The Cost-of-Capital XVA Appproach in Continuous Time

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Abstract

Since the 2008 crisis, derivative dealers charge to their clients various add-ons, dubbed XVAs, meant to account for counterparty risk and its capital and funding implications.

As banks cannot replicate jump-to-default related cash flows, deals trigger wealth transfers and shareholders need to set capital at risk. We devise an XVA policy, whereby so called contra-liabilities and cost of capital are sourced from bank clients at trade inceptions, on top of the fair valuation of counterparty risk, in order to guarantee to the shareholders a target hurdle rate h on their capital at risk.

The resulting all-inclusive XVA formula reads (CVA + FVA + KVA), where C sits for credit, F for funding, and where the KVA is a cost of capital risk premium. All these XVA metrics are portfolio-wide, nonnegative and, despite the fact that we include the default of the bank itself in our modeling, they are ultimately unilateral. This makes them naturally in line with the requirement that capital at risk and reserve capital should not decrease simply because the credit risk of the bank deteriorates.

Keywords: Counterparty risk, market incompleteness, credit valuation adjustment (CVA), funding valuation adjustment (FVA), capital valuation adjustment (KVA), wealth transfer.

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1 The Sustainable Pricing and Dividends Problem

We devise a pricing and dividend policy for a dealer bank, **sustainable** in the sense of ensuring to its shareholders a constant instantaneous return rate h on their capital at risk, even in the limiting case of a portfolio held on a run-off basis, i.e. without future deals.

Moreover, the corresponding policy of the bank should satisfy several **regulatory constraints**. Firstly, in order to comply with the Volcker rule that bans proprietary trading for a dealer bank, the market risk of the bank should be hedged. As a result, only counterparty risk remains. Secondly, reserve capital should be maintained by the bank at the level of its *expected* counterparty credit losses, in two lines: the credit valuation adjustment (CVA) of the bank, meant to cope with the counterparty risk of the bank clients, i.e. with the expected losses of the bank due to client defaults, and its funding valuation adjustment (FVA), meant to cope with the counterparty risk of the bank itself, i.e. with its expected risky funding expenses. Thirdly, capital should be set at risk by the bank to deal with its *exceptional* (above expected) *losses*. The above return rate h is then meant at a hurdle rate for the bank shareholders, i.e. a risk premium for their capital at risk within the bank.

Reserve capital, like capital at risk, should obviously be ***nonnegative***. Furthermore, it should not decrease simply because the credit risk of the bank itself has increased: see Albanese and Andersen (2014, Section 3.1) for the relevant wordings from Basel Committee on Banking Supervision (2012) and Federal Register (2014). We take this as a constraint that different lines of capital (as detailed later) should be nondecreasing with respect to the CDS spread of the bank (mathematically, the underlying default intensity process), ceteris paribus, a property that we refer to hereafter as ***monotonicity***.

Further requirements on a solution to the above sustainable pricing and dividends problem are **economic interpretability** and **logical consistency** (for intellectual adhesion by market participants), **numerical feasibility** and **robustness** at the level of a realistic banking portfolio (for practicality), and **minimality** in the sense of being, all things equal, as cheap as possible (for competitiveness).

The design of a pricing and dividend policy satisfying all the above requirements, revolving around a KVA specification which is minimal in the sense of Theorem 4.2, is the main achievement of this article. Although we can not claim for uniqueness, we will see in Section 6.2 that alternative XVA approaches in the literature breach several of the above requirements. Further contributions of the paper are:

- The notion of shareholder valuation (cf. Definition 2.1) as a systematic way to address the successive XVA layers;

- The solution of the ensuing XVA equations by the reduction of filtration methodology of Theorem 4.1, interpretable financially as computations “on a going concern” for the bank.

1.1 Solution Setup

The starting point of our solution to the sustainable pricing and dividends problem is an organizational and accounting separation between three kinds of business units within the bank: the ***CA desks***, the ***clean desks***, and the ***management*** of the bank.

The CA desks are the CVA desk and the FVA desk (or Treasury) of the bank, respectively in charge of the default risk of the clients and of the risky funding expenses of the bank. The corresponding cash flows are collectively called the contra-assets[1] of the bank. The CA desks fully guarantee the trading of the clean desks against client and bank defaults, through a clean margin collateral account, which also funds the trading of the clean desks at the risk-free rate. Thanks to this work accomplished by the CA desks, the clean desks can focus on the market risk of the contracts in their respective business lines, as if there was no counterparty risk. We denote by MtM the amount on the clean margin account of the bank (counted positively when posted by the CA desks) and we write

$$\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA} \tag{1}$$

for the overall amount of reserve capital of the bank, which will correspond to the valuation of its contra-assets.

Remark 1.1 The industry terminology distinguishes an FVA, in the specific sense of the cost of funding re-hypothecable collateral (variation margin), from an MVA defined as the cost of funding segregated collateral (initial margin, see Albanese et al. (2019, Section A)). In this paper, we merge the two in an overall FVA meant in the broad sense of the cost of funding the derivative business of the bank.

If (assumed all cash) collateral happens to be remunerated at some basis with respect to the risk-free rate, then this entails a further "liquidity valuation adjustment". However, the corresponding bases are typically small and the related adjustment negligible with respect to the XVA metrics considered in this paper. ∎

The management of the bank is in charge of the dividend distribution policy. We consider a level of capital at risk (CR) sufficient to make the bank resilient to a forty-year adverse event, i.e. greater than an economic capital (EC) defined as the expected shortfall of the losses of the bank over one year at the confidence level $\alpha = 97.5\% = 1 - \frac{1}{40}$. The implementation of a sustainable dividend remuneration policy requires a dedicated risk margin account, on which bank profits are initially retained so that they can then be gradually released as dividends at a hurdle rate h on shareholder capital at risk (as opposed to being readily distributed as day-one profit). Counterparty default losses, as also funding payments, are materialities for default if not paid. By contrast, risk margin payments are at the discretion of the bank management, hence they do not represent an actual liability to the bank. As a consequence, the capital valuation adjustment (KVA) amount on the risk margin account is also loss-absorbing, i.e. part

[1] Precisely defined in (64)

of capital at risk (CR). With minimality in view (see Section 1 and Theorem 4.2), we thus set

$$\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA}). \tag{2}$$

All bank accounts are marked-to-model, i.e. continuously and instantaneously readjusted to theoretical target levels, which will be defined in Sections 2-3 in view of yielding a solution to the sustainable pricing and dividends problem. All cash accounts of the bank, as well as all the collateral (assumed all cash for simplicity), are remunerated at the risk-free rate.

In line with the sustainability requirement edicted in Section 1, the portfolio is supposed to be held on a run-off basis between inception time 0 and its final maturity. The initial amounts MtM_0, CA_0, and KVA_0 are provided by the clients at the portfolio inception time 0. Mark-to-model readjustments of all bank accounts between time 0 and the bank default time τ (excluded) are on bank shareholders. If the bank defaults, the property of the residual amount on the reserve capital and risk margin accounts is transferred at time τ from the shareholders to the bondholders of the bank.

See Table 1 for a list of the main financial acronyms used in the paper.

CA	Contra-assets valuation	(1)
CL	Contra-liabilities valuation	Definition A.1
CR	Capital at risk	Section 1 and (2)
CVA	Credit valuation adjustment	Section 1, (11), (34), and example (57)
DVA	Debt valuation adjustment	Definition A.1 and example (60)
EC	Economic capital	Section 1 and Definition 3.2
FDA	Funding debt adjustment	Definition A.1 and example (61)
FV	Fair valuation of counterparty risk	Definition A.1 and (66)
FVA	Funding valuation adjustment	Section 1, (12), (35), and example (58)
KVA	Capital valuation adjustment	Section 1.1 and (16), (36), (43)
MtM	Mark-to-market	Section 1.1 and (10), (33)
SCR	Shareholder capital at risk	Definition 2.3

Table 1: Main financial acronyms and place where they are introduced conceptually and/or specified mathematically in the paper, as relevant.

2 The Cost-of-Capital XVA Equations

2.1 Probabilistic Setup

Let there be given, on a measurable space $(\Omega, \mathfrak{A})$, a stochastic basis $(\mathbb{G}, \mathbb{Q})$. The filtration $\mathbb{G} = (\mathfrak{G}_t)_{t\in\mathbb{R}_+}$ satisfies the usual conditions. All the processes in the paper are $\mathbb{G}$ adapted and all the random times of interest are $\mathbb{G}$ stopping times. The corresponding expectation and conditional expectation are denoted by $\mathbb{E}$ and $\mathbb{E}_t$. All cash flow and price processes are modeled as semimartingales.

We denote by T a finite and constant upper bound on the maturity of all claims in the portfolio, e.g. $T = 30$ years, also including the time of liquidating defaulted positions, such as two weeks. Hence (all cumulative) cash flow processes are stopped at T (and they start from 0 at time 0).

For any left-limited process Y, we denote by $Y^{\tau-}$ and ${}^{\tau-}Y$ the respective processes Y stopped before the bank default time τ and starting before τ, i.e.

$$Y^{\tau-} = JY + (1-J)Y_{\tau-},\ {}^{\tau-}Y = Y - Y^{\tau-},$$

where $J = \mathbb{1}_{[\![0,\tau[\![}$ is the survival indicator process of the bank.

The probability measure $\mathbb{Q}$ is used for the corresponding linear valuation task, using the risk-free asset as our numéraire everywhere[2]. In particular, the price processes[3] of the primary assets used by the bank for its hedging (of market risk) and risky funding purposes are assumed to be $\mathbb{Q}$ local martingales. In the case of the client derivative portfolio of the bank, however, pricing will depart from $\mathbb{Q}$ valuation of the contractually promised cash flows, in consideration of counterparty risk and of its funding and capital consequences. The notion of shareholder valuation below will be instrumental in this regard.

Definition 2.1 Given an optional, integrable process $\mathcal{Y}$ stopped at T (cumulative cash flow stream in the financial interpretation), we call value process Z of $\mathcal{Y}$ the optional projection of $(\mathcal{Y}_T - \mathcal{Y})$, i.e.

$$Z_t = \mathbb{E}_t(\mathcal{Y}_T - \mathcal{Y}_t),\ t \le T; \tag{3}$$

We call shareholder value process Y of $\mathcal{Y}$, any process Y vanishing on $[T, +\infty)$ if $T < \tau$ and such that

$$Y_t = \mathbb{E}_t(\mathcal{Y}_{\tau-} - \mathcal{Y}_t + Y_{\tau-}),\ t < \tau. \ \blacksquare \tag{4}$$

Note that the shareholder value equation (4), for a process Y vanishing on $[T, +\infty)$ if $T < \tau$, is equivalent to

$$Y_t^{\tau-} = \mathbb{E}_t(\mathcal{Y}^{\tau-}_{\tau\wedge T} - \mathcal{Y}^{\tau-}_t + \mathbb{1}_{\{\tau \le T\}} Y^{\tau-}_\tau),\ t \le \tau \wedge T. \tag{5}$$

In particular, $(\mathcal{Y} + Y)^{\tau-}$ is then a martingale (stopped before τ).

This makes it apparent that the shareholder valuation of $\mathcal{Y}$ is actually an equation for $Y^{\tau-}$. The corresponding backward stochastic differential equation (BSDE) is tantamount to the notion of recursive valuation of defaultable securities in Collin-Dufresne et al. (2004, Section 3.2), in the special case where $R_t(x) = x$ there. In their setup this notion is shown to be well posed in their Proposition 2, based on Schönbucher (2004)'s tool of the bank survival pricing measure. We will address the issue by a more comprehensive reduction of filtration methodology in Section 4.1 (yielding a more complete grasp on the related integrability issues).

[2]This numéraire choice simplifies equations by removing all terms related to the (risk-free, see after (2)) remuneration of the cash accounts and of the collateral.

[3]Risk-free discounted through the choice of the numéraire.

2.2 Abstract Trading Cash Flows

Unless explicitly specified, an amount paid means effectively paid if positive, but actually received if negative. A similar convention applies to the notions of loss and gain or cost and benefit.

At an abstract level, the (cumulative) trading cash flows of the bank consist of the contractually promised cash flows $\mathcal{P}$ from clients, the counterparty credit cash flows $\mathcal{C}$ to the clients (i.e., because of counterparty risk, the effective cash flows from the clients are $\mathcal{P} - \mathcal{C}$), the risky funding cash flows $\mathcal{F}$ to an external funder of the bank, and hedging cash flows $\mathcal{H}$ (inclusive of the cost of setting up the hedge) to the financial hedging markets. All these cumulative cash flow streams are assumed to be integrable (and stopped at T).

The concrete cash flows depend on the portfolio of the bank, of course, but also on the nature of the connections of the bank with the financial network, e.g. bilateral versus centrally cleared trading. In any case (see Sections 3.3 and 4.3 and Lemmas 5.1 and 5.2 for illustrations):

Assumption 2.1 The processes $\mathcal{C}^{\tau-}$ and $\mathcal{F}^{\tau-}$ are nondecreasing. The process $\mathcal{F}$ is stopped at τ and the hedging loss $\mathcal{H}$ is stopped before τ. The processes $\mathcal{H} = \mathcal{H}^{\tau-}$ and $\mathcal{F}$ are (zero-valued) martingales. ∎

For the bank, the funding issue ends at τ, which explains why $\mathcal{F}$ is stopped at τ. The assumption $\mathcal{H} = \mathcal{H}^{\tau-}$ is made for consistency with our premise that a bank cannot hedge its own jump-to-default exposure. Integrability aside, the martingale assumptions on $\mathcal{H}$ and $\mathcal{F}$ are in line with the view on $\mathbb{Q}$ provided before Definition 2.1.

2.3 Trading Losses

The risk of financial loss as a consequence of client default is hard to hedge, because single name credit default swaps that could in principle be used for that purpose are illiquid. The possibility for the bank of hedging its own jump-to-default is even more questionable, for practical but also legal reasons: For the bank, hedging its default would mean monetizing it beforehand, which goes against bondholder protection rules. Accordingly, we assume no XVA hedge (see however Remark 2.1 and Section A.1). The bank hedging loss $\mathcal{H}$ then collapses to the hedging loss of the clean desks.

In our marked-to-model framework (see the end of Section 1.1), the CVA and FVA desks trading losses are therefore given by

$$\mathcal{C} + \mathrm{CVA} - \mathrm{CVA}_0 \text{ and } \mathcal{F} + \mathrm{FVA} - \mathrm{FVA}_0, \tag{6}$$

for some theoretical target CVA and FVA levels specified later (see Table 1). Likewise, clean desks trading gains, inclusive of their hedging loss $\mathcal{H}$ and of the fluctuations of the mark-to-model of their position, sum up to

$$\mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0 - \mathcal{H}, \tag{7}$$

for some theoretical target MtM level specified later (see Table 1). In line with the fact that a dealer bank should not do proprietary trading (cf. Section 1), the clean desks are assumed to be perfectly hedged, in the sense that (7) vanishes identically. As, by Assumption 2.1, $\mathcal{H} = \mathcal{H}^{\tau-}$, i.e. ${}^{\tau-}\mathcal{H} = 0$, this perfect clean hedge condition splits into

$$\mathcal{P}^{\tau-} + \mathrm{MtM}^{\tau-} - \mathrm{MtM}_0 - \mathcal{H}^{\tau-} = 0, \; {}^{\tau-}\mathrm{MtM} = -{}^{\tau-}\mathcal{P}. \tag{8}$$

The second part is also consistent with the fact that, under the organization of the bank specified in Section 1.1, from bank default onward, the clean margin account is used for providing to the clean desks the contractually promised cash flows that cease to be exchanged between the client and the bank.

The overall trading loss of the bank results from the above as

$$\mathcal{L} = \mathcal{C} + \mathcal{F} + \mathrm{CA} - \mathrm{CA}_0 \tag{9}$$

(cf. (1)). The ensuing setup, where only the counterparty risk related cash flows remain, corresponds to the intuitive idea of a fully collateralized market hedge of its client portfolio by the bank.

Remark 2.1 One could include further an XVA hedge yielding any additional martingale hedging loss process stopped before τ into (9). Conversely, one could relax the perfect clean hedge assumption, i.e. the left hand side identity in (8) (with the right hand side still in force). Such extensions of the setup would change nothing to the qualitative conclusions of the paper, only implying additional terms in $\mathcal{L}$ and accordingly modified economic capital and KVA figures. ∎

2.4 MtM, CVA, and FVA

Shareholders are only hit by pre-bank default cash flows, and by the transfer to creditors of any residual value that the shareholders may still have within the bank right before bank default (cf. the end of Section 1.1). The latter directly applies, at least, if this residual value is positive, which will be the case of all our XVAs below. But it also applies if the corresponding value is negative provided it is guaranteed by a collateralization procedure, which corresponds to the MtM case (see Section 1.1). Accordingly (cf. Definition 2.1):

Definition 2.2 $\mathrm{MtM}^{\tau-}$, $\mathrm{CVA}^{\tau-}$, and $\mathrm{FVA}^{\tau-}$ are shareholder value processes, assumed to exist,[4] of $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{F}$. ∎

That is, MtM, CVA, and FVA are killed at T on $\{T < \tau\}$ and, for $t < \tau$,

$$\mathrm{MtM}_t = \mathbb{E}_t\big(\mathcal{P}_{\tau-} - \mathcal{P}_t + \mathrm{MtM}_{\tau-}\big), \tag{10}$$
$$\mathrm{CVA}_t = \mathbb{E}_t\big(\mathcal{C}_{\tau-} - \mathcal{C}_t + \mathrm{CVA}_{\tau-}\big), \tag{11}$$
$$\mathrm{FVA}_t = \mathbb{E}_t\big(\mathcal{F}_{\tau-} - \mathcal{F}_t + \mathrm{FVA}_{\tau-}\big). \tag{12}$$

[4]Explicit assumptions ensuring the latter will be provided in Sections 4.1 and 4.3.

Remark 2.2 By the observation made after (5), the processes CVA and FVA are such that the trading losses in (6), stopped before τ, are martingales. So is therefore their sum

$$\mathcal{L}^{\tau-} = \mathcal{C}^{\tau-} + \mathcal{F}^{\tau-} + \mathrm{CA}^{\tau-} - \mathrm{CA}_0 \tag{13}$$

(cf. (9)), which is interpreted financially as the trading loss of the bank shareholders. ∎

The processes CVA and FVA are so far unconstrained on $[\![\tau, +\infty[\![\bigcap \big(\{\tau \le T\} \times \mathbb{R}_+\big)$ (whereas MtM is determined on this set through the right-hand-side in (8)). We define them as zero there. As they already vanish on $[T, +\infty)$ if $T < \tau$, either of them, say Y, is in fact killed at $\tau \wedge T$, hence such that

$$^{\tau-}Y = \mathbb{1}_{[\![\tau,+\infty[\![}(Y_\tau - Y_{\tau-}) = -\mathbb{1}_{[\![\tau,+\infty[\![} Y_{\tau-}. \tag{14}$$

2.5 Shareholder Capital at Risk and KVA

Since contra-assets cannot be replicated, capital needs be set at risk by shareholders, who therefore deserve, in the cost-of-capital pricing approach of this paper, a further KVA add-on as a risk premium.

Economic capital (EC) is the level of capital at risk (CR) that a regulator would like to see on an economic basis. In our dynamic setup, EC and CR will be updated continuously. In particular, EC is assumed to be killed at $\tau \wedge T$, as will in turn be CR from what follows. In view of (2), where KVA is provided by the clients in the first place (see Section 1.1):

Definition 2.3 We define *shareholder* capital at risk (SCR), to be remunerated at the hurdle rate h, as

$$\mathrm{SCR} = \mathrm{CR} - \mathrm{KVA} = \max(\mathrm{EC}, \mathrm{KVA}) - \mathrm{KVA} = (\mathrm{EC} - \mathrm{KVA})^+, \tag{15}$$

where KVA is a shareholder value process, assumed to exist,[5] of $\int_0^{\cdot} h\mathrm{SCR}_s ds$, i.e.

$$\mathrm{KVA}_t = \mathbb{E}_t\Big[\int_t^\tau h\big(\mathrm{EC}_s - \mathrm{KVA}_s^{\tau-}\big)^+ ds + \mathrm{KVA}_{\tau-}\Big],\ t < \tau, \tag{16}$$

and KVA is killed at $\tau \wedge T$. ∎

Remark 2.3 The process $\mathrm{KVA}^{\tau-}$ is then a supermartingale with drift coefficient

$$-h\mathrm{SCR} = -h\big(\mathrm{EC} - \mathrm{KVA}\big)^+. \ ∎ \tag{17}$$

Note the following differential form of (16) (cf. (5)):

$$\begin{aligned} &\mathrm{KVA}_T^{\tau-} = 0 \text{ on } \{T < \tau\} \text{ and, for } t \le \tau \wedge T, \\ &d\mathrm{KVA}_t^{\tau-} = -h\mathrm{SCR}_t dt + d\nu_t, \\ &\text{for some martingale } \nu. \end{aligned} \tag{18}$$

[5] Explicit assumptions ensuring the latter will be provided in Section 4.5.

This formulation makes it apparent that the KVA corresponds financially to the amount to be maintained by the bank on its risk margin account in order to be in a position to deliver to its shareholders, dynamically into the future, a hurdle rate h on their capital at risk (SCR). This corresponds to the sustainability requirement in Section 1. Moreover the amount on the risk margin account should land off at $\mathrm{KVA}_T = 0$ on $\{T < \tau\}$. Indeed, ending up in the negative would mean an insufficient risk margin for ensuring the hurdle rate h to the shareholders. Conversely, ending up in the positive at $T < \tau$ would mean that the bank is unnecessarily expensive to its clients, which should be avoided for the sake of the minimality requirement in Section 1.

3 Technical Setup

This section yields a technical specification of the above abstract setup, in which the XVA equations are shown to be well posed in Section 4.

3.1 Reduction of Filtration Setup

In addition to the full model filtration $\mathbb{G} = (\mathfrak{G}_t)_{t\in\mathbb{R}_+}$, we introduce a smaller filtration $\mathbb{F} = (\mathfrak{F}_t)_{t\in\mathbb{R}_+}$ on $(\Omega, \mathfrak{A})$, satisfying like $\mathbb{G}$ the usual conditions and such that the bank default time τ is a $\mathbb{G}$, but not $\mathbb{F}$, stopping time.

Assumption 3.1 For any $\mathbb{G}$ predictable, resp. optional process Y, there exists an $\mathbb{F}$ predictable, resp. optional process Y', dubbed $\mathbb{F}$ reduction of Y, that coincides with Y on $]\!]0, \tau]\!]$, resp. on $[\![0, \tau[\![$. ∎

Assumption 3.2 There exists a probability measure $\mathbb{P}$ on $\mathfrak{F}_T$, equivalent to the restriction of $\mathbb{Q}$ to $\mathfrak{F}_T$, such that stopping before τ turns $(\mathbb{F}, \mathbb{P})$ local martingales on $[0, T]$ into $(\mathbb{G}, \mathbb{Q})$ local martingales on $[\![0, \tau \wedge T]\!]$ (stopped before τ); Conversely, the optional $\mathbb{F}$ reductions of $(\mathbb{G}, \mathbb{Q})$ local martingales on $[\![0, \tau \wedge T]\!]$ without jump at τ are $(\mathbb{F}, \mathbb{P})$ local martingales on $[0, T]$. ∎

Assumptions 3.1 and 3.2 mean that τ is an invariance time as per Crépey and Song (2017b), with so called invariance probability measure $\mathbb{P}$. Unless explicitly mentioned, probabilistic statements still refer to the stochastic basis $(\mathbb{G}, \mathbb{Q})$.

Remark 3.1 The "non-immersion" case where $\mathbb{P} \neq \mathbb{Q}$ corresponds to situations of hard wrong way risk (strong adverse dependence, see e.g. Crépey and Song (2016) and Crépey and Song (2017a)) between the defaults of the bank and a client, or between the default of the bank and its portfolio exposure with a client. ∎

The bank survival probability measure associated with $\mathbb{Q}$ below means the probability measure on $(\Omega, \mathfrak{A})$ with $(\mathbb{G}, \mathbb{Q})$ density process $Je^{\int_0^{\cdot} \gamma_s ds}$ (cf. Schönbucher (2004) and Collin-Dufresne et al. (2004)).

Lemma 2.3, Theorem 3.5, and Section 4.2 in Crépey and Song (2017b) *Under Assumption 3.1, if $\mathbb{Q}(\tau > T \,|\, \mathfrak{F}_T) > 0$ a.s., then*

$$\mathbb{F} \text{ optional reductions are uniquely defined on } [0,T]. \tag{19}$$

If, moreover, τ has a $(\mathbb{G},\mathbb{Q})$ intensity process $\gamma = \gamma J_-$ such that $e^{\int_0^\tau \gamma_s ds}$ is $\mathbb{Q}$ integrable, then there exists a unique invariance probability measure $\mathbb{P}$ on $\mathfrak{F}_T$. The measure $\mathbb{P}$ coincides with the restriction to $\mathfrak{F}_T$ of the bank survival probability measure associated with $\mathbb{Q}$. ∎

Hereafter we work under the corresponding specialization of Assumptions 3.1 and 3.2. In particular, any $\mathbb{G}$ stopping time θ then admits a unique $\mathbb{F}$ stopping time θ', dubbed $\mathbb{F}$ reduction of θ, such that $\theta \wedge \tau = \theta' \wedge \tau$. Moreover, as can be established by section theorem, for any $\mathbb{G}$ progressive Lebesgue integrand X such that the $\mathbb{G}$ predictable projection ${}^pX$ exists,[6] the indistinguishable equality $\int_0^\cdot {}^pX_s ds = \int_0^\cdot X_s ds$ holds. As a consequence, one can actually consider the $\mathbb{F}$ predictable reduction X' of any $\mathbb{G}$ progressive Lebesgue integrand X (even if this means replacing X by ${}^pX$).

The respective conditional expectations with respect to $(\mathfrak{G}_t, \mathbb{Q})$ and $(\mathfrak{F}_t, \mathbb{P})$ are denoted by $\mathbb{E}_t$ and $\mathbb{E}'_t$, or simply $\mathbb{E}$ and $\mathbb{E}'$ if $t = 0$. We will need the following spaces of processes:

- $\mathbb{S}_2$, the space of càdlàg $\mathbb{G}$ adapted processes Y over $[\![0, \tau \wedge T]\!]$ without jump at time τ and such that, denoting $Y^*_t = \sup_{s\in[0,t]} |Y_s|$:

$$\mathbb{E}\Big[Y_0^2 + \int_0^T J_s e^{\int_0^s \gamma_u du} d(Y^*)^2_s\Big] = \mathbb{E}'\Big[\sup_{t\in[0,T]} (Y')^2_t\Big] < \infty, \tag{20}$$

  where the equality was established as Lemma 5.2 in Crépey, Sabbagh, and Song (2020). Note that, for $Y \in \mathbb{S}_2$,

$$\mathbb{E}\Big[\sup_{t\in[0,\tau\wedge T]} Y_t^2\Big] \le \mathbb{E}\Big[Y_0^2 + \int_0^T J_s \, e^{\int_0^s \gamma_u du} d\big(Y^*\big)^2_s\Big] < \infty; \tag{21}$$

- $\mathbb{L}_2$, the space of $\mathbb{G}$ progressive processes X over $[0,T]$ such that

$$\mathbb{E}\big[\int_0^{\tau\wedge T} e^{\int_0^s \gamma_u du} X_s^2 ds\big] = \mathbb{E}'\big[\int_0^T (X'_s)^2 ds\big] < +\infty, \tag{22}$$

  where the equality follows from the formula (36) in Crépey, Sabbagh, and Song (2020) applied to the process $\int_0^\cdot e^{\int_0^t \gamma'_s ds} X'_t dt$;

- $\mathbb{S}'_2 \subset \mathbb{L}'_2$, the respective spaces of $\mathbb{F}$ adapted càdlàg and $\mathbb{F}$ progressive processes Y' and X' over $[0,T]$ that make the corresponding squared norm finite in the right-hand side of (20) or (22).

[6] For which σ integrability of X valued at any stopping time, e.g. X bounded or càdlàg, is enough.

In view of, in particular, (20):

The $\mathbb{F}$ optional reduction operator is an isometry from $\mathbb{S}_2$ onto $\mathbb{S}'_2$, with stopping before τ as the reciprocal operator. (23)

Finally, we postulate a standard weak martingale representation setup, driven by a multivariate Brownian motion and an integer valued random measure (see e.g. Crépey et al. (2020, Section 1.1)).

The following definition will be instrumental in what follows.

Definition 3.1 Given an $\mathbb{F}$ optional and $\mathbb{P}$ integrable process $\mathcal{X}$ stopped at T, we call clean value process of $\mathcal{X}$ the $\mathbb{F}$ adapted process X vanishing on $[T, +\infty)$ and such that

$$X_t = \mathbb{E}'_t(\mathcal{X}_T - \mathcal{X}_t), t \leq T. \blacksquare \tag{24}$$

3.2 Economic Capital

The value-at-risk and expected shortfall of a random loss ℓ, both at some confidence level $\alpha \in (0, 1)$ (with in practice α "close to 1"), respectively denote the left quantile of level α of ℓ, which we denote by $q^\alpha(\ell)$, and $(1-\alpha)^{-1} \int_\alpha^1 q^a(\ell)da$. As is well known, the expected shortfall operator is $(1-\alpha)^{-1}$ Lipschitz from the space of integrable losses ℓ to $\mathbb{R}$, and to $\mathbb{R}_+$ when restricted to centered losses ℓ.

Capital requirements are focused on the solvency issue, because it is when a regulated firm becomes insolvent that the regulator may choose to intervene, take over, or restructure a firm. Specifically, Basel II Pillar II defines economic capital as the $\alpha = 99\%$ value-at-risk of the depletion of core equity tier I capital (CET1) over one year. Moreover, the Fundamental review of the trading book required a shift from 99% value-at-risk to 97.5% expected shortfall as the reference risk measure in capital calculations.

In our setup, before bank default, CET1 depletions correspond to the shareholder trading loss process $\mathcal{L}^{\tau-}$ (see Albanese et al. (2019, Proposition 4.1) for more balance sheet details). In addition, economic capital calculations are typically made by a bank "on a going concern", hence assuming that the bank itself does not default. Accordingly (cf. the last sentence in the result recalled after (19)):

Definition 3.2 The economic capital of the bank at time t, EC_t, is the $(\mathfrak{F}_t, \mathbb{P})$ conditional expected shortfall of the random variable $(\mathcal{L}'_{(t+1)\wedge T} - \mathcal{L}'_t)$ (assumed $\mathbb{P}$ integrable) at the confidence level $\alpha = 97.5\%$, killed at $\tau \wedge T$. $\blacksquare$

Remark 2.2 and the converse part in Assumption 3.2 imply that the process $\mathcal{L}'$ is an $(\mathbb{F}, \mathbb{P})$ local martingale. Assuming its $\mathbb{P}$ integrability is not a practical restriction as, in concrete setups such as the one of Proposition 5.1, $\mathcal{L}^{\tau-}$ and $\mathcal{L}'$ are even square integrable $(\mathbb{G}, \mathbb{Q})$ and $(\mathbb{F}, \mathbb{P})$ martingales.

In particular, as the expected shortfall of a centered random variable is nonnegative:

Remark 3.2 EC is nonnegative. $\blacksquare$

3.3 Trading Cash Flows for Bilateral Portfolios

We now specify contractually promised cash flows $\mathcal{P}$, credit cash flows $\mathcal{C}$, and risky funding cash flows $\mathcal{F}$. Via (8) where $\mathcal{H} = \mathcal{H}^{\tau-}$, $\mathcal{P}$ also determines the hedging cash flows $\mathcal{H}$. For simplicity, we only consider European derivatives. Moreover, we assume bilateral trading with the clients, referring to Albanese, Armenti, and Crépey (2020) for the case of centrally cleared trading. The client portfolio of the bank is thus partitioned into netting sets of contracts which are jointly collateralized and liquidated upon client or bank default. Given each netting set c of the client portfolio, we denote by

- $\mathcal{P}^c$, its contractually promised cash flows;
- τ_c and R_c, the default time and recovery rate of the client corresponding to the netting set c, whereas τ and R are the analogous data regarding the bank itself;
- $\tau_c^\delta \geq \tau_c$ and $\tau^\delta \geq \tau$, the end of the so called close-out periods of the related client and of the bank itself, so that the liquidation of the netting set c happens at time $\tau_c^\delta \wedge \tau^\delta$;
- Γ^c and $\bar{\Gamma}^c$, the client collateral (cash) amounts received and posted by the bank in relation with the netting set c, assumed stopped before the first-default time of the involved parties;
- P^c, the clean value process of $(\mathcal{P}^c)'$, assumed $\mathbb{P}$ integrable. So, by Definition 3.1,

$$P_t^c = \mathbb{E}_t'\big((\mathcal{P}^c)_T' - (\mathcal{P}^c)_t'\big),\ t \leq T, \tag{25}$$

  and P^c vanishes on $[T, +\infty)$.

Note that, by linearity, (25) is the sum over the netting set c of the analogous quantity pertaining to each individual deal in c, which we call the clean valuation of the deal (recall that we restrict ourselves to European derivatives).

The rules regarding the settlement of contracts following defaults are that:

Assumption 3.3 At the time a party (a client or the bank itself) defaults, the property of the collateral posted on each involved netting set is transferred to the collateral receiver.

During the liquidation period of the corresponding netting sets c, the CA desks pay to the clean desks all the unpaid contractual cash flows.

At liquidation time, the property of an amount P^c on the clean margin amount is transferred from the CA desks to the clean desks; any positive value due by a non-defaulted party on this netting set is paid in full, whereas any positive value due by a defaulted counterparty on this netting set is only paid up to some recovery rate in $[0, 1]$. Here value is understood on a clean valuation basis as P^c (cf. (25)), net of the corresponding (already transferred) client collateral, but inclusive of all the promised contractual cash flows unpaid during the liquidation period. ∎

One is then in the abstract setup of the previous sections, for

$$
\begin{aligned}
\mathcal{P} &= \sum_{c} \Big((\mathcal{P}^c)^{\tau_c^\delta \wedge \tau^\delta} + \mathbb{1}_{[\![\tau_c^\delta \wedge \tau^\delta, \infty[\![} P^c_{\tau_c^\delta \wedge \tau^\delta} \Big) \text{ and} \\
\mathcal{C} &= \sum_{c; \tau_c \le \tau^\delta} (1 - R_c) \big(P^c_{\tau_c^\delta \wedge \tau^\delta} + \mathcal{P}^c_{\tau_c^\delta \wedge \tau^\delta} - \mathcal{P}^c_{(\tau_c \wedge \tau)-} - \Gamma^c_{(\tau_c \wedge \tau)-} \big)^+ \mathbb{1}_{[\![\tau_c^\delta \wedge \tau^\delta, \infty[\![} \\
&\quad - (1 - R) \sum_{c; \tau \le \tau_c^\delta} \big(P^c_{\tau^\delta \wedge \tau_c^\delta} + \mathcal{P}^c_{\tau^\delta \wedge \tau_c^\delta} - \mathcal{P}^c_{(\tau \wedge \tau_c)-} + \bar{\Gamma}^c_{(\tau \wedge \tau_c)-} \big)^- \mathbb{1}_{[\![\tau^\delta \wedge \tau_c^\delta, \infty[\![}
\end{aligned}
\tag{26}
$$

(cf. the proof of Lemma 5.2 for a detailed derivation regarding $\mathcal{C}$ and note that $\mathcal{C}^{\tau-}$ is nondecreasing, in line with the abstract specification of Assumption 2.1).

The risky funding cash flows $\mathcal{F}$ depend on the risky funding policy of the bank and on the actual decomposition of the collateral amounts Γ^c and $\bar{\Gamma}^c$ in terms of rehypothecable variation margin and/or segregated initial margin: see Lemma 5.1 for a basic variation margin illustration and see Albanese et al. (2019, Section A) for richer specifications, also involving initial margin.

In any case, $\mathcal{P}$ and $\mathcal{H}$ are additive over individual trades, whereas $\mathcal{C}$ is only additive over netting sets, and $\mathcal{F}$ only over at least as large funding sets. The variation margin is actually aggregatable throughout the overall derivative portfolio of the bank.

Remark 3.3 In practice, capital at risk (CR) can be used by the bank for its funding purposes. This induces an interference of CR with $\mathcal{F}$, hence an intertwining of the FVA and the KVA, which is the topic of Crépey, Sabbagh, and Song (2020). Instead, for simplicity hereafter, we assume that the bank does not use capital at risk (CR) for funding purposes. ∎

4 XVA Equations Well-Posedness and Comparison Results

In this section we show well-posedness results for the CVA, FVA, and KVA equations, whereas the MtM process is characterized in (37). We also establish a KVA minimality result.

4.1 Shareholder Valuation

Recall that the shareholder value equation (4), for a process Y vanishing on $[T, +\infty)$ if $T < \tau$, is equivalent to the BSDE (5) for $Y^{\tau-}$. This applies to each of the MtM, CVA, and FVA equations (10), (11), and (12). In the case of the KVA equation (16), the drift in the equation also depends on the KVA itself. To include this case as well as certain FVA specifications below, we extend the notion of shareholder valuation to cash flows including a component, depending on Y itself, of the form

$$\int_0^{\cdot} J_t j_t(Y_t) dt, \tag{27}$$

for some random function $j = j_t(y)$ measurable with respect to the product of the $\mathbb{F}$ predictable σ field by the Borel σ field on $\mathbb{R}$. We thus consider the following shareholder value equation, which generalizes (5):

$$Y_t^{\tau-} = \mathbb{E}_t\big(\mathcal{Y}_{\tau\wedge T}^{\tau-} - \mathcal{Y}_t^{\tau-} + \int_t^{\tau\wedge T} j_s(Y_s)ds + \mathbb{1}_{\{\tau\leq T\}}Y_\tau^{\tau-}\big),\ t \leq \tau\wedge T, \tag{28}$$

respectively the following clean value equation for Y' (cf. (24)):

$$Y'_t = \mathbb{E}'_t\big(\mathcal{Y}'_T - \mathcal{Y}'_t + \int_t^T j_s(Y'_s)ds\big), t \leq T, \tag{29}$$

and Y' vanishes on $[T, +\infty)$.

Definition 4.1 By $\mathbb{S}_2$ solution to the shareholder valuation equation (28) for $Y^{\tau-}$, we mean any $(\mathbb{G},\mathbb{Q})$ semimartingale solution $Y^{\tau-}$ in $\mathbb{S}_2$ to (28) with $(Y + \mathcal{Y} + \int_0^\cdot j_s(Y_s)ds)^{\tau-}$ in $\mathbb{S}_2$. By the equation (28) in $\mathbb{S}_2$, we mean this equation considered in terms of $\mathbb{S}_2$ solutions $Y^{\tau-}$. By well-posedness of this equation in $\mathbb{S}_2$, we mean existence and uniqueness of an $\mathbb{S}_2$ solution $Y^{\tau-}$.

By $\mathbb{S}'_2$ solution to the clean valuation equation (29) for Y', we mean any $(\mathbb{F},\mathbb{P})$ semimartingale solution Y' in $\mathbb{S}'_2$ to (29) with $(Y' + \mathcal{Y}' + \int_0^\cdot j_s(Y'_s)ds)$ in $\mathbb{S}'_2$. By the equation (29) in $\mathbb{S}'_2$, we mean this equation considered in terms of $\mathbb{S}'_2$ solutions Y'. By well-posedness of this equation in $\mathbb{S}'_2$, we mean existence and uniqueness of an $\mathbb{S}'_2$ solution Y'. ∎

Theorem 4.1 *The shareholder value equation* (28) *in* $\mathbb{S}_2$ *for* $Y^{\tau-}$ *is equivalent, through the bijection* (23)*, to the clean value equation* (29) *in* $\mathbb{S}'_2$ *for* Y'.

In the case where $\mathcal{Y}'$ *is in* $\mathbb{S}_2$*, if the random function* $z \mapsto j_t(z - \mathcal{Y}'_t)$ *is Lipschitz in the real* z *and such that* $j_\cdot(-\mathcal{Y}'_\cdot)$ *is in* $\mathbb{L}'_2$*, then the clean value equation* (29) *for* Y' *is well-posed in* $\mathbb{S}'_2$*, and so is the shareholder value equation* (28) *in* $\mathbb{S}_2$ *for* $Y^{\tau-}$.

Proof. To alleviate the notation, we show the stated equivalence in the base case $j = 0$, i.e. the one between (5) and

$$Y'_t = \mathbb{E}'_t(\mathcal{Y}'_T - \mathcal{Y}'_t), t \leq T. \tag{30}$$

First we show an equivalence between the following differential forms of (5) and (30):

$$\begin{aligned} &Y_T^{\tau-} = 0 \text{ on } \{T < \tau\} \text{ and, for } t \leq \tau\wedge T,\\ &dY_t^{\tau-} = -d\mathcal{Y}_t^{\tau-} + d\nu_t,\\ &\text{for some } (\mathbb{G},\mathbb{Q}) \text{ martingale } \nu \text{ in } \mathbb{S}_2, \end{aligned} \tag{31}$$

respectively

$$\begin{aligned} &Y'_T = 0 \text{ and, for } t \leq T,\\ &dY'_t = -d\mathcal{Y}'_t + d\mu_t,\\ &\text{for some } (\mathbb{F},\mathbb{P}) \text{ martingale } \mu \text{ in } \mathbb{S}'_2. \end{aligned} \tag{32}$$

By definition of $\mathbb{F}$ optional reductions, the terminal condition in (32) obviously implies the one in (31). Conversely, taking the $\mathcal{F}_T$ conditional expectation of the terminal condition in (31) yields

$$0 = \mathbb{E}[Y_T^{\tau-}\mathbb{1}_{\{T<\tau\}}|\mathcal{F}_T] = \mathbb{E}[Y'_T\mathbb{1}_{\{T<\tau\}}|\mathcal{F}_T] = Y'_T\mathbb{Q}(\tau > T\,|\,\mathfrak{F}_T),$$

hence $Y'_T = 0$ (as by assumption $\mathbb{Q}(\tau > T\,|\,\mathfrak{F}_T) > 0$, see above (19)), which is the terminal condition in (32).

For $Y^{\tau-}$ in $\mathbb{S}_2$, the martingale condition in (32) implies the one in (31), by stopping before τ and application to $\nu = \mu^{\tau-}$ of (23) and of the first part in Assumption 3.2. Conversely, the martingale condition in (31) implies that $(Y', \mu = \nu')$ satisfies the second line in (32) on $[\![0, \tau \wedge T]\!]$, hence on $[0, T]$, by (19). Moreover, by application of the second part in Assumption 3.2 and of (23), $\mu = \nu'$ is an $(\mathbb{F}, \mathbb{P})$ martingale in $\mathbb{S}'_2$.

Summarizing, if $Y^{\tau-}, \nu$ in $\mathbb{S}_2$ solve (31), then $Y', \mu = \nu'$ in $\mathbb{S}'_2$ solve (32); conversely, if Y', μ in $\mathbb{S}'_2$ solve (32), then $Y^{\tau-} = (Y')^{\tau-}, \nu = \mu^{\tau-}$ in $\mathbb{S}_2$ solve (31).

Now, if $Y^{\tau-}$ is an $\mathbb{S}_2$ solution to (16), then $Y^{\tau-}, \nu$ in $\mathbb{S}_2$ solve (31) (for some ν), hence $Y', \mu = \nu'$ in $\mathbb{S}'_2$ solve (32), therefore Y' is an $\mathbb{S}'_2$ solution to (36). Conversely, if Y' is an $\mathbb{S}'_2$ solution to (36), then Y', μ in $\mathbb{S}'_2$ solve (32) (for some μ), hence $Y^{\tau-} = (Y')^{\tau-}, \nu = \mu^{\tau-}$ in $\mathbb{S}_2$ solve (31), thus $Y^{\tau-}$ is an $\mathbb{S}_2$ solution to (16) (noting that $\nu \in \mathbb{S}_2$ is $\mathbb{Q}$ square integrable over $[\![0, \tau \wedge T]\!]$, by (21)).

This shows the first part of the theorem. Under the additional assumptions made in the second part, the well-posedness in $\mathbb{S}'_2$ of the clean value equation (29) follows from standard results (see e.g. Kruse and Popier (2016)) applied to the $(\mathbb{F}, \mathbb{P})$ BSDE for $Z' = Y' + \mathcal{Y}'$, i.e. the $(\mathbb{F}, \mathbb{P})$ BSDE with terminal condition $\mathcal{Y}'_T$ and coefficient $z \mapsto j(z - \mathcal{Y}')$. The well-posedness in $\mathbb{S}_2$ of the shareholder value equation (28) for $Y^{\tau-}$ then follows from the first part of the theorem. ∎

In particular, when all equations are stated within the corresponding spaces of square integrable solutions, then the equations (10), (11), (12), and (16) for $\mathrm{MtM}^{\tau-}$, $\mathrm{CVA}^{\tau-}$, $\mathrm{FVA}^{\tau-}$, and $\mathrm{KVA}^{\tau-}$ are respectively equivalent to the following more explicit formulations: For $t \leq T$,

$$\mathrm{MtM}'_t = \mathbb{E}'_t\big(\mathcal{P}'_T - \mathcal{P}'_t\big), \tag{33}$$

$$\mathrm{CVA}'_t = \mathbb{E}'_t\big(\mathcal{C}'_T - \mathcal{C}'_t\big), \tag{34}$$

$$\mathrm{FVA}'_t = \mathbb{E}'_t\big(\mathcal{F}'_T - \mathcal{F}'_t\big), \tag{35}$$

$$\mathrm{KVA}'_t = \mathbb{E}'_t \int_t^T h(\mathrm{EC}'_s - \mathrm{KVA}'_s)^+ ds, \tag{36}$$

and all four processes vanish on $[T, +\infty)$.

4.2 MtM and Clean Valuation

Proposition 4.1 *We have*

$$\mathrm{MtM}' = \sum_c P^c \mathbb{1}_{[\![0, (\tau_c^\delta)'[\![}, \tag{37}$$

which determines MtM *before* τ*, whereas, from* τ *onward,* MtM *proceeds from* $\mathcal{P}$ *by the second part in* (8)*.*

Proof. By (33), the specification of $\mathcal{P}$ in (26), and the computational properties of our reduction of filtration setup, we have, for $t \geq 0$,

$$\begin{aligned}\mathrm{MtM}'_t &= \mathbb{E}'_t\big(\mathcal{P}'_T - \mathcal{P}'_t\big)\\ &= \mathbb{E}'_t \sum_c \Big(\big((\mathcal{P}^c)'\big)_T^{(\tau_c^\delta)'} - \big((\mathcal{P}^c)'\big)_t^{(\tau_c^\delta)'} + \mathbb{1}_{t<(\tau_c^\delta)'\leq T} P^c_{(\tau_c^\delta)'} \Big)\\ &= \sum_c \mathbb{E}'_t \Big((\mathcal{P}^c)'_{(\tau_c^\delta)'\wedge T} - \big((\mathcal{P}^c)'\big)_{(\tau_c^\delta)'\wedge t} + \mathbb{1}_{t<(\tau_c^\delta)'\leq T} P^c_{(\tau_c^\delta)'} \Big).\end{aligned}$$

Moreover, for each c, (25) (where $(\mathcal{P}^c)'$ is assumed $\mathbb{P}$ integrable) and the fact that P^c vanishes on $[T, +\infty)$ yield, for $t < (\tau_c^\delta)' \wedge T$,

$$\begin{aligned}P^c_t + (\mathcal{P}^c)'_{(\tau_c^\delta)'\wedge t} &= P^c_t + (\mathcal{P}^c)'_t = \mathbb{E}'_t\big(P^c_T + (\mathcal{P}^c)'_T\big) = \mathbb{E}'_t\Big(\mathbb{E}'_{(\tau_c^\delta)'\wedge T}\big(P^c_T + (\mathcal{P}^c)'_T\big)\Big)\\ &= \mathbb{E}'_t\big(P^c_{(\tau_c^\delta)'\wedge T} + (\mathcal{P}^c)'_{(\tau_c^\delta)'\wedge T}\big) = \mathbb{E}'_t\Big((\mathcal{P}^c)'_{(\tau_c^\delta)'\wedge T} + \mathbb{1}_{t<(\tau_c^\delta)'\leq T} P^c_{(\tau_c^\delta)'}\Big),\end{aligned}$$

i.e., for $t < (\tau_c^\delta)' \wedge T$,

$$\mathbb{E}'_t\Big((\mathcal{P}^c)'_{(\tau_c^\delta)'\wedge T} - \big((\mathcal{P}^c)'\big)_{(\tau_c^\delta)'\wedge t} + \mathbb{1}_{t<(\tau_c^\delta)'\leq T} P^c_{(\tau_c^\delta)'} \Big) = P^c_t, \tag{38}$$

whereas, for $t \geq (\tau_c^\delta)' \wedge T$, the expression in the left-hand-side of (38) vanishes. Hence,

$$\mathrm{MtM}'_t = \sum_c P^c_t \mathbb{1}_{t<(\tau_c^\delta)'}. \blacksquare$$

In view of the observation about P^c made after (25), MtM computations thus reduce to clean valuations at the individual trade level. These are just standard pricing tasks at each individual claim level under the reduced stochastic basis $(\mathbb{F}, \mathbb{P})$, free from counterparty risk considerations.

Remark 4.1 The notion of clean valuation emerges from the above (see also after (25)) as the relevant notion of valuation for fully collateralized transactions. Hence, for consistency with the market, the model should be calibrated to market prices of fully collateralized transactions (prices of primary hedging or funding assets) by numerical identification, achieved by playing over the model parameters, between the latter and the corresponding model clean valuation formulas. ∎

4.3 CVA and FVA

We now consider the CVA and the FVA equations (11) and (12), as well as their reduced forms (34) and (35).

Regarding the risky funding cash flows, we postulate

$$d\mathcal{F}_t^{\tau-} = J_t f_t(\mathrm{FVA}_t^{\tau-})dt, \tag{39}$$

for some random function $f = f_t(y)$ measurable with respect to the product of the $\mathbb{F}$ predictable σ field by the Borel σ field on $\mathbb{R}$. This typically holds for the variation margin side of the risky funding cash flows: see (54) for a concrete example below.

Proposition 4.2 *The equation* (11) *in* $\mathbb{S}_2$ *for* $\mathrm{CVA}^{\tau-}$ *is equivalent to the formula* (34) *for a process* CVA' *in* $\mathbb{S}'_2$. *If* $\mathcal{C}'_T$ *is* $\mathbb{P}$ *square integrable, then the latter yields a well defined* CVA' *process in* $\mathbb{S}'_2$. *For* $\mathcal{F}^{\tau-}$ *as per* (39), *the equation* (12) *in* $\mathbb{S}_2$ *for* $\mathrm{FVA}^{\tau-}$ *is then in turn equivalent to the following equation in* $\mathbb{S}'_2$ *for* FVA' *(cf.* (35)*):*

$$\mathrm{FVA}'_t = \mathbb{E}'_t \int_t^T f_s(\mathrm{FVA}'_s)ds, \; t \le T. \tag{40}$$

Assuming f *Lipschitz in* y *and* $f_.(0)$ *in* $\mathbb{L}'_2$, *this equation is well posed in* $\mathbb{S}'_2$ *and the equation* (12) *for* $\mathrm{FVA}^{\tau-}$ *is well-posed in* $\mathbb{S}_2$.

Proof. By two successive applications of Theorem 4.1, with $j = 0$ in the CVA case and $\mathcal{Y} = 0$ in the FVA case. ∎

4.4 KVA in the Case of a Default-Free Bank

In this section we temporarily suppose the bank default free, i.e., formally,

$$\text{“}\tau = +\infty, \; (\mathbb{F}, \mathbb{P}) = (\mathbb{G}, \mathbb{Q}).\text{”}$$

The results are then extended to the case of a defaultable bank in Section 4.5.

In this part we use the “$\cdot'$” notation, not in the sense of $\mathbb{F}$ reduction (as $\mathbb{F} = \mathbb{G}$ here), but simply in order to distinguish the present equations from the ones in Section 4.5, where $\mathbb{F} \neq \mathbb{G}$. In Section 4.5 the present data will then be interpreted as the $\mathbb{F}$ reductions of the corresponding data there.

In particular, the process EC' is defined just like EC in Definition 3.2, except that $\tau = +\infty$ here. Given $C' \ge \mathrm{EC}' \ge 0$ representing a putative capital at risk process for the bank, we consider the auxiliary BSDE

$$K'_t = \mathbb{E}'_t \int_t^T h\big(C'_s - K'_s\big)ds, \; t \le T, \tag{41}$$

with the same interpretation as the KVA (cf. the comment following (18)), but relative to any putative capital at risk process C', and simplified to the present setup of a risk-free bank.

Lemma 4.1 *If* C' *is in* $\mathbb{L}'_2$, *then the equation* (41) *for* K' *has for unique* $\mathbb{S}'_2$ *solution*

$$K'_t = h\mathbb{E}'_t \int_t^T e^{-h(s-t)} C'_s ds, \; 0 \le t \le T. \tag{42}$$

If $\mathcal{L}'$ is in $\mathbb{L}_2'$, then EC' *is in* $\mathbb{L}_2'$ *and the* KVA' *equation* (36) *has a unique* $\mathbb{S}_2'$ *solution, such that*

$$\mathrm{KVA}'_t = h\mathbb{E}'_t \int_t^T e^{-h(s-t)} \max(\mathrm{EC}'_s, \mathrm{KVA}'_s) ds,\ 0 \le t \le T. \tag{43}$$

Proof. If $\mathcal{L}'$ is in $\mathbb{L}_2'$, then EC' is in $\mathbb{L}_2'$, by Definition 3.2 and $(1-\alpha)^{-1}$ Lipschitz property of the expected shortfall operator recalled in the beginning of Section 3.2. Moreover, the KVA' BSDE (36) has a Lipschitz coefficient

$$k_t(y) = h\big(\mathrm{EC}'_t - y\big)^+,\ y \in \mathbb{R}. \tag{44}$$

By the second part in Theorem 4.1 applied with $\mathcal{Y} = 0$, the KVA' equation (36) has therefore a unique $\mathbb{S}_2'$ solution. This also holds for the linear BSDE (41), by even simpler considerations. Moreover, the $\mathbb{S}_2'$ solution K' to (42) solves (41).

The process KVA' is in $\mathbb{S}_2'$ with martingale part in $\mathbb{S}_2'$ and, by (36), we have, for $0 \le t \le T$,

$$\mathrm{KVA}'_t = \mathbb{E}'_t \int_t^T h\big(\mathrm{EC}'_s - \mathrm{KVA}'_s\big)^+ ds = \mathbb{E}'_t \int_t^T h\Big(\max(\mathrm{EC}'_s, \mathrm{KVA}'_s) - \mathrm{KVA}'_s\Big) ds. \tag{45}$$

Hence the process KVA' solves in $\mathbb{S}_2'$ the linear BSDE (41) corresponding to the implicit data $C' = \max(\mathrm{EC}', \mathrm{KVA}') \in \mathbb{L}_2'$. Equation (43) is the corresponding instantiation of (42). ∎

Assuming $\mathcal{L}'$ is in $\mathbb{L}_2'$, let

$$\mathrm{CR}' = \max(\mathrm{EC}', \mathrm{KVA}'), \tag{46}$$

where KVA' is the $\mathbb{S}_2'$ solution to (36). Note that CR' is nonnegative, as this is already the case for EC' as seen in Remark 3.2. To emphasize the dependence on C', we henceforth denote by $K' = K'(C')$ the solution (42) to the linear BSDE (41). In particular, (43) and (46) read as

$$\mathrm{KVA}' = K'(\mathrm{CR}'). \tag{47}$$

We define the set of admissible capital at risk processes as

$$\mathrm{Adm}' = \{C' \in \mathbb{L}_2'; C' \ge \max\big(\mathrm{EC}', K'(C')\big)\}. \tag{48}$$

Here $C' \ge \mathrm{EC}'$ is the risk acceptability condition, while $C' \ge K'(C')$ expresses that the risk margin $K'(C')$, which would correspond through the constant hurdle rate h to the tentative capital at risk process C' (cf. the comment regarding the KVA made after (18)), is part of capital at risk (cf. the comment above (2)).

Proposition 4.3 *Assuming that $\mathcal{L}'$ is in $\mathbb{L}_2'$, then*
(i) $\mathrm{CR}' = \min \mathrm{Adm}', \mathrm{KVA}' = \min_{C' \in \mathrm{Adm}'} K'(C')$;
(ii) *The process* KVA' *is nondecreasing in the hurdle rate* h.

Proof. **(i)** By (47),

$$\mathrm{CR}' = \max(\mathrm{EC}', \mathrm{KVA}') = \max\big(\mathrm{EC}', K'(\mathrm{CR}')\big).$$

Therefore $\mathrm{CR}' \in \mathrm{Adm}'$. Moreover, for any $C' \in \mathrm{Adm}'$, we have (cf. (44)):

$$k_t(K'_t(C')) = h\big(\mathrm{EC}'_t - K'_t(C')\big)^+ \leq h(C'_t - K'_t(C')).$$

Hence the coefficient of the KVA′ BSDE (36) never exceeds the coefficient of the linear BSDE (41) when both coefficients are evaluated at the solution $K'_t(C')$ of (41). Since these are BSDEs with equal (null) terminal condition, the BSDE comparison principle of Proposition 4 in Kruse and Popier (2016)[7] applied to the BSDEs (41) and (36) yields $\mathrm{KVA}' \leq K'(C')$. Consequently, $\mathrm{KVA}' = \min_{C' \in \mathrm{Adm}'} K'(C')$ and, for any $C' \in \mathrm{Adm}'$,

$$C' \geq \max(\mathrm{EC}', K'(C')) \geq \max(\mathrm{EC}', \mathrm{KVA}') = \mathrm{CR}'.$$

Thus $\mathrm{CR}' = \min \mathrm{Adm}'$.
(ii) The coefficient (44) of the KVA′ BSDE (36) is nondecreasing in the parameter h. So is therefore the $\mathbb{S}'_2$ solution KVA′ to (36), by the BSDE comparison theorem of Kruse and Popier (2016, Proposition 4) applied to the BSDE (36) for different values of the parameter h. ∎

4.5 KVA in the Case of a Defaultable Bank

In the case of a defaultable bank, "·′" now denoting $\mathbb{F}$ reduction (predictable, optional, or progressive, as applicable), we have by applications of the first part in Theorem 4.1 (with $\mathcal{Y} = 0$ there):

Lemma 4.2 *The equation*

$$K^{\tau-}_t = \mathbb{E}_t\Big(\int_t^{\tau\wedge T} h\big(C_s - K_s\big)ds + \mathbb{1}_{\{\tau \leq T\}} K^{\tau-}_\tau\Big),\ t \leq \tau \wedge T \tag{49}$$

in $\mathbb{S}_2$ *for* $K^{\tau-}$ *is equivalent, through the bijection* (23)*, to the equation* (41) *in* $\mathbb{S}'_2$ *for* K' *.*

The equation (16) *in* $\mathbb{S}_2$ *for* $\mathrm{KVA}^{\tau-}$ *is equivalent, through the bijection* (23)*, to the equation* (36) *in* $\mathbb{S}'_2$ *for* KVA′*.* ∎

Hence, given also Lemma 4.1 :

Proposition 4.4 *If* $C' \in \mathbb{L}'_2$*, then the equation* (49) *for* $K^{\tau-}$ *is well posed in* $\mathbb{S}_2$ *and the* $\mathbb{F}$ *optional reduction* K' *of its* $\mathbb{S}_2$ *solution* K *is the* $\mathbb{S}'_2$ *solution to* (41)*.*

If $\mathcal{L}'$ *is in* $\mathbb{L}'_2$*, then the equation* (16) *for* $\mathrm{KVA}^{\tau-}$ *is well posed in* $\mathbb{S}_2$ *and the* $\mathbb{F}$ *optional reduction* KVA′ *of its* $\mathbb{S}_2$ *solution* $\mathrm{KVA}^{\tau-}$ *is the* $\mathbb{S}'_2$ *solution to* (36)*.* ∎

[7] Note that jumps are not an issue for comparison in our setup, where the coefficient k "only depends on y"; cf. Kruse and Popier (2016, Assumption (H3')).

In the case of a defaultable bank, writing $K = K(C)$ for the $\mathbb{S}_2$ solution to (49), the set of admissible capital at risk processes is defined by (cf. (48) and the following comments)

$$\mathrm{Adm} = \{C \in \mathbb{L}_2; C \geq \max\big(\mathrm{EC}, K(C)\big)\}. \tag{50}$$

The following result shows that $\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA})$ is in fact *the minimal and cheapest* capital at risk process C satisfying the risk admissibility condition $C \geq \mathrm{EC}$ and consistent with the target hurdle rate h on shareholder capital at risk.

Theorem 4.2 *Assuming that $\mathcal{L}'$ is in $\mathbb{L}_2'$:*
(i) *We have* $\mathrm{CR} = \min \mathrm{Adm}, \mathrm{KVA}^{\tau-} = \min_{C\in\mathrm{Adm}} K(C)$;
(ii) *The process* KVA *is nondecreasing in h.*

Proof. This follows by application of Propositions 4.3 and 4.4. ∎

4.6 From Replication to Balance Sheet Optimization

The KVA formula (43), where $\max(\mathrm{EC}', \mathrm{KVA}') = \mathrm{CR}'$ represents the capital at risk, appears as a continuous-time analog of the risk margin formula under the Swiss solvency test cost of capital methodology: See Swiss Federal Office of Private Insurance (2006, Section 6, middle of page 86 and top of page 88).

In a real-life environment where banks compete for clients (as opposed to our setup where only one bank is considered), an endogenous and stochastic implied hurdle rate arises from the competition between banks. See the last paragraph of Section 3.3 in Albanese et al. (2019) for the corresponding analysis in a one-period setup. How to extend such an analysis to the dynamic setup seems nontrivial and would be an interesting topic of further research.

In practice, the KVA formula (43) can be used either in the direct mode, for computing the KVA corresponding to a given target hurdle rate h set by the management of the bank, or in the reverse-engineering mode, like the Black–Scholes model with volatility, for defining the implied hurdle rate associated with the actual amount on the risk margin account of the bank. Cost of capital proxies have always been used to estimate return-on-equity. Whether it is used in the direct or in the implied mode, the KVA is a refinement, dynamic and fine-tuned for derivative portfolios, but the base concept is far older than even the CVA.

In the current state of the market, even when they are computed, the KVA and even the MVA (which is included in the FVA in this paper, see Remark 1.1) are not necessarily passed into entry prices. But they are strategically used for collateral and capital optimization purposes. This reflects a switch of paradigm in derivative management, from replication to balance sheet optimization.

5 Example

Let

$$U_0 = 1 \text{ and } dU_t = \lambda_t U_t dt + (1-R)U_{t-}dJ_t = U_{t-}d\mu_t,\ t \leq \tau \wedge T, \tag{51}$$

where $d\mu_t = \lambda_t dt + (1-R)dJ_t$, represent the martingale price process (see before Definition 2.1) of the risky funding asset used by the bank for its unsecured borrowing purposes, for a constant recovery rate R of the bank and the unsecured borrowing spread process $\lambda = (1-R)\gamma$. Note that, unless the bank would be allowed to sell default protection on itself, it can only be short in U.

We assume all re-hypothecable collateral (i.e. no initial margin) and we denote by D an optional process representing the difference between the collateral MtM posted by the CA desks to the clean desks and the collateral received by the CA desks from the clients. Note that the process $\mathcal{F}$ that arises from standard "self-financing calculus" below satisfies the related properties in Assumption 2.1.

Lemma 5.1 *For $t \le T$, we have*

$$d\mathcal{F}_t = (D_{t-} - \mathrm{CA}_{t-})^+ d\mu_t, \tag{52}$$

i.e.

$$\begin{aligned} d\mathcal{F}_t^{\tau-} &= J_t \lambda_t (D_t - \mathrm{CVA}_t - \mathrm{FVA}_t)^+ dt, \\ d\big({}^{\tau-}(-\mathcal{F})\big)_t &= (1-R)(D_{t-} - \mathrm{CVA}_{t-} - \mathrm{FVA}_{t-})^+(-dJ_t). \end{aligned} \tag{53}$$

If $\mathcal{C}'_T$ is $\mathbb{P}$ square integrable, then $\mathcal{F}^{\tau-}$ is of the form (39) *with*

$$f_\cdot(y) = \lambda'(D' - \mathrm{CVA}' - y)^+, \tag{54}$$

where CVA' *is defined by* (34)*.*

Proof. Assuming that capital at risk is not used by the bank for its funding purposes (cf. Remark 3.3), the funding strategy of the CA desks reduces to a splitting of the amount CA_t on the reserve capital account as

$$\begin{aligned} \mathrm{CA}_t &= \underbrace{D_t}_{\text{Posted collateral remunerated at the risk-free rate}} + \underbrace{(\mathrm{CA}_t - D_t)^+}_{\text{Cash invested at the risk-free rate}} - \underbrace{(\mathrm{CA}_t - D_t)^-}_{\text{Cash unsecurely funded}} \\ &= \underbrace{\big(D_t + (\mathrm{CA}_t - D_t)^+\big)}_{=:\, \xi_t,\ \text{invested at the risk-free rate}} - \underbrace{(\mathrm{CA}_t - D_t)^-}_{=:\, \eta_t U_t,\ \text{unsecurely funded}} \end{aligned} \tag{55}$$

(all risk-free discounted amounts). Given our use of the risk-free asset as numéraire, a standard self-financing equation yields[8]

$$d\left(\xi_t - \eta_t U_t\right) = -\eta_{t-} dU_t = -\eta_{t-} U_{t-} d\mu_t = -(D_{t-} - \mathrm{CA}_{t-})^+ d\mu_t,\ t \le \tau \wedge T.$$

[8]A left-limit in time is required in η because U jumps at time τ, so that the process η, which is defined through (55) as $\frac{(\mathrm{CA}-D)^-}{U}$, is not predictable.

As $\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}$, this yields (52), i.e. (53). If $\mathcal{C}'_T$ is $\mathbb{P}$ square integrable, then (54) follows by the first part in Proposition 4.2. ∎

In what follows we further assume that the bank portfolio involves a single client with default time denoted by τ_1, that $\mathbb{Q}(\tau_1 = \tau) = 0$, that the liquidation of a defaulted party is instantaneous, and that no contractual cash flows are promised at the exact times τ and τ_1.

Let J and J^1, respectively R and R_1, denote the survival indicator processes and constant recovery rates of the bank and its client toward each other (we assume that the bank has identical recovery rates toward its client and its external funder). In this case, D is of the form $J^1 Q$, where Q is the difference between the clean valuation P of the client portfolio and the amount VM of variation margin (re-hypothecable collateral) posted by the client to the bank.

In this setup, the application of Assumption 3.3 leads to the following shape of $\mathcal{C}$, in line with the generic bilateral trading specification of $\mathcal{C}$ in (26).

Lemma 5.2 *For $t \le T$,*

$$\begin{aligned} d\mathcal{C}^{\tau-}_t &= \mathbb{1}_{\{\tau_1 \le \tau\}}(1 - R_1) Q^+_{\tau_1}(-dJ^1_t),\\ d\big({}^{\tau-}(-\mathcal{C})\big)_t &= \mathbb{1}_{\{\tau \le \tau_1\}}(1 - R) Q^-_{\tau}(-dJ_t). \end{aligned} \tag{56}$$

Proof. Before the defaults of the bank or its client, the contractual cash flows are delivered as promised, hence there are no contributions to the process $\mathcal{C}$. Because of this, and since liquidations are instantaneous, it is enough to focus on the contributions to $\mathcal{C}$ at time $\tau \wedge \tau_1$. By symmetry, it is enough to prove the first line in (56). Let $\epsilon = Q^+_{\tau_1}$, where $Q = P - \mathrm{VM}$. By Assumption 3.3, if the counterparty defaults at $\tau_1 < \tau$, then (having excluded the possibility of contractual cash flows at times τ or τ_1):

- The property of the amount P_{τ_1} on the clean margin account is transferred from the CA desks to the clean desks;

- The following amount is transferred (property-wise, regarding VM_{τ_1}) from the clients to the clean desks:

$$\mathrm{VM}_{\tau_1} + R_1 Q^+_{\tau_1} - Q^-_{\tau_1} = \mathbb{1}_{\epsilon=0} P_{\tau_1} + \mathbb{1}_{\epsilon>0}(\mathrm{VM}_{\tau_1} + R_1 Q_{\tau_1}).$$

Combining both cash flows, the loss of the CA desks triggered by the default of the client amounts to

$$P_{\tau_1} - \big(\mathbb{1}_{\epsilon=0} P_{\tau_1} + \mathbb{1}_{\epsilon>0}(\mathrm{VM}_{\tau_1} + R_1 Q_{\tau_1})\big) = \mathbb{1}_{\epsilon>0}(P_{\tau_1} - \mathrm{VM}_{\tau_1} - R_1 Q_{\tau_1}) = (1 - R_1) Q^+_{\tau_1},$$

which shows the first line in (56). ∎

Proposition 5.1 *If* $\mathbb{1}_{\{\tau_1'<T\}}(Q'_{\tau_1'})^+$ *is* $\mathbb{P}$ *square integrable and* $\lambda'(J^1)'(Q')^+$ *is in* $\mathbb{L}_2'$, *then the* CVA *and* FVA *equations* (11)–(12) *are well-posed in* $\mathbb{S}_2$ *and we have, for* $t \le T$,

$$\mathrm{CVA}'_t = \mathbb{E}'_t\big[\mathbb{1}_{\{t<\tau_1'<T\}}(1-R_1)(Q'_{\tau_1'})^+\big], \quad (57)$$

$$\mathrm{FVA}'_t = \mathbb{E}'_t\int_t^T \lambda'_s\big((J^1)'_s Q'_s - \mathrm{CVA}'_s - \mathrm{FVA}'_s\big)^+ ds, \quad (58)$$

$$d\mathcal{L}'_t = (1-R_1)(Q'_{\tau_1'})^+(-d(J^1)'_t) + d\mathrm{CVA}'_t \quad (59)$$
$$+\lambda'_t(J^1)'_t(Q'_t - \mathrm{CVA}'_t - \mathrm{FVA}'_t)^+dt + d\mathrm{FVA}'_t.$$

Moreover, $\mathcal{L}'$ *is in* $\mathbb{S}_2'$ *and the ensuing* KVA *implications of Proposition 4.4 are in force, i.e. the* KVA *equation* (16) *is well posed in* $\mathbb{S}_2$ *and the* KVA$'$ *formula* (43) *holds.*

Regarding the contra-liabilities introduced in Definition A.1, we have

$$\mathrm{DVA}_t = \mathbb{E}_t\big[\mathbb{1}_{\{t<\tau\le\tau_1\wedge T\}}(1-R)Q^-_\tau\big] + \mathbb{E}_t\big[\mathbb{1}_{\{t<\tau\le T\}}\mathrm{CVA}_{\tau-}\big], \quad (60)$$

$$\mathrm{FDA}_t = \mathbb{E}_t\big[\mathbb{1}_{\{t<\tau\le T\}}(1-R)(J^1_{\tau-}Q_{\tau-} - \mathrm{CA}_{\tau-})^+\big] + \mathbb{E}_t\big[\beta_t^{-1}\beta_\tau\mathbb{1}_{\{t<\tau\le T\}}\mathrm{FVA}_{\tau-}\big]. \quad (61)$$

Proof. The CVA and FVA related statements follow by application of Proposition 4.2. The dynamics (59) for $\mathcal{L}'$ are obtained by plugging the first lines of (56) and (53) into (13) and then taking $\mathbb{F}$ reductions of all the data. This process $\mathcal{L}'$ belongs to $\mathbb{S}_2'$ as the sum (modulo a constant) between the $(\mathbb{F},\mathbb{P})$ optional projection of $\mathbb{1}_{\{\tau_1'<T\}}(Q'_{\tau_1'})^+$, assumed $\mathbb{P}$ square integrable, and the $(\mathbb{F},\mathbb{P})$ martingale part of FVA$'$.

The DVA and FDA formulas readily follow from Definition A.1 and (56). ∎

6 Discussion

We revisit, in the light of the requirements of Section 1, the cost-of-capital XVA solution to the sustainable pricing and dividends problem.

6.1 Regulatorily Admissible and Sustainable

The cost-of-capital, all XVA-inclusive, pricing formula is

$$\mathrm{MtM} - (\mathrm{CVA} + \mathrm{FVA} + \mathrm{KVA}), \quad (62)$$

which should be charged to the clients of the deals at portfolio inception time 0. In line with the spirit of the regulation recalled in the second paragraph of Section 1, the mark-to-market amount MtM can and should be used by the clean desks for maintaining a fully collateralized hedge of their market risk as per (8). The reserve capital amount CA = CVA + FVA can and should be used by the CVA traders and the Treasury for coping with the expected client default losses and risky funding expenses of the bank. The KVA risk margin amount can and should be used by the management of the bank for gradually releasing a dividend risk premium to the shareholders of the bank, at a hurdle rate h on their capital at risk.

Shareholder trading gains and KVA risk margin payments then result in a $-(\mathcal{L}^{\tau-} + \mathrm{KVA}^{\tau-} - \mathrm{KVA}_0)$ dividend stream. The following result is an immediate consequence of the martingale and supermartingale properties of the respective $\mathcal{L}^{\tau-}$ and $\mathrm{KVA}^{\tau-}$ processes observed in Remarks 2.2-2.3:

Corollary 6.1 *Shareholder dividends* $-(\mathcal{L}^{\tau-} + \mathrm{KVA}^{\tau-} - \mathrm{KVA}_0)$ *are a submartingale stopped before* τ*, with drift coefficient* $h\mathrm{SCR}$. ∎

Hence, a cost-of-capital XVA pricing and dividend policy ensures to the shareholders of a dealer bank a submartingale equity process, with average growth rate corresponding to a hurdle rate h on their capital at risk. This holds even in the case of a portfolio held on a run-off basis, i.e. without the need to enter new deals for generating new profits. This feature addresses the sustainability requirement in Section 1. Moreover, as shown in Albanese et al. (2019, Section 4.2), the sustainability property of Corollary 6.1 can be extended to the more realistic case of a trade incremental portfolio, by application of a suitable trade incremental XVA policy at every new deal.

Even though our setup is bilateral, in the sense that it (crucially) includes the default of the bank itself (which is the essence of the contra-liabilities wealth transfer issue detailed in Section A), we end up with nonnegative and monotone (portfolio-wide) CVA, FVA, and KVA. These properties are apparent on the reduced formulas (34)–(36), which price the related (all nonnegative) cash flows until the final maturity T of the portfolio, as opposed to $\tau \wedge T$ (which breaks monotonicity) in the case of naively bilateral XVA formulas.

Monotonicity of economic capital with respect to the bank funding (i.e. credit) spread is expected to hold in view of Definition 3.2 and of the concrete specifications of the process $\mathcal{L}'$, such as (59) (with CVA and FVA as in (57)-(58)). However, this monotonicity seems difficult to formally establish mathematically. At least, if it is indeed satisfied by economic capital, then it also holds for capital at risk, by (2), where KVA is already monotone as seen above.

6.2 Economically Credible and Logically Consistent

Whereas counterparty jump-to-default risk risk can fundamentally not be hedged, a large part of the XVA literature relies on a replication paradigm. As established in Proposition A.2 below, in a theoretical, complete counterparty risk market, the all-inclusive XVA formula should simply be CVA − DVA (instead of CVA + FVA + KVA when market incompleteness is accounted for, cf. (62)). This means that an XVA replication approach is not only economically unrealistic, but also necessarily internally inconsistent as soon as its all-inclusive XVA formula differs from CVA − DVA. In particular, the Burgard and Kjaer (2011, 2013, 2017) FVA approach was pioneering but it breaches four of the requirements stated in Section 1, namely: nonnegativity, monotonicity, economic realism (which is lacking to an "XVA replication paradigm"), and logical (internal) consistency. Likewise, the Green et al. (2014) KVA approach was pioneering but it breaches monotonicity, economic realism, internal consistency, and (see below) minimality. Kjaer (2019) is closer to the spirit of the present paper.

With respect to the competing XVA replication framework, our approach results in materially different XVA formulas and balance sheet implications. In particular:

- Despite the fact that we include the default of the bank itself in our modeling, our (portfolio-wide) XVA metrics are, ultimately, unilateral (hence monotone), and they are always nonnegative;
- Our KVA is loss-absorbing, hence, by contrast with the KVA of Green, Kenyon, and Dennis (2014), it does not appear as a liability in the balance sheet;
- As a consequence of the previous point, the KVA that arises from our theory discounts future capital at risk projections at the hurdle rate h; given the very long time horizon of XVA computations, this discounting makes our KVA materially cheaper.

In addition, instead of working with economic capital, Green, Kenyon, and Dennis (2014) use approximations in the form of scriptural regulatory capital specifications. This is done for simplicity but it is less satisfying economically. It is also less self-consistent: By contrast, under the cost-of-capital, economic capital based, XVA approach, clean valuation MtM computations flow into CVA computations, which in turn flow into FVA computations, which all flow into KVA computations. These connections then make the MtM, $\text{CA} = \text{CVA} + \text{FVA}$, and KVA equations, thus the derivative pricing problem as a whole, a self-contained and self-consistent problem.

6.3 Numerically Feasible and Robust

From Section 3 onward, we considered a dealer bank involved into bilateral derivative portfolios with clients. We refer the reader to Albanese et al. (2017) and Albanese et al. (2019, Section 5) for numerical applications on realistically large bilateral trade portfolios, based on respective nested Monte Carlo and enhanced (neural nets and quantile) regression computational strategies. In the current regulatory environment, bilateral exotic trades are typically hedged by vanilla portfolios that are cleared through central counterparties. The abstract equations of Section 2 can also be applied to the case of centrally cleared derivative portfolios, which is done in Albanese, Armenti, and Crépey (2020). A cost-of-capital XVA approach, thus extended, can then be applied to the situation of a bank involved into an arbitrary combination of bilateral and centrally cleared derivative portfolios.

The model risk inherent to XVA computations in general, and to economic capital based KVA computations more specifically, can be addressed by a Bayesian variant of our baseline cost-of-capital XVA approach. Toward this end, we combine, in a global simulation, paths of the risk factors obtained in several models, all (econometrically realistic and) calibrated to the market in the sense specified in Remark 4.1. Drawing scenarios equally from each, tails are more leptokurtotic and risk measures are typically greater as they are when one picks just a single good model from among the many that are equally valid. The difference between the resulting enhanced KVA and a baseline, reference KVA, can be used as a reserve against model risk.

6.4 Minimal

Nonnegative and ultimately unilateral XVAs (cf. Section 6.1) are more conservative than the naively (but then non monotone) bilateral, sometimes even negative portfolio-wide, XVAs that appear in part of the related literature (see Section 6.2).

On the other hand, an FVA computed at the level of a unique funding set as in (58), across possibly multiple netting sets as shown in Albanese et al. (2019), avoids the over-conservatism of FVAs sometimes calculated for simplicity by netting set and aggregated. Indeed, such simplification misses the FVA markdown corresponding to the re-hypothecability of variation margin across netting sets. One should in fact also account for the further FVA markdown due to the possibility for a bank to use its capital at risk as variation margin, which is done in Crépey, Sabbagh, and Song (2020).

Our KVA is minimal in the sense provided by Theorem 4.2. An even cheaper, bilateral KVA as in Albanese, Caenazzo, and Crépey (2017, Proposition 4.2(v)) results from a variation of our approach whereby, upon bank default, notwithstanding the bankruptcy rules recalled in the last paragraph of Section 1.1, the residual risk margin flows back into equity capital and not to bondholders However, via the participation of the KVA to capital at risk (cf. (2)), such a bilateral KVA may lead to violations of the capital at risk monotonicity requirement.

Likewise, a cheaper, bilateral FVA as in Albanese et al. (2017, Proposition 4.2(i)) follows from asserting that, upon bank default, the residual reserve capital of the FVA desk[9] flows back into equity capital and not to bondholders. Such a bilateral FVA may still satisfy the FVA reserve capital monotonicity requirement, by a compensation between two opposite effects when the default intensity of the bank increases in the bilateral FVA formula: increased funding spread versus shortened time integration interval. However, in both this FVA and the above KVA cases, the corresponding violations of the usual bankruptcy rules induce "shareholder arbitrage", in the sense of a riskless profit to shareholders in the case where the bank would default instantaneously at time 0, right after the client portfolio has been setup and the corresponding reserve capital and risk margin amounts have been sourced from the clients.

Hence, "local departures" from our cost-of-capital XVA solution to the sustainable pricing and dividends problem of Section 1 may be a bit cheaper, but they are less self-consistent. Moreover, as seen in Section 6.2, more radically different approaches to the problem suffer from more severe shortcomings with respect to the requirements of Section 1. In an intuitive formulation, we conclude that the cost of capital XVA solution to the sustainable pricing and dividends problem may not be the only solution, nor is it necessarily "globally minimal", but it has some "locally minimizing properties, at least in certain directions of the search space", and we are not aware of any other "distant solution".

[9]Disentangling for this purpose the CA desks into a CVA desk and an FVA desk, each endowed with their own reserve capital account.

A Wealth Transfer Analysis

In this appendix we consider the XVA issue from the point of view of the bank bondholders, as opposed to the shareholders mainly in the above. This brings to light the symmetrical companions of the contra-assets, i.e. the so called contra-liabilities. Put together, contra-assets and contra-liabilities will allow us to analyze the wealth transfers triggered by the trading of the bank due to the impossibility for the latter of hedging out counterparty risk. Note that a view on DVA and FDA as wealth transfers is consistent with the conclusions drawn in a structural default model of the bank by Andersen, Duffie, and Song (2019) (who do not deal with the KVA).

Using (14) that applies to $Y =$ CVA and FVA, Figure 1 details the split of the overall trading cash flows, $\mathcal{L}$ in (9), between the pre-bank default trading cash flows, i.e. the shareholder trading cash flows $\mathcal{L}^{\tau-}$ as per (13), and the bondholder trading cash flows,

$$^{\tau-}(-\mathcal{L}) =^{\tau-}(-\mathcal{C}) +^{\tau-}(-\mathcal{F}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{CA}_{\tau-}. \quad (63)$$

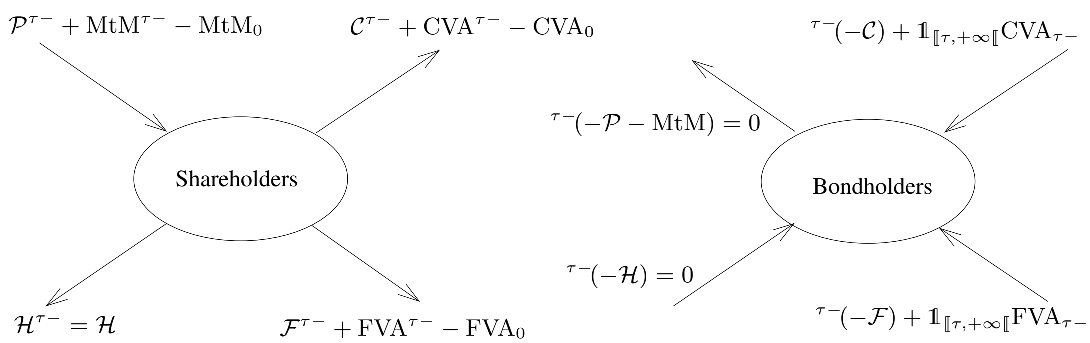

Figure 1: *Left*: Pre-bank-default trading cash flows $\mathcal{L}^{\tau-}$. *Right*: Trading cash flows from bank default onward $^{\tau-}(-\mathcal{L})$.

The notion of valuation referred to below is the one of Definition 2.1.

Definition A.1 Assuming the integrability of the related cash flows, we call

- contra-assets, the cash-flows valued by the CA process, i.e. in view of (1), (11), and (12),

$$\mathcal{C}^{\tau-} + \mathcal{F}^{\tau-} + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{CA}_{\tau-}; \quad (64)$$

- contra-liabilities, the bondholder trading cash flows (63);
- debt valuation adjustment (DVA), the value process of $(^{\tau-}(-\mathcal{C}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{CVA}_{\tau-})$;
- funding debt adjustment (FDA), the value process of $(^{\tau-}(-\mathcal{F}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{FVA}_{\tau-})$;

- fair valuation of counterparty risk (FV), the value of $(\mathcal{C}+\mathcal{F})$;
- contra-liability valuation (CL), the value process of the contra-liabilities ${}^{\tau-}(-\mathcal{L})$;
- shareholder and bondholder risk margin cash flows, the respective pre-bank-default and bank-default-onward cash flows valued by the KVA in (16), i.e.

$$\int_0^{\cdot\wedge\tau} h\big(\mathrm{EC}_s - \mathrm{KVA}_s\big)^+ ds \text{ and } \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{KVA}_{\tau-}; \tag{65}$$

- KVA^{sh} and KVA^{bh}, the value processes of the shareholder and bondholder risk margin cash flows, i.e. KVA^{sh}_t and KVA^{bh}_t are the conditional expectations of the integral term and of $\mathrm{KVA}_{\tau-}$ in (16). ∎

Lemma A.1 *We have* $\mathrm{CL} = \mathrm{DVA} + \mathrm{FDA}$, *which is also the value process of* $(-\mathcal{L})$. *Moreover,*

$$\mathrm{FVA} = \mathrm{FDA} \textit{ and } \mathrm{FV} = \mathrm{CA} - \mathrm{CL} = \mathrm{CVA} - \mathrm{DVA} \tag{66}$$

hold before τ.

Proof. The first part holds by Definition A.1 and the zero-valued martingale property of $\mathcal{L}^{\tau-}$. The second part holds by Definition A.1, (11), (12), (1), and the zero-valued martingale property of $\mathcal{F}$. ∎

The notion of wealth below refers to the sum between the valuation of the future cash flows that affect the concerned economic agent (shareholders, bondholders, or bank as a whole), plus all the cash flows accumulated by the agent until valuation time, starting from the initial wealth of the agent at time 0.

Proposition A.1 The shareholder wealth is given by

$$\mathrm{SHC}_0 - \big(\mathcal{L}^{\tau-} + \mathrm{KVA}^{\tau-} - \mathrm{KVA}_0\big) + \mathrm{KVA}^{sh}, \tag{67}$$

where SHC_0 is the initial shareholder capital. The bondholder wealth is given by

$$\mathrm{CL} + \mathrm{KVA}^{bh} +^{\tau-}(-\mathcal{L}) +^{\tau-}(-\mathrm{KVA}). \tag{68}$$

The bank wealth equals

$$\mathrm{SHC}_0 + \mathrm{KVA}_0 + \mathrm{CL} - \mathcal{L}. \tag{69}$$

Shareholder, bondholder and bank wealths are martingales.

Proof. The formulas (67) and (68) follow from the definition of wealth by inspection of the cash flows that affect the shareholders and the bondholders. Regarding (67), the initial shareholder capital SHC_0 is eroded by the shareholder trading losses $\mathcal{L}^{\tau-}$ and replenished by the risk margin payments to shareholders $\mathrm{KVA}_0 - \mathrm{KVA}^{\tau-}$, whereas KVA^{sh} corresponds to the expected future risk margin payments to shareholders. The sum between (67) and (68) yields (69). Bank and bondholder wealths are Doob martingales and so is their difference which is shareholder wealth. ∎

A.1 What-If Analysis

Lastly, we examine the consequences of an assumption that the bank could, both practically and legally, hedge out counterparty risk. As explained in Section 2.3, this assumption is counterfactual. However, we endorse it here for the sake of the argument, in order to understand better the role of the opposite assumption, that counterparty risk is *not* hedged, before in the paper.

We recall from the first part of Lemma A.1 that the time 0 value of $(-\mathcal{L})$ is CL_0. In line with the view on $\mathbb{Q}$ provided before Definition 2.1, the counterparty risk hedge of the bank must trade at the price corresponding to its value process.

Proposition A.2 *On top of the market hedging loss $\mathcal{H} = \mathcal{H}^{\tau-}$ as before, we assume that the bank setups a counterparty risk hedge insuring the payment of a cash flow stream $\mathcal{L}$ to the bank, along with a time 0 premium CL_0. Then $\mathrm{MtM}_0 - \mathrm{FV}_0 = \mathrm{MtM}_0 - (\mathrm{CVA}_0 - \mathrm{DVA}_0)$ (by (66)) is a replication price for the derivative portfolio of the bank. The bondholder and shareholder wealth processes that ensue from the corresponding replication strategy are respectively given by the constants 0 and SHC_0, so that SHC_0 is then also the wealth of the bank as a whole.*

Proof. Here is the corresponding replication strategy. The clean desks do the same as before. So do the CA desks, except that they pass to the client (at time 0) and to the bank shareholders (on $(0, \tau)$) a diminished add-on $\mathrm{FV} = \mathrm{CA} - \mathrm{CL}$, instead of CA before without the counterparty risk hedge. A new business unit within the bank (cf. Section 1.1), which we call the CL desk, puts the upfront premium CL_0 of the counterparty risk hedge on a dedicated cash account, along with a matching liability of CL_0 on the bank balance sheet. The CL desk cash account, like all the other ones within the bank, is market-to-model, i.e. reset in continuous time to the value process of the corresponding liability (see Section 1.1), namely to the value CL of the cash flow $(-\mathcal{L})$ due by the bank under the terms of the counterparty risk hedge. Before bank default, the resets to the cash account of the CL desk, which accumulate to $\mathrm{CL}_0 - \mathrm{CL}$, are passed in real time to the shareholders, as is the $\mathcal{L}^{\tau-}$ component from the cash flows of the counterparty risk hedge (which thus do not stay on the balance sheet of the bank). Finally, from time τ onward, the ${}^{\tau-}\mathcal{L}$ component of the cash flows of the counterparty risk hedge is absorbed by the bondholders as on offset on their realized recovery, previously ${}^{\tau-}(-\mathcal{L})$.

As a result, the trading loss of the bank starting before τ vanishes, whereas before τ the trading loss of the bank coincides with

$$\underbrace{\mathcal{C} + \mathcal{F} + \mathrm{FV} - \mathrm{FV}_0}_{\text{modified trading loss of the CA desks}} + \underbrace{-\mathcal{L}^{\tau-} + \mathrm{CL} - \mathrm{CL}_0}_{\text{counterparty risk hedging loss components passed to shareholders}}$$
$$= \mathcal{C} + \mathcal{F} + \mathrm{CA} - \mathrm{CA}_0 - \mathcal{L} = 0,$$

where (66) was used in the first equality and (9) in the second one. ∎

Remark A.1 Before τ, the amount available to the bank free of charge for its risky funding purposes is $\mathrm{FV} + \mathrm{CL} = \mathrm{CA}$ as before. Hence the risky funding cash-flows $\mathcal{F}$ are not modified by the counterparty risk hedge. The client default cash-flows $\mathcal{C}$ are not affected by the counterparty risk hedge either. ∎

Under the counterparty risk hedge of Proposition A.2, shareholders bear no risk, hence require no risk premium. Accordingly, the ensuing KVA vanishes. The recovery of the bank at its own default becomes zero, leaving the bondholders entirely wiped out by the hedge. The clients are better off by the amount CL_0 plus the previous, nonvanishing KVA_0 amount.

However, again, in reality, jump-to-default exposures (own jump-to-default, in particular) are not hedged by the bank. The difference between the wealths (69) and SHC_0 of the bank without and with the counterparty risk hedge is equal to $\mathrm{KVA}_0 + \mathrm{CL} - \mathcal{L} = (\mathrm{CL}_0 + \mathrm{KVA}_0) + (\mathrm{CL} - \mathrm{CL}_0 - \mathcal{L})$. The first term $\mathrm{CL}_0 + \mathrm{KVA}_0$ is the additional pricing rebate, coming on top of the complete counterparty risk market rebate FV_0, which is required from the clients in order to realign shareholders to the target hurdle rate h on their capital at risk, given the unhedged counterparty risk that arises in the form of the zero-valued martingale $\mathrm{CL} - \mathrm{CL}_0 - \mathcal{L}$ in the second term. The bondholder wealth (68) can then be interpreted as the wealth transferred to the bondholders by the trading of the bank, due to the inability of the bank to hedge counterparty risk. The market incompleteness wealth transfer to the shareholders, i.e. the difference between their wealths (67) without the counterparty risk hedge and SHC_0 with the hedge, is given by

$$\big(- \mathcal{L}^{\tau-} + \mathrm{KVA}_0 - \mathrm{KVA}^{\tau-} \big) + \mathrm{KVA}^{sh}. \tag{70}$$

The terms in parenthesis corresponds to their accumulated dividends, i.e. to the appreciation of their capital (at a hurdle rate h on their capital at risk as seen in Corollary 6.1). The last term KVA^{sh} is the conditional expectation of their dividends in the future.

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