θ before θ is an eligible process h in their setup), we have

$$\mathbb{E}[\mathbb{1}_{\theta < T}G(\theta, m_\theta)] = V_0 - \mathbb{E}\Delta_\theta V. \quad (\text{A.13})$$

In a basic immersed setup, $\mathbb{E}\Delta_\theta V$ vanishes and equality holds in (A.11): see the comments before Section 3 in Duffie et al. (1996), page 1379 in Collin-Dufresne et al. (2004),

or following (3.22), (H.3) and Proposition 6.1 in Bielecki and Rutkowski (2001a)). But, in general, $\mathbb{E}\Delta_\theta V$ is nonnull and intractable.

By contrast, one specific instance of (A.6), which corresponds to the approach by Collin-Dufresne et al. (2004, Theorem 1), consists in using $m^* = m$ and $W^* = W$, taking for \mathbb{Q}^* the so-called survival measure⁷ with (\mathbb{G}, \mathbb{Q}) density process $e^{\int_0^\cdot \gamma(u, m_u, (0, u)) du} \mathbb{1}_{(0, \theta]}$. This fixes the discrepancy in (A.11) by singularly changing the probability measure, while sticking to the original model filtration \mathbb{G} (or, more precisely, resorting to the \mathbb{Q}^* augmentation \mathbb{G}^* of \mathbb{G} , obtained by adding to \mathbb{G} all the \mathbb{Q}^* null sets A such that $A \subseteq \{\theta \leq T\}$).

Another instance of (A.6) consists in using

$$m^* = m \text{ and } dW_t^* = dB_t - \beta(t, m_t, (0, t))dt.$$

As it follows from Lemma 3.5 and Section 4.4 in Crépey and Song (2017a), this process W^* is a $(\mathbb{G}^* = \mathbb{F} = \mathbb{B}, \mathbb{Q}^* = \mathbb{P})$ Brownian motion, for some probability measure \mathbb{P} , distinct from \mathbb{Q} but equivalent to it on \mathcal{F}_T , on which \mathbb{P} is uniquely determined through (2.6). The corresponding formula (A.7) is none other than our expectation formula (3.2). This approach fixes the discrepancy in (A.11) by reducing the filtration from \mathbb{G} to $\mathbb{F} = \mathbb{B}$, while changing the probability measure “as little as possible”, i.e. equivalently on \mathcal{F}_T (in a basic immersive setup, an invariance time approach does not change \mathbb{Q} at all, whereas Collin-Dufresne et al. (2004)’s measure change is still singular).

We emphasize that Collin-Dufresne et al. (2004)’s approach only provides a transfer of conditional expectation formulas (because of the singularity of their measure change), as opposed to a transfer of semimartingale calculus as a whole under the approach of this paper. See also the comments in Crépey and Song (2017b, Section 4.2) and Crépey and Song (2015, Section 3.3). From a more general and theoretical perspective, these different approaches can be related via the generalized Girsanov formulas of Kunita (1976) and Yoeurp (1985) (cf. Song (2013)).

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⁷The “survival measure” idea and terminology were first introduced and used for various purposes in Schönbucher (1999, 2004)).

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