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Counterparty Risk and Funding: Immersion and Beyond

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Abstract In Crépey (2015, Part II), a basic reduced-form counterparty risk modeling approach was introduced under a rather standard immersion hypothesis between a reference filtration and the filtration progressively enlarged by the default times of the two parties, also involving the continuity of some of the data at default time. This basic approach is too restrictive for application to credit derivatives, which are characterized by strong wrong-way risk, i.e. adverse dependence between the exposure and the credit riskiness of the counterparties, and gap risk, i.e. slippage between the portfolio and its collateral during the so-called cure period that separates default from liquidation. This paper shows how a suitable extension of the basic approach can be devised so that it can be applied in dynamic copula models of counterparty risk on credit derivatives. More generally, this extended approach is applicable in any marked default times intensity setup satisfying a suitable integrability condition. The integrability condition expresses that no mass is lost in a related measure change.

Keywords: Counterparty risk, funding, BSDE, reduced-form credit modeling, immersion, wrong-way risk, gap risk, collateral, credit derivatives, marked default times, Gaussian copula, Marshall-Olkin copula, dynamic copula.

Mathematics Subject Classification: 91G40, 60H10, 60G07.

1 Introduction

Counterparty risk is the risk of default of a party in an OTC derivative transaction, a topical issue since the global financial crisis. As a significant part of the market is moving to central counterparties (also-called clearing houses), nowadays the problem must be analysed in two different setups, bilateral versus centrally cleared. In this paper, we focus on the bilateral case. See Brigo, Morini, and Pallavicini (2013) and Crépey and an Introductory Dialogue by D. Brigo) (2014), respectively in a more financial and mathematical perspective, for recent bilateral counterparty risk references in book form, and see Armenti and Crépey (2015) for the case of centrally cleared trading. The case of centrally cleared trading is also considered in Brigo and Pallavicini (2014), but from the point of view of a client, as opposed to a member, of a clearing house. Then the analysis of the present paper applies as well, as explained in the remark 52. With respect to Brigo and Pallavicini (2014), the present paper is more mathematical and yields a special focus on credit derivatives.

To mitigate counterparty risk, a margining procedure is set up according to what is called in the context of bilateral trading a master agreement between the two parties, also referred to as a CSA, for credit support annex. However, accounting for various frictions and delays, notably the so-

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called cure period that separates default and liquidation, there is gap risk, i.e. risk of slippage between the portfolio and its collateral. This is why another layer of collateralization, called initial margins as opposed to the variation margin that only accounts for market risk, is now maintained in both centrally cleared transactions and bilateral transactions under a sCSA (standard CSA). Gap risk is magnified in the presence of wrong-way risk, i.e. adverse dependence between the underlying exposure and the credit risk of the counterparties. This is a special case of concern regarding counterparty risk on credit derivatives, given the default contagion and frailty effects between the two parties and the underlying credit names. In fact, to properly deal with counterparty risk embedded in credit derivatives, one needs a credit portfolio model with the following features. First, the model should be calibratable to relevant data sets: CDS data if the targeted application consists of counterparty risk computations on CDS contracts and, additionally, tranches data for computations involving CDO contracts. In particular, one needs a bottom-up model of portfolio credit risk, with efficient pricing schemes for vanillas (CDS contracts and/or CDO tranches) as well as a copula separation property between the individual and the dependence model parameters. Second, as counterparty risk and funding valuation adjustments price options on future values of the underlyings, a dynamic model is required.

One possibility is to use dynamic copula models resulting from the introduction of a suitable filtration on top of a static copula model for the default times of the two parties and of underlying credit names. The dynamic Gaussian copula (DGC) model of Crépey, Jeanblanc, and Wu (2013) can suffice to deal with counterparty risk on CDS contracts. If there are also CDO tranches in the portfolio, then a Gaussian copula dependence structure is not rich enough for calibration purposes. Instead, one can use the dynamic Marshall-Olkin (DMO) common-shock model of Bielecki, Cousin, Crépey, and Herbertsson (2014b,2014a).

However, this dynamic copula methodology, reviewed in Crépey and an Introductory Dialogue by D. Brigo) (2014, Part IV), does not immediately extend to the case of bilateral counterparty risk combined with the related funding issue. As is well known since the seminal papers by Korn (1995), Cvitanic and Karatzas (1993) or El Karoui, Peng, and Quenez (1997), in presence of different borrowing and lending rates, pricing rules become nonlinear. This question has known a revival of interest in recent years in connection with the post-crisis multi-curve issue (see Piterbarg (2010), Mercurio (2014) and Bielecki and Rutkowski (2015)). Accordingly, the pricing equations for the corresponding valuation adjustment, dubbed TVA for total valuation adjustment (inclusive of counterparty risk and funding costs), become nonlinear (see Crépey (2015, Part I), Brigo and Pallavicini (2014) or Bichuch, Capponi, and Sturm (2015)). Moreover, they are posed over random time intervals and may involve nonstandard, implicit terminal conditions at the first default time of a party. To deal with such equations, a first reduced-form counterparty risk modeling approach was introduced in Crépey (2015, Part II), in a rather basic immersion setup between the reference filtration of the underlying market exposure and the full model filtration progressively enlarged by the default times of the two parties. But this basic immersion setup, with a related continuity assumption on some of the data at the first default time of the two parties, is too restrictive for wrong-way and gap risk applications such as counterparty risk on credit derivatives, in which case one also faces specific dependence and dimensionality challenges.

1.1 Contributions and Outline

To tackle this issue, this paper shows how an extended reduced-form approach can be applied, beyond the first approach of Crépey (2015, Part II) dubbed “basic approach” henceforth, in the above-mentioned dynamic copula models of portfolio credit risk. In the first part of the paper (Sect. 2 through 6), we generalize, resorting to the notion of invariance time in Crépey and Song (2015a), the basic reduced-form approach of Crépey (2015, Part II). With respect to our previous works, we also introduce a positive cure period (also considered in Brigo and Pallavicini (2014)). The TVA is modeled in terms of solutions to backward stochastic differential equations (BSDEs): the exact BSDE (3.5), the full BSDE (3.6) and the reduced BSDE (1). The exact BSDE (3.5) is derived from risk-neutral valuation and hedging principles. It is then approximated by the full BSDE (3.6), which can also be viewed as the exact BSDE for slightly simplified data. The reduced BSDE is an auxiliary, simpler equation that, if solved, yields a solution to the full BSDE. In this sense, solving the reduced BSDE is sufficient in practice. This is Theorem 1, which can also be digged out from the results of Crépey and Song (2014a), but in a more abstract setup there, mainly motivated by the converse to this theorem, i.e. any solution to the full BSDE is based on a solution to the reduced BSDE. Beyond self-containedness, the main reason why we include a direct proof of Theorem 1 in this paper is to show how easily it flows once the right framework has been set up, namely the condition (C) in this paper. In the easiness of the result lies its power here. However, this raises the issues of the strength and practicality of the condition (C). To demonstrate the viability of the approach, the second part of this paper (Sect. 7 through 9) implements it through marked default times in dynamic extensions of two well known copula models: the Gaussian copula and the exponential (or Marshall Olkin) copula.

The detailed outline of the paper is as follows. In Sect. 2 we present the bilateral counterparty risk and funding setup. In Sect. 3 we derive the exact and full TVA BSDEs with respect to the full model filtration $\mathbb{G}$. Sect. 4 develops an extended reduced-form approach for the full BSDE, which results in the reduced BSDE, applicable whenever the first default time of the two parties satisfies the condition (C) in this paper (i.e., essentially, is an invariance time satisfying the condition (A) in Crépey and Song (2015a), where the condition (A) means (C) but also (B) in the present paper). In Sect. 5 we establish the well-posedness of the reduced BSDE under a standard CSA specification of the data. In the marked default times framework of Sect. 6, we derive a CVA/DVA (credit/debit valuation adjustment) and LVA (funding liquidity valuation adjustment) decomposition of an all-inclusive TVA. Sect. 7 and Sect. 8 apply the proposed approach in the DGC and DMO models of counterparty risk on credit derivatives. In Sect. 9 numerical results are presented to illustrate the respective wrong-way and gap risk flavor of these models.

1.2 Standing Notation and Terminology

We write $\int_a^b = \int_{(a,b]}$ with, in particular, $\int_a^b = 0$ whenever $a \geq b$; $x^+ = \max(x, 0)$, $x^- = \max(-x, 0) = (-x)^+$. Any function involving discrete arguments is viewed as continuous with respect to these, in reference to the discrete topology. We denote by $\boldsymbol{\lambda}$ the Lebesgue measure on $\mathbb{R}_+$, by $\mathcal{B}(\mathbb{S})$ the Borel σ field on a topological space $\mathbb{S}$, by $\mathcal{P}(\mathbb{F})$, $\mathcal{O}(\mathbb{F})$ and $\mathcal{R}(\mathbb{F})$ the predictable, optional and progressive σ fields with respect to a filtration $\mathbb{F}$. When a process f_t can be represented in terms of a function of some factor process X, we typically write $f(t, X_t)$, i.e. the function is denoted by the same letter as the process. Order relationships between random variables (resp. processes) are meant almost surely (resp. in the indistinguishable sense).

Throughout the paper we consider an OTC derivative (or a netted portfolio of OTC derivatives) traded between two defaultable counterparties, generically referred to as the "contract between the bank and its counterparty".

Part I

Wrong Way and Gap Risks Modeling: A Marked Default Times Perspective

Here is a non-exhaustive list of notations introduced in the course of the first part of the paper.

CSA, CVA, DVA, LVA, TVA Credit support annex, Credit valuation adjustment, debit valuation adjustment, liquidity funding valuation adjustment, total valuation adjustment.

$\mathbb{G}, \mathbb{F}$ Full model filtration (including the information related to the default of the two counterparties), market reference filtration.

$\mathbb{E}, \widetilde{\mathbb{E}}$ Expectations under the probability measures $\mathbb{Q}$ and $\mathbb{P}$.

$V, I^c \geq 0, I^b \leq 0, C^c = V + I^c, C^b = V + I^b, C = V + I^c + I^b$ Variation margin (counted positively when posted by the counterparty and negatively when posted by the bank), initial margin posted by the counterparty, negative of the initial margin posted by the bank, total collateral guarantee for the bank, negative of the total collateral guarantee for the counterparty, negative of the collateral funded by the bank.

$R_c, R_b, \Lambda \in [0, 1]$ Recovery rate of the counterparty to the bank, of the bank to the counterparty, one minus the recovery rate of the bank to its funder.

T, δ Time horizon of the CSA, length of the cure (or liquidation) period.

$\tau_c, \tau_b, \tau, \bar{\tau}, \tau^\delta, \bar{\tau}^\delta$ Default time of the counterparty, default time of the bank, first default time of the two parties, $\tau \wedge T$, $\tau + \delta$, time horizon $\bar{\tau}^\delta = \mathbb{1}_{\tau<T}\tau^\delta + \mathbb{1}_{\{\tau \geq T\}}T$ of the TVA pricing problem.

$\gamma^c, \gamma^b, \gamma$ Intensities of τ_c, τ_b and τ, hence $\max(\gamma^b, \gamma^c) \le \gamma \le \gamma^b + \gamma^c$, with indistinguishable equality in the right hand side if $\tau_b \neq \tau_c$ a.s..

J, S $\mathbb{1}_{[0,\tau)}$, Azéma supermartingale of τ

U' $\mathbb{F}$ predictable reduction of a $\mathbb{G}$ predictable process U

$r, r+c, r+\lambda, r+\bar{\lambda}, \widetilde{\lambda} = \bar{\lambda} - \gamma^b \Lambda$ OIS (risk-free) rate, rate of remuneration of the posted collateral, investing rate of the bank, unsecured funding rate of the bank, liquidity funding spread of the bank.

$P, \Delta = \int_{[\tau,\cdot]} e^{\int_s^\cdot r_u du}, Q, \Pi, \Theta = Q - \Pi$ Reference or "clean" price process of the contract from the bank's perspective (mark-to-market ignoring counterparty risk and assuming a risk-free funding rate), cumulative contractual dividends capitalized at the risk-free rate that fail to be paid from time τ onwards, $P + \Delta$, all-inclusive price process of the contract (inclusive of counterparty risk and funding costs, as opposed to the clean price P), TVA process.

$\varepsilon_c = (Q_{\tau^\delta} - C^c_\tau)^+,\ \varepsilon_b = (Q_{\tau^\delta} - C^b_\tau)^-$ Liquidation debts of the counterparty to the bank, of the bank to the counterparty

$\chi, \xi = Q_{\tau^\delta} - \chi = \mathbb{1}_{\{\tau_c \le \tau_b^\delta\}}(1 - R_c)\varepsilon_c - \mathbb{1}_{\{\tau_b \le \tau_c^\delta\}}(1 - R_b)\varepsilon_b$ Close-out cashflow of the bank, counterparty risk exposure of the bank.

$\mathcal{W}, (-\mathcal{W}_{\tau_b -} - C_{\tau_b -})^+$ Value process of the hedging, collateralization and funding portfolio of the bank, debt of the bank to its funder right before τ_b.

$g_t(\vartheta), f_t(\vartheta) = g_t(Q_t - \vartheta) - r_t\vartheta, \widetilde{f}_t(\vartheta)$ Funding coefficient such that $(-r_t\mathcal{W}_t + g_t(-\mathcal{W}_t))dt$ represents the bank's funding cost over $(t, t+dt)$ (hence $g = 0$ corresponds to linear funding at the OIS risk-free rate r), coefficient of the exact and full BSDEs, coefficient of the reduced BSDE.

$\tau_e, \gamma^e; E; E_b, E_c$ Stopping time with mark e and intensity γ^e; finite set of marks such that $\tau = \min_{e\in E} \tau_e$; subsets of E such that $\tau_b = \min_{e\in E_b} \tau_e, \tau_c = \min_{e\in E_c} \tau_e$.

2 Counterparty Risk and Funding Setup

In this section we present all the cashflows involved in the bilateral counterparty risk and funding problem, adopting the perspective of the bank. In particular, a cashflow of ± 1 means ± 1 to the bank. We assume that the bank, having obtained ("bought") the contract from its counterparty at time 0 in exchange of some premium Π_0, sets-up a collateralization, hedging and funding portfolio. Collateral consists of cash or various possible eligible securities posted through margin calls as default guarantee by the two parties. The margin requirements are specified by the CSA. Regarding hedging, for simplicity, we restrict ourselves to a securely funded hedge, entirely implemented by means of swaps, short sales or repurchase agreements, at no upfront payment. As explained in Crépey and an Introductory Dialogue by D. Brigo) (2014, Section 4.2.1 page 87)[1], this assumption encompasses the vast majority of hedges that are used in practice. In particular, it includes (counterparty-risk-free) CDS contracts that can be used for hedging the counterparty jump-to-default exposure. We call funder of the bank a third party insuring funding of the bank's strategy. Assumed default-free for simplicity, it plays the role of lender/borrower after exhaustion of the implicit sources of funding provided to the bank through its hedge and its collateral. Typically, the funder is the treasury of the bank. Alternatively, the bank can get some funding directly in the market, in various ways (in practice, the funder can be composed of several entities or devices). See Pallavicini, Perini, and Brigo (2012, Sect. 4.2) for a detailed description of different funding policies.

Let $(\Omega, \mathbb{G}, \mathbb{Q})$, with $\mathbb{G} = (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ satisfying the usual conditions and $\mathbb{Q}$ expectation denoted by $\mathbb{E}$, represent a risk-neutral pricing stochastic basis, such that all our processes are $\mathbb{G}$ adapted and all the random times of interest are $\mathbb{G}$ stopping times. The meaning of a risk-neutral pricing measure in our setup, with different funding rates in particular, is specified by a martingale condition that will be introduced in the form of the all-inclusive price BSDE (3.1) for the contract. But, in the first place, a pricing measure in our sense must be such that the gain processes related to the trading of the hedging assets, processes denoted in vector form by $\mathcal{M}$, are local martingales. As explained in the comments following Assumption 4.4.1 in Crépey and an Introductory Dialogue by D. Brigo) (2014, page 96)[2], this rules out arbitrage opportunities in the market of hedging instruments (provided one restricts attention to hedging strategies resulting in a wealth process bounded from below; see Bielecki and Rutkowski (2015, Corollary 3.1) for a formal statement).

[1] Or Crépey (2015, Part I, Section 2.1) in the journal version.

[2] Or Crépey (2015, Part I, Assumption 4.1) in the journal version.

For reasons pertaining to division and specialisation of tasks in banks, the all-inclusive price Π of the contract is obtained as the difference between a reference or "clean" price P and a counterparty risk and funding adjustment (TVA) Θ (essentially; this will be formalized in detail in Definition 32). The clean price P is the mark-to-market ignoring counterparty risk and assuming a risk-free funding rate. Specifically, we denote by r_t a progressively measurable and Lebesgue integrable OIS rate process and by $\beta_t = e^{-\int_0^t r_s ds}$ the corresponding discount factor. The OIS rate, where OIS stands for overnight indexed swap, is together the best market proxy of a risk-free rate and the typical reference rate for the remuneration of the collateral. Let a finite variation process D represent the cumulative promised dividend process of the contract (contractual cashflows ignoring counterparty risk). The formula for P reads

$$\beta_t P_t = \mathbb{E}\left(\int_t^T \beta_s dD_s \,\Big|\, \mathcal{G}_t\right), \quad t \in [0, T]. \tag{2.1}$$

Here T is a relevant time horizon, typically the one of the CSA. If there is some residual value in the contract at that time, it is treated as a dividend $(D_T - D_{T-})$ at time T.

But the two parties are defaultable. Let τ_b and τ_c stand for the default times of the bank and of the counterparty, modeled as $\mathbb{G}$ stopping times with $(\mathbb{G}, \mathbb{Q})$ (predictable) intensities γ^b and γ^c. As a consequence, the first default time of the two parties, $\tau = \tau_b \wedge \tau_c$, is a stopping time with intensity γ such that $\max(\gamma^b, \gamma^c) \le \gamma \le \gamma^b + \gamma^c$, with indistinguishable equality in the right hand side if $\tau_b \neq \tau_c$ a.s.. Note that in such an intensity setup, any event $\{\tau_b = t\}$ or $\{\tau_c = t\}$, for any fixed time t, has zero probability and can be ignored in the analysis. An additional feature is a time lag $\delta \ge 0$, called the cure period, typically taken as ten (resp. five) days in the case of bilateral (resp. centrally cleared) transactions, between the first default time τ of the two parties and the liquidation of the contract. For any time t, we write

$$\bar{t} = t \wedge T, \;\; t^\delta = t + \delta, \;\; \bar{t}^\delta = \mathbb{1}_{t<T} t^\delta + \mathbb{1}_{\{t \ge T\}} T.$$

If $\tau < T$, the contractual dividends dD_t cease to be paid from time τ onwards and a close-out cashflow χ paid to the bank at time τ^δ closes out its position. Hence, the liquidation procedure results in an effective time horizon $\bar{\tau}^\delta$ of the pricing problem.

Until $\bar{\tau}$, the bank needs to fund its position, i.e. the contract and its collateral (the cost of funding the hedge is already accounted for in the hedging assets gain martingale $\mathcal{M}$). We denote by $g = g_t(\pi)$ an $\mathcal{R}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R})$ measurable funding coefficient such that

$$\big(-r_t \mathcal{W}_t + g_t(-\mathcal{W}_t)\big) dt \tag{2.2}$$

represents the bank's funding cost over $(t, t + dt)$, where $\mathcal{W}$ is the value process of the hedging, collateralization and funding portfolio of the bank. In addition, the bank may receive a funding windfall benefit at its own default, modeled as a cashflow

$$(-\mathcal{W}_{\tau_b -} - C_{\tau_b -})^+ \Lambda \text{ at } \tau_b \text{ if } \tau_b < \bar{\tau}^\delta. \tag{2.3}$$

Here the process $(-C)$ represents the amount of collateral funded by the bank, so that $(-\mathcal{W}_{\tau_b-} - C_{\tau_b-})^+$ represents the debt of the bank to its funder right before τ_b, and $\Lambda \in [0, 1]$ corresponds to the fractional loss of the funder in case of default of the bank (one minus the recovery rate of the bank to its funder). In Sect. 5 we will provide a typical specification of all the data, in particular χ, g and C.

2.1 Dynamics of the Wealth Process of the Bank

After having bought the contract from the counterparty at time 0 in exchange of some premium Π_0, the bank sets up a hedge $(-\zeta)$, which is a left-continuous row-vector process of the same dimension as $\mathcal{M}$. The "short" negative sign notation in $(-\zeta)$ is used for consistency with the idea, just to fix the mindset, that the contract is "bought" by the bank at time 0. Let

$$J = \mathbb{1}_{[0,\tau)}, \;\; \tau^\star = \tau_b \wedge \tau_c^\delta, \;\; \bar{\tau}^\star = \mathbb{1}_{\tau<T} \tau^\star + \mathbb{1}_{\{\tau \ge T\}} T, \;\; J^\star = \mathbb{1}_{[0,\tau^\star)}. \tag{2.4}$$

The collateralization, hedging and funding portfolio of the bank, with wealth process $\mathcal{W}$ (depending on Π_0 and ζ), is supposed to be held by the bank itself before $\bar{\tau}^\star$ and, if $\tau_b \le \tau_c^\delta$ (i.e. $\tau^\star = \tau_b$) and $\tau < T$, taken over by a risk-free liquidator on $[\bar{\tau}^\star, \bar{\tau}^\delta]$.

Lemma 21 *Ignoring the close-out cashflow χ at $\bar{\tau}^{\delta}$ if $\tau < T$, which will be added separately later (see Lemma 31), we have $\mathcal{W}_0 = -\Pi_0$ and, for $0 < t \le \bar{\tau}^{\delta}$,*

$$d\mathcal{W}_t = r_t \mathcal{W}_t dt + J_t dD_t - J^{\star}_t g_t(-\mathcal{W}_t)dt - (-\mathcal{W}_{\tau_b-} - C_{\tau_b-})^+ \Lambda \mathbb{1}_{\{\tau^\star = \tau_b < T\}} dJ^{\star}_t - \zeta_t d\mathcal{M}_t. \quad (2.5)$$

Proof. Collecting all terms in the above-described collateralization, hedging and funding scheme, we obtain $\mathcal{W}_0 = -\Pi_0$ and, for $0 < t \le \bar{\tau}^{\delta}$:

$$\begin{aligned} d\mathcal{W}_t = &\underbrace{J_t dD_t}_{\text{bank gets dividends}} - \underbrace{J^{\star}_{t-}\zeta_t d\mathcal{M}_t}_{\text{bank pays on its hedge}} \\ &+ \underbrace{J^{\star}_t (r_t \mathcal{W}_t - g_t(-\mathcal{W}_t))dt}_{\text{funding benefits / costs to bank}} \\ &- \underbrace{(-\mathcal{W}_{\tau_b-} - C_{\tau_b-})^+ \Lambda \mathbb{1}_{\{\tau^\star = \tau_b < T\}} dJ^{\star}_t}_{\text{windfall funding benefit of the bank at its own default time } \tau_b \text{ (if } \tau_b \le \tau_c^{\delta})} \\ &- \underbrace{(1 - J^{\star}_{t-})\zeta_t d\mathcal{M}_t}_{\text{liquidator pays on the hedge of the bank during the cure period}} \\ &+ \underbrace{(1 - J^{\star}_t) r_t \mathcal{W}_t dt,}_{\text{risk-free funding benefits/costs of the liquidator during the cure period}} \end{aligned}$$

which yields (2.5). ∎

Remark 21 Our assumption of a securely funded hedge is reflected by the fact that the hedge ζ doesn't enter the g and $(-\mathcal{W}_{\tau_b-} - C_{\tau_b-})^+$ terms in (2.5). This can be compared with Crépey and an Introductory Dialogue by D. Brigo) (2014, Example 4.4.3 page 97)[3], where a more general funding policy for the hedge is considered.

3 Derivation of the Exact and Full TVA BSDEs

In this section we derive the exact and full TVA BSDEs with respect to the full model filtration $\mathbb{G}$. Our starting point is the notion of all-inclusive price Π, which is the value of the contract inclusive of counterparty risk and funding costs (as opposed to the clean price P). The justification for the following definition is provided by Lemma 31.

Definition 31 An all-inclusive price of the contract for the bank is a $(\mathbb{G}, \mathbb{Q})$ semimartingale Π that satisfies the following price BSDE on $[0, \bar{\tau}^{\delta}]$:

$$\begin{aligned} &\Pi_{\bar{\tau}^{\delta}} = \mathbb{1}_{\{\tau < T\}} \chi, \\ &d\nu_t := d\Pi_t - r_t \Pi_t dt + (\Pi_{\tau_b-} - C_{\tau_b-})^+ \Lambda \mathbb{1}_{\{\tau^\star = \tau_b < T\}} dJ^{\star}_t + J_t dD_t - J^{\star}_t g_t(\Pi_t) dt \qquad (3.1) \\ &\text{defines a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale on } [0, \bar{\tau}^{\delta}]. \end{aligned}$$

Lemma 31 *If an all-inclusive price Π can be found with $d\nu_t = \zeta_t d\mathcal{M}_t$ for some hedge ζ, assuming $g_t(\pi)$ Lipschitz in π with a Lipschitz constant that is Lebesgue integrable on $[0, T]$ (ω-wise), then (Π_0, ζ) yields an exact replication price and hedge for the bank, i.e. the resulting wealth process of the bank satisfies $\mathcal{W} = -\Pi$ on $[0, \bar{\tau}^{\delta}]$. In particular, we have*

$$\mathcal{W}_{\bar{\tau}^{\delta}} = -\Pi_{\bar{\tau}^{\delta}} = -\mathbb{1}_{\{\tau < T\}} \chi,$$

so that after the close-out cashflow $\mathbb{1}_{\{\tau < T\}}\chi$ has been paid to the bank (or its liquidator if $\tau < T$ and $\tau_b \le \tau_c^{\delta}$) at time $\bar{\tau}^{\delta}$, the bank's position is closed break-even.

[3] Or the equation (3.4) in the journal version Crépey (2015, Part I).

Proof. Under the assumptions of the lemma, the process $Z = \beta\Pi + \beta\mathcal{W}$ satisfies $Z_0 = 0$ and $dZ_t = \alpha_t Z_t dt$ on $[0, \bar{\tau})$, where $\alpha_t := \mathbb{1}_{\{\Pi_t \neq \mathcal{W}_t\}} \frac{g_t(\Pi_t) - g_t(-\mathcal{W}_t)}{\Pi_t - (-\mathcal{W}_t)}$ is Lebesgue integrable over $[0, T]$. Hence,

$$d\big(e^{-\int_0^t \alpha_s ds} Z_t\big) = e^{-\int_0^t \alpha_s ds}\big(dZ_t - \alpha_t Z_t dt\big) = 0,$$

i.e. $e^{-\int_0^t \alpha_s ds} Z_t$ is constant on $[0, \bar{\tau})$, equal to 0 in view of the initial condition for Z, i.e. $\mathcal{W} = -\Pi$ holds on $[0, \bar{\tau})$. This is followed by a jump of the two processes $\mathcal{W}$ and $(-\Pi)$ by the same amount

$$(-\mathcal{W}_{\tau_b-} - C_{\tau_b-})^+ \Lambda = (\Pi_{\tau_b-} - C_{\tau_b-})^+ \Lambda$$

at $\tau_b = \tau^\star = \bar{\tau}^\star$ if $\tau < T$ and $\tau_b \leq \tau_c^\delta$, after which $\mathcal{W}$ and $(-\Pi)$ coincide again on $[\bar{\tau}^\star, \bar{\tau}^\delta]$ by the same argument as above. Hence, $\mathcal{W} = -\Pi$ holds on $[0, \bar{\tau}^\delta]$. ∎

More broadly, if an all-inclusive price can be found with $d\nu_t = \zeta_t d\mathcal{M}_t + d\varepsilon_t$ for some hedge ζ and a "small" cost martingale ε, then the hedging error $\rho = \mathcal{W} + \Pi$, which starts from 0 at time 0, remains "small" all the way through. In particular,

$$\mathcal{W}_{\bar{\tau}^\delta} \approx -\Pi_{\bar{\tau}^\delta} = -\mathbb{1}_{\{\tau<T\}}\chi,$$

so that after the close-out cashflow $\mathbb{1}_{\{\tau<T\}}\chi$ at $\bar{\tau}^\delta$, the bank's position is closed with a "small" hedging error.

Let

$$Q_t = P_t + \Delta_t, \text{ where } \beta_t\Delta_t = \int_{[\tau,t]} \beta_s dD_s \tag{3.2}$$

(in particular, $\Delta_t = 0$ and $Q_t = P_t$ for $t < \tau$). In words, Δ_t represents the cumulative contractual dividends capitalized at the risk-free rate that fail to be paid by the counterparty to the bank from time τ onwards, so that Δ_t belongs to Q_t, the debt (if positive and before consideration of the collateral) of the counterparty to the bank at time $t \geq \tau$.

Definition 32 *Given an all-inclusive price Π, the corresponding TVA is the process defined on $[0, \bar{\tau}^\delta]$ as $\Theta = Q - \Pi$.*

3.1 Exact TVA BSDE

Let

$$f_t(\vartheta) = g_t(Q_t - \vartheta) - r_t\vartheta \quad (\vartheta \in \mathbb{R})$$

and

$$\xi = Q_{\tau^\delta} - \chi, \;\; \bar{\xi}_t = \mathbb{E}(\beta_t^{-1}\beta_{\tau^\delta}\xi \,|\, \mathcal{G}_t) \;(t \leq \bar{\tau}^\delta), \tag{3.3}$$

assuming integrability of the so-called counterparty risk exposure ξ.

Lemma 32 *Let there be given $\mathbb{G}$ semimartingales Π and Θ such that $\Theta = Q - \Pi$ on $[0, \bar{\tau}^\delta]$. The process Π is an all-inclusive price of the contract for the bank if and only if the process Θ satisfies the following $(\mathbb{G}, \mathbb{Q})$ TVA BSDE on $[0, \bar{\tau}^\delta]$:*

$$\begin{aligned}
&\Theta_{\bar{\tau}^\delta} = \mathbb{1}_{\{\tau<T\}}\xi, \\
&d\mu_t = d\Theta_t - r_t\Theta_t dt + J_t^\star g_t(Q_t - \Theta_t)dt + (Q_{\tau_b-} - \Theta_{\tau_b-} - C_{\tau_b-})^+ \Lambda \mathbb{1}_{\{\tau^\star=\tau_b<T\}} dJ_t^\star
\end{aligned} \tag{3.4}$$

defines a $(\mathbb{G}, \mathbb{Q})$ local martingale on $[0, \bar{\tau}^\delta]$.

Proof. Assuming Θ defined in terms of an all-inclusive price Π as $(Q-\Pi)$ on $[0,\bar{\tau}^{\delta}]$, the terminal condition for Θ in (3.4) follows, by definition of the exposure ξ in the left hand side of (3.3), from the terminal condition for Π in (3.1). Moreover, for $t\in[0,\bar{\tau}^{\delta}]$, we have

$$-\beta_t\Theta_t=-\beta_tQ_t+\beta_t\Pi_t=-(\beta_tP_t+\int_0^t\beta_sdD_s)+(\beta_t\Pi_t+\int_0^t\beta_sJ_sdD_s).$$

Hence,

$$\begin{aligned}&-\beta_t\Theta_t-\int_0^t\beta_sJ_s^{\star}g_s(Q_s-\Theta_s)ds+\int_0^t\beta_{\tau_b}(Q_{\tau_b-}-\Theta_{\tau_b-}-C_{\tau_b-})^{+}\Lambda\mathbb{1}_{\{\tau^{\star}=\tau_b<T\}}dJ_s^{\star}\\&=-\Big(\beta_tP_t+\int_0^t\beta_sdD_s\Big)+\beta_0\Pi_0+\\&\quad\int_0^t\Big(d(\beta_s\Pi_s)+\beta_{\tau_b}(\Pi_{\tau_b-}-C_{\tau_b-})^{+}\Lambda\mathbb{1}_{\{\tau^{\star}=\tau_b<T\}}dJ_s^{\star}+\beta_sJ_sdD_s-\beta_sJ_s^{\star}g_s(\Pi_s)ds\Big)\\&=-\Big(\beta_tP_t+\int_0^t\beta_sdD_s\Big)+\beta_0\Pi_0+\int_0^t\beta_sd\nu_s\end{aligned}$$

(cf. (3.1)). Since $(\beta P+\int_0^{\cdot}\beta_sdD_s)$ (cf. (2.1)) and ν are $(\mathbb{G},\mathbb{Q})$ local martingales, this establishes the martingale condition in (3.4). Hence, (3.1) implies (3.4). The converse implication is proven similarly. ∎

As reflected in (3.4), the TVA pricing problem, as the all-inclusive pricing problem (3.1), is originally posed over the domain $[0,\bar{\tau}^{\delta}]$, which corresponds to the union of the three subdomains in Figure 1, with respective dividend and funding data abbreviated as (D,g), $(0,g)$ and $(0,0)$. But since data $(0,0)$ simply means, in terms of pricing, taking conditional expectation of the terminal condition discounted at the risk-free rate, the proposition that follows shows that the BSDE (3.4) can be reformulated as the BSDE (3.5) on the smaller time interval $[0,\bar{\tau}^{\star}]\subseteq[0,\bar{\tau}^{\delta}]$, modulo a modified terminal condition $\bar{\xi}$ at $\bar{\tau}^{\star}$ instead of ξ at $\bar{\tau}^{\delta}$.

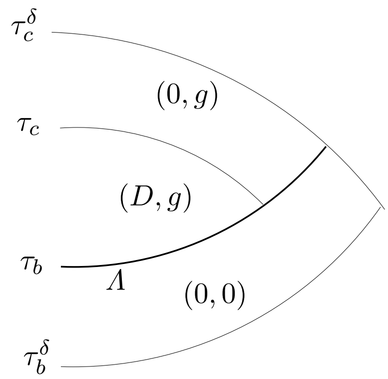

Fig. 1 Representation of the data of the TVA pricing problem in a (t,ω) state space representation, focusing on the default and liquidation times and ignoring T to alleviate the picture, i.e. "for $T=\infty$". The data in parentheses refer to the effective dividends and funding costs of the bank depending on the time t and scenario ω.

Proposition 31 *Let there be given $\mathbb{G}$ semimartingales Π and Θ such that $\Theta=Q-\Pi$ on $[0,\bar{\tau}^{\delta}]$. The process Π is an all-inclusive price of the contract for the bank if and only if $\Theta=\mathbb{1}_{\{\tau<T\}}\bar{\xi}$ on $(\bar{\tau}^{\star},\bar{\tau}^{\delta}]$ and Θ satisfies the following $(\mathbb{G},\mathbb{Q})$ "exact TVA BSDE" on $[0,\bar{\tau}^{\star}]$:*

$$\begin{aligned}&\Theta_{\bar{\tau}^{\star}}=\mathbb{1}_{\{\tau<T\}}\left(\bar{\xi}_{\tau^{\star}}-(Q_{\tau_b-}-C_{\tau_b-}-\Theta_{\tau_b-})^{+}\mathbb{1}_{\{\tau^{\star}=\tau_b\}}\Lambda\right)\ \textit{and}\\&d\mu_t:=d\Theta_t+f_t(\Theta_t)dt\ \textit{is a}\ (\mathbb{G},\mathbb{Q})\ \textit{local martingale on}\ [0,\bar{\tau}^{\star}].\end{aligned}\qquad(3.5)$$

Proof. As explained before the proposition, (3.4) is equivalent to $\Theta=\mathbb{1}_{\{\tau<T\}}\bar{\xi}$ on $[\bar{\tau},\bar{\tau}^{\delta}]$ and (3.5) on $[0,\bar{\tau}]$. ∎

Remark 31 This equivalence holds up to the value of Θ at $\tau^\star$ if $\tau^\star = \tau_b < T$, with a windfall funding benefit at default $(Q_{\tau_b-} - C_{\tau_b-} - \Theta_{\tau_b-})^+ \Lambda \mathbb{1}_{\{\tau=\tau_b<T\}}$ considered as a dividend at the intermediate time τ_b in (3.4) and as part of the valuation adjustment at the terminal time $\tau_b = \tau^\star$ in (3.5). But this difference is immaterial in practice for the bank, which only risk manages Θ before τ_b.

3.2 Full TVA BSDE

The risk-neutral pricing approach that underlies the TVA BSDE (3.5) (or (3.6) for slightly modified data below) implies that the bank actively risk manages its position. Otherwise, these equations are not enough conservative. In view of the incompleteness of the TVA market, our approach is only a first step. However, any incomplete market approach, adding in some way or another one layer of optimization to the analysis (e.g. utility maximisation), would most likely be hardly feasible in practice, especially on real-life portfolios with tens to hundreds of thousands of contracts. The TVA problem entails some nonlinearities but banks can't really deal with nonlinear pricing rules, because these are too complicated to manage at the portfolio level. Hence, one should aim in the end for "the best linear TVA approximation".

Toward this aim, it is useful to derive an equivalent but simpler reduced BSDE, which will be the topic of Sect. 4. But the reduction of the exact TVA BSDE (3.5) would lead to nested BSDEs, where the coefficient of the corresponding reduced BSDE on $[0, T]$ would be defined in terms of solutions to auxiliary reduced BSDEs "conditional on $\mathcal{G}_{\bar{\tau}}$" (see Crépey and Song (2014b, Section 4) for such a treatment in a preprint version of this paper). The nested feature arises because, in the scenarios where $\tau_c \le \tau_b \wedge T$, the nonlinear funding coefficient g of the bank is still in force, hence the funding data of the problem remain nonlinear, on the time interval $[\bar{\tau}, \bar{\tau}^\star] = [\tau_c, \bar{\tau}^\star]$ (upper right subdomain in Figure 1). However, because the cure period δ is only a few days, the quantitative impact on the TVA Θ of the coefficient g on the time interval $[\bar{\tau}, \bar{\tau}^\star]$ can only be very limited. Hence, in order to avoid the numerical burden of nested BSDEs, we work henceforth with the following full TVA BSDE:

$$\begin{aligned} &\Theta_{\bar{\tau}} = \mathbb{1}_{\{\tau<T\}} \left(\bar{\xi}_\tau - (P_{\tau_b-} - C_{\tau_b-} - \Theta_{\tau_b-})^+ \mathbb{1}_{\{\tau=\tau_b\}} \Lambda \right), \\ &d\mu_t := d\Theta_t + f_t(\Theta_t)dt \text{ is a } (\mathbb{G}, \mathbb{Q}) \text{ local martingale on } [0, \bar{\tau}]. \end{aligned} \tag{3.6}$$

This equation is obtained by replacing g and Λ by 0 on the time interval $[\bar{\tau}, \bar{\tau}^\star]$ in the exact TVA BSDE (3.5), which means replacing $(0, g)$ by $(0, 0)$ and Λ by 0 in the upper right subdomain in Figure 1. Indeed, proceeding in this way, the argument already used for deriving Proposition 31 shows that the time interval on which the TVA needs be computed can be reduced further, from $[0, \bar{\tau}^\delta]$ initially to $[0, \bar{\tau}^\star]$ as done in Proposition 31 regarding the exact TVA BSDE, to even $[0, \bar{\tau}]$ now for the appropriately modified terminal condition regarding the full TVA BSDE (3.6), with all data set to zero outside $[0, \bar{\tau}]$. Reformulating the problem on the smaller domain $[0, \bar{\tau}]$ makes it simpler in view of the reduced-form analysis of Sect. 4, because τ, as opposed to $\tau^\star$, has an intensity. Note that (3.6) is also, essentially, the so-called master equation in Brigo and Pallavicini (2014).

Of course, a TVA BSDE modeling approach based on (3.6) supposes the existence (at the very least) of a solution to (3.6), which is still a quite nonstandard BSDE. Even if a bit simpler than (3.5), (3.6) does not fall in the classical scope of BSDEs with random but explicit terminal conditions (see e.g. Kruse and Popier (2015, Sect. 5)). But the results of the next section allow reducing the $(\mathbb{G}, \mathbb{Q})$ BSDE (3.6) to the simpler BSDE (4.5) with a null terminal condition at the fixed time horizon T, stated with respect to a smaller filtration $\mathbb{F}$ and a possibly changed probability measure $\mathbb{P}$. Hence, the well-posedness of the full BSDE (3.6) will follow from the well-posedness of the reduced BSDE (4.5), which will be established under a typical specification of the data in Theorem 2. Even in the case where $\Lambda = 0$ and the terminal condition of (3.6) is explicit (does not depend on Θ_{τ_b-}), the null terminal condition of (4.5) is a key feature for the numerical scheme used in Sect. 9.

4 Reduced Form Approach

In this section we develop an extended reduced-form approach for the full TVA BSDE (3.6), beyond the basic immersion setup of Crépey (2015, Part II), in view of the wrong-way and gap risk applications of the second part of the paper.

By Corollary 3.23 2) in He, Wang, and Yan (1992), for any $\mathcal{G}_\tau$ measurable random variable ζ, there exists a $\mathbb{G}$ predictable process $\widehat{\zeta}$ such that

$$\mathbb{1}_{\{\tau<\infty\}}\mathbb{E}[\zeta|\mathcal{G}_{\tau-}] = \mathbb{1}_{\{\tau<\infty\}}\widehat{\zeta}_\tau. \tag{4.1}$$

If ζ is integrable, then the time integral $\gamma_t\widehat{\zeta}_t dt$ exists (and is independent of the choice of a version of $\widehat{\zeta}$) and

$$\zeta_t dJ_t + \gamma_t\widehat{\zeta}_t dt \tag{4.2}$$

is a $(\mathbb{G},\mathbb{Q})$ local martingale on $\mathbb{R}_+$(cf. Corollary 5.31 1) in He et al. (1992)). In particular let $\widehat{\xi}$ be a $\mathbb{G}$ predictable process such that, on $\{\tau<\infty\}$,

$$\widehat{\xi}_\tau = \mathbb{E}(\bar{\xi}_\tau \,|\, \mathcal{G}_{\tau-}) = \mathbb{E}(\beta_\tau^{-1}\beta_{\tau+\delta}\xi \,|\, \mathcal{G}_{\tau-}) \tag{4.3}$$

For $t\in[0,\bar{\tau}]$ and $\vartheta\in\mathbb{R}$, we write

$$\begin{aligned} \widehat{f}_t(\vartheta) &= f_t(\vartheta) + \gamma_t\widehat{\xi}_t - (P_t - C_t - \vartheta)^+\gamma_t^b\Lambda - \gamma_t\vartheta \\ &= \gamma_t\widehat{\xi}_t + g_t(P_t-\vartheta) - (P_t - C_t - \vartheta)^+\gamma_t^b\Lambda - (r_t+\gamma_t)\vartheta. \end{aligned} \tag{4.4}$$

Let $Y^{\tau-} = JY + (1-J)Y_-$ represent the process Y stopped at $(\tau-)$ (i.e. right before τ), for any left-limited process Y. Let S denote the Azéma supermartingale of τ, i.e. the process such that $S_t = \mathbb{Q}(\tau > t\,|\,\mathcal{F}_t)$, $t\geq 0$. Conditions of the following kind are studied at the theoretical level in Crépey and Song (2014a,2015a).

Condition (B). $\mathbb{F}$ is a subfiltration of $\mathbb{G}$ satisfying the usual conditions such that any $\mathbb{G}$ predictable process U admits an $\mathbb{F}$ predictable reduction, i.e. an $\mathbb{F}$ predictable process, denoted by U', that coincides with U on $(0,\tau]$.

Remark 41 If $S_T > 0$ (almost surely), then any inequality between two $\mathbb{G}$ predictable processes on $(0,\tau]$ implies the same inequality between their $\mathbb{F}$ predictable reductions on $(0,T]$ (see (Song 2014, Lemma 6.1)); in particular (see also Crépey and Song (2015a, Lemma A.1), the process U' is uniquely determined on $(0,T]$.

Condition (C). There exist:

- **(C.1)** a subfiltration $\mathbb{F}$ of $\mathbb{G}$ satisfying the usual conditions such that $\mathbb{F}$ semimartingales stopped at τ are $\mathbb{G}$ semimartingales,
- **(C.2)** a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that $(\mathbb{F},\mathbb{P})$ local martingales stopped at $(\tau-)$ are $(\mathbb{G},\mathbb{Q})$ local martingales on $[0,T]$,
- **(C.3)** an $\mathbb{F}$ progressive reduction $\widetilde{f}_t(\vartheta)$ of $\widehat{f}_t(\vartheta)$, i.e. an $\mathcal{R}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R})$ function $\widetilde{f}_t(\vartheta)$ such that $\int_0^\cdot \widehat{f}_t(\vartheta)dt = \int_0^\cdot \widetilde{f}_t(\vartheta)dt$ on $[0,\bar{\tau}]$.

The condition (C.1) is related to the so-called $(\mathcal{H}')$ hypothesis between $\mathbb{F}$ and $\mathbb{G}$, i.e. $\mathbb{F}$ semimartingales are $\mathbb{G}$ semimartingales (see Bielecki, Jeanblanc, and Rutkowski (2009)). Both conditions (C.1) and (C.3) hold under the condition (B) (see Crépey and Song (2015a)). The condition (C) with (C.1) and (C.3) reinforced as (B) yields the condition (A) in Crépey and Song (2015a). Hence, assuming (C), the random time τ is essentially an invariance time in the sense of the condition (A) in Crépey and Song (2015a). If $(\mathbb{F},\mathbb{P})$ local martingales don't jump at τ, then the condition (C.2) says that $(\mathbb{F},\mathbb{P})$ local martingales stopped at τ are $(\mathbb{G},\mathbb{Q})$ local martingales. In the case where $\mathbb{P}=\mathbb{Q}$, this property is related to the notions of immersion of $\mathbb{F}$ into $\mathbb{G}$, i.e. $(\mathbb{F},\mathbb{Q})$ local martingales are $(\mathbb{G},\mathbb{Q})$ local martingales (see Bielecki et al. (2009)), and of an $\mathbb{F}$ pseudo-stopping time τ, i.e. $(\mathbb{F},\mathbb{Q})$ local martingales stopped at τ are $(\mathbb{G},\mathbb{Q})$ local martingales (see Nikeghbali and Yor (2005)). However, even in this "immersion" case where $\mathbb{P}=\mathbb{Q}$, the condition (C) offers a richer framework than a standard reduced-form intensity model of credit risk, where the full model filtration $\mathbb{G}$ is given as the reference filtration $\mathbb{F}$ progressively enlarged by τ, i.e. "$\mathbb{G}=\mathbb{F}\vee\mathbb{H}$" in the standard notation of Bielecki et al. (2009). In fact, under (C.1-2-3), the full filtration $\mathbb{G}$ can be bigger than $\mathbb{F}\vee\mathbb{H}$. In particular, even for $\mathbb{P}=\mathbb{Q}$, the conditions (C.1-2-3) do not exclude a jump of an $\mathbb{F}$ adapted càdlàg process at time τ, which happens for instance with a nonvanishing dividend $\Delta_\tau = D_\tau - D_{\tau-}$ at time τ in the DMO model of Sect. 8. By contrast, a jump of an $\mathbb{F}$ adapted càdlàg process at time τ cannot happen in a basic reduced-form credit risk setup (see Crépey and an Introductory Dialogue by D. Brigo) (2014, Lemma 13.7.3(ii) page 331)[4]). In addition, the flexibility of the condition (C) comes from the possibility to choose $(\mathbb{F},\mathbb{P})$ ensuring (C.1-2-3). See Sect. 7 and Sect. 8 for concrete examples, with $\mathbb{P} \neq \mathbb{Q}$ and $\Delta_\tau = 0$ in the first case and $\mathbb{P} = \mathbb{Q}$ but $\Delta_\tau \neq 0$ in the second case, which we view as respective wrong-way risk and gap risk stylized examples (as our concluding figure 7 will illustrate).

The result that follows can also be retrieved from Crépey and Song (2014a), but in a much more abstract setup there, mainly motivated by the converse to this result. Hence, we provide a self-contained proof, valid under the "minimal condition" (C).

Theorem 1 *Under the condition (C), assume that an $(\mathbb{F},\mathbb{P})$ semimartingale $\widetilde{\Theta}$ satisfies the following reduced TVA BSDE on $[0,T]$:*

$$\widetilde{\Theta}_T = 0 \text{ and } d\widetilde{\mu}_t := d\widetilde{\Theta}_t + \widetilde{f}_t(\widetilde{\Theta}_t)dt \text{ is an } (\mathbb{F},\mathbb{P}) \text{ local martingale on } [0,T]. \tag{4.5}$$

Let $\Theta = \widetilde{\Theta}$ on $[0,\bar{\tau})$ and $\Theta_{\bar{\tau}} = \mathbb{1}_{\{\tau<T\}}\Big(\bar{\xi}_\tau - (P_{\tau-} - C_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{\tau=\tau_b\}}\Lambda\Big)$. Then Θ is a $(\mathbb{G},\mathbb{Q})$ semimartingale satisfying the full TVA equation (3.6) on $[0,\bar{\tau}]$ and we have, for $t \in [0,\bar{\tau}]$,

$$\begin{aligned} d\mu_t = d\widetilde{\mu}_t^{\tau-} &- \big(\bar{\xi}_\tau - (P_{\tau-} - C_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{\tau=\tau_b\}}\Lambda - \widetilde{\Theta}_{\tau-}\big)dJ_t \\ &- \big((\widehat{\xi}_t - \widetilde{\Theta}_t)\gamma_t - (P_t - C_t - \widetilde{\Theta}_t)^+ \gamma_t^b \Lambda\big)dt. \end{aligned} \tag{4.6}$$

Proof. By definition of Θ here, a $(\mathbb{G},\mathbb{Q})$ semimartingale by (C.1), we have, for $t \in [0,\bar{\tau}]$:

$$\begin{aligned} d\Theta_t &= d(J_t\widetilde{\Theta}_t) - \Big(\bar{\xi}_\tau - (P_{\tau-} - C_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{\tau=\tau_b\}}\Lambda\Big)\,dJ_t \\ &= d\widetilde{\Theta}_t^{\tau-} + \widetilde{\Theta}_{\tau-}\,dJ_t - \Big(\bar{\xi}_\tau - (P_{\tau-} - C_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{\tau=\tau_b\}}\Lambda\Big)\,dJ_t. \end{aligned} \tag{4.7}$$

Then by (4.5), for $t \in [0,\bar{\tau}]$:

$$\begin{aligned} -d\Theta_t &= \widetilde{f}_t(\widetilde{\Theta}_t)dt - d\widetilde{\mu}_t^{\tau-} + \Big(\bar{\xi}_\tau - (P_{\tau-} - C_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{\tau=\tau_b\}}\Lambda - \widetilde{\Theta}_{\tau-}\Big)\,dJ_t \\ &= f_t(\Theta_t)dt - d\widetilde{\mu}_t^{\tau-} + \Big(\bar{\xi}_\tau - (P_{\tau-} - C_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{\tau=\tau_b\}}\Lambda - \widetilde{\Theta}_{\tau-}\Big)\,dJ_t \\ &\quad + \Big((\widehat{\xi}_t - \widetilde{\Theta}_t)\gamma_t - (P_t - C_t - \Theta_t)^+ \gamma_t^b \Lambda\Big)\,dt, \end{aligned}$$

by (C.3). By (C.2), $\widetilde{\mu}_t^{\tau-}$ is a $(\mathbb{G},\mathbb{Q})$ local martingale, as is also on $[0,\bar{\tau}]$

$$\Big(\bar{\xi}_\tau - (P_{\tau-} - C_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{\tau=\tau_b\}}\Lambda - \widetilde{\Theta}_{\tau-}\Big)\,dJ_t + \Big((\widehat{\xi}_t - \widetilde{\Theta}_t)\gamma_t - (P_t - C_t - \Theta_t)^+ \gamma_t^b \Lambda\Big)\,dt$$

(cf. (4.2)). This yields the decomposition (4.6) of $d\mu_t := d\Theta_t + f_t(\Theta_t)dt$, which implies the martingale condition in (3.6), where the terminal condition holds by definition of $\Theta_{\bar{\tau}}$ in the theorem. ∎

Summarizing so far, assuming the condition (C) on the default time τ, based on Theorem 1, we can design a TVA process in terms of a solution to the reduced BSDE (4.5). Moreover, this approach is not arbitrary. In fact, under the additional condition (B) that is latent in (C.1) and (C.3), the results of Crépey and Song (2014a) show that the full and reduced BSDEs are equivalent (assuming a positive Azéma supermartingale S of τ, as typically verified in applications). In particular, as will be seen in the final statements of Theorems 5 and 8, the condition (B) holds and S is positive in the concrete models of Sect. 7 and 8, so that this equivalence applies to each of the reduced TVA BSDEs considered in Corollaries 71, 81 and 82. Moreover, in Theorem 2, the reduced BSDE (4.5) will be seen to well-posed under a typical specification of the data. Hence, as a consequence of the equivalence between (4.5) and (3.6), the full BSDE (3.6) is well-posed too.

[4] Or Lemma 2.1(ii) in the journal version Crépey (2015, Part II).

5 Well-Posedness of the Reduced BSDE under a Typical Specification of the Data

In view of applications, we need to specify the close-out cashflow χ, hence the counterparty risk exposure $\xi = Q_{\tau^\delta} - \chi$, and the funding coefficient $g_t(\pi)$, conformly with usual CSA specifications. Let V denote the variation margin process, where $V \geq 0$ (resp. ≤ 0) means collateral posted by the counterparty and received by the bank (resp. posted by the bank and received by the counterparty). Let a process $I^c \geq 0$ (resp. $I^b \leq 0$) represent the initial margin posted by the counterparty (resp. the negative of the initial margin posted by the bank). Let

$$C^c = V + I^c \text{ and } C^b = V + I^b, \tag{5.1}$$

respectively the total collateral guarantee for the bank and the negative of the total collateral guarantee for the counterparty.

Remark 51 In practice, the variation and initial margin calls are executed according to a discrete schedule (t_l). Specifically, the variation margin V tracks the mark-to-market P at grid times $t_l < \tau$. The initial margins I^c and I^b are also updated at some of the times $t_l < \tau$, based on risk measures of the profit-and-loss of the variation-margined position at the horizon δ of the cure period (see Brigo and Pallavicini (2014) and Armenti and Crépey (2015) for detailed specifications). Hence, the margin processes are stopped at the greatest $t_l < \tau$ (or $= \tau$, but the corresponding event has zero probability in a default intensity setup).

In particular, we assume all the margin processes are stopped at τ. The bank's close-out cashflow χ is derived from the liquidation debts of the counterparty to the bank and vice versa, respectively modeled at time τ^δ as

$$\varepsilon_c = (Q_{\tau^\delta} - C^c_\tau)^+, \ \ \varepsilon_b = (Q_{\tau^\delta} - C^b_\tau)^-. \tag{5.2}$$

Note that $\varepsilon_c \times \varepsilon_b \equiv 0$, by nonnegativity of I^c and $(-I^b)$. The close-out cashflow is modeled as

$$\chi = \begin{cases} C^c_\tau + R_c \varepsilon_c \text{ if } \varepsilon_c > 0 \text{ and } \tau_c \leq \tau^\delta_b, \\ C^b_\tau - R_b \varepsilon_b \text{ if } \varepsilon_b > 0 \text{ and } \tau_b \leq \tau^\delta_c, \\ \qquad Q_{\tau^\delta} \text{ otherwise.} \end{cases} \tag{5.3}$$

Here R_c and R_b stand for constant recovery rates of the counterparty to the bank and vice versa. The ensuing counterparty risk exposure of the bank results from the left hand side in (3.3) as

$$\xi = Q_{\tau^\delta} - \chi = \mathbb{1}_{\{\tau_c \leq \tau^\delta_b\}}(1 - R_c)\varepsilon_c - \mathbb{1}_{\{\tau_b \leq \tau^\delta_c\}}(1 - R_b)\varepsilon_b. \tag{5.4}$$

Remark 52 The expression (5.4) for the counterparty risk exposure ξ is consistent with the one in Brigo and Pallavicini (2014), where it is also explained that, if the bank is a client of a counterparty acting as member of a clearing house, the above formalism can still be used just by setting $R_c = 1$ and $I^c = 0$. In fact, as members of a clearing house are backed-up by other members if they default, their actual recovery rate is immaterial and everything happens for their clients as if R_c would be one. In addition, as clearing members don't post initial margins to their clients but to the clearing house, the initial margin posted by the counterparty has no impact on the funding costs of the bank in this case, so that everything happens for the bank as if I^c would be zero. The situation where the bank itself is a member (as opposed to a client) of a clearing house is dealt with in Armenti and Crépey (2015).

We assume that the posted collateral is remunerated at a rate $(r_t + c_t)$ and that the bank can invest (resp. get some unsecured funding) at a rate $(r_t + \lambda_t)$ (resp. $(r_t + \bar{\lambda}_t)$), for $\mathbb{G}$ predictable processes $c, \lambda, \bar{\lambda}$ (and also r, noting that predictability is not a true restriction with respect to progressive measurability for all these processes that are time integrated). The collateral funded by the bank is $(-C)$, where $C = C^b + I^c = V + I^c + I^b$ (assuming all the margins re-hypothecable, i.e. reusable by the receiving party). This results in the following dt-funding cost of the bank, for $t < \bar{\tau}^\star$:

$$\underbrace{J_t(r_t + c_t)C_t}_{\text{remuneration of the collateral}} + \underbrace{(r_t + \bar{\lambda}_t)\,(-\mathcal{W}_t - C_t)^+ - (r_t + \lambda_t)\,(-\mathcal{W}_t - C_t)^-}_{\text{funding costs / benefits}},$$

which is of the form (2.2) with in particular, for $t < \tau$,

$$g_t(\pi) = c_t C_t + \bar{\lambda}_t (\pi - C_t)^+ - \lambda_t (\pi - C_t)^- . \quad (5.5)$$

Remark 53 This expression for g corresponds to the case of re-hypothecable margins, as is typical for the variation margin, whereas the initial margins are sometimes segregated. In the case of segregated initial margins, the expression for g changes slightly, but the overall structure (2.2) of the funding costs is still valid.

Assuming henceforth (5.5), $\widehat{f}_t(\vartheta)$ in (4.4) is such that, on $[0, \bar{\tau}]$,

$$\begin{aligned}\widehat{f}_t(\vartheta) + (r_t + \gamma_t)\vartheta &= \gamma_t \widehat{\xi}_t + c_t C_t + \bar{\lambda}_t (P_t - C_t - \vartheta)^+ - \lambda_t (P_t - C_t - \vartheta)^- - (P_t - C_t - \vartheta)^+ \gamma_t^b \Lambda \\ &= \underbrace{\gamma_t \widehat{\xi}_t}_{cdva_t} + \underbrace{c_t C_t + \widetilde{\lambda}_t (P_t - C_t - \vartheta)^+ - \lambda_t (P_t - C_t - \vartheta)^-}_{lva_t(\vartheta)},\end{aligned} \quad (5.6)$$

where $\widetilde{\lambda}_t = \bar{\lambda}_t - \gamma_t^b \Lambda$ can be interpreted as a liquidity borrowing spread of the bank, net of its credit spread to its funder. From the perspective of the bank, the term $\gamma_t \widehat{\xi}_t$ represents the counterparty risk component of $\widehat{f}_t(\vartheta)$, whereas the remaining terms can be interpreted as a funding liquidity component. The positive (resp. negative) components of $\widehat{f}_t(\vartheta)$ can be considered as deal adverse (resp. deal friendly) as they increase (resp. decrease) the TVA Θ of the bank. Depending on the sign of $\Pi = Q - \Theta$, a "less positive" Π is interpreted as a lower buyer price by the bank and a "more negative" Π as a higher seller price by the bank.

Remark 54 The materiality of a debit benefit at own default, represented by a DVA cashflow $\mathbb{1}_{\{\tau_b \le \tau_c^\delta\}}(1 - R_b)\varepsilon_b$ in (3.3), or of a funding benefit at own default (2.3), corresponding in the FVA debate to Hull and White (2012a,2012b,2014)'s DVA2 cashflow, are clearly subject to caution, unless corresponding hedges allow the bank to monetize these benefits before it defaults. The most conservative practice for the bank is to set $R_b = 1$ and $\Lambda = 0$ in order to disregard these cashflows. This avoids reckoning "fake benefits" or benefits to bondholders only, whereas a sound management should only consider the interest of the shareholders (see Albanese, Brigo, and Oertel (2013) and Albanese and Iabichino (2013)).

The assumptions in the following reduced BSDE well-posedness result, even though not minimal, are sufficient for our later purposes in this paper. In particular, the boundedness assumption on P, from which the one on the collateral C follows naturally, will be satisfied in the case of vanilla credit derivatives such as CDS contracts and CDO tranches (cf. (6.12)). Note that, on $[0, \bar{\tau}]$, we have

$$\begin{aligned}\widehat{f}_t(\vartheta) &= \gamma_t \widehat{\xi}_t + c_t C_t + \widetilde{\lambda}_t (P_t - C_t - \vartheta)^+ - \lambda_t (P_t - C_t - \vartheta)^- - (r_t + \gamma_t)\vartheta \\ &= \gamma_t \widehat{\xi}_t + c_t C_t + \bar{\lambda}_t (P_t - C_t - \vartheta)^+ - \lambda_t (P_t - C_t - \vartheta)^- - \gamma_t (P_t - C_t) \\ &\quad + (\gamma_t - \gamma_t^b \Lambda)(P_t - C_t - \vartheta)^+ - \gamma_t (P_t - C_t - \vartheta)^- - r_t \vartheta.\end{aligned} \quad (5.7)$$

Assuming the condition (B) and $S_T > 0$, we may and do henceforth choose for $\widetilde{f}_t(\vartheta)$ the process obtained from $\widehat{f}_t(\vartheta)$ by replacing each process U involved in (5.7) by its $\mathbb{F}$ predictable reduction U' or, in the case of $U = P$ or C, by $(P_-)'$ and $(C_-)'$. We write $\|U\|^p_{\widetilde{\mathcal{H}}_p} = \widetilde{\mathbb{E}}[\int_0^T U_t^p dt]$ ($p > 0$), where $\widetilde{\mathbb{E}}$ means $\mathbb{P}$ expectation.

Theorem 2 *Under the condition (B), assuming $S_T >$, λ', $\bar{\lambda}'$ and r' bounded from below on $[0, T]$, $(P_-)'$ and $(C_-)'$ bounded on $[0, T]$ and c', λ', $\bar{\lambda}'$, r', γ' in $\widetilde{\mathcal{H}}_2$, the reduced BSDE (4.5) (with $\widetilde{f}$ as specified above) is well-posed in $\widetilde{\mathcal{H}}_2$, where well-posedness includes existence, uniqueness, comparison and the standard BSDE a priori bound and error estimates. The $\widetilde{\mathcal{H}}_2$ solution $\widetilde{\Theta}$ to (4.5) satisfies*

$$\widetilde{\Theta}_t = \widetilde{\mathbb{E}}\left[\int_t^T \widetilde{f}_s(\widetilde{\Theta}_s) ds \,\Big|\, \mathcal{F}_t\right] = \widetilde{\mathbb{E}}\left[\int_t^T e^{-\int_t^s \gamma'_r dr} \bar{f}_s(\widetilde{\Theta}_s) ds \,\Big|\, \mathcal{F}_t\right], \ t \in [0, T], \quad (5.8)$$

where we set $\bar{f}_t(\vartheta) = \widetilde{f}_t(\vartheta) + \gamma'_t \vartheta$.

Proof. Assuming λ', $\bar{\lambda}'$ and r' bounded from below, it follows from (5.7), where $\gamma_t \geq \gamma^b_t \Lambda \geq 0$, that $\widetilde{f}_t(\vartheta)$ satisfies the classical BSDE monotonicity assumption

$$\big(\widetilde{f}_t(\vartheta) - \widetilde{f}_t(\vartheta')\big)(\vartheta - \vartheta') \leq C(\vartheta - \vartheta')^2 \tag{5.9}$$

on $(0, T]$, (cf. the remark 41), for some constant C. Hence, by application of the results of Kruse and Popier (2015, Sect. 4), the reduced BSDE (4.5) with coefficient $\widetilde{f}_t(\vartheta)$ is well-posed in $\widetilde{\mathcal{H}}_2$ (which includes existence, uniqueness, comparison and the standard BSDE bound and error estimates) as soon as the following auxiliary integrability conditions hold:

$$\sup_{|\vartheta| \leq \bar{\vartheta}} |\widetilde{f}_\cdot(\vartheta) - \widetilde{f}_\cdot(0)| \in \widetilde{\mathcal{H}}_1 \ (\bar{\vartheta} > 0), \ \ \widetilde{f}_\cdot(0) \in \widetilde{\mathcal{H}}_2. \tag{5.10}$$

Since

$$|\widetilde{f}(\vartheta) - \widetilde{f}(0)| \leq \big(|\bar{\lambda}'| + |\lambda'| + |r'| + \gamma'\big)|\vartheta|,$$

(5.10) holds in particular for $(P_-)'$, $(C_-)'$ bounded and c', λ', $\bar{\lambda}'$, r', γ' in $\widetilde{\mathcal{H}}_2$. The left hand side identity in (5.8) is the usual integral representation for an $\widetilde{\mathcal{H}}_2$ solution $\widetilde{\Theta}$ to the reduced BSDE (4.5). It implies the right hand side identity in (5.8) by treating the $(-\gamma'\vartheta)$ term in $\widetilde{f}$ as a discount factor at the rate γ'. ∎

6 Marked Default Times Framework

A residual issue left open by Theorem 2 is the specification of a concrete but general enough framework where $cdva = \gamma\widehat{\xi}$ in (5.6) can be computed in practice. Toward this aim, this section implements the extended reduced-form approach of the previous sections based on a default time τ obtained as a $\mathbb{G}$ stopping time with a mark. The role of the mark is to convey some additional information about the default, e.g. to encode wrong-way and gap risk features that would be out-of-reach in the basic immersion setup of Crépey (2015, Part II).

We assume in the sequel that τ is endowed with a mark e in a finite set E, i.e.

$$\tau = \min_{e \in E} \tau_e, \tag{6.1}$$

where the τ_e are $\mathbb{G}$ stopping times avoiding each other, with respective $(\mathbb{G}, \mathbb{Q})$ intensities γ^e_t and such that

$$\mathcal{G}_\tau = \mathcal{G}_{\tau-} \vee \sigma(\iota), \tag{6.2}$$

in which $\iota = \mathrm{argmin}_{e \in E} \tau_e$ is the "identity" of the mark of τ. We denote by $\mathcal{E}$ the powerset of E.

Lemma 61 *Assuming a marked stopping time τ in the above sense, for any $\mathcal{G}_\tau$ measurable random variable ζ, there exists a $\mathcal{P}(\mathbb{G}) \otimes \mathcal{E}$ measurable function $\widetilde{\zeta} = \widetilde{\zeta}^e_t$ such that*

$$\mathbb{1}_{\{\tau = \tau_e\}} \zeta = \mathbb{1}_{\{\tau = \tau_e\}} \widetilde{\zeta}^e_\tau, \ \ e \in E. \tag{6.3}$$

For any such function $\widetilde{\zeta}$, a $\mathbb{Q} \times \boldsymbol{\lambda}$ a.e. version of $\gamma\widehat{\zeta}$ is given by $J_- \sum_E \gamma^e \widetilde{\zeta}^e$. In particular, the intensity of τ satisfies $\gamma = J_- \sum_{e \in E} \gamma^e$, $\mathbb{Q} \times \boldsymbol{\lambda}$ a.e., and we have

$$cdva_t = J_- \sum_{e \in E} \gamma^e \widetilde{\xi}^e, \ \ \mathbb{Q} \times \boldsymbol{\lambda} \ a.e., \tag{6.4}$$

for any $\mathcal{P}(\mathbb{G}) \otimes \mathcal{E}$ measurable function $\widetilde{\xi} = \widetilde{\xi}^e_t$, which exists, such that, for each $e \in E$,

$$\bar{\xi}_\tau = \widetilde{\xi}^e_\tau \ \text{on the event} \ \{\tau = \tau_e\}. \tag{6.5}$$

Proof. The existence of the processes $\widetilde{\zeta}^e(\pi)$ follows from (6.2). By $\mathbb{G}$ predictability of these processes, the $\widetilde{\zeta}^e_\tau$ are $\mathcal{G}_{\tau-}$ locally integrable. Hence, by localization, one can assume the $\widetilde{\zeta}^e_\tau$ integrable. Then, on $\{\tau < \infty\}$,

$$\mathbb{E}[\zeta|\mathcal{G}_{\tau-}] = \mathbb{E}[\textstyle\sum_{e\in E} \mathbb{1}_{\{\tau=\tau^e\}}\widetilde{\zeta}^e_\tau|\mathcal{G}_{\tau-}] = \sum_{e\in E}\widetilde{\zeta}^e_\tau\mathbb{E}[\mathbb{1}_{\{\tau=\tau^e\}}|\mathcal{G}_{\tau-}].$$

Let q^e_t denote a $\mathcal{P}(\mathbb{G})\otimes\mathcal{E}$ measurable function, which exists by Corollary 3.23 2) in He et al. (1992), such that $q^e_\tau\mathbb{1}_{\{\tau<\infty\}} = \mathbb{E}[\mathbb{1}_{\{\tau=\tau^e\}}|\mathcal{G}_{\tau-}]\mathbb{1}_{\{\tau<\infty\}}$ $(e\in E)$. For bounded $Z\in\mathcal{P}(\mathbb{G})$, we compute $\mathbb{E}[Z_\tau\mathbb{1}_{\{\tau=\tau^e<\infty\}}]$ in two ways:

$$\mathbb{E}[Z_\tau\mathbb{1}_{\{\tau=\tau^e<\infty\}}] = \mathbb{E}[Z_\tau q^e_\tau\mathbb{1}_{\{\tau<\infty\}}] = \mathbb{E}[\int_0^\infty Z_s q^e_s\gamma_s ds],$$

and

$$\mathbb{E}[Z_\tau\mathbb{1}_{\{\tau=\tau^e<\infty\}}] = \mathbb{E}[Z_{\tau^e}\mathbb{1}_{\{\tau=\tau^e<\infty\}}] = \mathbb{E}[Z_{\tau^e}\mathbb{1}_{\{\tau^e\le\tau<\infty\}}] = \mathbb{E}[\int_0^\infty Z_s\mathbb{1}_{\{s\le\tau\}}\gamma^e_s ds].$$

Hence, $\mathbb{Q}$ almost surely: $q^e_t\gamma_t = \mathbb{1}_{\{t\le\tau\}}\gamma^e_t$, dt almost everywhere, so that

$$\mathbb{Q}[q^e_\tau\gamma_\tau \neq \gamma^e_\tau,\ \tau<\infty] = \mathbb{E}[\mathbb{1}_{\{q^e_\tau\gamma_\tau\neq\gamma^e_\tau,\ \tau<\infty\}}] = \mathbb{E}[\int_0^\infty \mathbb{1}_{\{q^e_t\gamma_t\neq\gamma^e_t\}}\gamma_t dt] = 0.$$

Therefore, on $\{\tau<\infty\}$,

$$\gamma_\tau\widehat{\zeta}_\tau = \gamma_\tau\mathbb{E}[\zeta|\mathcal{G}_{\tau-}] = \sum_{e\in E}\widetilde{\zeta}^e_\tau\gamma_\tau q^e_\tau = \sum_{e\in E}\widetilde{\zeta}^e_\tau\gamma^e_\tau.$$

This implies that

$$\gamma\widehat{\zeta} = J_-\sum_E\gamma^e\widetilde{\zeta}^e,\ \ \mathbb{Q}\times\boldsymbol{\lambda}\text{-a.e.}$$

In particular, (6.4) follows by definition (5.6) of *cdva*. ∎

The assumption of a finite set E in (6.1) ensures tractability of the setup, while offering a sufficiently large playground for applications, as the second part of the paper demonstrates. However, as developed in Crépey and Song (2015b), from a theoretical point of view, this assumption can be essentially eliminated and the approach remains in essence valid, modulo an integral instead of a sum over E in (6.4) and similar expressions later in the paper.

We now give a concrete specification ensuring (6.2), in the case where $\mathbb{G}$ is the progressive enlargement of a reference filtration $\mathbb{F}$ by n random times $\eta_1,\ldots,\eta_n$ avoiding each other. Let the $\eta_{(i)}$ be the increasing ordering of the η_i, with also $\eta_{(0)}=0$ and $\eta_{(n+1)}=\infty$. The optional splitting formula of Song (2013b) means, for any $\mathbb{G}$ optional process Y, the existence of $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}([0,\infty]^n)$-measurable functions $Y^{(0)},Y^{(1)},\ldots,Y^{(n)}$ such that

$$Y = \sum_{i=0}^n Y^{(i)}(\eta_1 \nmid \eta_{(i)},\ldots,\eta_n \nmid \eta_{(i)})\mathbb{1}_{[\eta_{(i)},\eta_{(i+1)})}, \tag{6.6}$$

where $a\nmid b$ denotes a if $a\le b$ and ∞ if $a>b$, for $a,b\in[0,\infty]$. This holds, in particular, in any recursively immersed or multivariate density model of default times. By recursively immersed model of default times, we mean a model where a reference filtration is successively progressively enlarged by random times, such that each successive enlargement has the immersion property. By multivariate density model, we mean a model with a conditional density of the default times given some reference market filtration, in the sense of the condition (DH) in page 1800 of Pham (2010). This is the multivariate extension of the notion of a density time, first introduced in an initial enlargement setup in Jacod (1987) and revisited in a progressive enlargement setup, under the name of initial time, in Jeanblanc and Le Cam (2009) and El Karoui, Jeanblanc, and Jiao (2010).

Lemma 62 *Assuming the optional splitting formula in force, e.g. in any recursively immersed or multivariate density model of default times, let* $\eta=\eta_1\wedge\eta_2$. *If the* η_i *avoid each other and* $\mathbb{F}$ *stopping times, then*

$$\mathcal{G}_\eta = \mathcal{G}_{\eta-}\vee\sigma(\{\eta=\eta_1\},\{\eta=\eta_2\}).$$

Proof. Let $N = \{1,2,3,\ldots,n\}$. By the optional splitting formula (6.6), for any $\mathbb{G}$ optional process Y and $i \in N$, we have

$$\begin{aligned} Y_\eta \mathbb{1}_{\{\eta=\eta_{(i)}\}} &= Y^{(i)}_\eta(\eta_1 \dagger \eta, \ldots, \eta_n \dagger \eta)\mathbb{1}_{\{\eta=\eta_{(i)}\}} \\ &= \textstyle\sum_{I\subseteq N;|I|=i-1} Y^{(i)}_\eta(\eta_1 \dagger \eta, \eta_2 \dagger \eta, \ldots, \eta_n \dagger \eta)\mathbb{1}_{\{\forall j\in I, \eta_j<\eta\}}\mathbb{1}_{\{\forall j\in N\setminus I, \eta\leq\eta_j\}} \\ &= \textstyle\sum_{k=1}^{2} \mathbb{1}_{\{\eta=\eta_k\}} \sum_{I\subseteq N;|I|=i-1} Y^{(i,I,k)}_\eta(\eta;\eta_j, j\in I)\mathbb{1}_{\{\forall j\in I,\eta_j<\eta\}}\mathbb{1}_{\{\forall j\in N\setminus I,\eta\leq\eta_j\}}, \end{aligned}$$

where $Y^{(i,I,k)}_t(\omega, y; y_j, j \in I)$ is $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}([0,\infty])\otimes\mathcal{B}([0,\infty]^{i-1})$ measurable. Moreover, as η avoids $\mathbb{F}$ stopping times, He et al. (1992, Theorem 3.20) and the monotone class theorem imply that $Y^{(i,I,k)}_\eta(\eta; \eta_j, j \in I)$ is $\mathcal{G}_{\eta-}$ measurable. So, on each event $\{\eta = \eta_k\}$, $Y_\eta\mathbb{1}_{\{\eta=\eta_{(i)}\}}$ is $\mathcal{G}_{\eta-}$ measurable. As $\mathcal{G}_\eta$ is generated by the Y_η for the $\mathbb{G}$ optional processes Y, this proves the result. ∎

6.1 No Cure Period

If $\delta = 0$, then $\bar{\xi} = \xi$, for which the expression in (5.4) reduces to

$$\xi = \mathbb{1}_{\{\tau=\tau_c\}}(1 - R_c)\big(P_\tau + \Delta_\tau - C^c_\tau\big)^+ - \mathbb{1}_{\{\tau=\tau_b\}}(1 - R_b)\big(P_\tau + \Delta_\tau - C^b_\tau\big)^-, \tag{6.7}$$

where $\Delta_\tau = D_\tau - D_{\tau-}$. Moreover, for every process $U = P, (D - D_-)$, C^c and C^b, there exists by (6.2) a $\mathcal{P}(\mathbb{G})\otimes\mathcal{E}$ measurable function $\widetilde{U} = \widetilde{U}^e_t$ such that $U_\tau = \widetilde{U}^e_\tau$ holds on the event $\{\tau = \tau_e\}$, for each $e \in E$. In the present case where $\delta = 0$, the process $(D - D_-)$ plays the role of the process Δ when $\delta \neq 0$. Accordingly, in the case of $U = (D - D_-)$, we write $\widetilde{U}^e_t = \widetilde{\Delta}^e_t$. To alleviate the notation, we also rewrite $(\widetilde{C^c})^e$ as $\widetilde{C}^{c,e}$ and $(\widetilde{C^b})^e$ as $\widetilde{C}^{b,e}$.

Remark 61 As will be illustrated in the DGC and DMO models (see Sect. 9, Figure 7 in particular, and the explanations following (7.11) and (8.6)), the dependence of $\widetilde{P}^e$ and $\widetilde{\Delta}^e$ on e can be used to embed respective wrong way and gap risk effects. The dependence of $\widetilde{C}^{c,e}$ and $\widetilde{C}^{b,e}$ on e could also be used to render further gap risk features such as a jump of the collateral at the default of a counterparty, e.g. in the case of a collateral posted in a currency strongly dependent on this counterparty (cf. Ehlers and Schönbucher (2006)).

Consistent with (6.1), let's assume $\tau_b = \min_{e\in E_b}\tau_e$ and $\tau_c = \min_{e\in E_c}\tau_e$, where $E = E_b \cup E_c$ (not necessarily a disjoint union, as will be exploited in Sect. 8). We may then take in (6.5) (where $\bar{\xi} = \xi$ when $\delta = 0$)

$$\widetilde{\xi}^e_t = \mathbb{1}_{e\in E_c}(1 - R_c)(\widetilde{P}^e_t + \widetilde{\Delta}^e_t - \widetilde{C}^{c,e}_t)^+ - \mathbb{1}_{\{e\in E_b\}}(1 - R_b)(\widetilde{P}^e_t + \widetilde{\Delta}^e_t - \widetilde{C}^{b,e}_t)^-, \tag{6.8}$$

so that by (6.4), we have on $[0, \bar{\tau}]$:

$$cdva_t = (1 - R_c)\sum_{e\in E_c}\gamma^e_t\left(\widetilde{P}^e_t + \widetilde{\Delta}^e_t - \widetilde{C}^{c,e}_t\right)^+ - (1 - R_b)\sum_{e\in E_b}\gamma^e_t\left(\widetilde{P}^e_t + \widetilde{\Delta}^e_t - \widetilde{C}^{b,e}_t\right)^-, \tag{6.9}$$

where the two terms have clear respective CVA and DVA interpretation. Hence, in the no cure period $\delta = 0$ case, (5.6) is rewritten, on $[0, \bar{\tau}]$, as

$$\begin{aligned} \widehat{f}_t(\vartheta) + (r_t + \gamma_t)\vartheta = &\underbrace{(1 - R_c)\sum_{e\in E_c}\gamma^e_t\left(\widetilde{P}^e_t + \widetilde{\Delta}^e_t - \widetilde{C}^{c,e}_t\right)^+}_{cva_t} - \underbrace{(1 - R_b)\sum_{e\in E_b}\gamma^e_t\left(\widetilde{P}^e_t + \widetilde{\Delta}^e_t - \widetilde{C}^{b,e}_t\right)^-}_{dva_t} \\ &+ \underbrace{c_tC_t + \widetilde{\lambda}_t\left(P_t - C_t - \vartheta\right)^+ - \lambda_t\left(P_t - C_t - \vartheta\right)^-}_{lva_t(\vartheta)}, \end{aligned} \tag{6.10}$$

where we set $\widetilde{\lambda}_t = \bar{\lambda}_t - \Lambda\sum_{e\in E_b}\gamma^e_t$. We can then choose the coefficient $\widetilde{f}$ of the reduced TVA BSDE (4.5) based on (6.10) in the way described before Theorem 2. Once stated in a Markov setup where

$$\widetilde{f}_t(\vartheta) = \widetilde{f}(t, \widetilde{X}_t, \vartheta), \;\; t \in [0, T], \tag{6.11}$$

for some $(\mathbb{F}, \mathbb{P})$ jump diffusion $\widetilde{X}$, all the ingredients in the coefficient $\widetilde{f}$ can be computed and the reduced BSDE (4.5) can be solved numerically, which will be our approach in the second part of the paper. This is only for $\delta = 0$ here but we'll study in Sect. 8.3 a case where $\delta > 0$.

Part II

Application to Credit Derivatives

We write $N = \{-1, 0, 1, \ldots, n\}$ and $N^\star = \{1, \ldots, n\}$, for some nonnegative integer n. In the second part of the paper, we apply the above approach to counterparty risk on credit derivatives traded between the bank and the counterparty respectively labeled as -1 and 0, i.e. for $\tau_b = \tau_{-1}$ and $\tau_c = \tau_0$, and referencing the names in N (typically $N^\star$, for one does not trade credit protection on oneself in practice, but this makes no difference mathematically). The examples in this part are important, not only to provide some insights regarding the condition (C) and the marked default time framework introduced abstractly in the first part of the paper, but also as concrete models addressing challenges posed by counterparty risk and funding costs on credit derivatives. This will be accomplished in two different setups, the dynamic Gaussian copula (DGC) model of Crépey et al. (2013) and the dynamic Marshall-Olkin (DMO) copula or common-shock model of Bielecki et al. (2014b,2014a). These two models are dynamic extensions of the perhaps best known copula models, namely the Gaussian copula and the exponential (or Marshall-Olkin) copula. We chose them as prototypes of multivariate density and recursively immersed models of portfolio credit risk, respectively (see the explanations preceding Lemma 62). Other respectively related models include the one-period Merton model of Fermanian and Vigneron (2013, Section 6) and the multivariate Poisson model of Brigo, Pallavicini, and Torresetti (2007).

Regarding financial products, we will consider stylized CDS contracts and protection legs of CDO tranches corresponding to cumulative dividend processes of the respective form, for $0 \le t \le T$:

$$
\begin{aligned}
D^i_t &= \Big((1-R_i)\mathbb{1}_{\{t\ge\tau_i\}} - (t\wedge\tau_i)S_i\Big)Nom_i \\
D_t &= \Big(\Big((1-R)\sum_{j\in N}\mathbb{1}_{\{t\ge\tau_j\}} - (n+2)a\Big)^+ \wedge (n+2)(b-a)\Big)Nom,
\end{aligned}
\tag{6.12}
$$

where all the recoveries R_i and R (resp. nominals Nom_i and Nom) will be set in the numerics of Sect. 9 to 40% (resp. to 100). The contractual spreads S_i of the CDS contracts will be set so that the corresponding prices are equal to 0 at time 0. Protection legs of CDO tranches, where the attachment and detachment points a and b are such that $0 \le a \le b \le 100\%$, can also be seen as CDO tranches with upfront payment. Note that credit derivatives traded as swaps or with upfront payment coexist since the crisis.

Until Sect. 8.3, it is assumed that $\delta = 0$.

7 Dynamic Gaussian Copula TVA Model

Here is an non-exhaustive list of notations introduced in the course of the study of the DGC model.

$B^i; \varsigma, h_i; \alpha$ Brownian motions with correlation ϱ used for constructing the DGC default times; other DGC model primitives such that $\tau_i = h_i^{-1}\big(\int_0^{+\infty}\varsigma(u)dB^i_u\big)$; $\alpha^2(t) = \int_t^{+\infty}\varsigma^2(v)dv$.

$J, \mathbf{z} = (z_j)_{j\in J}, \Phi_{\rho,\sigma}(\mathbf{z}), {\psi^j}_{\rho,\sigma}(\mathbf{z})$ Subset of N representing a set of alive obligors in the financial interpretation, $|J|$-variate real vector, $|J|$-variate normal survival function with homogeneous marginal variances σ^2 and pairwise correlations ρ valued at $\mathbf{z}$, logarithmic density $-\frac{\partial_{z_j}\Phi_{\rho,\sigma}}{\Phi_{\rho,\sigma}}(\mathbf{z})$.

$W^i, \widetilde{W}^i, \bar{W}^i$ $(\mathbb{G},\mathbb{Q})$, $(\mathbb{F},\mathbb{P})$ and $(\mathbb{F},\mathbb{Q})$ Brownian motions.

$\beta^i, \widetilde{\beta}^i, \bar{\beta}^i$ $(\mathbb{G},\mathbb{Q})$, $(\mathbb{F},\mathbb{P})$ and $(\mathbb{F},\mathbb{Q})$ drift coefficients of the B^i.

$\gamma^i, \widetilde{\gamma}^i, \bar{\gamma}^i$ $(\mathbb{G},\mathbb{Q})$, $(\mathbb{F},\mathbb{P})$ and $(\mathbb{F},\mathbb{Q})$ default intensities of name i.

$m^i_t = \int_0^t \varsigma(s)dB^i_s$, $k^i_t = (\mathbb{1}_{\{\tau_i\le t\}}, \tau_i\mathbb{1}_{\{\tau_i\le t\}})$ DGC Markov primitives.

$X_t = (\mathbf{m}_t, \mathbf{k}_t)$ where $\mathbf{m}_t = (m^i_t)_{i\in N}$, $\mathbf{k}_t = (k^i_t)_{i\in N}$ Full DGC model Markov factor process.

$\widetilde{X}_t = (\mathbf{m}_t, \widetilde{\mathbf{k}}_t)$ where $\mathbf{m}_t = (m^i_t)_{i\in N}$, $\widetilde{\mathbf{k}}_t = (\mathbb{1}_{i\in N^\star} k^i_t)_{i\in N}$ Reduced DGC model Markov factor process.

7.1 Model of Default Times

We consider a multivariate Brownian motion $\mathbf{B} = (B^i)_{i\in N}$, with pairwise correlation $\varrho \in [0,1)$, in its own completed filtration $\mathbb{B} = (\mathcal{B})_{t\geq 0}$. Specifically, for $i \in N$, we assume

$$B^i_t = \sqrt{\varrho} Z_t + \sqrt{1-\varrho} Z^i_t, \tag{7.1}$$

where Z and the Z^i are independent Brownian motions. For any $i \in N$, let h_i be a continuously differentiable increasing function from $\mathbb{R}^*_+$ to $\mathbb{R}$, with derivative denoted by $\dot{h}_i$, such that $\lim_0 h_i(s) = -\infty$ and $\lim_{+\infty} h_i(s) = +\infty$, and let

$$\tau_i = h_i^{-1}\Big(\int_0^{+\infty} \varsigma(u) dB^i_u\Big), \tag{7.2}$$

where ς is a square integrable function with unit L^2 norm. As a consequence, the $(\tau_i)_{i\in N}$ follow the standard one-factor Gaussian copula model with correlation parameter ϱ and with marginal survival function $\Phi \circ h_i$ of τ_i, where Φ is the standard normal survival function. In particular, as $\varrho < 1$, the τ_i avoid each other. In order to make the model dynamic as required by counterparty risk applications, we introduce the model filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ given as the Brownian filtration $\mathbb{B}$ progressively enlarged by the τ_i, augmented so as to satisfy the usual conditions. Note that the τ_i are $\mathcal{B}_\infty$ measurable, totally inaccessible $\mathbb{G}$ stopping times. Let

$$m^i_t = \int_0^t \varsigma(u) dB^i_u,\ \ k^i_t = (\mathbb{1}_{\{\tau_i\leq t\}}, \tau_i \mathbb{1}_{\{\tau_i\leq t\}}),\ \ \mathbf{m}_t = (m^i_t)_{i\in N},\ \ \mathbf{k}_t = (k^i_t)_{i\in N}.$$

The reason why we consider $\tau_i \mathbb{1}_{\{\tau_i\leq t\}}$ on top of $\mathbb{1}_{\{\tau_i\leq t\}}$ in k^i_t is because of a dependence of prices on past default times in the DGC model. Augmenting the factor process in this way allows to take care of this path-dependence. Conversely, only having $\tau_i \mathbb{1}_{\{\tau_i\leq t\}}$ in k^i_t would lead to time discontinuous functions in the representations (7.8), which would be nonstandard from the point of view of PDE (continuous) viscosity solutions in Corollaries 71, 81 and 82. Note that $\tau_i \mathbb{1}_{\{\tau_i\leq t\}} \leq t$, hence the process $\mathbf{k}$ is bounded on $[0,T]$.

Additional notation is required for stating a number of explicit formulas that hold in the DGC model. Let $\alpha^2(t) = \int_t^{+\infty} \varsigma^2(v)dv$, assumed positive for all t for simplicity. Denoting by I and J generic subsets of N, meant to represent sets of defaulted and alive obligors in the financial interpretation, and for $j \in N$, we write:

$$Z^{j,I}_t(u) = \frac{h_j(u) - m^j_t}{\alpha(t)} - \frac{\varrho}{(|I|-1)\varrho + 1} \sum_{i\in I} \frac{h_i(\tau_i) - m^i_t}{\alpha(t)},$$

$$\rho^I = \frac{\varrho}{|I|\varrho + 1}, \quad (\sigma^I)^2 = \frac{(|I|-1)\varrho + 1 - \varrho^2|I|}{(|I|-1)\varrho + 1}.$$

In addition, we define

$$Z^j_t = Z^{j,I}_t(t),\ \ \rho_t, \sigma_t \ = \ \rho^I, \sigma^I \ \text{ on } \ \{I = \mathcal{I}_t\},$$

where $\mathcal{I}_t$ represents the set of obligors in N that are in default at time t. Let also

$$\mathcal{J}_t = N \setminus \mathcal{I}_t,\ \ \mathcal{Z}_t = \big(Z^j_t,\, j \in \mathcal{J}_t\big).$$

For any $\sigma > 0$ and $\rho \in [0,1]$, we write:

$$\Phi_{\rho,\sigma}(\mathbf{z}) = \mathbb{Q}(Z_j > z_j, j \in J),\ \ \psi^j_{\rho,\sigma}(\mathbf{z}) = -\frac{\partial_{z_j}\Phi_{\rho,\sigma}}{\Phi_{\rho,\sigma}}(\mathbf{z}), \tag{7.3}$$

where $\mathbf{z} = (z_j)_{j\in J}$ is a real vector and $\mathbf{Z} = (Z_j)_{j\in J}$ is a $|J|$-dimensional centered Gaussian vector under $\mathbb{Q}$, with homogeneous marginal variances σ^2 and pairwise correlations ρ. The standard normal survival and density functions are respectively denoted by Φ and ϕ.

Theorem 3 *The dynamic Gaussian copula model is a multivariate density model of default times, with conditional Lebesgue density* $p_t(t_i, i \in N) = \partial_{t_{-1}} \dots \partial_{t_n} \mathbb{Q}(\tau_i < t_i,\, i \in N \,|\, \mathcal{B}_t)$ *of the* τ_i, $i \in N$, *given, for any nonnegative* t_i, $i \in N$, *and* $t \in \mathbb{R}_+$, *by*

$$p_t(t_i,\, i \in N) = \int_{\mathbb{R}} \phi(y) \prod_{i \in N} \phi\left(\frac{h_i(t_i) - m_t^i + \alpha(t)\sqrt{\varrho} y}{\alpha(t)\sqrt{1-\varrho}}\right) \frac{\dot{h}_i(t_i)}{\alpha(t)\sqrt{1-\varrho}} dy. \tag{7.4}$$

For any $j \in N$, τ_j *admits a* $(\mathbb{G}, \mathbb{Q})$ *intensity given, for* $t \in \mathbb{R}_+$, *by:*

$$\mathbb{1}_{\{\tau_j \geq t\}} \big(\frac{\dot{h}_j}{\alpha}\big)(t) \psi^j_{\rho_t, \sigma_t}(\mathcal{Z}_t) = \gamma_j(t, \mathbf{m}_t, \mathbf{k}_t), \tag{7.5}$$

with predictable version

$$\gamma_t^j = \gamma_j(t, \mathbf{m}_t, \mathbf{k}_{t-}). \tag{7.6}$$

Proof. The expression for the conditional density p of the τ_i given $\mathcal{B}_t$ is obtained by differentiation of their conditional survival function, given in Crépey and an Introductory Dialogue by D. Brigo) (2014, page 172)[5]. The existence of the $(\mathbb{G}, \mathbb{Q})$ intensity of τ_j and its expression

$$-\mathbb{1}_{\{\tau_j \geq t\}} \frac{\partial_u \Phi_{\rho_t, \sigma_t}\Big(Z_t^{j,I}(u); Z_t^l,\, l \in \mathcal{J}_t \setminus \{j\}\Big) |_{u=t, I=\mathcal{I}_t}}{\Phi_{\rho_t, \sigma_t}\Big(Z_t^j; Z_t^l,\, l \in \mathcal{J}_t \setminus \{j\}\Big)},$$

which yields (7.5), can be established based on the definition of the intensity as derivative of the dual predictable projection (or by application of the Laplace formula). ∎

Theorem 4 *For every* $t \geq 0$, *we have*

$$\mathcal{G}_t = \mathcal{B}_t \vee \bigvee_{i \in N} \big(\sigma(\tau_i \wedge t) \vee \sigma(\{\tau_i > t\})\big). \tag{7.7}$$

There exist processes β_t^i *and* γ_t^i *of the form*

$$\beta_t^i, \;\; \gamma_t^i = \beta_i, \gamma_i\,(t, \mathbf{m}_t, \mathbf{k}_{t-}), \tag{7.8}$$

for continuous functions β_i *and* γ_i *with linear growth in* $\mathbf{m}$, *such that the* $dW_t^i = dB_t^i - \beta_t^i dt$ *are* $(\mathbb{G}, \mathbb{Q})$ *Brownian motions and the* $(\mathbb{G}, \mathbb{Q})$ *compensated default indicator processes are written as* $dM_t^i = d\mathbb{1}_{\tau_i \leq t} - \gamma_t^i dt$, $i \in N$. *The* W^i *and the* M^i, $i \in N$, *have the* $(\mathbb{G}, \mathbb{Q})$ *martingale representation property. The process* $X = (\mathbf{m}, \mathbf{k})$ *is a* $(\mathbb{G}, \mathbb{Q})$ *jump diffusion.*

Proof. The functions and processes γ_i and γ^i are the ones in (7.5) and (7.6). The continuity of the function γ_i is apparent on the formula (7.5), which, in view of Lemma A2, also shows the linear growth of γ_i in $\mathbf{m}$. The β^i can be computed by making use of the probability measure $\mathbb{Q}^T$, classical in the study of density models, such that $\frac{d\mathbb{Q}^T}{d\mathbb{Q}} \propto p_T(\tau_j,\, j \in N)$, for every $T > 0$. More precisely, the τ_i are $\mathbb{Q}^T$ independent between them and from $\mathcal{B}_T$ (cf. Theorem 4.7 in Song (2013a)), so that the B^i are $(\mathbb{G}, \mathbb{Q}^T)$ Brownian motions on $[0, T]$, for every $T > 0$. Hence, their $(\mathbb{G}, \mathbb{Q})$ drifts β^i can be obtained by application of a Girsanov formula from $(\mathbb{G}, \mathbb{Q}^T)$ to $(\mathbb{G}, \mathbb{Q})$, $T > 0$, which reveals the functional dependence in (7.8) with the claimed properties for the β^i. The martingale representation property and (7.7) are proved by induction over the cardinality of N as follows. We write $\mathbb{G} = \mathbb{G}^N$. If N is reduced to a singleton, then the density property of τ given $\mathcal{B}_t$ implies the results, by the optional splitting formula (6.6) for (7.7) and by Jeanblanc and Song (2013, Theorem 6.4) for the martingale representation property. If N' is obtained by adding a new name, say $(n+1)$, to N, then the density properties of τ_{n+1} and of $(\tau_i)_{i \in N}$ given $\mathcal{B}_t$ imply the density property of τ_{n+1} given $\mathcal{B}_t \vee \bigvee_{i \in N} \big(\sigma(\tau_i \wedge t) \vee \sigma(\{\tau_i > t\})\big)$. Hence, the results for $\mathbb{G}^{N'}$ follow likewise from those, if assumed, for $\mathbb{G}^N$. Finally, the Itô formula that applies to the semimartingale $X = (\mathbf{m}, \mathbf{k})$ shows that X is a $(\mathbb{G}, \mathbb{Q})$ jump diffusion (for the details see Crépey and an Introductory Dialogue by D. Brigo) (2014, Corollary 7.2.7 page 177)[6] and see also Heath and Schweizer (2000), Becherer and Schweizer (2005)). ∎

[5] Or Crépey et al. (2013, page 3) in the journal version.

[6] Or Crépey et al. (2013, Corollary 2.2) in the journal version.

7.2 DGC TVA Model

A DGC setup can be used as a TVA model for credit derivatives, with mark $i=-1,0$ and $E_b = \{-1\},\ E_c=\{0\}$. Since there are no joint defaults in this model, it is harmless to assume that the contract promises no cashflow at τ, i.e. $\Delta_\tau = 0$, so that (cf. (5.2) with currently $\delta=0$)

$$\varepsilon_c = (P_\tau - C^c_\tau)^+,\ \ \varepsilon_b = (P_\tau - C^b_\tau)^-.$$

The results of Crépey and an Introductory Dialogue by D. Brigo) (2014, Corollaries 7.3.1 page 178 and 7.3.3 page 181)[7] show that in the case of a portfolio of vanilla credit derivatives on names in N, e.g. CDS contracts and CDO tranches as of (6.12), we have a semi-explicit formula for P of the form

$$P_t = P(t, \mathbf{m}_t, \mathbf{k}_t), \tag{7.9}$$

for a continuous function P.

Remark 71 In the function $P(t,\mathbf{m},\mathbf{k})$, the domain of definition of the variables t_i corresponding to the second components of the $k_i \in \{0,1\}\times\mathbb{R}_+$, $i\in N$, is given as the union of all the strict order sets of t_i for $i\in I$, i.e. the sets of the form $\{(t_i)_{i\in I}; t_{\pi(1)} < \cdots < t_{\pi(|I|)}\}$, for all the subsets $I\subseteq N$ and all the bijections π of $\{1,\cdots,|I|\}$ to I. Indeed, the τ_i, which correspond to the second components of the k_i in the probabilistic interpretation, only take their values in this union. Here and below any continuity statement with respect to the t_i is meant in the sense of the corresponding domain and topology.

We assume that for every process $U = P$, C^c and C^b, there exists a continuous function $\widetilde{U} = \widetilde{U}_i(t,\mathbf{m},\mathbf{k})$ such that

$$U_\tau = \widetilde{U}_i(\tau, \mathbf{m}_\tau, \mathbf{k}_{\tau-}), \tag{7.10}$$

or $\widetilde{U}^i_\tau$ in a shorthand notation[8], holds on the event $\{\tau=\tau_i\}$, $i=-1,0$. For vanilla credit derivatives on names in N satisfying (7.9), e.g. CDS contracts and CDO tranches as of (6.12), the above condition always holds regarding $U=P$ for vanilla credit derivatives on names in N, since we have

$$P_\tau = P(\tau,\mathbf{m}_\tau,\mathbf{k}_\tau) = P(\tau,\mathbf{m}_\tau,\mathbf{k}^{i,\tau}_{\tau-}) \text{ on } \{\tau=\tau_i\}, \tag{7.11}$$

where $\mathbf{k}^{i,t}$ denotes the vector obtained from $\mathbf{k}$ by replacing the component with index i by $(1,t)$. Note that, since the i^{th} component of $\mathbf{k}^{i,\tau}_{\tau-}$ equals $(1,\tau)$ and not $(0,0)$, the DGC functions $\widetilde{P}^i$ incorporate a wrong-way risk effect through the spike of intensities of surviving names at other names' defaults (this will be illustrated numerically by the left graph in Figure 7). The conditions (7.10) regarding $U=C^c$ and C^b may be satisfied or not depending on the CSA. In view of (6.8) and (6.10), we have

$$\begin{aligned}
\widetilde{\xi}^i_t &= \mathbb{1}_{i=0}(1-R_c)(\widetilde{P}^i_t - \widetilde{C}^{c,i}_t)^+ - \mathbb{1}_{i=-1}(1-R_b)(\widetilde{P}^i_t - \widetilde{C}^{b,i}_t)^-,\ \ i=-1,0, \\
\widehat{f}_t(\vartheta) + (r_t+\gamma_t)\vartheta &= (1-R_c)\gamma^0_t\big(\widetilde{P}^0_t - \widetilde{C}^{c,0}_t\big)^+ - (1-R_b)\gamma^{-1}_t\big(\widetilde{P}^{-1}_t - \widetilde{C}^{b,-1}_t\big)^- \\
&\quad + c_tC_t + \widetilde{\lambda}_t\,(P_t - C_t - \vartheta)^+ - \lambda_t\,(P_t-C_t-\vartheta)^-,\ \ t\in[0,\bar{\tau}],
\end{aligned} \tag{7.12}$$

where $\gamma_t = \gamma^0_t + \gamma^{-1}_t$ and $\widetilde{\lambda}_t = \bar{\lambda}_t - \Lambda\gamma^{-1}_t$. We assume that the processes $r, c, \lambda, \bar{\lambda}, P$ and C are given before τ as continuous functions of $(t,\widetilde{X}_t)$, where $\widetilde{X}_t = (\mathbf{m}_t, \widetilde{\mathbf{k}}_t)$ with $\widetilde{\mathbf{k}}_t = (\mathbb{1}_{i\in N^\star}k^i_t)_{i\in N}$. Regarding P, (7.9) shows that this property always holds in the case of vanilla credit derivatives on names in N. Note that, in view of (7.8), this property is also satisfied by the process $\gamma(t,\mathbf{m}_t,\mathbf{k}_t)$, where we define the function γ as $\gamma_0+\gamma_{-1}$.

Remark 72 In the DGC or in the DMO model without cure period, the only modification required to deal with path-dependent margins tracking the mark-to-market P at discrete grid times as described in the remark 51 would be to augment the pre-default factor process $\widetilde{X}_t$ by additional margin components, as already done for reasons pertaining to the cure period in the DMO setup of Sect. 8.3.

[7] Or Crépey et al. (2013, Corollaries 3.1 and 3.2) in the journal version.

[8] And to alleviate the notation we rewrite $(\widetilde{C^c})^i$ as $\widetilde{C}^{c,i}$, $(\widetilde{C^c})_i$ as $\widetilde{C}^c_i$, etc..

In addition to the notation introduced before Theorem 3, we write, for $j \in N$:

$$d\zeta_t^{j,I}, \text{ the } (\mathbb{F},\mathbb{Q}) \text{ martingale component of } \left(-\frac{1}{\alpha(t)}dm_t^j + \frac{\varrho}{(|I|-1)\varrho+1}\sum_{i\in I}\frac{1}{\alpha(t)}dm_t^i\right),$$

$$Z_t^{\star,j} = Z_t^{j,I}(t),\ \ d\zeta_t^j = d\zeta_t^{j,I} \ \text{ on } \ \{I = \mathcal{I}_t^\star\},$$

where $\mathcal{I}_t^\star$ represents the set of obligors in $N^\star$ that are in default at time t. Let also

$$\mathcal{J}_t^\star = N^\star \setminus \mathcal{I}_t^\star,\ \ \mathcal{Z}_t^\star = \big(Z_t^{\star,j},\, j\in\mathcal{J}_t^\star\big),\ \ \widetilde{\mathcal{Z}}_t = \big(Z_t^{\star,-1}, Z_t^{\star,0}; Z_t^{\star,j},\, j\in\mathcal{J}_t^\star\big).$$

Note that $\mathbb{P}\neq\mathbb{Q}$ in (DGC.2) below, hence the DGC model is a case of "no immersion" in the sense of the comments following the statement of the condition (C).

Theorem 5 *The condition (C) holds, for:*

(DGC.1) *a reference filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ in (C.1) given as $\mathbb{B}$ progressively enlarged by the default times for $i\in N^\star$, which satisfies*

$$\mathcal{F}_t = \mathcal{B}_t \vee \bigvee_{i\in N^\star}\big(\sigma(\tau_i\wedge t)\vee\sigma(\{\tau_i>t\})\big),\ \ t\geq 0, \tag{7.13}$$

(DGC.2) *a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that a family of $(\mathbb{F},\mathbb{P})$ martingales with the $(\mathbb{F},\mathbb{P})$ martingale representation property is given by the*

$$d\widetilde{W}_t^i = dB_t^i - \widetilde{\beta}_t^i dt,\ i\in N \text{ and } d\widetilde{M}_t^i = d\mathbb{1}_{\tau_i\leq t} - \widetilde{\gamma}_t^i dt,\ i\in N^\star, \tag{7.14}$$

where

$$\widetilde{\beta}_t^i := \beta_i(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_{t-}) = \beta_i(t,\widetilde{X}_{t-}),\ \ \widetilde{\gamma}_t^i := \gamma_i(t,\mathbf{m}_t,\widetilde{\mathbf{k}}_{t-}) = \gamma_i(t,\widetilde{X}_{t-}), \tag{7.15}$$

(DGC.3) *a Markov specification*

$$\widetilde{f}_t(\vartheta) = \widetilde{f}(t,\widetilde{X}_t,\vartheta) \tag{7.16}$$

in (C.3), for the $(\mathbb{F},\mathbb{P})$ jump diffusion $\widetilde{X}_t = (\mathbf{m}_t,\widetilde{\mathbf{k}}_t)$ and for the function $\widetilde{f}=\widetilde{f}(t,\widetilde{x},\vartheta)$ given, writing $\widetilde{x}=(\mathbf{m},\widetilde{\mathbf{k}})$ for every $\mathbf{m}=(m_i)_{i\in N}\in\mathbb{R}^N$ and $\widetilde{\mathbf{k}}=(k_i)_{i\in N}\in\{(0,0)\}\times\{(0,0)\}\times(\{(0,0)\}\cup(\{1\}\times\mathbb{R}_+))^{N^\star}$, by:

$$\begin{aligned}\widetilde{f}(t,\widetilde{x},\vartheta)+(r(t,\widetilde{x})+\gamma(t,\widetilde{x}))\vartheta &= (1-R_c)\gamma_0\left(\widetilde{P}_0-\widetilde{C}_0^c\right)^+(t,\widetilde{x}) - (1-R_b)\gamma_{-1}\left(\widetilde{P}_{-1}-\widetilde{C}_{-1}^b\right)^-(t,\widetilde{x})\\ &+\left(cC+\widetilde{\lambda}\,(P-C-\vartheta)^+ - \lambda\,(P-C-\vartheta)^-\right)(t,\widetilde{x}),\end{aligned} \tag{7.17}$$

where $\gamma(t,\widetilde{x})=\gamma_0(t,\widetilde{x})+\gamma_{-1}(t,\widetilde{x})$ and $\widetilde{\lambda}(t,\widetilde{x})=\bar{\lambda}(t,\widetilde{x})-\gamma_{-1}(t,\widetilde{x})\Lambda$.

In addition, $(\mathbb{F},\mathbb{P})$ local martingales don't jump at τ, τ avoids $\mathbb{F}$ stopping times, the condition (B) is satisfied and the Azéma supermartingale S of τ is given, for $t\in\mathbb{R}_+$, by

$$S_t = \frac{\Phi_{\rho_t^\star,\sigma_t^\star}\big(\widetilde{\mathcal{Z}}_t\big)}{\Phi_{\rho_t^\star,\sigma_t^\star}\big(\mathcal{Z}_t^\star\big)} > 0. \tag{7.18}$$

In particular, the DGC model is a marked default times setup where the full and reduced BSDEs are equivalent.

Proof. Whatever needs be proven in (DGC.1) and (DGC.3), namely, the formula (7.13) in (DGC.1) and the Markov property of $\widetilde{X}$ in (DGC.3), can be addressed as the analogous statements in Theorem 4, whereas the existence of a probability measure satisfying (DGC.2) will be proven separately in Sect. 7.3.

The expression (7.18) for the Azéma supermartingale S of τ results from the following "multiname key lemma formula" (cf. Crépey and an Introductory Dialogue by D. Brigo) (2014, Lemma 13.7.6 page

333)[9]), which can be established in any density model of default times thanks to the optional splitting formula (6.6):

$$S_t = \frac{\mathbb{Q}\big(\tau > t; \tau_j > t,\ j \in \mathcal{J}_t^\star \,|\, \mathcal{B}_t \vee \bigvee_{i\in\mathcal{I}_t^\star} \sigma(\tau_i)\big)}{\mathbb{Q}\big(\tau_j > t,\ j \in \mathcal{J}_t^\star \,|\, \mathcal{B}_t \vee \bigvee_{i\in\mathcal{I}_t^\star} \sigma(\tau_i)\big)}.$$

(DGC.1-2-3) obviously imply (C.1) and (C.3). In view of (7.15), each $(\mathbb{F},\mathbb{P})$ martingale in (7.14), stopped at $(\tau-)$ or, equivalently by avoidance between the DGC times τ_i, stopped at τ, is a $(\mathbb{G},\mathbb{Q})$ local martingale. Hence, (DGC.2) implies (C.2) via the $(\mathbb{F},\mathbb{P})$ martingale representation property that is included in (DGC.2). This also shows that $(\mathbb{F},\mathbb{P})$ local martingales don't jump at τ. By He et al. (1992, Theorem 5.27 1)) applied to the indicator process $\mathbb{1}_{\{\tau=\nu\}}$, where ν is an arbitrary $\mathbb{F}$ predictable stopping time, we have $\mathbb{Q}(\tau=\nu)=0$ as soon as the $(\mathbb{F},\mathbb{Q})$ drift of S is continuous, as it is in the DGC model in view of the formula (7.18) for S. Besides, by the $(\mathbb{F},\mathbb{Q})$ martingale representation property in this model (see the end of the proof of Theorem 6), the $(\mathbb{F},\mathbb{Q})$ compensated martingale of the default indicator process of an $\mathbb{F}$ totally inaccessible stopping time ν only jumps at the τ_i, $i\in N^\star$. But, by the same argument as for $(\mathbb{F},\mathbb{P})$ above, $(\mathbb{F},\mathbb{Q})$ local martingales don't jump at τ. Hence, $\mathbb{Q}(\tau=\nu)=0$. We conclude that τ avoids all $\mathbb{F}$ stopping times. To check the condition (B), by the monotone class theorem, we only need consider the elementary $\mathbb{G}$ predictable processes of the form $U=\nu f(\tau_c\wedge s,\tau_b\wedge s)\mathbb{1}_{(s,t]}$, for an $\mathcal{F}_s$ measurable random variable ν and a Borel function f. Since $U\mathbb{1}_{(0,\tau]}=\nu f(s,s)\mathbb{1}_{(s,t]}\mathbb{1}_{(0,\tau]}$, we may take $U'=\nu f(s,s)\mathbb{1}_{(s,t]}$ in the condition (B). The statements in the last line of the theorem follow from the other results by Lemma 62 and in view of the explanations given in the last paragraph of Sect. 4. ∎

Remark 73 For merely establishing the condition (C) in the DGC model, a shortcut would be to use the sufficiency condition of Crépey and Song (2015a, Theorem 5.1), namely, proving the exponential integrability of $\int_0^\tau \gamma_s ds$. However, ultimately, just like the proof of the existence of a probability measure satisfying (DGC.2) in Sect. 7.3, this alternative proof would rely on the Gaussian estimates of Sect. A. Moreover, Theorem 5 yields additional results of independent interest, such as the complete description of the $(\mathbb{F},\mathbb{P})$ local martingales in (DGC.2) (which is useful for BSDE applications) and the final statements in the theorem.

Corollary 71 *In the DGC model we have $\gamma'=\gamma(\cdot,\widetilde{X}_{\cdot-})\in\widetilde{\mathcal{H}}_2$. Assuming all the other conditions in Theorem 2 and no cure period, so for $\delta=0$ and $\widetilde{f}=\widetilde{f}(t,\widetilde{X},\vartheta)=\widetilde{f}(t,\mathbf{m},\widetilde{\mathbf{k}},\vartheta)$ as of (7.17), then the corresponding reduced TVA BSDE (4.5) admits a unique square integrable solution $\widetilde{\Theta}_t=\widetilde{\Theta}(t,\widetilde{X}_t)$, where the function $\widetilde{\Theta}(t,\widetilde{x})$ is a continuous viscosity solution to the corresponding semilinear PIDE[10]. A solution Θ to the full TVA BSDE (3.6) is obtained by setting $\Theta=\widetilde{\Theta}$ on $[0,\bar{\tau})$ and*

$$\Theta_{\bar\tau}=\mathbb{1}_{\{\tau<T\}}\left(\widetilde{\xi}^{\imath}_\tau-(P_{\tau-}-C_{\tau-}-\widetilde{\Theta}_{\tau-})^+\mathbb{1}_{\{\tau=\tau_{-1}\}}\Lambda\right),$$

where $\imath=0$ or -1 denotes the identity of the defaulting counterparty (cf. (7.12)). The $(\mathbb{G},\mathbb{Q})$ local martingale component μ of Θ satisfies, for $t\in[0,\bar\tau]$:

$$\begin{aligned} d\mu_t = d\widetilde{\mu}_t &- \Big(\widetilde{\xi}^{\imath}_\tau-(P_{\tau-}-C_{\tau-}-\widetilde{\Theta}_{\tau-})^+\mathbb{1}_{\{\tau=\tau_{-1}\}}\Lambda-\widetilde{\Theta}_{\tau-}\Big)dJ_t \\ &-\Big(\sum_{i=-1,0}(\widetilde{\xi}^i_t-\widetilde{\Theta}_t)\gamma^i_t-(P_t-C_t-\widetilde{\Theta}_t)^+\gamma_t^{-1}\Lambda\Big)dt. \end{aligned} \tag{7.19}$$

Proof. We use the notation $\boldsymbol{.}$ for stochastic integration. By (DGC.2) in Theorem 5, $\widetilde{W}^i=B^i-\widetilde{\beta}^i\boldsymbol{.}\boldsymbol{\lambda}$ is an $(\mathbb{F},\mathbb{P})$ Brownian motion, for $i\in N^\star$. Set $\widetilde{m}^i=\varsigma\boldsymbol{.}\widetilde{W}^i$, hence $m^i=\widetilde{m}^i+\varsigma\widetilde{\beta}^i\boldsymbol{.}\boldsymbol{\lambda}$. For fixed $t\le T$, for any $p\ge1$, there exist constants, all denoted by the same symbol C, such that

$$\begin{aligned}\widetilde{\mathbb{E}}[\textstyle\sum_i\sup_{s\le t}|m^i_s|^p] &\le C\widetilde{\mathbb{E}}\Big[\ \textstyle\sum_i\sup_{s\le t}|\widetilde{m}^i_s|^p+\sum_i(|\varsigma\widetilde{\beta}^i|\boldsymbol{.}\boldsymbol{\lambda}_t)^p\ \Big]\\ &\le C\widetilde{\mathbb{E}}\Big[\ \textstyle\sum_i\sup_{s\le t}|\widetilde{m}^i_s|^p\ \Big]+C\widetilde{\mathbb{E}}\Big[\ (1+\sum_j\sup_{s\le\cdot}|m^j_s|^p)\boldsymbol{.}\boldsymbol{\lambda}_t\ \Big]\\ &= C\widetilde{\mathbb{E}}\Big[\ \textstyle\sum_i\sup_{s\le t}|\widetilde{m}^i_s|^p\ \Big]+C+C\widetilde{\mathbb{E}}\Big[\textstyle\sum_j\sup_{s\le\cdot}|m^j_s|^p\Big]\boldsymbol{.}\boldsymbol{\lambda}_t,\end{aligned}$$

[9] Or Crépey et al. (2013, Lemma 2.5) in the journal version.

[10] Not written as not directly used in the paper.

where the definition of the process $\widetilde{\beta}^i$ in (7.15) was used in conjunction with the linear growth of the function β_i in (7.8) to pass to the second line. Hence, by virtue of the Gronwall inequality,

$$\widetilde{\mathbb{E}}[\sum_i \sup_{s\le t} |m^i_s|^p] \le \left(1 + \widetilde{\mathbb{E}}\left[\sum_i \sup_{s\le t} |\widetilde{m}^i_s|^p\right]\right) Ce^{Ct} < \infty,$$

from which $\gamma' = \gamma(\cdot, \widetilde{X}_{\cdot-}) = \gamma_0(\cdot, \widetilde{X}_{\cdot-}) + \gamma_{-1}(\cdot, \widetilde{X}_{\cdot-}) \in \widetilde{\mathcal{H}}_2$ follows by linear growth in $\mathbf{m}$ of the functions γ_i in (7.8). As a consequence, well-posedness in $\widetilde{\mathcal{H}}_2$ of the corresponding reduced TVA BSDE (4.5) follows from Theorem 2 (assuming all the other conditions there). Since well-posedness in the sense of Theorem 2 includes comparison and the usual a priori bound and error BSDE estimates, the representation of the solution $\widetilde{\Theta}.$ to the BSDE in terms of a continuous viscosity solution to the corresponding semilinear PIDE (with continuous coefficients) follows from standard arguments (see e.g. Delong (2013) or Crépey (2013, Chapter 13)). By Theorem 5, the reduced stochastic basis $(\mathbb{F}, \mathbb{P})$ satisfies the condition (C), so that the remaining statements in the corollary follow from Theorem 1. ∎

Presumably, reinforcing if need be the assumptions on $\widetilde{f}$, one could show that the function $\widetilde{\Theta}(t, \widetilde{x})$ in Corollary 71 is a classical (not only viscosity) solution to the corresponding PIDE (see e.g. Becherer and Schweizer (2005)). In this case, an application of the Itô formula related to the $(\mathbb{F}, \mathbb{P})$ jump diffusion $\widetilde{X}_t = (\mathbf{m}_t, \widetilde{\mathbf{k}}_t)$ yields the following functional representation of $\widetilde{\mu}$ in (7.19):

$$d\widetilde{\mu}_t = \varsigma(t) \sum_{i\in N} \partial_{m_i}\widetilde{\Theta}(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_t) d\widetilde{W}^i_t + \sum_{i\in N^\star} \delta_i \widetilde{\Theta}(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_{t-}) d\widetilde{M}^i_t, \tag{7.20}$$

where $\partial_{m_i}\widetilde{\Theta}$ denotes the partial derivative of the function $\widetilde{\Theta}$ with respect to m_i and where

$$\delta_i\widetilde{\Theta}(t, \mathbf{m}, \widetilde{\mathbf{k}}) = \widetilde{\Theta}(t, \mathbf{m}, \widetilde{\mathbf{k}}^{i,t}) - \widetilde{\Theta}(t, \mathbf{m}, \widetilde{\mathbf{k}}). \tag{7.21}$$

This formula shows the nature of the TVA Greeks in the DGC model, namely the integrands $\partial_{m_i}\widetilde{\Theta}(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_t)$ and $\delta_i\widetilde{\Theta}(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_{t-})$ in (7.20). For length care, we do not conduct the above-mentioned regularity analysis. Similar comments apply and will not be repeated regarding Corollaries 81 and 82.

7.3 Proof of the Existence of a Probability Measure $\mathbb{P}$ Satisfying (DGC.2)

In this section we use the notation $\centerdot$ for stochastic integration. Let Q^c denote the continuous martingale component of the $(\mathbb{F}, \mathbb{Q})$ Azéma supermartingale S of τ.

Lemma 71 *The process $\nu^c = \frac{1}{S} \centerdot Q^c$ satisfies, for $t \in [0, T]$:*

$$d\nu^c_t = \sum_{j\in\widetilde{\mathcal{J}}_{t-}} \psi^j_{\rho^\star_t,\sigma^\star_t}(\widetilde{\mathcal{Z}}_t) d\zeta^j_t - \sum_{j\in\mathcal{J}^\star_{t-}} \psi^j_{\rho^\star_t,\sigma^\star_t}(\mathcal{Z}^\star_t) d\zeta^j_t. \tag{7.22}$$

For any $j \in N^\star$, τ_j admits an $(\mathbb{F}, \mathbb{Q})$ intensity given, for $t \in \mathbb{R}_+$, by

$$\mathbb{1}_{\{\tau_j\ge t\}}(\frac{\dot{h}_j}{\alpha})(t)\psi^j_{\rho^\star_t,\sigma^\star_t}(\mathcal{Z}^\star_t) = \bar{\gamma}_j(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_t), \tag{7.23}$$

with predictable version

$$\bar{\gamma}^j_t = \bar{\gamma}_j(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_{t-}). \tag{7.24}$$

Proof. The formula (7.22) is obtained by Itô calculus applied to the expression (7.18) of S. The statements about the $(\mathbb{F}, \mathbb{Q})$ intensity of τ_j are proven similarly to the analogous $(\mathbb{G}, \mathbb{Q})$ statements in Theorem 3. ∎

Corollary 72 *There exists a constant $C > 0$ such that, for $0 \le r \le t$ and $j \in N^\star$,*

$$\langle\nu^c\rangle_t \le C(\sum_{i\in N} \sup_{0<s\le t} |m^i_s|^2 + 1)t, \tag{7.25}$$

$$\widetilde{\gamma}^j_r \vee \bar{\gamma}^j_r \le C(\sum_{i\in N} \sup_{0<s\le t} |m^i_s| + 1), \;\; \widetilde{\gamma}^j_r \ln(\widetilde{\gamma}^j_r \vee \bar{\gamma}^j_r) \le C \sum_{i\in N} \sup_{0<s\le t} (|m^i_s| + 1)\ln(|m^i_s| + 1). \tag{7.26}$$

Proof. Applying Lemma A2 to the formula (7.22) and noting that the function α, continuous and assumed positive, is bounded away from 0 on $[0,T]$, we obtain, for positive constants C that may change from place to place:

$$\langle \nu^c \rangle_t \le C \int_0^t (\sum_{I\subseteq N} \sum_{j\in N\setminus I} |Z_s^{j,I}(s)| + 1)^2 ds \le C(\sum_{I\subseteq N} \sum_{j\in N\setminus I} \sup_{0<s\le t} |Z_s^{j,I}(s)| + 1)^2 t,$$

which yields (7.25). Applying Lemma A2 to the formulas (7.23)–(7.24) and (7.5)–(7.6) (recalling also $\widetilde{\gamma}^j_t = \gamma_j(t, \mathbf{m}_t, \widetilde{\mathbf{k}}_{t-})$), we obtain the left hand side inequality in (7.26)), whence the right hand side inequality follows from

$$\begin{aligned}\widetilde{\gamma}^j_r \ln(\widetilde{\gamma}^j_r \vee \bar{\gamma}^j_r) &\le C(\max_{i\in N} \sup_{0<s\le t} |m^i_s| + 1) \ln\big(C(\max_{i\in N} \sup_{0<s\le t} |m^i_s| + 1)\big) \\ &= \max_{i\in N} \sup_{0<s\le t} C(|m^i_s| + 1) \ln(C|m^i_s| + 1). \blacksquare\end{aligned}$$

Our strategy for constructing a probability measure $\mathbb{P}$ satisfying (DGC.2) is as follows: a tentative $\mathbb{Q}$ density of a probability measure $\mathbb{P}$ will be defined in the form of the stochastic exponential $\mathcal{E}(\nu)$ in (7.27) so that, on the one hand, the drift in the $(\mathbb{F},\mathbb{Q})$ to $(\mathbb{F},\mathbb{P})$ Girsanov measure change compensates the drift in the $(\mathbb{F},\mathbb{Q})$ to $(\mathbb{G},\mathbb{Q})$ Jeulin progressive enlargement of filtration formula for the $(\mathbb{F},\mathbb{Q})$ Brownian motions ($\bar{W}^i$ in the proof of Theorem 6) and, on the other hand, the default intensities, given by $\bar{\gamma}^i$ as of (7.24) under $(\mathbb{F},\mathbb{Q})$, become $\widetilde{\gamma}^i$ under $(\mathbb{F},\mathbb{P})$ (where, by definition in (7.15), $\widetilde{\gamma}^i$ coincides with the $(\mathbb{G},\mathbb{Q})$ intensity γ^i before τ). The proof that this strategy works is essentially a matter of checking that the tentative measure change density is a valid one, i.e. a positive $(\mathbb{F},\mathbb{Q})$ martingale, for which the estimates of Corollary 72 will be useful.

Let $\bar{M}^i = \mathbb{1}_{[\tau_i,+\infty)} - \bar{\gamma}^i \centerdot \boldsymbol{\lambda}$. We consider the $(\mathbb{F},\mathbb{Q})$ local martingale given as the Doléans-Dade exponential $\mathcal{E}(\nu)$, where $\nu = \mathbb{1}_{(0,T]} \frac{1}{S} \centerdot Q^c + \sum_{i\in N^\star} \mathbb{1}_{(0,T]}(\frac{\widetilde{\gamma}^i}{\bar{\gamma}^i} - 1) \centerdot \bar{M}^i$, i.e.

$$\mathcal{E}(\nu) = \mathcal{E}\left(\mathbb{1}_{(0,T]}\nu^c\right) \prod_{i\in N^\star} \left(1 + (\frac{\widetilde{\gamma}^i_{\tau_i}}{\bar{\gamma}^i_{\tau_i}} - 1)\mathbb{1}_{\{\tau_i\le T\}}\mathbb{1}_{[\tau_i,+\infty)}\right) \exp \int_0^{\cdot\wedge\tau_i\wedge T} \left(\bar{\gamma}^i_s - \widetilde{\gamma}^i_s\right) ds \qquad (7.27)$$

(recall $\nu^c = \frac{1}{S} \centerdot Q^c$).

Lemma 72 *There exists $\epsilon > 0$ such that, for any $s \in [0,T]$:*

$$\mathbb{E}\big[\mathcal{E}\big(\mathbb{1}_{(s,t]}\nu\big) \,\big|\, \mathcal{F}_s\big] = 1, \;\; t \in [s, s+\epsilon]. \qquad (7.28)$$

Proof. For notational simplicity, we only prove it for $s = 0$, i.e. we prove that, for t small enough, $\mathcal{E}(\nu)$ is an $(\mathbb{F},\mathbb{Q})$ martingale on $[0,t]$.

By Corollary 72, multivariate Hölder inequality and Lemma A3,

$$\exp\Big(\langle \nu^c \rangle_t + \sum_{i\in N^\star} \int_0^t (\widetilde{\gamma}^i_s \ln(\widetilde{\gamma}^i_s) - \widetilde{\gamma}^i_s \ln(\bar{\gamma}^i_s) - \widetilde{\gamma}^i + \bar{\gamma}^i_s) ds\Big)$$

is $\mathbb{Q}$ integrable for sufficiently small t. Hence the result follows by an application of Lepingle and Mémin (1978, Theorem IV.3). $\blacksquare$

Theorem 6 *$\mathcal{E}(\nu)$ is a positive $(\mathbb{F},\mathbb{Q})$ martingale and the probability measure $\mathbb{P}$ with $\mathbb{Q}$ density process $\mathcal{E}(\nu)$ satisfies the condition (DGC.2).*

Proof. Note that $\widetilde{\gamma}^i$, defined through γ^i by (7.14), is positive in view of (7.5), for any $i \in N^\star$. Hence, $\mathcal{E}(\nu) > 0$. If $T \le \epsilon$ in (7.28), then the $(\mathbb{F},\mathbb{Q})$ martingale property of $\mathcal{E}(\nu)$ directly follows from Lemma 72. Otherwise, we write

$$\mathbb{E}[\mathcal{E}(\nu)] = \mathbb{E}\left[\mathcal{E}\left(\mathbb{1}_{(0,T-\epsilon]}\nu\right)\mathbb{E}\left[\mathcal{E}\left(\mathbb{1}_{(T-\epsilon,T]}\nu\right) \,\middle|\, \mathcal{F}_{T-\epsilon}\right]\right] = \mathbb{E}\left[\mathcal{E}\left(\mathbb{1}_{(0,T-\epsilon]}\nu\right)\right],$$

by (7.28) applied with $s = T - \epsilon$ and $t = T$, so that $(\mathbb{F},\mathbb{Q})$ martingale property of $\mathcal{E}(\nu)$ follows by induction. Since $\mathcal{E}(\nu)$ is a positive $(\mathbb{F},\mathbb{Q})$ martingale, we can define a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ by the $\mathbb{Q}$ density process $\mathcal{E}(\nu)$. By the Girsanov theorem, the $(\mathbb{F},\mathbb{P})$ intensity of τ_i is $\widetilde{\gamma}^i$, $i \in N^\star$, and, denoting by $\bar{W}^i = B^i - \bar{\beta}^i \boldsymbol{\cdot} \boldsymbol{\lambda}$ the $(\mathbb{F},\mathbb{Q})$ Brownian motion obtained as the $(\mathbb{F},\mathbb{Q})$ martingale component of B^i, the process

$$\widehat{W}^i = \bar{W}^i - \langle \nu^c, B^i \rangle = B^i - (\bar{\beta}^i \boldsymbol{\cdot} \boldsymbol{\lambda} + \langle \nu^c, B^i \rangle)$$

is an $(\mathbb{F},\mathbb{P})$ Brownian motion, for each $i \in N$. Moreover, by the Jeulin formula (see e.g. Dellacherie, Maisonneuve, and Meyer (1992, no 77 Remarques b))), $\widehat{W}^i$ is a $(\mathbb{G},\mathbb{Q})$ Brownian motion until time τ, as is also $\widetilde{W}^i = B^i - \widetilde{\beta}^i \boldsymbol{\cdot} \boldsymbol{\lambda}$ in (7.14), because $\widetilde{\beta}^i_{\cdot\wedge\tau} = \beta^i_{\cdot\wedge\tau}$. Hence, $\widetilde{W}^i_{\cdot\wedge\tau} = W^i_{\cdot\wedge\tau}$. Therefore, the $\mathbb{F}$ predictable processes $\bar{\beta}^i \boldsymbol{\cdot} \boldsymbol{\lambda} + \langle \nu^c, B^i \rangle$ and $\widetilde{\beta}^i \boldsymbol{\cdot} \boldsymbol{\lambda}$ coincide until τ, hence on $[0,T]$ by positivity of S_T given by (7.18) (cf. the remark 41). In conclusion, $\widetilde{W}^i = \widehat{W}^i$, an $(\mathbb{F},\mathbb{P})$ Brownian motion, for any $i \in N^\star$. The $(\mathbb{F},\mathbb{P})$ martingale representation property of the $\widetilde{W}^i, i \in N$ and $\widetilde{M}^i, i \in N^\star$ follows by equivalent change of measure from the $(\mathbb{F},\mathbb{Q})$ martingale representation property of the $\bar{W}^i, i \in N$ and $\bar{M}^i, i \in N^\star$, which can be proven as the $(\mathbb{G},\mathbb{Q})$ martingale representation property of the $W^i, i \in N$ and $M^i, i \in N$ in Theorem 4. Thus, all the conditions hold in (DGC.2). ∎

Remark 74 $\bar{W}^i$ yields an $(\mathbb{F},\mathbb{Q})$ local martingale that, stopped at τ, or equivalently $(\tau-)$ by continuity of $\bar{W}^i$, fails to be a $(\mathbb{G},\mathbb{Q})$ local martingale. In fact, by the above computations, we have

$$\bar{W}^i_{\cdot\wedge\tau} = \widetilde{W}^i_{\cdot\wedge\tau} + J_-(\widetilde{\beta}^i - \bar{\beta}^i) \boldsymbol{\cdot} \boldsymbol{\lambda},$$

where $(\widetilde{\beta}^i - \bar{\beta}^i) \boldsymbol{\cdot} \boldsymbol{\lambda} = \langle \nu^c, B^i \rangle$ is not null on $[0,\tau]$, in view of the expression of ν^c in (7.22). This justifies the “no immersion” statement before Theorem 5.

8 Dynamic Marshall-Olkin Copula TVA Model

The above dynamic Gaussian copula model can suffice to deal with TVA on CDS contracts. But a Gaussian copula dependence structure is not rich enough for a joint calibration to CDS and tranches data. If CDO tranches are also present in a portfolio, a possible alternative is the dynamic Marshall-Olkin (DMO) copula model.

Here is an non-exhaustive list of notations introduced in the course of the study of the DMO model.

- $\mathcal{Y}$ Family of “shocks”, i.e. subsets $Y \subseteq N$ of names likely to default together in the financial interpretation.
- η_Y, γ^Y, B^Y Shock time η_Y with intensity γ^Y driven by an independent Brownian driven B^Y (or with γ^Y deterministic in an elementary DMO model specification).
- $\mathcal{Y}_b = \{Y \in \mathcal{Y};\, -1 \in Y\}, \mathcal{Y}_c = \{Y \in \mathcal{Y};\, 0 \in Y\}, \mathcal{Y}_\bullet = \mathcal{Y}_b \cup \mathcal{Y}_c, \mathcal{Y}_\circ = \mathcal{Y} \setminus \mathcal{Y}_\bullet$ Collection of the shocks triggering the default of the bank, of the counterparty, of at least one of them, of none of them.
- $\tau_i = \min_{\{Y \in \mathcal{Y}; i \in Y\}} \eta_Y, \tau_b = \tau_{-1} = \min_{Y \in \mathcal{Y}_b} \eta_Y,\ \tau_c = \tau_0 = \min_{Y \in \mathcal{Y}_c} \eta_Y$ Default time of name i in the DMO model, of the bank (i.e. name -1), of the counterparty (i.e. name 0).
- $\gamma^Y_t, H^Y_t = \mathbb{1}_{\{\eta_Y \le t\}}, K^Y_t = (\mathbb{1}_{\{\eta_Y \le t\}}, \eta_Y \mathbb{1}_{\{\eta_Y \le t\}})$ DMO Markov primitives.
- $X_t = (\boldsymbol{\Gamma}_t, \mathbf{H}_t)$ where $\boldsymbol{\Gamma} = (\gamma^Y)_{Y \in \mathcal{Y}}$, $\mathbf{H} = (H^Y)_{Y \in \mathcal{Y}}$ Full DMO model Markov factor process (case where $\delta = 0$).
- $\widetilde{X}_t = (\boldsymbol{\Gamma}_t, \widetilde{\mathbf{H}}_t)$ where $\boldsymbol{\Gamma} = (\gamma^Y)_{Y \in \mathcal{Y}}$, $\widetilde{\mathbf{H}} = (\mathbb{1}_{Y \in \mathcal{Y}_\circ} H^Y)_{Y \in \mathcal{Y}}$. Reduced DMO model Markov factor process (case where $\delta = 0$).
- $X_t = (t, \boldsymbol{\Gamma}_t, \mathbf{K}_t)$ where $\boldsymbol{\Gamma} = (\gamma^Y)_{Y \in \mathcal{Y}}$, $\mathbf{K} = (K^Y)_{Y \in \mathcal{Y}}$ Full DMO model Markov factor process in the cure period case where $\delta > 0$.

8.1 Model of Default Times

We define a family $\mathcal{Y}$ of “shocks”, i.e. subsets $Y \subseteq N$ of obligors, usually consisting of the singletons $\{-1\}$, $\{0\}$, $\{1\}$, …, $\{n\}$, and of a few “common shocks” representing simultaneous defaults. The shock intensities are given in the form of extended CIR processes as, for every $Y \in \mathcal{Y}$,

$$d\gamma^Y_t = a(b_Y(t) - \gamma^Y_t)dt + c\sqrt{\gamma^Y_t}\,dB^Y_t, \tag{8.1}$$

for nonnegative constants a and c, continuous functions $b_Y(t)$ and independent Brownian motions B^Y in their own completed filtration $\mathbb{B} = (\mathcal{B}_t)_{t\geq 0}$, under the risk-neutral measure $\mathbb{Q}$. In fact, one could use any independent Markov processes γ^Y with semi-analytic formulas for $\mathbb{E}e^{-\int_0^t \gamma_s^Y ds}$, for calibration purposes, and square integrable, for the sake of (5.10). The case of deterministic intensities $\gamma_t^Y = b_Y(t)$ is treated in a similar fashion. We emphasize that, even if we don't engage into any calibration exercise in this paper, the empirical study in Bielecki et al. (2014a, Part II) shows that this model is efficiently calibratable to CDS and CDO market data, including at the peak of the 2007–2008 credit crisis. Shock random times (avoiding each other) and their indicator processes are defined by

$$\eta_Y = \inf\{t > 0; \int_0^t \gamma_s^Y ds > \epsilon_Y\} \text{ and } H_t^Y = \mathbb{1}_{\{\eta_Y \leq t\}},\ \ Y \in \mathcal{Y}, \tag{8.2}$$

where the ϵ_Y are i.i.d. standard exponential random variables. The full model filtration $\mathbb{G}$ is given as $\mathbb{B}$ progressively enlarged by the random times η_Y, $Y \in \mathcal{Y}$. Let M^Y denote the compensated martingale $dM_t^Y = dH^Y - (1-H_t^Y)\gamma_t^Y dt,\ t \geq 0$. We define $\boldsymbol{\Gamma} = (\gamma^Y)_{Y\in\mathcal{Y}}, \mathbf{H} = (H^Y)_{Y\in\mathcal{Y}}$ and $\tau_i = \min_{\{Y\in\mathcal{Y}; i\in Y\}} \eta_Y$, $i \in N$.

Theorem 7 *The dynamic Marshall-Olkin (or common-shock) model is a recursively immersed model of default times. For* $t \geq 0$,*we have*

$$\mathcal{G}_t = \mathcal{B}_t \vee \bigvee_{Y\in\mathcal{Y}} \big(\sigma(\eta_Y \wedge t) \vee \sigma(\{\eta_Y > t\})\big). \tag{8.3}$$

The B^Y *and the* M^Y, $Y \in \mathcal{Y}$, *have the* $(\mathbb{G}, \mathbb{Q})$ *martingale representation property.*

Proof. We prove the martingale representation property and (8.3) by induction as follows. We write $\mathbb{G} = \mathbb{G}^{\mathcal{Y}}$. If $\mathcal{Y}$ is a singleton (case of a Cox time in view of (8.2)), then the immersion of $\mathbb{B}$ into $\mathbb{G}^{\mathcal{Y}}$ implies the results, by the optional splitting formula (6.6) for (8.3) and by Jeanblanc and Song (2013, Theorem 6.4) for the martingale representation property. Moreover, if $\mathcal{Z}$ is obtained by addition of a new $Z \subseteq N$ to $\mathcal{Y}$, then the independence of the ϵ_Y implies that η_Z is a Cox time with intensity in $\mathbb{G}^{\mathcal{Y}}$, hence immersion of $\mathbb{G}^{\mathcal{Y}}$ into $\mathbb{G}^{\mathcal{Z}}$ follows (this is the recursively immersed feature stated in the lemma) and the results for $\mathbb{G}^{\mathcal{Z}}$ are implied likewise from those, if assumed, for $\mathbb{G}^{\mathcal{Y}}$. ∎

8.2 DMO TVA Model

A DMO setup can be used as a TVA model for credit derivatives, with

$$E_b = \mathcal{Y}_b := \{Y \in \mathcal{Y};\, -1 \in Y\},\ \ E_c = \mathcal{Y}_c := \{Y \in \mathcal{Y};\, 0 \in Y\},\ \ E = \mathcal{Y}_\bullet := \mathcal{Y}_b \cup \mathcal{Y}_c.$$

In particular,

$$\tau_b = \tau_{-1} = \min_{Y\in\mathcal{Y}_b} \eta_Y,\ \ \tau_c = \tau_0 = \min_{Y\in\mathcal{Y}_c} \eta_Y,\ \text{hence } \tau = \min_{Y\in\mathcal{Y}_\bullet} \eta_Y,\ \ \gamma = J_- \sum_{Y\in\mathcal{Y}_\bullet} \gamma^Y. \tag{8.4}$$

The results of Crépey and an Introductory Dialogue by D. Brigo) (2014, Corollary 8.3.1 page 205)[11] show that in the case of a portfolio of vanilla credit derivatives on names in N, e.g. CDS contracts and CDO tranches as of (6.12), we have a semi-explicit formula for P of the form

$$P_t = P(t, \boldsymbol{\Gamma}_t, \mathbf{H}_t), \tag{8.5}$$

for a continuous function P. We assume that for every process $U = P$, $(D - D_-)$, C^c and C^b, there exists a continuous function $\widetilde{U} = \widetilde{U}_Y(t, \boldsymbol{\gamma}, \mathbf{k})$, rewritten for brevity $\widetilde{\Delta}$ in the case of $(D - D_-)$ (noting that, in the present case where $\delta = 0$, the process $(D - D_-)$ plays the role of the process Δ when $\delta \neq 0$), such that

$$U_\tau = \widetilde{U}_Y(\tau, \boldsymbol{\Gamma}_\tau, \mathbf{H}_{\tau-}), \tag{8.6}$$

[11] Or Bielecki et al. (2014a, Part II, Corollary 3.1) in the journal version.

or $\widetilde{U}^Y_\tau$ in a shorthand notation[12], holds on every event of the form $\{\tau = \eta_Y\}$, $Y \in \mathcal{Y}_\bullet$. For vanilla credit derivatives on names in N, e.g. CDS contracts and CDO tranches as of (6.12), the above condition always holds regarding $U = P$, by (8.5) and the DMO analog of the DGC identity (7.11), and in view of (6.12) it also holds for $U = D - D_-$. As will be illustrated by the right graph in our concluding figure 7, the $\widetilde{\Delta}^Y$ convey the gap risk effect in the DMO model. The conditions (8.6) on $U = C^c$ and C^b may be satisfied or not depending on the CSA (see the remark 72). In view of (6.8), the coefficient $\widetilde{\xi}$ (in the present no cure period case where $\delta = 0$) is given as

$$\widetilde{\xi}^Y_t = \mathbb{1}_{Y\in\mathcal{Y}_c}(1-R_c)\big(\widetilde{P}^Y_t + \widetilde{\Delta}^Y_t - \widetilde{C}^{c,Y}_t\big)^+ - \mathbb{1}_{Y\in\mathcal{Y}_b}(1-R_b)\big(\widetilde{P}^Y_t + \widetilde{\Delta}^Y_t - \widetilde{C}^{b,Y}_t\big)^-,\ \ Y \in \mathcal{Y}_\bullet. \tag{8.7}$$

The coefficient $\widehat{f}_t(\vartheta)$ in (6.10) is given, on $[0,\bar{\tau}]$, by

$$\begin{aligned}\widehat{f}_t(\vartheta) + (r_t+\gamma_t)\vartheta = \quad & (1-R_c)\sum_{Y\in\mathcal{Y}_c}\gamma^Y_t\big(\widetilde{P}^Y_t + \widetilde{\Delta}^Y_t - \widetilde{C}^{c,Y}_t\big)^+ - (1-R_b)\sum_{Y\in\mathcal{Y}_b}\gamma^Y_t\big(\widetilde{P}^Y_t + \widetilde{\Delta}^Y_t - \widetilde{C}^{b,Y}_t\big)^- \\ & + c_tC_t + \widetilde{\lambda}_t\,(P_t - C_t - \vartheta)^+ - \lambda_t\,(P_t - C_t-\vartheta)^-,\end{aligned} \tag{8.8}$$

where $\widetilde{\lambda}_t = \bar{\lambda}_t - \Lambda\sum_{Y\in\mathcal{Y}_b}\gamma^Y_t$. Let $\mathcal{Y}_\circ = \mathcal{Y}\setminus\mathcal{Y}_\bullet$ and let $\widetilde{X}_t = (\boldsymbol{\Gamma}_t, \widetilde{\mathbf{H}}_t)$, where $\widetilde{\mathbf{H}} = (\mathbb{1}_{Y\in\mathcal{Y}_\circ}H^Y)_{Y\in\mathcal{Y}}$. We assume that the processes $r, c, \lambda, \bar{\lambda}, P$ and C are given before τ as continuous functions of $(t, \widetilde{X}_t)$. Regarding P, (8.5) shows that this property always holds in the case of vanilla credit derivatives on names in N. Note that, in view of the last identity in (8.4), this property is also verified by the process γ. The next result, stated without proof, is the DMO analog of the DGC Theorem 5. There the main difficulty, related to (DGC.2), came from the fact that we had to use $\mathbb{P}\neq\mathbb{Q}$, as opposed to $\mathbb{P}=\mathbb{Q}$ simply here. This is an easier case of immersion in the sense of the comments following the statement of the condition (C) (note that the Azéma supermartingale given by (8.10) is continuous and nonincreasing, consistent with this immersion property of the setup). Hence we state the result without proof.

Theorem 8 *The condition (C) holds, for:*

(DMO.1) *a reference filtration $\mathbb{F} = (\mathcal{F}_t)$ in (C.1) given as $\mathbb{B}$ progressively enlarged by the $\eta_Y, Y \in \mathcal{Y}_\circ$, which satisfies*

$$\mathcal{F}_t = \mathcal{B}_t \vee \bigvee_{Y\in\mathcal{Y}_\circ}\big(\sigma(\eta_Y\wedge t)\vee\sigma(\{\eta_Y > t\})\big),\ \ t\geq 0,$$

(DMO.2) *$\mathbb{P}=\mathbb{Q}$ in (C.2), where the $B^Y, Y\in\mathcal{Y}$, and the $M^Y, Y\in\mathcal{Y}_\circ$, have the $(\mathbb{F},\mathbb{P}=\mathbb{Q})$ martingale representation property,*

(DMO.3) *a Markov specification $\widetilde{f}_t(\vartheta) = \widetilde{f}(t,\widetilde{X}_t,\vartheta)$ of (C.3), for the $(\mathbb{F},\mathbb{P}=\mathbb{Q})$ jump diffusion $\widetilde{X}_t = (\boldsymbol{\Gamma}_t,\widetilde{\mathbf{H}}_t)$ and the function $\widetilde{f} = \widetilde{f}(t,\widetilde{x},\vartheta)$ given, writing $\widetilde{x} = (t,\boldsymbol{\gamma},\widetilde{\mathbf{k}})$ for $\boldsymbol{\gamma} = (\gamma_Y)_{Y\in\mathcal{Y}}\in\mathbb{R}^{\mathcal{Y}}_+$ and $\widetilde{\mathbf{k}} = (k_Y)_{Y\in\mathcal{Y}}\in\{0,1\}^{\mathcal{Y}}$ with $k_Y = 0$ if $Y\in\mathcal{Y}_\bullet$, by:*

$$\begin{aligned}&\widetilde{f}(t,\widetilde{x},\vartheta) + (r(t,\widetilde{x}) + \gamma(t,\widetilde{x}))\vartheta = \\ &\quad (1-R_c)\sum_{Y\in\mathcal{Y}_c}\gamma_Y\big(\widetilde{P}_Y + \widetilde{\Delta}_Y - \widetilde{C}^c_Y\big)^+(t,\widetilde{x}) - (1-R_b)\sum_{Y\in\mathcal{Y}_b}\gamma_Y\big(\widetilde{P}_Y + \widetilde{\Delta}_Y - \widetilde{C}^b_Y\big)^-(t,\widetilde{x}) \\ &\qquad + \big(cC + \widetilde{\lambda}\big(P-\vartheta-C\big)^+ - \lambda\big(P-\vartheta-C\big)^-\big)(t,\widetilde{x}),\end{aligned} \tag{8.9}$$

where $\gamma(t,\widetilde{x}) = \sum_{Y\in\mathcal{Y}_\bullet}\gamma_Y$, $\widetilde{\lambda} = \bar{\lambda} - \Lambda\sum_{Y\in\mathcal{Y}_b}\gamma_Y$.

In addition, $(\mathbb{F},\mathbb{P}=\mathbb{Q})$ local martingales don't jump at τ, τ avoids $\mathbb{F}$ stopping times, the condition (B) is satisfied and the Azéma supermartingale S of τ is given, for $t\in[0,T]$, by

$$S_t = e^{-\sum_{Y\in\mathcal{Y}_\bullet}\int_0^t\gamma^Y_s ds} > 0. \tag{8.10}$$

In particular, the DMO model is a marked default times setup, where the full and reduced BSDEs are equivalent.

Since the condition (C) is satisfied, we can derive the following DMO analog, stated without proof, of the DGC corollary 71.

[12] And to alleviate the notation we rewrite $(\widetilde{C^c})^Y$ as $\widetilde{C}^{c,Y}$, $(\widetilde{C^c})_Y$ as $\widetilde{C}^c_Y$, etc..

Corollary 81 *In the DMO model, we have* $\gamma' = \gamma(\cdot, \widetilde{X}_\cdot) = \sum_{Y\in\mathcal{Y}_\bullet}\gamma^Y \in \widetilde{\mathcal{H}}_2$. *Assuming all the other conditions in Theorem 2 and without cure period, so for* $\delta = 0$ *and* $\widetilde{f} = \widetilde{f}(t, \boldsymbol{\gamma}, \widetilde{\mathbf{k}}, \vartheta)$ *as of* (8.9)*, the corresponding reduced TVA BSDE* (4.5) *admits a unique square integrable solution* $\widetilde{\Theta}_t = \widetilde{\Theta}(t, \widetilde{X}_t)$*, where the function* $\widetilde{\Theta}(t, \widetilde{x})$ *is a continuous viscosity solution to the corresponding semilinear PIDE. A solution* Θ *to the full TVA BSDE* (3.6) *is obtained by setting* $\Theta = \widetilde{\Theta}$ *on* $[0, \bar{\tau})$ *and*

$$\Theta_{\bar{\tau}} = \mathbb{1}_{\{\tau<T\}}\Big(\widetilde{\xi}^{\imath}_{\tau} - (P_{\tau-} - C_{\tau-} - \widetilde{\Theta}_{\tau-})^{+}\mathbb{1}_{\{-1\in\imath\}}\Lambda\Big),$$

where $\imath \in \mathcal{Y}_\bullet$ *is the identity of the shock triggering the first default of a party. The* $(\mathbb{G},\mathbb{Q})$ *local martingale component* μ *of* Θ *satisfies, for* $t \in [0, \bar{\tau}]$*:*

$$\begin{aligned} d\mu_t = d\widetilde{\mu}_t &- \Big(\widetilde{\xi}^{\imath}_{\tau} - \widetilde{\Theta}_{\tau-} - (P_{\tau-} - C_{\tau-} - \widetilde{\Theta}_{\tau-})^{+}\mathbb{1}_{\{-1\in\imath\}}\Lambda\Big)dJ_t \\ &- \Big(\sum_{Y\in\mathcal{Y}_\bullet}(\widetilde{\xi}^{Y}_t - \widetilde{\Theta}_t)\gamma^Y_t - (P_t - C_t - \widetilde{\Theta}_t)^{+}\Lambda\sum_{Y\in\mathcal{Y}_b}\gamma^Y_t\Big)dt. \ \blacksquare \end{aligned} \tag{8.11}$$

8.3 Cure Period

In our DGC and DMO examples so far, we postulated no cure period, i.e. $\delta = 0$. Let's now assume, in a DMO setup, a positive cure period δ, with a $\mathcal{P}(\mathbb{G})$ measurable $\mathbf{C} = (V, I^c, I^b)$ and with deterministic interest rates r_t for simplicity. Similar considerations would apply in a DGC setup. In the case of vanilla credit derivatives (CDS contracts and CDO tranches with promised dividends given by (6.12)), in a deterministic interest rate environment, the process $\Delta_t = \int_{[\tau,t]} e^{\int_s^t r_u du} dD_s$ is a function of the default times in $[\tau, t]$. Writing $K^Y_t = (\mathbb{1}_{\{\eta_Y\le t\}}, \eta_Y\mathbb{1}_{\{\eta_Y\le t\}}) = (H^Y_t, \eta_Y H^Y_t)$, $\mathbf{K} = (K^Y)_{Y\in\mathcal{Y}}$, $\Delta^\star_t = \int_{[0,t]} e^{-\int_t^s r_u du} dD_s$ (so that $\beta_t\Delta_t = \beta_t\Delta^\star_t - \beta_\tau\Delta^\star_{\tau-}$), we consider a cure period $(\mathbb{G},\mathbb{Q})$ factor process $X_t = (t, \boldsymbol{\Gamma}_t, \mathbf{K}_t)$. By application of the results of Bielecki, Jakubowski, and Nieęglowski (2012) (or by a direct proof based on Heath and Schweizer (2000, Theorem 1) and Becherer and Schweizer (2005, Corollary 2.3)), X is a $(\mathbb{G},\mathbb{Q})$ homogeneous strong Markov process. Recall from (5.4) and (5.2) that

$$\xi = \mathbb{1}_{\{\tau_c\le\tau_b^\delta\}}(1-R_c)(Q_{\tau^\delta} - C^c_\tau)^{+} - \mathbb{1}_{\{\tau_b\le\tau_c^\delta\}}(1-R_b)(Q_{\tau^\delta} - C^b_\tau)^{-}, \tag{8.12}$$

where

$$\mathbb{1}_{\{\tau_c\le\tau_b^\delta\}} = \mathbb{1}_{\{\tau_c\le\tau^\delta\}} = 1 - \prod_{Y\in\mathcal{Y}_c}(1 - H^Y_{\tau^\delta}),\ \ \mathbb{1}_{\{\tau_b\le\tau_c^\delta\}} = \mathbb{1}_{\{\tau_b\le\tau^\delta\}} = 1 - \prod_{Y\in\mathcal{Y}_b}(1 - H^Y_{\tau^\delta}).$$

Moreover, in the case of vanilla credit derivatives such as CDS and CDO contracts as of (6.12), we have in the DMO model:

$$Q_{\tau^\delta} = P_{\tau^\delta} + \Delta_{\tau^\delta} = P_{\tau^\delta} + \Delta^\star_{\tau^\delta} - e^{\int_\tau^{\tau^\delta} r_u du}\Delta^\star_{\tau-} = P(\tau^\delta, \boldsymbol{\Gamma}_{\tau^\delta}, \mathbf{K}_{\tau^\delta}) + \Delta_\star(\tau^\delta, \boldsymbol{\Gamma}_{\tau^\delta}, \mathbf{K}_{\tau^\delta}) - e^{\int_\tau^{\tau^\delta} r_u du}\Delta^\star_{\tau-},$$

for continuous functions P and $\Delta_\star$.

Remark 81 Similar as in the DGC case (see the remark 71), any continuity statement with respect to the second components t_Y of the values k_Y of the K^Y_t, $Y \in \mathcal{Y}$, has to be understood in the sense of the corresponding "order sets" domain and topology.

Hence, ξ can be written in functional form as

$$\xi = \xi_\star(\tau^\delta, \boldsymbol{\Gamma}_{\tau^\delta}, \mathbf{K}_{\tau^\delta}, \mathbf{C}_\tau, \Delta^\star_{\tau-}) = \xi_\star(X_{\tau^\delta}, \mathbf{C}_\tau, \Delta^\star_{\tau-}), \tag{8.13}$$

for some function $\xi_\star$ continuous in the values x of X and where $\mathbf{C}_\tau$ and $\Delta^\star_{\tau-}$ are considered as $\mathcal{G}_\tau$ measurable parameters. We consider the $(\mathbb{F}, \mathbb{P} = \mathbb{Q})$ reduced factor process $\widetilde{X}_t = (\boldsymbol{\Gamma}_t, \widetilde{\mathbf{K}}_t, \mathbf{C}'_t, (\Delta^\star_-)'_t)$, where $\widetilde{\mathbf{K}} = (\mathbb{1}_{Y\in\mathcal{Y}_\circ}K^Y)_{Y\in\mathcal{Y}}$. Note that there exist unique $\mathbb{F}$ predictable reductions $\mathbf{C}'$ of $\mathbf{C}$ (assumed $\mathbb{G}$ predictable) and $(\Delta^\star_-)'$ of $\Delta^\star_-$ on $[0, T]$, by virtue of the condition (B) and of the positivity of S_T established in the DMO model in Theorem 5. We denote by $\mathbf{k}$ and $\widetilde{x}$ respective values of $\mathbf{K}$ (or $\widetilde{\mathbf{K}}$) and $\widetilde{X}$ and we write $\mathbf{k}^{Y,t}$ for the vector obtained from $\mathbf{k}$ by replacing the component with index Y by $(1, t)$.

Lemma 81 *We have* $cdva_t = J_{t-}cdva(t, \widetilde{X}_t)$, $\mathbb{Q} \times \boldsymbol{\lambda}$ *a.e., where*

$$cdva(t, \widetilde{X}_t) = \sum_{Y \in \mathcal{Y}_\bullet} \gamma_t^Y \bar{\xi}\big(t, \boldsymbol{\Gamma}_t, \widetilde{\mathbf{K}}_t^{Y,t}, \mathbf{C}'_t, (\Delta^\star_-)'_t\big), \tag{8.14}$$

for a continuous function $\bar{\xi}(t, \widetilde{x})$ *such that*

$$\bar{\xi}_\tau = \bar{\xi}(\tau, \widetilde{X}_\tau). \tag{8.15}$$

Proof. In view of (8.13), the $(\mathbb{G}, \mathbb{Q})$ Markov property of X yields

$$\bar{\xi}_\tau = \mathbb{E}(\beta_\tau^{-1}\beta_{\tau^\delta}\xi \,|\, \mathcal{G}_\tau) = \mathbb{E}(\beta_\tau^{-1}\beta_{\tau^\delta}\xi_\star(X_{\tau^\delta}, \mathbf{C}_\tau, \Delta^\star_{\tau-}) \,|\, \mathcal{G}_\tau) = \bar{\xi}(X_\tau, \mathbf{C}_\tau, \Delta^\star_{\tau-}), \tag{8.16}$$

for some continuous function $\bar{\xi}(x, \mathbf{c}, d)$. On $\{\tau = \eta_Y < T\}$, we have

$$\mathbf{K}_\tau = (\widetilde{\mathbf{K}}_{\tau-})^{Y,\tau}$$

and hence

$$\begin{aligned} &\bar{\xi}\big(X_\tau, \mathbf{C}_\tau, \Delta^\star_{\tau-}\big) = \bar{\xi}\big(\tau, \boldsymbol{\Gamma}_\tau, \mathbf{K}_\tau, \mathbf{C}_\tau, \Delta^\star_{\tau-}\big) \\ &\quad = \bar{\xi}\big(\tau, \boldsymbol{\Gamma}_\tau, (\mathbf{K}_{\tau-})^{Y,\tau}, \mathbf{C}_\tau, \Delta^\star_{\tau-}\big) = \bar{\xi}\big(\tau, \boldsymbol{\Gamma}_\tau, (\widetilde{\mathbf{K}}_{\tau-})^{Y,\tau}, \mathbf{C}'_\tau, (\Delta^\star_-)'_\tau\big), \end{aligned} \tag{8.17}$$

from which the results follows by an application of Lemma 61. ∎

In view of (5.6) and of Lemma 81, postulating that $lva_t(\vartheta)$ in (5.6) is given before τ as a continuous function $lva(t, \widetilde{X}_t, \vartheta)$, the condition (6.11) holds for the function $\widetilde{f}(t, \widetilde{x}, \vartheta)$ such that, with $\widetilde{x} = (\boldsymbol{\gamma}, \widetilde{\mathbf{k}}, \mathbf{c}, d)$ and $\gamma(t, \widetilde{x}) = \sum_{Y \in \mathcal{Y}_\bullet} \gamma_Y$:

$$\widetilde{f}(t, \widetilde{x}, \vartheta) + (r(t) + \gamma(t, \widetilde{x}))\vartheta = cdva(t, \widetilde{x}) + lva(t, \widetilde{x}, \vartheta). \tag{8.18}$$

Therefore, Theorem 8 still holds for $\delta > 0$ with $\widetilde{f}$ as of (8.18) instead of (8.9) in (DMO.3). Hence, we can derive the following cure period analog of Corollary 81.

Corollary 82 *In the DMO model, we have* $\gamma' = \gamma(\cdot, \widetilde{X}_\cdot) = \sum_{Y \in \mathcal{Y}_\bullet} \gamma^Y \in \widetilde{\mathcal{H}}_2$. *Assuming all the other conditions in Theorem 2 and for a cure period* $\delta > 0$, *hence for* $\widetilde{f} = \widetilde{f}(t, \boldsymbol{\gamma}, \widetilde{\mathbf{k}}, \mathbf{c}, d, \vartheta)$ *as of* (8.18), *the corresponding reduced TVA BSDE* (4.5) *admits a unique square integrable solution* $\widetilde{\Theta}_t = \widetilde{\Theta}(t, \widetilde{X}_t)$, *where the function* $\widetilde{\Theta}(t, \widetilde{x})$ *is a continuous viscosity solution to the corresponding semilinear PIDE. A solution* Θ *to the full TVA BSDE* (3.6) *is obtained by setting* $\Theta = \widetilde{\Theta}$ *on* $[0, \bar{\tau})$ *and*

$$\Theta_{\bar{\tau}} = \mathbb{1}_{\{\tau<T\}} \Big(\bar{\xi}\big(\tau, \boldsymbol{\Gamma}_\tau, (\widetilde{\mathbf{K}}_{\tau-})^{\imath,\tau}, \mathbf{C}'_\tau, (\Delta^\star_-)'_\tau\big) - (P_{\tau-} - C_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{-1 \in \imath\}} \Lambda\Big),$$

where $\imath \in \mathcal{Y}_\bullet$ *is the identity of the shock triggering the first default of a party. The* $(\mathbb{G}, \mathbb{Q})$ *local martingale component* μ *of* Θ *satisfies, for* $t \in [0, \bar{\tau}]$:

$$\begin{aligned} d\mu_t = d\widetilde{\mu}_t &- \Big(\bar{\xi}\big(\tau, \boldsymbol{\Gamma}_\tau, (\widetilde{\mathbf{K}}_{\tau-})^{\imath,\tau}, \mathbf{C}'_\tau, (\Delta^\star_-)'_\tau\big) - \widetilde{\Theta}_{t-} - (P_{\tau-} - C_{\tau-} - \widetilde{\Theta}_{\tau-})^+ \mathbb{1}_{\{\tau=\tau_b\}} \Lambda\Big) dJ_t \\ &- \Big(\sum_{Y \in \mathcal{Y}_\bullet} (\bar{\xi}\big(t, \boldsymbol{\Gamma}_t, (\widetilde{\mathbf{K}}_t)^{Y,t}, \mathbf{C}'_t, (\Delta^\star_-)'_t\big) - \widetilde{\Theta}_t)\gamma_t^Y - (P_t - C_t - \widetilde{\Theta}_t)^+ \Lambda \sum_{Y \in \mathcal{Y}_b} \gamma_t^Y\Big) dt. \ \blacksquare \end{aligned}$$

9 Numerical Implementation and Results

Due to funding costs, the TVA equations are nonlinear BSDEs. In the case of credit derivatives, they are also very high-dimensional. For nonlinear and very high-dimensional problems, any numerical scheme based, even to some extent, on dynamic programming, such as purely backward deterministic PDE schemes, but also forward/backward simulation/regression BSDE schemes, are ruled out by the curse of dimensionality (see e.g. Crépey (2013, Part IV)). Now, in any bottom-up credit portfolio model such as the DGC or the DMO model, the dimension is at least the number of credit names. Hence, for n greater than a few units, the only feasible TVA schemes are purely forward simulation schemes, such as the branching particles scheme of Henry-Labordère (2012) or a Monte Carlo estimation of the successive terms of the linear expansion of Fujii and Takahashi (2012a,2012b), respectively dubbed "PHL scheme" and "FT scheme" below. In our setup, the PHL scheme involves a nontrivial and rather sensitive fine-tuning for finding a polynom in ϑ that approximates the terms $(P_t - C_t - \vartheta)^{\pm}$ in $lva_t(\vartheta)$ in a suitable range for ϑ. Ideally, such a polynom should be adaptive and depend on $(P_t - C_t)$, but in a PHL scheme the approximating polynom has to be fixed once for all in the simulation. The only way we were able to achieve a good fine-tuning is by using a preliminary knowledge on the solution obtained by running the FT scheme or a linear approximation in the first place. Since a numerical scheme that can be run automatically and does not involve any fine-tuning is preferable, we focus on the FT scheme in the sequel.

9.1 Fujii and Takahashi's TVA Linear Expansion

The FT scheme is based on an expansion of the coefficient, hence of the solution $\widetilde{\Theta}$, to a Markovian BSDE on $[0,T]$ such as (4.5)/(6.11), as a series $\widetilde{\Theta} \approx \widetilde{\Theta}^{(0)} + \widetilde{\Theta}^{(1)} + \widetilde{\Theta}^{(2)} + \widetilde{\Theta}^{(3)} + \cdots$ of solutions to linear BSDEs, where the next BSDE in the series uses the solution to the previous one as input data. Let

$$\bar{f}(t,\widetilde{x},\vartheta) := \widetilde{f}(t,\widetilde{x},\vartheta) + \gamma(t,\widetilde{x})\vartheta = cdva(t,\widetilde{x}) + lva(t,\widetilde{x},\vartheta) - r(t,\widetilde{x})\vartheta \tag{9.1}$$

(cf. (5.6)). In order to exploit some cancellation between related discount factors in Lemma 91 below, it is preferable to use an FT expansion of the coefficient $\bar{f}$ rather than $\widetilde{f}$, treating the $\gamma(t,\widetilde{x})\vartheta$ term in $\widetilde{f}$ as a discount factor (cf. the right identity in (5.8)). In terms of $\bar{f}$, the reduced BSDE (4.5) is written as: $\widetilde{\Theta}_T = 0$ and, for $t \in [0,T]$,

$$-d\widetilde{\Theta}_t = \widetilde{f}(t,\widetilde{X}_t,\widetilde{\Theta}_t)dt - d\widetilde{\mu}_t = \big(\bar{f}(t,\widetilde{X}_t,\widetilde{\Theta}_t) - \gamma(t,\widetilde{X}_t)\widetilde{\Theta}_t\big)dt - d\widetilde{\mu}_t.$$

The corresponding FT expansion reads (cf. Fujii and Takahashi (2012a, Equations (2.4), (2.6) and (2.7) in the arXiv version)): $\widetilde{\Theta}^{\epsilon}_T = 0$ and, for $t \in [0,T]$,

$$\begin{aligned}
-d\widetilde{\Theta}^{\epsilon}_t &= \big(\epsilon\bar{f}(t,\widetilde{X}_t,\widetilde{\Theta}^{\epsilon}_t) - \widetilde{\gamma}(t,\widetilde{X}_t)\widetilde{\Theta}^{\epsilon}_t\big)dt - d\widetilde{\mu}^{\epsilon}_t\\
\widetilde{\Theta}^{\epsilon}_t &= \widetilde{\Theta}^{(0)}_t + \epsilon\widetilde{\Theta}^{(1)}_t + \epsilon^2\widetilde{\Theta}^{(2)}_t + \epsilon^3\widetilde{\Theta}^{(3)}_t + \cdots\\
\bar{f}(t,\widetilde{X}_t,\widetilde{\Theta}^{\epsilon}_t) &= \bar{f}(t,\widetilde{X}_t,\widetilde{\Theta}^{0}_t) + \big(\epsilon\widetilde{\Theta}^{(1)} + \epsilon^2\widetilde{\Theta}^{(2)} + \epsilon^3\widetilde{\Theta}^{(3)} + \cdots\big)\partial_\vartheta\bar{f}(t,\widetilde{X}_t,\widetilde{\Theta}^{(0)}_t) + \ldots .
\end{aligned}$$

Quoting Fujii and Takahashi (2012a, page 4 in the arXiv version), "by putting $\epsilon = 1$, $\widetilde{\Theta}^{(0)} + \widetilde{\Theta}^{(1)} + \widetilde{\Theta}^{(2)} + \widetilde{\Theta}^{(3)} + \cdots$ is expected to provide a reasonable approximation for the original $\widetilde{\Theta}$ as long as the residual term is small enough to allow the perturbative treatment." This is studied mathematically in a diffusive setup in Takahashi and Yamada (2015). Collecting all terms in ϵ^i, the resulting first $\widetilde{\Theta}^{(i)}$ terms are written as $\widetilde{\Theta}^{(0)} = 0$, due to the null terminal condition of the reduced TVA BSDE (4.5), and

$$\begin{aligned}
\widetilde{\Theta}^{(1)}_t &= \widetilde{\mathbb{E}}\Big[\int_t^T e^{-\int_t^s \gamma(r,\widetilde{X}_r)dr}\bar{f}\Big(s,\widetilde{X}_s,\widetilde{\Theta}^{(0)}_s = 0\Big)ds \,\Big|\, \mathcal{F}_t\Big],\\
\widetilde{\Theta}^{(2)}_t &= \widetilde{\mathbb{E}}\Big[\int_t^T e^{-\int_t^s \gamma(r,\widetilde{X}_r)dr}\partial_\vartheta\bar{f}\Big(s,\widetilde{X}_s,\widetilde{\Theta}^{(0)}_s = 0\Big)\widetilde{\Theta}^{(1)}_s ds \,\Big|\, \mathcal{F}_t\Big],\\
\widetilde{\Theta}^{(3)}_t &= \widetilde{\mathbb{E}}\Big[\int_t^T e^{-\int_t^s \gamma(r,\widetilde{X}_r)dr}\partial_\vartheta\bar{f}\Big(s,\widetilde{X}_s,\widetilde{\Theta}^{(0)}_s = 0\Big)\widetilde{\Theta}^{(2)}_s ds \,\Big|\, \mathcal{F}_t\Big].
\end{aligned} \tag{9.2}$$

The first two lines correspond to the identities (2.19) and (2.22) in the arXiv version of Fujii and Takahashi (2012a). Compared with the third line, the complete third order term comprises another component based on $\partial^2_{\vartheta^2}\widetilde{f}$. In our case, $\partial^2_{\vartheta^2}\widetilde{f}$ involves a Dirac measure via the terms $(P_t - C_t - \vartheta)^{\pm}$ in $lva_t(\vartheta)$ (cf. (5.6)), so that we truncate the expansion to the term $\widetilde{\Theta}^{(3)}_t$ as above. Moreover, we use the interacting particles implementation of these formulas provided in Fujii and Takahashi (2015) and Fujii, Sato, and Takahashi (2014). Namely, we randomize, based on independent exponential draws ζ_j with parameters μ_j, each time integral that intervenes in (9.2) either explicitly or implicitly through the terms $\widetilde{\Theta}^{(1)}_s$ and $\widetilde{\Theta}^{(2)}_s$. Specifically, the following identities follow from (9.2) by the tower rule :

$$\begin{aligned}
\widetilde{\Theta}^{(1)}_0 &= \widetilde{\mathbb{E}}\Big[\mathbb{1}_{\zeta_1<T}\frac{e^{\mu_1\zeta_1}}{\mu_1}e^{-\int_0^{\zeta_1}\gamma(r,\widetilde{X}_r)dr}\bar{f}\Big(\zeta_1,\widetilde{X}_{\zeta_1},0\Big)\Big],\\
\widetilde{\Theta}^{(2)}_0 &= \widetilde{\mathbb{E}}\Big[\mathbb{1}_{\zeta_1+\zeta_2<T}\frac{e^{\mu_1\zeta_1+\mu_2\zeta_2}}{\mu_1\mu_2}e^{-\int_0^{\zeta_1+\zeta_2}\gamma(r,\widetilde{X}_r)dr}\partial_\vartheta\bar{f}\Big(\zeta_1,\widetilde{X}_{\zeta_1},0\Big)\bar{f}\Big(\zeta_1+\zeta_2,\widetilde{X}_{\zeta_1+\zeta_2},0\Big)\Big],\\
\widetilde{\Theta}^{(3)}_0 &= \widetilde{\mathbb{E}}\Big[\mathbb{1}_{\zeta_1+\zeta_2+\zeta_3<T}\frac{e^{\mu_1\zeta_1+\mu_2\zeta_2+\mu_3\zeta_3}}{\mu_1\mu_2\mu_3}e^{-\int_0^{\zeta_1+\zeta_2+\zeta_3}\gamma(r,\widetilde{X}_r)dr}\times\\
&\qquad \partial_\vartheta\bar{f}\Big(\zeta_1,\widetilde{X}_{\zeta_1},0\Big)\partial_\vartheta\bar{f}\Big(\zeta_1+\zeta_2,\widetilde{X}_{\zeta_1+\zeta_2},0\Big)\bar{f}\Big(\zeta_1+\zeta_2+\zeta_3,\widetilde{X}_{\zeta_1+\zeta_2+\zeta_3},0\Big)\Big].
\end{aligned} \tag{9.3}$$

In the case of dynamic $(\mathbb{G},\mathbb{Q})$ copula models as considered in this second part of the paper, $(\mathbb{F},\mathbb{P})$ simulation may be nontrivial. That would for instance be an issue in the DGC model (in the DMO model $(\mathbb{F},\mathbb{P}=\mathbb{Q})$ simulation is equally easy as $(\mathbb{G},\mathbb{P}=\mathbb{Q})$ simulation due to the immersion properties of the setup). But, as explained after the next result, it is always possible to reformulate the $\mathbb{P}$ expectations in (9.3) as $\mathbb{Q}$ expectations, by a direct formula not involving any Girsanov weights, which allows estimating the $\widetilde{\Theta}^{(i)}_0$ simply by $(\mathbb{G},\mathbb{Q})$ simulation.

Lemma 91 *Assuming the conditions (B) and (C) and the positivity of S_T, then, for any $\mathbb{F}$ progressively measurable nonnegative process h and any independent nonnegative random variable ζ, it holds:*

$$\widetilde{\mathbb{E}}\left[\mathbb{1}_{\{\zeta<T\}}e^{-\int_0^\zeta\gamma'_s ds}h_\zeta\right] = \mathbb{E}\left[\mathbb{1}_{\{\zeta<\bar{\tau}\}}h_\zeta\right]. \tag{9.4}$$

Proof. By Fubini's theorem, we only need to prove the lemma for $\zeta = s$ constant positive. Let $H_s = \mathbb{1}_{\{s<T\}}h_s$. Since h is $\mathbb{F}$ adapted, $\mathbb{E}[H_s\mathbb{1}_{\{s<\tau\}}] = \mathbb{E}[H_sS_s]$. We recall from Crépey and Song (2015a, Lemma 2.2 5)) the following multiplicative stochastic exponential decomposition of S on $[0,T]$ (having assumed $S_T > 0$):

$$S = S_0\mathcal{E}(-\frac{1}{S_-}\boldsymbol{\cdot}\mathcal{D})\mathcal{E}(\frac{1}{pS}\boldsymbol{\cdot}\mathcal{Q}),$$

where $S = \mathcal{Q} - \mathcal{D}$ is the $(\mathbb{F},\mathbb{Q})$ canonical Doob-Meyer decomposition of S. By Crépey and Song (2015a, Theorem 3.1), the $(\mathbb{F},\mathbb{Q})$ density process of $\mathbb{P}$ is given by $\mathcal{E}(\frac{1}{pS}\boldsymbol{\cdot}\mathcal{Q})$ on $[0,T]$. Therefore,

$$\mathbb{E}[H_s\mathbb{1}_{\{s<\tau\}}] = \mathbb{E}[H_sS_s] = \mathbb{E}[H_sS_0\mathcal{E}(-\frac{1}{S_-}\boldsymbol{\cdot}\mathcal{D})_s\mathcal{E}(\frac{1}{pS}\boldsymbol{\cdot}\mathcal{Q})_s] = \widetilde{\mathbb{E}}[H_sS_0\mathcal{E}(-\frac{1}{S_-}\boldsymbol{\cdot}\mathcal{D})_s]. \tag{9.5}$$

Note that $S_0 = 1$ and $\mathcal{D}$ is the drift of the Azema supermartingale of τ in $(\mathbb{F},\mathbb{Q})$, so that $\mathcal{D}$ is absolutely continuous, i.e. $\mathcal{D} = \nu\boldsymbol{\cdot}\boldsymbol{\lambda}$ for some density process ν. Hence, the stochastic exponential $\mathcal{E}(-\frac{1}{S_-}\boldsymbol{\cdot}\mathcal{D})$ is a usual exponential. By Jeulin (1980, Remark 4.5), $\mathbb{1}_{(0,\tau]}\frac{\nu}{S_-}\boldsymbol{\cdot}\boldsymbol{\lambda}$ is the $(\mathbb{G},\mathbb{Q})$ predictable dual projection of $\mathbb{1}_{[\tau,\infty)}$. Therefore,

$$\mathbb{1}_{(0,\tau]}\frac{\nu}{S_-} = \gamma'\mathbb{1}_{(0,\tau]},$$

so that, in view of the remark 41, $\frac{\nu}{S_-} = \gamma'$ holds on $[0,T]$. As a consequence, after substitution of ζ for s, the identity (9.5) is rewritten as (9.4). ∎

Applications of Lemma 91 to (9.3) yield

$$
\begin{aligned}
\widetilde{\Theta}_0^{(1)} &= \mathbb{E}\Big[\mathbb{1}_{\zeta_1<\bar{\tau}}\frac{e^{\mu_1\zeta_1}}{\mu_1}\bar{f}\Big(\zeta_1,\widetilde{X}_{\zeta_1},0\Big)\Big],\\
\widetilde{\Theta}_0^{(2)} &= \mathbb{E}\Big[\mathbb{1}_{\zeta_1+\zeta_2<\bar{\tau}}\frac{e^{\mu_1\zeta_1+\mu_2\zeta_2}}{\mu_1\mu_2}\partial_\vartheta\bar{f}\Big(\zeta_1,\widetilde{X}_{\zeta_1},0\Big)\bar{f}\Big(\zeta_1+\zeta_2,\widetilde{X}_{\zeta_1+\zeta_2},0\Big)\Big],\\
\widetilde{\Theta}_0^{(3)} &= \mathbb{E}\Big[\mathbb{1}_{\zeta_1+\zeta_2+\zeta_3<\bar{\tau}}\frac{e^{\mu_1\zeta_1+\mu_2\zeta_2+\mu_3\zeta_3}}{\mu_1\mu_2\mu_3}\\
&\qquad \partial_\vartheta\bar{f}\Big(\zeta_1,\widetilde{X}_{\zeta_1},0\Big)\partial_\vartheta\bar{f}\Big(\zeta_1+\zeta_2,\widetilde{X}_{\zeta_1+\zeta_2},0\Big)\bar{f}\Big(\zeta_1+\zeta_2+\zeta_3,\widetilde{X}_{\zeta_1+\zeta_2+\zeta_3},0\Big)\Big].
\end{aligned}
\tag{9.6}
$$

In the two cases considered with $\delta=0$ in this second part of the paper, the reduced factor process $\widetilde{X}$ consists of some components of a full factor process X, namely $X_t=(\mathbf{m}_t,\mathbf{k}_t)$ in the DGC setup without cure period of Sect. 7.2 and $X_t=(\boldsymbol{\Gamma}_t,\mathbf{H}_t)$ in the DMO setup without cure period of Sect. 8.2. Hence, for $\delta=0$, we can compute the $\widetilde{\Theta}_0^{(i)}$ based on (9.6) by $(\mathbb{G},\mathbb{Q})$ simulation of X, which is fast and easy in both cases based on the copula properties of each model. In the DMO setup with positive cure period $\delta>0$ of Sect. 8.3, one residual difficulty with the formulas (9.6) is that $\bar{f}(t,\widetilde{x},\vartheta)$ in (9.1) involves nontrivial $cdva(t,\widetilde{x})$ terms as of (8.14). But the computation of these can be avoided by resorting to the following add-on to Lemma 91, (9.7) implying that the $\widetilde{\Theta}_0^{(i)}$ in (9.6) can be computed by $(\mathbb{G},\mathbb{Q})$ simulation of X, $\mathbf{C}$ and $\Delta^\star$. Recall $\xi_\star$ from (8.13).

Lemma 92 *In the DMO setup with positive cure period $\delta>0$ of Sect. 8.3, the notation of which we use here, for any $\mathbb{F}$ predictable process h and for any independent diffuse random variable ζ such that h_ζ is $\mathbb{Q}$ integrable, we have:*

$$
\mathbb{E}[\mathbb{1}_{\{\zeta<\bar{\tau}\}}h_\zeta cdva(\zeta,\widetilde{X}_\zeta)]=\mathbb{E}\big[\mathbb{1}_{\{\zeta<\bar{\tau}\}}h_\zeta\textstyle\sum_{Y\in\mathcal{Y}_\bullet}\gamma_\zeta^Y e^{-\int_\zeta^{\zeta+\delta}r(s)ds}\xi_\star\big(X_{\zeta^\delta}^{Y,\zeta},\mathbf{C}_\zeta,\Delta_\zeta^\star\big)\big],
\tag{9.7}
$$

where we write $X_t^{Y,s}=(t,\boldsymbol{\Gamma}_t,(\mathbf{K}_t)^{Y,s})$, for any $0\le s\le t$.

Proof. By the recursively immersed feature of the DMO model (Markov copula properties of Bielecki et al. (2014a, Part I) and Bielecki, Jakubowski, and Nieęglowski (2012)), the process X^{-Y} obtained as X deprived from the Y^{th} component of $\mathbf{K}$ is a $(\mathbb{G},\mathbb{Q})$ homogeneous strong Markov process, for any $Y\in\mathcal{Y}$. We denote by $\mathcal{T}_\delta^Y$ the transition function of the process X^{-Y} killed at the rate r over the time horizon δ, i.e., $\mathbf{k}^{-Y}$ representing a value of $\mathbf{K}$ deprived from its Y^{th} component,

$$
\begin{aligned}
(\varphi,(t,\boldsymbol{\gamma},\mathbf{k}^{-Y}))\to\mathcal{T}_\delta^Y[\varphi](t,\boldsymbol{\gamma},\mathbf{k}^{-Y})&=\mathbb{E}\big[e^{-\int_t^{t^\delta}r(s)ds}\varphi(X_{t^\delta}^{-Y})|X_t^{-Y}=(t,\boldsymbol{\gamma},\mathbf{k}^{-Y})\big]\\
&=\mathbb{E}\big[e^{-\int_t^{t^\delta}r(s)ds}\varphi(X_{t^\delta}^{-Y})|\mathcal{G}_t\big].
\end{aligned}
$$

On $\{\tau=\eta_Y\}$, we have $\mathbf{K}_{\tau^\delta}=(\mathbf{K}_{\tau^\delta})^{Y,\tau}$, hence (cf. (8.13))

$$
\xi=\xi_\star(\tau^\delta,\boldsymbol{\Gamma}_{\tau^\delta},(\mathbf{K}_{\tau^\delta})^{Y,\tau},\mathbf{C}_\tau,\Delta_{\tau-}^\star)=\xi_Y(X_{\tau^\delta}^{-Y},\mathbf{C}_\tau,\Delta_{\tau-}^\star),
$$

for functions ξ_Y such that, for any $x=(s,\boldsymbol{\gamma},\mathbf{k})$, $\mathbf{c},d$,

$$
\xi_Y(x^{-Y},\mathbf{c},d)=\xi_\star(s,\boldsymbol{\gamma},\mathbf{k}^{Y,s-\delta},\mathbf{c},d).
$$

Viewing $\mathbf{C}_\tau$ and $\Delta_{\tau-}^\star$ as $\mathcal{G}_{\tau-}$ measurable parameters, we compute:

$$
\begin{aligned}
\bar{\xi}_\tau &=\mathbb{E}[e^{-\int_\tau^{\tau^\delta}r(s)ds}\xi|\mathcal{G}_\tau]=\mathbb{E}[e^{-\int_\tau^{\tau^\delta}r(s)ds}\sum_{Y\in\mathcal{Y}_\bullet}\xi_Y(X_{\tau^\delta}^{-Y},\mathbf{C}_\tau,\Delta_{\tau-}^\star)\mathbb{1}_{\{\tau=\eta_Y\}}|\mathcal{G}_\tau]\\
&=\sum_{Y\in\mathcal{Y}_\bullet}\mathbb{E}[e^{-\int_\tau^{\tau^\delta}r(s)ds}\xi_Y(X_{\tau^\delta}^{-Y},\mathbf{C}_\tau,\Delta_{\tau-}^\star)\mathbb{1}_{\{\tau=\eta_Y\}}|\mathcal{G}_\tau]\\
&=\sum_{Y\in\mathcal{Y}_\bullet}\mathcal{T}_\delta^Y[\xi_Y(\cdot,\mathbf{C}_\tau,\Delta_{\tau-}^\star)](X_\tau^{-Y})\mathbb{1}_{\{\tau=\eta_Y\}}=\sum_{Y\in\mathcal{Y}_\bullet}\mathcal{T}_\delta^Y[\xi_Y(\cdot,\mathbf{C}_\tau,\Delta_{\tau-}^\star)](X_{\tau-}^{-Y})\mathbb{1}_{\{\tau=\eta_Y\}},
\end{aligned}
$$

where the fact that $\{\tau=\eta_Y\}=\{\tau\geq\eta_Y\}\in\mathcal{G}_\tau$ (resp. X^{-Y} doesn't jump at τ on $\{\tau=\eta_Y\}$) was used to pass to the third line (resp. in the last identity). Hence, Lemma 61 yields

$$cdva_t = J_{t-}\sum_{Y\in\mathcal{Y}_\bullet}\gamma_t^Y\mathcal{T}_\delta^Y[\xi_Y(\cdot,\mathbf{C}_t,\Delta^\star_{t-})](X^{-Y}_{t-}),\ \ \mathbb{Q}\times\boldsymbol{\lambda}\text{ a.e.} \tag{9.8}$$

As a consequence, given an independent random variable ζ with density p such that h_ζ is $\mathbb{Q}$ integrable, we can write, using respectively the formula (9.8) and the definition of $\mathcal{T}_\delta^Y$ to pass to the second and third line:

$$\begin{aligned}
\mathbb{E}[h_\zeta\,\mathbb{1}_{\{\zeta\leq\bar\tau\}}cdva(\zeta,\widetilde X_\zeta)] &= \textstyle\int_0^T\mathbb{E}\big[h_t\mathbb{1}_{\{t<\bar\tau\}}cdva(t,\widetilde X_t)\big]\,p(t)dt=\int_0^T\mathbb{E}\big[h_t\mathbb{1}_{\{t<\bar\tau\}}cdva_t\big]\,p(t)dt\\
&= \textstyle\int_0^T\mathbb{E}\Big[h_t\mathbb{1}_{\{t\leq\tau\}}\sum_{Y\in\mathcal{Y}_\bullet}\gamma_t^Y\mathcal{T}_\delta^Y[\xi_Y(\cdot,\mathbf{C}_t,\Delta^\star_{t-})](X_t^{-Y})\Big]p(t)dt\\
&= \textstyle\int_0^T\ \mathbb{E}\left[h_t\mathbb{1}_{\{t\leq\tau\}}\sum_{Y\in\mathcal{Y}_\bullet}\gamma_t^Y\mathbb{E}\big[e^{-\int_t^{t^\delta}r(s)ds}\xi_Y(X^{-Y}_{t^\delta},\mathbf{C}_t,\Delta^\star_{t-})|\mathcal{G}_t\big]\right]\ p(t)dt\\
&= \textstyle\int_0^T\ \mathbb{E}\big[h_t\mathbb{1}_{\{t\leq\tau\}}\sum_{Y\in\mathcal{Y}_\bullet}\gamma_t^Y e^{-\int_t^{t^\delta}r(s)ds}\xi_\star(t^\delta,\boldsymbol{\Gamma}_{t^\delta},(\mathbf{K}_{t^\delta})^{Y,t},\mathbf{C}_t,\Delta^\star_{t-})\big]\ p(t)dt\\
&= \textstyle\mathbb{E}\big[\mathbb{1}_{\{\zeta\leq T\}}h_\zeta\mathbb{1}_{\{\zeta\leq\tau\}}\sum_{Y\in\mathcal{Y}_\bullet}\gamma_\zeta^Y e^{-\int_\zeta^{\zeta^\delta}r(s)ds}\xi_\star(X^{Y,\zeta}_{\zeta^\delta},\mathbf{C}_\zeta,\Delta^\star_\zeta)\big].\ \blacksquare
\end{aligned}$$

Note that this proof exploits the specific immersion and Markov copula properties of the DMO model, through which each process X^{-Y} is Markov in the full model filtration $\mathbb{G}$.

To conclude this paper we present TVA computations in the above DGC and DMO models, on CDS contracts and protection legs of CDO tranches corresponding to dividend processes of the respective forms D^i and D as of (6.12). For these computations we follow the most conservative TVA approach of ignoring windfall benefits at own default, setting $R_b=1$ and $\Lambda=0$ (see the remark 54), which allows numerical validation of the results based on the reduced BSDE (1) by results based on the full BSDE (3.6). Namely, further setting $c=\lambda=0$, we have the following linear approximation formula for the time-0 value of the solution to the full TVA BSDE:

$$\begin{aligned}
\Theta_0 &\approx \mathbb{E}\Big[\mathbb{1}_{\{\tau<T\}}\beta_{\tau^\delta}\xi+\int_t^{\bar\tau}\beta_s g_s(P_s)ds\Big]\\
&= \mathbb{E}\Big[\mathbb{1}_{\{\tau<T\}}\beta_{\tau^\delta}\mathbb{1}_{\{\tau_c\leq\tau_b^\delta\}}(1-R_c)(P_{\tau^\delta}+\Delta_{\tau^\delta}-C_\tau)^+ +\int_t^{\bar\tau}\beta_s\bar\lambda_s\,(P_s-C_s)^+\,ds\Big].
\end{aligned} \tag{9.9}$$

For $\bar\lambda=0$, this approximation is exact and a Monte Carlo loop based on the second line in (9.9) yields an unbiased estimate for $\Theta_0=\widetilde\Theta_0$ alternative to Monte Carlo estimates for $\widetilde\Theta_0^{(1)}+\widetilde\Theta_0^{(2)}+\widetilde\Theta_0^{(3)}$ in (9.3) or (9.6). For $\bar\lambda\neq0$, (9.9) is only a linear approximation to Θ_0.

Unless stated otherwise, the following numerical values are used (on top of $R_b=1$ and $\Lambda=c=\lambda=0$, for consistency with (9.9), and of the values set after (6.12) regarding the parameters of the credit derivative contracts):

$$r=0,\ \ R_c=40\%,\ \ \delta=0,\ \ V=I^c=I^b=0,\ \ \bar\lambda=100\text{ bp }=0.01,\ \ \mu_j=\frac{2}{T},\ \ m=10^4, \tag{9.10}$$

where m is the number of runs that are used in all the Monte Carlo estimates. In particular, for these data, Θ is nonnegative.

9.2 Numerical Results in the DGC Model

We start by TVA computations on CDS contracts with maturity $T=10$ years in a DGC model with $\varsigma=\frac{\mathbb{1}_{[0,T+1]}}{\sqrt{T+1}}$ and $\varrho=0.6$ unless otherwise stated. The functions h_i are chosen so that the τ_i follow exponential distributions calibrated to the fair contractual CDS spreads that appear in Table 1. In Figure 2, the left graph shows the TVA on a CDS on name 1, computed in a DGC model with $n=1$ by FT schemes of order 1 to 3, for different levels of nonlinearity represented by the value of the unsecured borrowing spread $\bar\lambda$. The right graph shows similar results regarding a portfolio of one CDS contract on each name $i=1,\dots,10$. The time-0 clean value of the default leg of the CDS in case $n=1$, respectively the sum of the ten default legs in case $n=10$, is 4.52, respectively 40.78 (of course $P_0=0$ in both

i	-1	0	1	i	-1	0	1	2	3	4	5	6	7	8	9	10
S_i	36	41	47	S_i	39	40	47	36	41	48	54	54	27	30	36	50

Table 1 Time-0 bp CDS spreads of names -1 (the bank), 0 (the counterparty) and of the reference names 1 to n used when $n = 1$ (*left*) and $n = 10$ (*right*).

cases by definition of the fair contractual CDS spreads). Hence, in relative terms, the TVA numbers visible in Figure 2 are quite high, much greater for instance than in the cases of counterparty risk on interest rate derivatives considered in Crépey, Gerboud, Grbac, and Ngor (2013). This is explained by the wrong-way risk feature of the DGC model, namely, the default intensities of the surviving names and the value of the CDS protection spike at defaults in this model (as will be demonstrated by the left graph in our concluding figure 7). When $\bar{\lambda}$ increases (for $\bar{\lambda} = 0$ that's a case of linear TVA where FT higher order terms equal 0), the second (resp. third) FT term may represent in each case up to 5% to 10% of the first (resp. second) FT term, from which we conclude that the first FT term can be used as a first order linear estimate of the TVA, with a nonlinear correction that can be estimated by the second FT term.

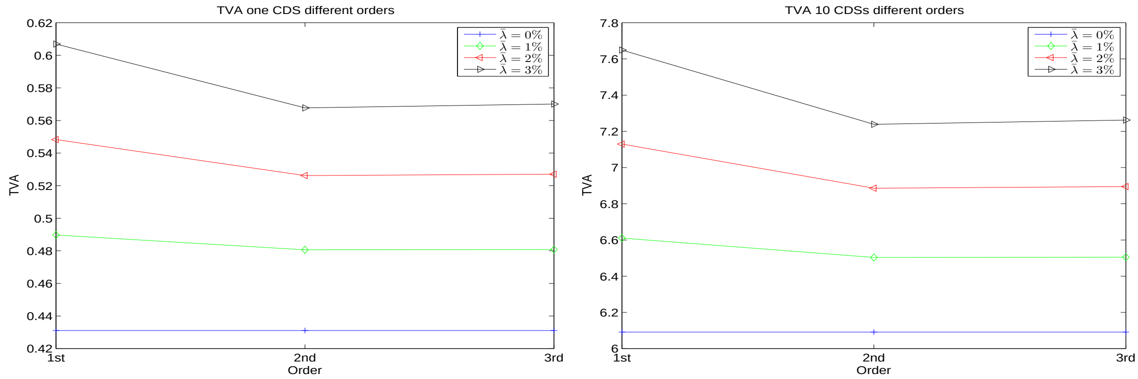

Fig. 2 *Left*: DGC TVA on one CDS computed by FT scheme of order 1 to 3, for different levels of nonlinearity (unsecured borrowing spread $\bar{\lambda}$). *Right*: Similar results regarding the portfolio of CDS contracts on ten names.

In Figure 3, the left graph shows the TVA on one CDS computed by FT schemes of order 3 as a function of the DGC correlation parameter ϱ, with other parameters set as before. The right graph shows the analogous results regarding the portfolio of ten CDS contracts. In both cases, the TVA numbers increase (roughly linearly) with ϱ, as desirable from the financial interpretation point of view. For more about this (regarding also the change of monotonicity that would occur at very high ϱ in case of a reference entity of the CDS significantly riskier than the counterparty), see Brigo and Chourdakis (2008), Brigo and Capponi (2010) and Brigo, Capponi, and Pallavicini (2014).

In Figure 4, the left graph shows that the errors, in the sense of the % relative standard errors (% rel. SE), of the different orders of the FT scheme don't explode with the dimension (number of credit names that underlie the CDS contracts). The middle graph, produced with $n = 1$, shows that the errors don't explode with the level of nonlinearity represented by the unsecured borrowing spread $\bar{\lambda}$. Consistent with the fact that the successive FT terms are computed by purely forward Monte Carlo schemes, their computation times are essentially linear in the number of names, as visible in the right graph.

Table 2 illustrates the statement made after (9.9). Namely, for $\bar{\lambda} = 0$, the 95% confidence interval of the FT scheme based on (9.6) is included into (in particular, fully consistent with) the 95% confidence interval of the Monte Carlo based on the formula (9.9), formula which is unbiased for $\bar{\lambda} = 0$. But, as $\bar{\lambda}$ increases (i.e. from bottom to top in the table), the approximation (9.9) reveals a significant upward bias increasing with $\bar{\lambda}$, for (9.9) uses $(P_s - C_s)^+$ instead of what should be $(P_s - C_s - \Theta_s)^+$ under the time integral, with $\Theta \geq 0$ here. On top of this bias, one can see in Table 2 that the confidence intervals of the Monte Carlo estimates based on (9.9) are substantially broader than the FT ones. In addition,

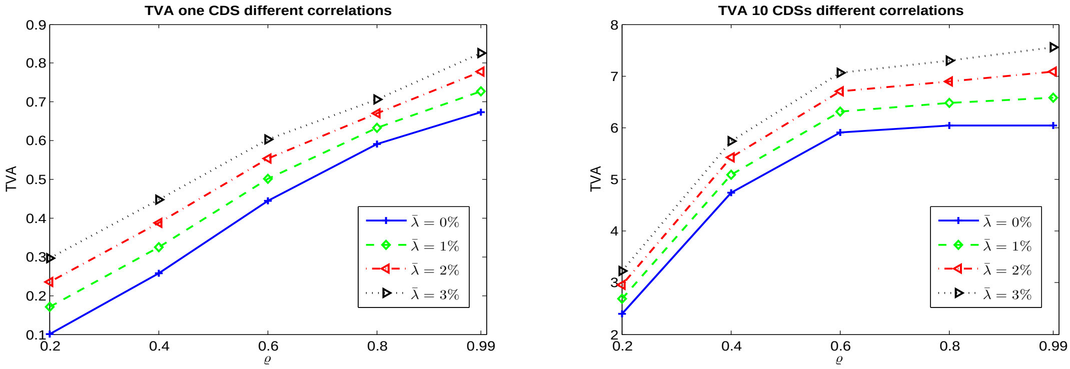

Fig. 3 *Left*: TVA on one CDS computed by FT scheme of order 3 as a function of the DGC correlation parameter ϱ. *Right*: Similar results regarding a portfolio of CDS contracts on ten different names.

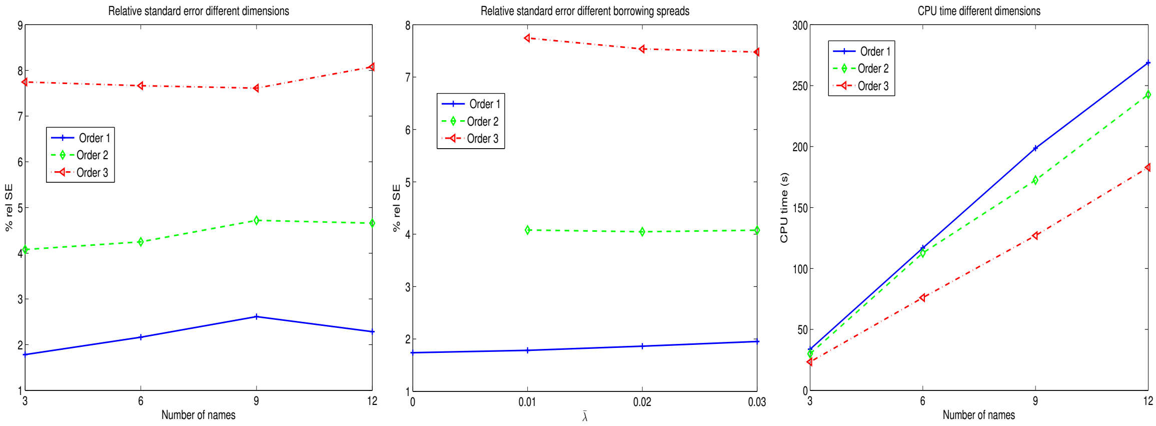

Fig. 4 *Left*: The % relative standard errors of the different orders of the expansions don't explode with the number of names ($\bar{\lambda} = 100$ bp). *Middle*: The % relative standard errors of the different orders of the expansions don't explode with the level of nonlinearity represented by the unsecured borrowing spread $\bar{\lambda}$ ($n = 1$). *Right*: Since FT terms are computed by purely forward Monte Carlo schemes, their computation times are linear in the number of names ($\bar{\lambda} = 100$ bp).

to evaluate the time integral in (9.9) with the required accuracy, we used time discretisation with 500 time points (i.e. a time mesh of $T/500 \approx$ one week for $T = 10y$ here), so that the computation times with (9.9) are much larger than the FT ones, typically a few minutes for FT versus a few hours with (9.9) (all of course proportional to the number of CDS contracts that are used). An alternative would be to randomize the time-integral in (9.9) as we do in the FT schemes, but this would result in an even greater variance than the one reflected in the last column of Table 2 (for computation times that would be similar to the ones of the FT scheme). Summarizing, in the DGC model, the TVA numbers based on the full BSDE take significantly more time and/or are significantly less accurate than the TVA numbers based on the reduced BSDE.

$\bar{\lambda}$ (bps)	FT3	95% CI	(9.9)	95% CI
300	0.60	$[0.58, 0.63]$	1.17	$[1.11, 1.22]$
200	0.54	$[0.52, 0.56]$	0.91	$[0.85, 0.96]$
100	0.48	$[0.47, 0.50]$	0.65	$[0.59, 0.70]$
0	0.43	$[0.41, 0.44]$	0.40	$[0.34, 0.45]$

Table 2 TVA computations on one CDS ($\varrho = 0.6$). *Columns 2 and 3:* by the FT scheme based on (9.6). *Columns 4 and 5:* by Monte Carlo based on the formula (9.9).

9.3 Numerical Results in the DMO Model

We consider a DMO model with constant shock intensities and $n = 120$ credit names (unless stated otherwise). Note that the dependence between names is all in the common shocks in this model. The stochasticity of the intensities is not crucial for the gap risk feature that we want to investigate here. Using deterministic (constant in this case) intensities allows speeding up the simulations. We take individual shock intensities $\gamma^{\{i\}} = 10^{-4} \times (100+i)$, which increases from 101 bp to 220 bp as i increases from 1 to 120. We consider four nested groups of joint defaults, respectively consisting of the riskiest 3%, 9%, 21% and 100% (i.e. all) names, with respective shock intensities of 20, 10, 6.67 and 5 bp. The counterparty (resp. the bank) is taken as the eleventh (resp. tenth) safest name in the portfolio. In this model, we consider CDO tranches with upfront payment (dividend process D as of (6.12)), for a maturity $T = 2$ years and attachment (resp. detachment) points 0%, 3% and 14% (resp. 3%, 14% and 100%). Figure 5 shows the corresponding TVA computed by FT scheme of order 1 to 3 for different levels of nonlinearity (unsecured borrowing spread $\bar{\lambda}$). The respective values of P_0 (upfront payment) for the equity, mezzanine and senior tranche are 229.65, 5.68 and 2.99. Compared with these, the TVA numbers of Figure 5 are very high, especially for the higher tranches, considerably greater again (cf. Figure 2 and related comments) than the TVA numbers computed on interest rate derivatives in Crépey et al. (2013). This is explained by the gap risk feature of the DMO model, namely, the joint default dividend $\Delta_\tau \neq 0$. By comparison, in the DGC model we have $\Delta_\tau = 0$ but the default intensities of surviving names and the cost of credit protection spike at defaults (whereas default intensities of surviving names are not affected by defaults in the DMO model). It is in this sense that we view the DGC and the DMO model as respective wrong-way and gap risk setups.

The second (resp. third) FT term never exceeds in each case, depending on $\bar{\lambda}$ increasing from 0 to 300 bp (for $\bar{\lambda} = 0$ that's a case of linear TVA with higher order FT terms all equal to 0), more than 5% of the first (resp. second) FT term in Figure 5, from which we conclude that the first FT term can be used as a first order linear estimate of the TVA, with a nonlinear correction that can be estimated by the second FT term. Figure 6 is the analog of the DGC CDS Figure 4, but for the DMO CDO tranches of Figure 5, with similar conclusions. Table 3 compared with Figures 5 and 6 shows that

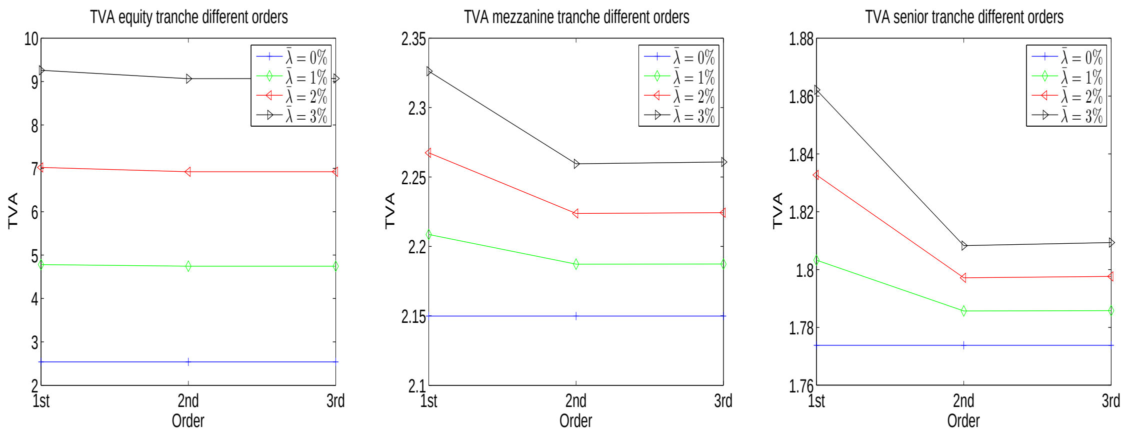

Fig. 5 TVA on CDO tranches with 120 underlying names computed by FT scheme of order 1 to 3, for different levels of nonlinearity (unsecured borrowing spread $\bar{\lambda}$). *Left*: Equity tranche. *Middle*: Mezzanine tranche. *Right*: Senior Tranche.

on top of being biased (depending on $\bar{\lambda}$, equal to 100 bp in Table 3), a Monte Carlo estimate based on the linear approximation formula (9.9) has a large variance, especially for higher tranches. In fact, for higher tranches, nonzero payoffs become quite rare events, so that exploiting the knowledge of the explicit formulas for the intensities in an FT scheme greatly improves the variance by comparison with a crude simulation based on (9.9). In addition, the simulations for (9.9) take considerably more time, due to the discretisation that is used for valuing the time integral (or, if the integral was randomized as in the FT scheme, then this would increase the variance even further).

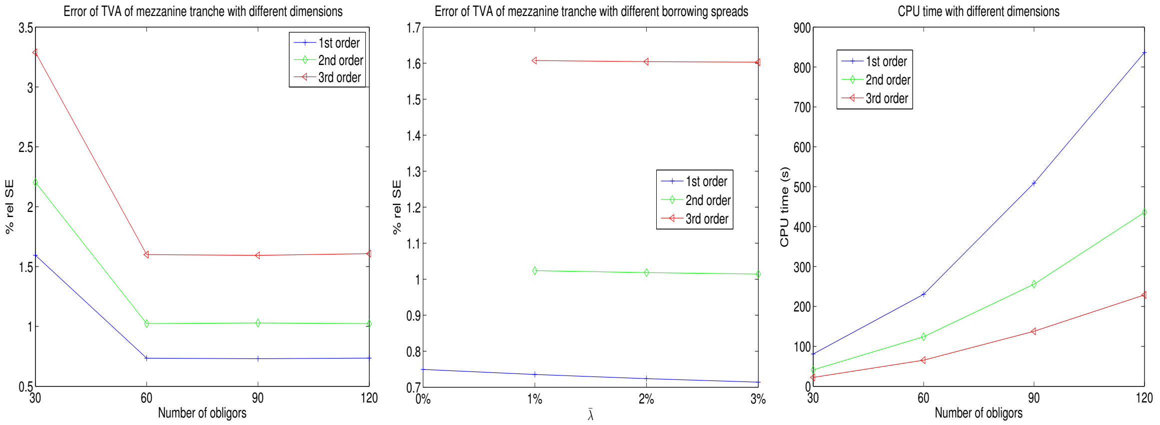

Fig. 6 Analog of Figure 4 for the mezzanine CDO tranche of the middle panel of Figure 5 in the DMO model.

Tranche	TVA	Rel. SE	95% CI
Eq.	5.00	7.43%	[4.63 , 5.37]
Mezz.	2.05	63.25%	[0.75 , 3.34]
Sen.	1.67	64.59 %	[0.59 , 2.75]

Tranche	TVA	Rel. SE	95% CI
Eq.	4.94	2.40%	[4.82 , 5.06]
Mezz.	2.14	19.60%	[1.72 , 2.55]
Sen.	1.74	20.02%	[1.39 , 2.09]

Table 3 Linearized DMO TVA on CDO tranches computed by a Monte Carlo based on (9.9) ($\bar{\lambda}$ = 100 bp). *Left*: $m = 10^4$. *Right*: $m = 10^5$.

In the remaining paragraphs of this section, we show some results computed in the DMO setup in the continuous variation-margining case where $V_\tau = P_{\tau-}$ (cf. the remark 51), as opposed to $V = 0$ before.

Table 4 compares the performance of the FT scheme based on the reduced BSDE and of Monte Carlo simulations based on the formula (9.9) to compute the TVA in the continuous variation-margining case where $V_\tau = P_{\tau-}$. As already repeatedly found above, the FT scheme has significantly less variance, crucially so for higher tranches. The FT scheme also takes considerably less computation time, due to the need of computing the time integral in the case of (9.9) (unless this integral is randomized but this would increase the variance even more). Note that even in this continuous variation-margining case,

Tranche	TVA	Rel. SE	95% CI
Eq.	0.99	5.02%	[0.96 , 0.99]
Mezz.	2.12	4.94%	[2.09 , 2.15]
Sen.	1.76	4.94%	[1.74 , 1.79]

Tranche	TVA	Rel. err.	95% CI
Eq.	1.02	17.02%	[0.84 , 1.19]
Mezz.	1.95	66.51%	[0.65 , 3.24]
Sen.	1.62	66.64%	[0.54 , 2.70]

Table 4 TVA computations in the continuous variation-margining case $V_\tau = P_{\tau-}$ ($\bar{\lambda} = 100$ bp, $\delta = 0$, $m = 10^4$). *Left*: FT scheme based on (9.3). *Right* : Monte Carlo based on the formula (9.9).

we have

$$Q_{\tau^\delta} - C^c_\tau = (P_{\tau^\delta} - P_{\tau-}) + \Delta_{\tau^\delta} - I^c_\tau, \tag{9.11}$$

where the wrong-way and gap terms $(P_{\tau^\delta} - P_{\tau-})$ and $\Delta_{\tau^\delta} = \beta^{-1}_{\tau^\delta} \int_{[\tau,\tau^\delta]} \beta_s dD_s$, in which Δ_{τ^δ} includes the "joint default dividend" $D_\tau - D_{\tau-} = \Delta_\tau$, can be quite substantial. Accordingly, observe from the left panel in Table 4 that, even though $V_\tau = P_{\tau-}$ and $\delta = 0$, the TVA numbers are still important relatively to the corresponding values of P_0 especially for higher tranches. By comparison with Table 3, we see that it's only for the equity tranche that the TVA is substantially reduced by the variation margins, whereas the TVA of higher tranches, essentially due to common shocks, cannot be mitigated by variation margining.

These findings motivate the need for initial margins I^c. But the left panel of Table 5, computed with $\delta = 0$ based on (9.3), shows that the amount of initial margins I^c, assumed a constant proportional to Θ_0 for simplicity here, that is required to balance the DMO gap risk term (joint default dividend $\Delta_\tau = D_\tau - D_{\tau-}$) is huge, especially for higher tranches. This is consistent with the extreme tail event feature of CVA on long protection (especially higher) CDO tranches. Namely, most of the CVA comes from the few joint default scenarios giving rise to the joint default dividends $\Delta_\tau = D_\tau - D_{\tau-}$. To compete with these, initial margins must be of the same level of magnitude, i.e. very large, and this at every point in time of every possible scenario, as DMO default times are totally inaccessible. Such levels of initial margins would represent a huge funding charge for the counterparty. We only consider the perspective of the bank here, but a huge funding charge for the counterparty means that the bank could hardly claim such levels of initial margins. About the difficulty of mitigating joint to default risk by collateralization, see also Bo and Capponi (2015, last paragraph of Sect. 3.1).

Back to $I^c = 0$, the right panel in Table 5 shows the impact of the cure period δ, using Lemma 92 to compute the $\widetilde{\Theta}_0^{(i)}$ by $(\mathbb{G}, \mathbb{Q})$ simulation based on (9.7). Regarding long default protection positions of the bank corresponding to payoffs D as of (6.12), most of the CVA is due to the joint default dividend Δ_τ in the common-shock model, especially for higher tranches. This gap risk is already there for $\delta = 0$, instantaneously realized in the joint default dividend $\Delta_\tau = D_\tau - D_{\tau-}$, rather than developing progressively through $\delta > 0$. Accordingly, the right panel in Table 5 shows that the impact of δ is very limited, even if a bit less for the equity tranche. In the DGC model, we would have a similar effect through a large term $(P_{\tau^\delta} - P_{\tau-})$ in (9.11), already large for $\delta = 0$ as revealed by the comparison between the left graphs in Figures 3 and 7. However, this limited impact of $\delta > 0$ (or instantaneous impact of τ, already present for $\delta = 0$) is restricted to this particular situation of counterparty risk on credit derivatives that we consider here. For counterparty risk on other kinds of derivatives, the methodology of the first part of this paper (Sect. 2 through 6) is equally relevant and the impact of δ is significant. In fact, the continuously variation-margined TVA for $\delta = 0$ is then equal to zero. Hence, once a position is fully collateralized in terms of variation margins, the gap risk related to the cure period becomes the first order residual risk. The reader is referred in this regard to Armenti and Crépey (2015, Sect. 8.5), where an impact in $\sqrt{\delta}$ is observed in the context of a model of counterparty risk on interest rate derivatives.

Tranche/I^c	Θ_0	$10\Theta_0$	$10^2\Theta_0$	$10^3\Theta_0$	$10^4\Theta_0$
Eq.	0.97	0.85	0.16	0.00	0.00
Mezz.	2.12	2.11	1.99	0.87	0.00
Sen.	1.77	1.76	1.66	0.73	0.00

Tranche/ δ	0	2 weeks	1 year
Eq.	0.99	0.99	1.49
Mezz.	2.12	2.12	2.14
Sen.	1.76	1.76	1.76

Table 5 Impact on Θ_0 of the initial margin I^c posted by the counterparty *(left panel with $\delta = 0$)* and of the cure period $\delta \geq 0$ *(right panel with $I^c = 0$)* in the continuous variation-margining case where $V_\tau = P_{\tau-}$.

9.4 Conclusions

To put a final point on the respective wrong-way and gap risk features of the DGC and DMO models, our concluding figure 7 shows the analogs of the left graph in Figure 3 and of the middle graph in Figure 5 using flawed simulations where we replace $\widetilde{P}_t^e + \widetilde{\Delta}_t^e$ by P_{t-} in all the coefficients $\widetilde{f}$ (cf. (6.10), (7.17) and (8.9)), thus artificially removing the respective wrong-way and gap risk from the DGC and DMO models. We can see from the figure that the corresponding fake TVA numbers are up to five (resp. ten) times smaller than the "true" TVA levels that can be seen in Figure 3 (resp. 5). In addition of being much smaller, the fake DGC TVA numbers in the left panel are now mostly decreasing with ϱ, showing by comparison with the previous DGC results that it is actually the wrong-way risk that explains the "systemic" increasing feature observed in the "true" DGC model. As for the fact that the fake DMO TVA numbers are so small, it confirms that most of the TVA in the genuine DMO model is due to joint defaults, which is consistent with the findings in Bo and Capponi (2015, Figure 1).

From a broader numerical perspective, independent of the particular credit derivative models that are used in the second part of this paper, let's recap the advantages of the reduced TVA BSDE (4.5) with respect to the full TVA BSDE (3.6) from a numerical point of view. First, in case $\Lambda > 0$, no direct simulation approach for the full TVA BSDE (3.6) seems possible. Second, when $\Lambda = 0$ and a

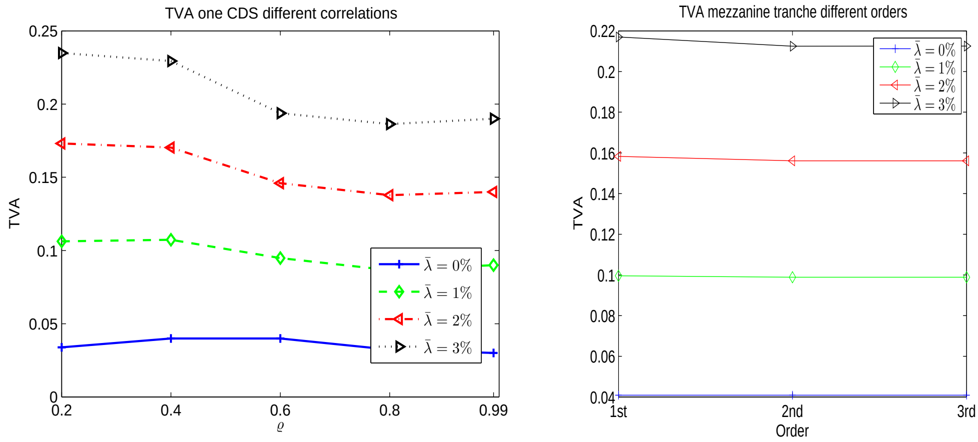

Fig. 7 *Left*: Analog of the left graph of Figure 3 in a fake DGC model without wrong-way-risk. *Right*: Analog of the middle graph of Figure 5 in a fake DMO model without gap risk.

direct simulation approach for the full BSDE (3.6) could be possible, in the case of high dimensional applications where only purely forward simulation schemes are feasible, it's only a first order linear approximation (9.9) that can be estimated directly based on (3.6). By contrast, a nonlinear correction can be computed for (4.5) based on the FT expansion (9.2) and its particle implementation (9.3) or (9.6). The tractability of the FT schemes (9.3) or (9.6) is due to the null terminal condition $\widetilde{\Theta}_T = 0$ in (3.6), implying that $\widetilde{\Theta}^{(0)}_s = 0$ in (9.2), (9.3) or (9.6), which would not be the case in a tentative adaptation of the FT scheme to the full BSDE (3.6). Third, even in cases where one can neglect the nonlinearity in (3.6) and solve it by standard Monte Carlo, using the reduced BSDE (4.5) improves the variance of the simulation.

A Gaussian Estimates

In this appendix we derive Gaussian estimates used in the study of the DGC model of Sect. 7.

Lemma A1 *Given a positive decreasing continuously differentiable function Γ on $\mathbb{R}_+$ such that*

$$\int_{\mathbb{R}_+} t^d \Gamma(t)dt < \infty \quad \text{and} \quad \lim_{t\uparrow\infty} t^{d-1}\Gamma(t) \to 0,$$

for some integer $d \geq 0$, we write $g(y) = -\frac{\Gamma'(y)}{\Gamma(y)}$, $G(y) = \int_y^\infty t^d \Gamma(t)dt$. Let $\bar{y} \geq 0$ and $\alpha, \epsilon > 0$.
(i) *If $g(y) \geq \alpha y$ for $y > \bar{y}$, then*

$$G(y) \leq (\frac{1}{\alpha} + \epsilon) y^{d-1}\Gamma(y) \quad \text{for} \quad y > \bar{y} \vee \sqrt{|d-1|(\frac{1}{\epsilon\alpha^2} + \frac{1}{\alpha})}.$$

(ii) *If $g(y) \leq \alpha y$ for $y > \bar{y}$, then*

$$G(y) \geq (\frac{1}{\alpha} - \epsilon) y^{d-1}\Gamma(y) \quad \text{for} \quad y > \bar{y} \vee \sqrt{|d-1|(\frac{1}{\epsilon\alpha^2} - \frac{1}{\alpha})}.$$

Proof. We only prove (i), for (ii) is similar. For every positive continuously differentiable function φ on $(0, +\infty)$,

$$\begin{aligned}(G(y) - \varphi(y)\Gamma(y))' &= -y^d\Gamma(y) - \varphi'(y)\Gamma(y) + \varphi(y)g(y)\Gamma(y)\\ &= (\varphi(y)g(y) - y^d - \varphi'(y))\Gamma(y) \geq (\alpha y\varphi(y) - y^d - \varphi'(y))\Gamma(y)\end{aligned}$$

for $y \geq \bar{y}$. For $\varphi(y) = (\frac{1}{\alpha} + \epsilon)y^{d-1}$,

$$\alpha y\varphi(y) - y^d - \varphi'(y) = (1+\epsilon\alpha)y^d - y^d - (\frac{1}{\alpha}+\epsilon)(d-1)y^{d-2} =$$
$$\epsilon\alpha y^d - (\frac{1}{\alpha}+\epsilon)(d-1)y^{d-2} = (\epsilon\alpha y^2 - (\frac{1}{\alpha}+\epsilon)(d-1))y^{d-2}.$$

Therefore, if $y > \bar{y} \vee \sqrt{|d-1|(\frac{1}{\epsilon\alpha^2}+\frac{1}{\alpha})}$, then $(G(y)-\varphi(y)\Gamma(y))' \geq \alpha y\varphi(y) - y^d - \varphi'(y) \geq 0$. But $\lim_{y\uparrow\infty}(G(y)-\varphi(y)\Gamma(y)) = 0$, hence $G(y)-\varphi(y)\Gamma(y) \leq 0$. ∎

We use the notation (7.3) as well as Φ and ϕ for the standard normal survival and density functions. By a first application of Lemma A1, to the standard normal density $\Gamma = \phi$, we recover the following classical inequalities on $\psi = \frac{\phi}{\Phi}$: for any constant $c > 1$,

$$c^{-1}y \leq \psi(y) \leq cy, \ y > y_0, \tag{A.1}$$

for some $y_0 > 0$ depending on c. The following estimate, where c and y_0 are as here, can be seen as a multivariate extension of the right hand side inequality in (A.1).

Lemma A2 *There exist constants a and b such that, for every $j \in J$,*

$$0 \leq \psi^j_{\rho,\sigma}(\mathbf{z}) \leq a + b||\mathbf{z}||_\infty. \tag{A.2}$$

Proof. By conditional independence of the components of a multivariate Gaussian vector with homogeneous pairwise correlation ϱ, we have $\Phi_{\rho,\sigma}(\mathbf{z}) = \int_{\mathbb{R}} \Gamma(y)dy$, where $\Gamma(y) = \prod_{l\in J}\Phi\left(\frac{z_l+\sigma\sqrt{\rho}\,y}{\sigma\sqrt{1-\rho}}\right)\phi(y)$. Hence

$$\psi^j_{\rho,\sigma}(\mathbf{z}) = \frac{1}{\sigma\sqrt{1-\rho}}\int_{\mathbb{R}} w_{\rho,\sigma}(\mathbf{z},y)\psi\left(\frac{z_j+\sigma\sqrt{\rho}\,y}{\sigma\sqrt{1-\rho}}\right)dy, \tag{A.3}$$

where $w_{\rho,\sigma}(\mathbf{z},y) = \frac{\Gamma(y)}{\Phi_{\rho,\sigma}(\mathbf{z})}$. Straightforward computations yield

$$g(t) = -\frac{\Gamma'(t)}{\Gamma(t)} = \sum_{l\in J}\psi(\frac{z_l+\sigma\sqrt{\rho}t}{\sigma\sqrt{1-\rho}})\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} + t \geq t,$$

whereas for $t > \max_{l\in J}\frac{1}{\sigma\sqrt{\rho}}(\sigma\sqrt{1-\rho}y_0 - z_l)$ and $t > \frac{1}{\sigma\sqrt{\rho}}\max_{l\in J} z_l$, we have

$$g(t) \leq \textstyle\sum_{l\in J} c\frac{z_l+\sigma\sqrt{\rho}t}{\sigma\sqrt{1-\rho}}\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} + t \leq \bar{\alpha}t,$$

with $\bar{\alpha} := \sum_{l\in J} 2c\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} + 1 \geq 1$. Applying Lemma A1(i) with $d=1, \alpha=1$ and $\epsilon=1$, respectively (ii) with $d=0$, $\alpha=\bar{\alpha}$ and $\epsilon=\frac{1}{2\bar{\alpha}}$, yields

$$\int_y^\infty t\Gamma(t)dt \leq 2\Gamma(y), \ y>0, \quad \text{respectively} \int_y^\infty \Gamma(t)dt \geq \frac{1}{2\bar{\alpha}y}\Gamma(y), \ y > \bar{y}\vee\frac{1}{\sqrt{\bar{\alpha}}},$$

where $\bar{y} = \frac{1}{\sigma\sqrt{\rho}}\max_{l\in J}|z_l| + \frac{1}{\sigma\sqrt{\rho}}\sigma\sqrt{1-\rho}y_0$. Thus, setting $y_1 = \bar{y}+1 = \frac{1}{\sigma\sqrt{\rho}}\max_{l\in J}|z_l| + \frac{1}{\sigma\sqrt{\rho}}\sigma\sqrt{1-\rho}y_0 + 1$,

$$\int_0^\infty t\Gamma(t)dt = \int_0^{y_1} t\Gamma(t)dt + \int_{y_1}^\infty t\Gamma(t)dt \leq y_1\int_0^{y_1}\Gamma(t)dt + 2\Gamma(y_1)$$
$$\leq y_1\int_0^{y_1}\Gamma(t)dt + 4\bar{\alpha}y_1\int_{y_1}^\infty\Gamma(t)dt \leq (1+4\bar{\alpha})\int_0^\infty\Gamma(t)dt,$$

i.e.

$$\int_0^\infty t w_{\rho,\sigma}(\mathbf{z},t)dt \leq (1+4\bar{\alpha})y_1. \tag{A.4}$$

Now, by (A.3) and the right hand side inequality in (A.1),

$$\begin{aligned}0 &\le \sigma\sqrt{1-\rho}\,\psi^{j}_{\rho,\sigma}(\mathbf{z})\\ &\le \int_{\mathbb{R}}\left(\frac{1}{\Phi(y_0)}\mathbb{1}_{\{\frac{z_j+\sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}\le y_0\}}+c\frac{z_j+\sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}\mathbb{1}_{\{\frac{z_j+\sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}>y_0\}}\right)w_{\rho,\sigma}(\mathbf{z},y)dy\\ &= \left(\frac{1}{\Phi(y_0)}+\frac{cz_j}{\sigma\sqrt{1-\rho}}\right)+\frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}\int_{\mathbb{R}}\mathbb{1}_{\{\sigma\sqrt{\rho}y>\sigma\sqrt{1-\rho}y_0-z_j\}}yw_{\rho,\sigma}(\mathbf{z},y)dy\\ &\le \left(\frac{1}{\Phi(y_0)}+\frac{cz_j}{\sigma\sqrt{1-\rho}}\right)+\frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}\int_0^{\infty}yw_{\rho,\sigma}(\mathbf{z},y)dy,\end{aligned} \tag{A.5}$$

so that by substitution of (A.4) into (A.5)

$$0\le \sigma\sqrt{1-\rho}\,\psi^{j}_{\rho,\sigma}(\mathbf{z})\le \left(\frac{1}{\Phi(y_0)}+\frac{cz_j}{\sigma\sqrt{1-\rho}}\right)+\frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}(1+4\bar{\alpha})y_1. \ ∎$$

Lemma A3 *Let $m_t=\int_0^t\varsigma(s)dB_s$, where B is a a univariate standard Brownian motion and ς is a square integrable function with unit L^2 norm. For any constant $q>0$, $e^{q\sup_{0\le s\le t}m_s^2}$ is integrable for sufficiently small t.*

Proof. The process $(m_t)_{t\ge0}$ is equal in law to a time changed Brownian motion $(W_{\bar{t}})_{t\ge0}$, where W is a a univariate standard Brownian motion and $\bar{t}=\int_0^t\varsigma^2(s)ds$ goes to 0 with t. Thus, it suffices to show the result with m replaced by W. Let r_t be the density function of the law of $\sup_{0\le s\le t}|W_s|$ and let $R_t(y)=\int_y^{\infty}r_t(x)dx, y>0$, so that

$$\mathbb{E}[e^{q\sup_{0\le s\le t}W_s^2}]=\int_0^{\infty}e^{qy^2}r_t(y)dy=-[R_t(y)e^{qy^2}]_0^{\infty}+2q\int_0^{\infty}yR_t(y)e^{qy^2}dy \tag{A.6}$$

and (using the reflection principle of the Brownian motion)

$$\begin{aligned}R_t(y)&=\mathbb{Q}[\sup_{0\le s\le t}(W_s^++W_s^-)>y]\le\mathbb{Q}[\sup_{0\le s\le t}W_s^+>\tfrac{y}{2}]+\mathbb{Q}[\sup_{0\le s\le t}W_s^->\tfrac{y}{2}]\\ &=2\mathbb{Q}[\sup_{0\le s\le t}W_s>\tfrac{y}{2}]=2\mathbb{Q}[|W_t|>\tfrac{y}{2}]=2\mathbb{Q}[|W_1|>\tfrac{y}{2\sqrt{t}}]=4\Phi(\tfrac{y}{2\sqrt{t}}),\end{aligned}$$

where by the left hand side in (A.1)

$$\Phi(\frac{y}{2\sqrt{t}})\frac{y}{2\sqrt{t}}\le c\phi(\frac{y}{2\sqrt{t}})=\frac{c}{\sqrt{2\pi}}e^{-\frac{y^2}{8t}},\quad \frac{y}{2\sqrt{t}}>y_0.$$

Therefore, for $\frac{1}{8t}>q$, both terms are finite in the right hand side of (A.6), which shows the result. ∎

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