Capital Valuation Adjustment and Funding Valuation Adjustment*

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October 23, 2019

Abstract

Since the global financial crisis of 2008–09, derivative dealers charge to their clients various add-ons, dubbed XVAs, meant to account for counterparty risk and its capital and funding implications.

As banks cannot replicate jump-to-default related cash flows, deals trigger wealth transfers from bank shareholders to bondholders and shareholders need to set capital at risk. In view of this, we devise an XVA strategy, whereby the so-called contra-liabilities and cost of capital are sourced from bank clients at trade inceptions, on top of the fair valuation of counterparty risk, in order to compensate shareholders for wealth transfers and risk on their capital.

The resulting all-inclusive XVA formula reads (CVA + FVA + KVA), where C sits for credit, F for funding, and where the KVA is a cost of capital risk premium. All these (portfolio-wide) XVA metrics are nonnegative and, despite the fact we include the default of the bank itself in our modeling, unilateral. This makes them immediately in line with the requirement that the reserve capital and capital at risk of a bank should not decrease uniquely because the credit risk of the bank worsens.

Keywords: Counterparty risk, market incompleteness, wealth transfer, cost of capital, credit valuation adjustment (CVA), funding valuation adjustment (FVA), capital valuation adjustment (KVA).

Mathematics Subject Classification: 91B25, 91B26, 91B30, 91G20, 91G40.

JEL Classification: D52, G13, G24, G28, G33, M41.

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Acknowledgement: We are grateful for useful discussions to Yannick Armenti, Agostino Capponi, Jaksa Cvitanic, Karl-Theodor Eisele, Rodney Hoskinson, Monique Jeanblanc, Marek Rutkowski, Michael Schmutz, Martin Schweizer, and Shiqi Song.

*First, very preliminary version of this paper published in March 2016 on arXiv:1603.03012 and ssrn.2745909.

1 Introduction

Since the global financial crisis of 2008–09, investment banks charge to their clients, in the form of rebates with respect to the mark-to-market (MtM) of financial derivatives, various add-ons meant to account for counterparty risk and its capital and funding implications. These add-ons are generically termed XVAs, where VA stands for valuation adjustment and X is a catch-all letter to be replaced by C for credit, D for debt, F for funding, M for margin, or K for capital.

Paradoxically, whereas counterparty jump-to-default risk risk can fundamentally not be hedged, most of the XVA literature relies on a replication paradigm. In this paper, switching from a replication to a cost-of-capital approach, we devise an XVA pricing, accounting, and dividend policy of a dealer bank, ensuring to the bank shareholders a target hurdle rate on their capital at risk. In particular, our KVA is a risk premium, in the line of the risk margin in the Solvency II insurance regulation (but devised in a consistent continuous-time framework): It devises entry prices which keep the position of a derivative market maker on an "efficient frontier" corresponding to a given return on the shareholder capital at risk that is earmarked for coping with trading losses. The resulting policy can be seen as a banking and continuous-time counterpart to the cost-of-capital Solvency II insurance regulatory framework.

Related papers are Burgard and Kjaer (2011, 2013, 2017), Castagna (2012, 2013, 2014), Albanese and Andersen (2015), Albanese, Andersen, and Iabichino (2015), Andersen, Duffie, and Song (2019), and Green, Kenyon, and Dennis (2014). However, except for the last one (which is still in a semi-replication mindset), these papers only consider FVA (or its avatar MVA); they do not propose an approach to KVA.

We view the counterparty risk market incompleteness tenet as much more credible and realistic than the competing XVA replication paradigm. Besides divergent implications, detailed in Albanese, Crépey, Hoskinson, and Saadeddine (2019), regarding inclusion or not of the different XVAs in entry prices and in the balance sheet, our approach also results in materially modified XVA formulas. In particular:

- Despite the fact that we include the default of the bank itself in our modeling, our portfolio-wide XVA metrics are, ultimately, unilateral, and they are always nonnegative; this makes them naturally in line with the requirement that reserve capital and capital at risk of a bank should not diminish as an effect of the sole deterioration of the bank credit spread;

- As opposed to the KVA in Green, Kenyon, and Dennis (2014), the KVA that arises from our theory discounts future capital at risk projections at the hurdle rate h; this makes an important difference given the very long time horizon of XVA computations;

- Our XVA suite is self-contained and self-consistent: our main KVA input data consists of future projections of an economic capital based on the CVA and FVA desks trading loss processes, as opposed to future projections of (scriptural) regulatory capital in Green, Kenyon, and Dennis (2014).

1.1 Outline

Sections 2 and 3 deliver the conceptual backbone of our dynamic cost-of-capital XVA approach. Section 4 sets a technical stage, where the XVA equations are proven to be well posed in Section 5. Section 6 compares the two competing XVA paradigms: replication or so-called "semi-replication", in most of the XVA literature, versus cost-of-capital in this work. Section 7 concludes.

The ***main contributions*** of the paper are:

- The notion of shareholder valuation (of pre-bank default cash flows and of residual shareholder value itself before the bank default time τ) as a systematic way to address the successive XVA layers (starting with MtM at the bottom);

- The solution of the ensuing XVA equations by a reduction of filtration methodology, which can be seen as a way to address the idea of computations "on a going concern" for a bank, also related to the Schönbucher (2004) and Collin-Dufresne, Goldstein, and Hugonnier (2004) notion of bank survival probability measure;

- The KVA specification as per Definition 3.3, formula (47), and the corresponding optimality result of Proposition 5.1 and Theorem 5.2;

- The comparison of Section 6 between the semi-replication and the cost-of-capital XVA approaches;

- The conclusion of Section 7.1 according to which, duly assessing the XVA originating cash flows in terms of shareholder valuation in order to account for all wealth transfers involved, all the (portfolio-wide) XVA metrics are nonnegative and, ultimately, unilateral.

2 General Setup

We consider a dealer bank, which is a market maker, involved into bilateral derivative portfolios with clients. The client portfolio of the bank is assumed to be held on a run-off basis, i.e. set up at time 0 and such that no new unplanned trades enter the portfolio in the future (we refer the reader to Albanese et al. (2019, Section 4.2) for the reconciliation of this setup with the realistic case of a trade incremental portfolio). For simplicity, we only consider European derivatives.

The bank has two kinds of stakeholders: ***shareholders***, who have the control of the bank and are solely responsible for investment decisions before bank default, and ***bondholders*** representing the junior creditors of the bank, which have no decision power until bank default, but are protected by laws, of the pari-passu type, forbidding trades that would trigger value away from them to shareholders during the default resolution process of the bank. The bank also has senior creditors, represented in our framework by an ***external funder*** that can lend unsecured to the bank and is assumed to enjoy an exogenously given recovery rate in case of default of the bank.

There are three kinds of business units within the bank: the ***CA desks***, i.e. the CVA desk and the FVA desk, in charge of contra-assets, i.e. of counterparty risk and its funding implications for the bank; the ***clean desks***, who focus on the market risk of the contracts in their respective business lines; the ***management*** of the bank, in charge of the dividend release policy of the bank.

Collateral means cash or liquid assets that are posted to guarantee a netted set of transactions against defaults of the counterparties. We assume that the CA desks fully guarantee the trading of the clean desks against clients and bank defaults, through a ***clean margin account***, which can be seen as collateral exchanged between the CA desks and the clean desks. In addition, the CA desks value the contra-assets, charge them to the clients at deal inception, deposit the corresponding payments in a ***reserve capital account***, and then are exposed to the corresponding payoffs. As time proceeds, contra-assets realize (counterparty default losses and funding expenditures occur) and are covered by the CA desks with the reserve capital account.

On top of reserve capital, the so-called risk margin is sourced by the management of the bank from the clients at deal inception, deposited into a ***risk margin account***, and then gradually released as KVA payments into the shareholder dividend stream. Another account contains the ***shareholder capital at risk*** earmarked by the bank to deal with exceptional trading losses (beyond the expected losses that are already accounted for by reserve capital).

Assumption 2.1 We write MtM, CA, KVA, and SCR for the respective (risk-free discounted) amounts on the clean margin, reserve capital, risk margin, and shareholder capital at risk accounts of the bank.

All these amounts are continuously, instantaneously reset to theoretical target levels defined in Sections 3-4.

All cash accounts are remunerated at the risk-free rate. ∎

Assumption 2.2 The initial amounts MtM_0, CA_0, and KVA_0 are provided by the clients at portfolio inception time 0. Resets between time 0 and the bank default time τ (excluded) are on bank shareholders.

At the (positive) bank default time τ, the property of the residual amount on the reserve capital and risk margin accounts is transferred from the shareholders to the bondholders of the bank. ∎

See Table 1 for a list of the main financial acronyms used in the paper.

2.1 Pricing Setup

We consider, on a measurable space $(\Omega, \mathfrak{A})$, a pricing stochastic basis $(\mathbb{G}, \mathbb{Q})$, with model filtration $\mathbb{G} = (\mathfrak{G}_t)_{t \in \mathbb{R}_+}$ and risk-neutral pricing measure $\mathbb{Q}$, such that all the processes of interest are $\mathbb{G}$ adapted and all the random times of interest are $\mathbb{G}$ stopping times. The corresponding expectation and conditional expectation are denoted by $\mathbb{E}$ and $\mathbb{E}_t$. All cash flow and price processes are modeled as semimartingales.

CA	Contra-assets valuation	Assumption 2.1 and (5)
CL	Contra-liabilities valuation	Definition 3.2
CR	Capital at risk	(19)
CVA	Credit valuation adjustment	(10), (38), and example (60)
DVA	Debt valuation adjustment	Definition 3.2 and example (62)
EC	Economic capital	Section 3.6 and Definition 4.2
FDA	Funding debt adjustment	Definition 3.2 and example (63)
FV	Fair valuation of counterparty risk	Definition 3.2 and example (64)
FVA	Funding valuation adjustment	(11), (39), and example (61)
KVA	Capital valuation adjustment	Assumption 2.1 and (20), (40), (47)
MtM	Mark-to-market	Assumption 2.1 and (9), (37)
SCR	Shareholder capital at risk	Assumption 2.1 and (18)
XVA	Generic "X" valuation adjustment	First paragraph of Section 1

Table 1: Main financial acronyms and place where they are introduced conceptually and/or specified mathematically in the paper, as relevant.

We denote by T a finite and constant upper bound on the maturity of all claims in the portfolio, also including the time (such as two weeks) of liquidating defaulted positions, so that all (cumulative) cash flow processes are stopped at T (starting from 0 at time 0); all prices and valuation adjustments are supposed to vanish on $[T, +\infty)$ if $T < \tau$.

We use the risk-free asset as a numéraire. To retrieve undiscounted equations, one just needs to capitalize all cash flows and values (as well as the amounts on the different banking accounts, economic capital, etc.) at the risk-free rate.

For any left-limited process Y, we denote by $Y^{\tau-}$ and ${}^{\tau-}Y$ the processes Y stopped before τ and starting before τ, i.e.

$$Y^{\tau-} = JY + (1-J)Y_{\tau-},\ {}^{\tau-}Y = Y - Y^{\tau-},$$

where $J = \mathbb{1}_{[\![0,\tau[\![}$ is the survival indicator process of the bank.

Valuation below corresponds to the standard notion of risk-neutral expectation of future (risk-free discounted) cash flows; shareholder valuation corresponds to the valuation of pre-bank default cash flows and of the residual (shareholder) value at τ.

Definition 2.1 Given an optional, integrable process $\mathcal{Y}$ stopped at T (cumulative cash flow stream in the financial interpretation), we call (risk-neutral) value process Z of $\mathcal{Y}$ the optional projection of $(\mathcal{Y}_T - \mathcal{Y})$, such that

$$Z_t = \mathbb{E}_t(\mathcal{Y}_T - \mathcal{Y}_t),\ t \le T; \tag{1}$$

we call shareholder value process Y of $\mathcal{Y}$, any process Y vanishing on $[T, +\infty)$ if $T < \tau$ and such that

$$Y_t = \mathbb{E}_t(\mathcal{Y}_{\tau-} - \mathcal{Y}_t + Y_{\tau-}),\ t < \tau. \ \blacksquare \tag{2}$$

Note that the shareholder value equation (2), for a process Y vanishing on $[T,+\infty)$ if $T<\tau$, is equivalent to

$$Y_t^{\tau-}=\mathbb{E}_t(\mathcal{Y}_{\tau\wedge T}^{\tau-}-\mathcal{Y}_t^{\tau-}+\mathbb{1}_{\{\tau\le T\}}Y_\tau^{\tau-}),\ t\le\tau\wedge T. \tag{3}$$

In particular, $(\mathcal{Y}+Y)^{\tau-}$ is then a martingale (stopped before τ).

This makes it apparent that shareholder valuation is actually an equation for $Y^{\tau-}$. The corresponding backward stochastic differential equation (BSDE) is tantamount to the notion of recursive valuation of defaultable securities in the special case where $R_t(x)=x$ in Collin-Dufresne et al. (2004, Section 3.2). In their setup this notion is shown to be well posed in their Proposition 2, based on Schönbucher (2004)'s technical tool of the bank survival pricing measure. We will address the issue by reduction of filtration in Section 5.1.

Remark 2.1 In the XVA context, valuation corresponds to valuation from the point of view of the bank as a whole; shareholder valuation indeed corresponds to valuation from the bank shareholders point of view, because shareholders are only hit by pre-bank default cash flows, as well as by the wealth transfer to creditors of any residual value that shareholders may still have right before bank default. The latter directly applies, at least, if this residual value is positive, which will be the case of all our XVAs below; But it also applies if the corresponding value is negative, provided it is guaranteed by a collateralization procedure, which will correspond to the MtM case. ∎

3 Derivation of the XVA Equations

Unless explicitly specified, an amount paid (received) means effectively paid (received) if positive, but actually received (paid) if negative. A similar convention applies to the notions of cost vs. benefit, loss vs. gain, etc..

3.1 Trading Cash Flows

The (cumulative) trading cash flows of the bank consist of the contractually promised cash flows $\mathcal{P}$ from clients, counterparty credit cash flows $\mathcal{C}$ to clients (i.e., because of counterparty risk, the effective cash flows from clients are $\mathcal{P}-\mathcal{C}$), risky funding cash flows $\mathcal{F}$ to the external funder, and hedging cash flows $\mathcal{H}$ to the financial hedging markets (see Sections 4.2 and 5.4 for concrete specifications). All these cumulative cash flow streams are assumed to be integrable (and stopped at T).

In practice, $\mathcal{C}^{\tau-}$ is made of nonnegative default losses of the bank upon client defaults (see e.g. Lemma 5.3). Accordingly:

Assumption 3.1 The process $\mathcal{C}^{\tau-}$ is nondecreasing. ∎

The risky funding cash flows of the bank arise as the stochastic integral of predictable funding ratios against wealth processes of buy-and-hold strategies into funding assets related to the default of the bank. These wealth processes are assumed to be local martingales. Accordingly (also assuming integrability of $\mathcal{F}$):

Assumption 3.2 The risky funding cash flow process $\mathcal{F}$ is a finite variation martingale with nondecreasing $\mathcal{F}^{\tau-}$ component, stopped at τ. ∎

See Lemma 5.2 for concrete illustration. The assumption that $\mathcal{F}^{\tau-}$ is nondecreasing rules out models where the bank could invest (not only borrow) at its unsecured borrowing spread. Indeed, as a consequence on ${}^{\tau-}\mathcal{F}$ through the martingale condition on the process $\mathcal{F}$ as a whole, such a possibility would imply that the bank can hedge its own jump-to-default exposure, which we exclude in our setup (see the second paragraph of Section 2 and Section 3.5). Finally, for the bank, the funding issue ends at τ, which explains why $\mathcal{F}$ is stopped at τ.

Similarly, regarding now hedging losses:

Assumption 3.3 The hedging loss $\mathcal{H}$ of the clean desks, inclusive of the cost of setting their hedge, is a martingale stopped before τ, i.e. $\mathcal{H} = \mathcal{H}^{\tau-}$. ∎

The assumption $\mathcal{H} = \mathcal{H}^{\tau-}$ is made for consistency with our premise that a bank cannot hedge its own jump-to-default exposure.

By Assumptions 3.2 and 3.3:

Remark 3.1 The processes $\mathcal{F}$ and $\mathcal{H} = \mathcal{H}^{\tau-}$ have zero value. ∎

3.2 Trading Losses

Counterparty jump-to-default risk cannot be replicated: The risk of financial loss as a consequence of client default is hard to hedge, because single name credit default swaps (CDS instruments) that could in principle be used for that purpose are illiquid. The possibility for the bank of hedging its own jump-to-default is even more questionable, for practical but also legal reasons: For the bank, hedging its default means 'monetizing' it beforehand (cf. Section 3.5), which goes against the bondholder protection rules. Accordingly, we conservatively assume no XVA hedge (see however Remark 3.2 and cf. the discussion section 6), i.e. the bank hedging loss $\mathcal{H}$ is in fact the hedging loss of the clean desks. Through our mark-to-model assumption 2.1 on all bank accounts, the CVA and FVA desks trading losses are respectively given by

$$\mathcal{C} + \mathrm{CVA} - \mathrm{CVA}_0 \text{ and } \mathcal{F} + \mathrm{FVA} - \mathrm{FVA}_0, \tag{4}$$

for some theoretical target CVA and FVA levels, further specified in Section 3.3, such that

$$\mathrm{CVA} + \mathrm{FVA} = \mathrm{CA}. \tag{5}$$

Likewise, clean desks trading gains, inclusive of their mark-to-model fluctuations and hedging loss $\mathcal{H}$, sum up to

$$\mathcal{P} + \mathrm{MtM} - \mathrm{MtM}_0 - \mathcal{H}, \tag{6}$$

for some theoretical target MtM level to be specified in Section 3.3. In line with the fact that a dealer bank should not do proprietary trading (cf. the so-called Volcker rule):

Assumption 3.4 The clean desks are perfectly hedged, in the sense that (6) vanishes identically. ∎

As $\mathcal{H} = \mathcal{H}^{\tau-}$, i.e. ${}^{\tau-}\mathcal{H} = 0$, this perfect clean hedge assumption splits into

$$\mathcal{P}^{\tau-} + \mathrm{MtM}^{\tau-} - \mathrm{MtM}_0 - \mathcal{H}^{\tau-} = 0,\ {}^{\tau-}\mathrm{MtM} = -{}^{\tau-}\mathcal{P}. \tag{7}$$

The second part is also consistent with the fact that, from bank default onward, the clean margin account is used for providing to the clean desks the contractually promised cash flows that cease to be exchanged between the client and the bank (see before Assumption 2.1).

The overall trading loss of the bank results from the above as

$$L = \mathcal{C} + \mathcal{F} + \mathrm{CA} - \mathrm{CA}_0. \tag{8}$$

The ensuing setup corresponds to a fully collateralized market hedge of its client portfolio by the bank, so that only the counterparty risk related cash flows remain.

Remark 3.2 The inclusion of an XVA hedge yielding any additional martingale hedging loss process stopped before τ into (8), or a relaxation of the perfect clean hedge assumption in the form of the left hand side in (7) becoming a (nonnecessarily vanishing) martingale (with the right hand side in (7) still in force), would change nothing to the qualitative conclusions of the paper, only implying possibly smaller economic capital and KVA. ∎

Remark 3.3 The industry terminology distinguishes an FVA, in the specific sense of the cost of funding re-hypothecable collateral (variation margin), from an MVA defined as the cost of funding segregated collateral (initial margin, see e.g. Albanese, Caenazzo, and Crépey (2017)). In this paper, we merge the two in an overall FVA meant in the broad sense of the cost of funding the derivative business of the bank.

If (assumed all cash) collateral happens to be remunerated at some basis with respect to the risk-free rate, then this entails a further "liquidity valuation adjustment". However, the corresponding bases are typically small and the related adjustment negligible with respect to the XVA metrics considered in this paper. ∎

3.3 MtM, CVA, and FVA

In accordance with Remark 2.1:

Definition 3.1 $\mathrm{MtM}^{\tau-}$, $\mathrm{CVA}^{\tau-}$, and $\mathrm{FVA}^{\tau-}$ are the shareholder value processes of $\mathcal{P}$, $\mathcal{C}$, and $\mathcal{F}$. ∎

That is (see Definition 2.1), MtM, CVA, and FVA are killed at T on $\{T < \tau\}$ and, for $t < \tau$,

$$\mathrm{MtM}_t = \mathbb{E}_t\big(\mathcal{P}_{\tau-} - \mathcal{P}_t + \mathrm{MtM}_{\tau-}\big), \tag{9}$$

$$\mathrm{CVA}_t = \mathbb{E}_t\big(\mathcal{C}_{\tau-} - \mathcal{C}_t + \mathrm{CVA}_{\tau-}\big), \tag{10}$$

$$\mathrm{FVA}_t = \mathbb{E}_t\big(\mathcal{F}_{\tau-} - \mathcal{F}_t + \mathrm{FVA}_{\tau-}\big). \tag{11}$$

Remark 3.4 The trading loss of the shareholders,

$$L^{\tau-} = \mathcal{C}^{\tau-} + \mathcal{F}^{\tau-} + \mathrm{CA}^{\tau-} - \mathrm{CA}_0 \tag{12}$$

(cf. (8)), is a martingale (stopped before τ). ∎

Proof. By the observation made after (3), the processes CVA and FVA are such that the trading losses in (4), stopped before τ, are martingales. So is therefore their sum $L^{\tau-}$. ∎

3.4 Contra-liabilities

The processes CVA and FVA (and KVA later on) are so far unconstrained on $[\![\tau, +\infty[\![\bigcap \big(\{\tau \leq T\} \times \mathbb{R}_+\big)$. We define the three XVA processes as zero there. As they already vanish on $[T, +\infty)$ if $T < \tau$, hence each of them, say Y (hence also $Y = \mathrm{CA}$ in (5)), is in fact killed at τ. Therefore, in particular,

$$^{\tau-}Y = \mathbb{1}_{[\![\tau,+\infty[\![}(Y_\tau - Y_{\tau-}) = -\mathbb{1}_{[\![\tau,+\infty[\![}Y_{\tau-}. \tag{13}$$

The above then yields the split of the overall trading cash flows L in (8), depicted in Figure 1, between the pre-bank default trading cash flows, i.e. the shareholder trading cash flows, $L^{\tau-}$, and the bondholder trading cash flows, dubbed contra-liabilities, $^{\tau-}(-L)$.

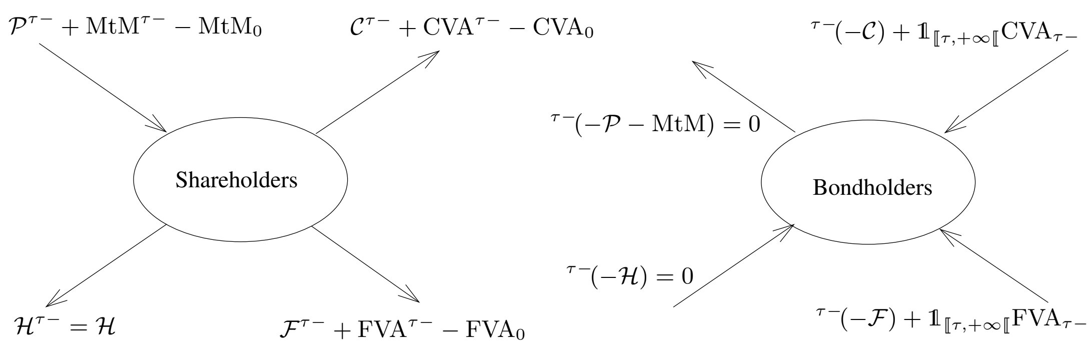

Figure 1: *Left*: Pre-bank-default trading cash flows $L^{\tau-}$. *Right*: Trading cash flows from bank default onward $^{\tau-}(-L)$.

Definition 3.2 We call

- DVA (debt valuation adjustment), the value process of $(^{\tau-}(-\mathcal{C}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{CVA}_{\tau-})$;
- FDA (funding debt adjustment), the value process of $(^{\tau-}(-\mathcal{F}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{FVA}_{\tau-})$;

- CL = DVA + FDA, i.e. the value of the contra-liabilities

$$^{\tau-}(-L) = ^{\tau-}(-\mathcal{C}) + ^{\tau-}(-\mathcal{F}) + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{CA}_{\tau-}; \tag{14}$$

- FV (fair valuation of counterparty risk), the value of $(\mathcal{C}+\mathcal{F})$, i.e. of $\mathcal{C}$ (by Remark 3.1 regarding $\mathcal{F}$). ∎

Lemma 3.1 *We have*

$$\mathrm{FVA} = \mathrm{FDA} \tag{15}$$

$$\mathrm{FV} = \mathrm{CA} - \mathrm{CL} = \mathrm{CVA} - \mathrm{DVA}. \tag{16}$$

Proof. By (10), (11), (5), Definition 3.2, and Remark 3.1 regarding $\mathcal{F}$. ∎

3.5 Wealth Transfer Analysis

Let us temporarily assume, for the sake of the argument, that the bank would be able to hedge its own jump-to-default risk, i.e. the contra-liabilities $^{\tau-}(-L)$, by selling a contract delivering the cash flows (zero valued martingale) $\mathrm{CL} - \mathrm{CL}_0 + ^{\tau-}(-L)$. Accounting for this hedge and assuming that the CA desks would pass to the client (at time 0) and shareholders (through resets later on) the modified add-on CA − CL = FV (instead of CA before without the hedge, see the first part in Assumption 2.2), then (the amount that needs by borrowed by the CA desks for the trading requirements of the bank is the same as before and) the trading loss of the bank would become (using (16) in the first equality and (8) in the second one)

$$\begin{aligned}&\mathcal{C} + \mathcal{F} + \mathrm{FV} - \mathrm{FV}_0 + \mathrm{CL} - \mathrm{CL}_0 + ^{\tau-}(-L) = \\ &\qquad \mathcal{C} + \mathcal{F} + \mathrm{CA} - \mathrm{CA}_0 + ^{\tau-}(-L) = L + ^{\tau-}(-L) = L^{\tau-}.\end{aligned}$$

Hence the recovery of the bank at its own default becomes zero (as the process $L^{\tau-}$ is stopped before τ), leaving the bondholders entirely wiped out by the hedge.

Assume, additionally, that the CA desks would have access to a further hedge of the contra-assets

$$\mathcal{C}^{\tau-} + \mathcal{F}^{\tau-} + \mathbb{1}_{[\![\tau,+\infty[\![}\mathrm{CA}_{\tau-} \tag{17}$$

(the cash flows valued by CA itself, by (10), (11), and (5)), i.e. an hedge that would generate (inclusive of the cost of funding the hedge) the proceeds $L^{\tau-}$ (a risk-neutral martingale, as seen in Proposition 3.4). Then the trading loss process of the bank would vanish entirely.

Thus, FV = CA − CL is the cost of replication of counterparty risk in a theoretical, complete counterparty risk market. However, in reality, jump-to-default exposures (own jump-to-default, in particular) cannot be hedged by the bank. The difference CL = CA − FV between the valuations CA and FV of counterparty risk making shareholder trading losses a martingale in the respective realistic incomplete and theoretically complete markets is therefore interpreted as the wealth transfer from clients and shareholders to bondholders triggered by the derivative portfolio of the bank, due to the inability of the bank to hedge jump-to-default exposures.

3.6 Shareholder Capital at Risk and Capital Valuation Adjustment

Since contra-assets (not even talking about contra-liabilities) cannot be replicated, capital needs be set at risk by shareholders, who therefore deserve, in the cost-of-capital pricing approach of this paper, a further KVA add-on as a risk premium.

Capital at risk (CR) is the resource of the bank devoted to cope with losses beyond their expected levels, the latter being already taken care of by reserve capital (CA). Economic capital (EC) is the level of capital at risk that a regulator would like to see on an economic basis. In our dynamic setup, EC and CR will be updated continuously. In particular, EC will be assumed to be killed at $\tau \wedge T$ (as will in turn be CR).

In the context of XVA computations that entail projections over decades, the main source of information is market prices of liquid instruments, which allow the bank to calibrate a risk-neutral pricing measure $\mathbb{Q}$, whereas there is little of relevance that can be said about the historical probability measure. Accordingly:

Assumption 3.5 The estimate of the historical probability measure used by the bank in its economic capital and cost of capital computations is set equal to the risk-neutral pricing measure. ∎

Any discrepancy between the historical and the risk-neutral pricing measures is left to model risk.

Remark 3.5 We calibrate the pricing measure to derivative market prices, including the corresponding credit premia, and then we perform all our (including capital) calculations based on these. If we were able to estimate the historical probability measure reliably for forward projection over time horizons that, in the XVA context, can be as long as fifty years into the future, then we could have a more sophisticated, hybrid setup, with historical measure distinct from the risk-neutral measure. In the absence of such a reliable methodology, we do all the computations under the risk-neutral measure. This is also the choice advocated by Solvency II insurance regulators. As, in particular, implied CDS spreads are typically larger than statistical estimates of default probabilities, we believe that this approach is conservative. Moreover, the hurdle rate h that will appear in the KVA below can be interpreted as a risk aversion parameter of the bank shareholders and the KVA as a corresponding risk premium (see Albanese et al. (2019, Section 3.3)). ∎

Counterparty default losses, as also funding payments, are materialities for default if not paid. In contrast, risk margin (KVA) payments are at the discretion of the bank management, hence they do not represent an actual liability to the bank. Accordingly (see Section 6.2 for discussion):

Assumption 3.6 The risk margin is loss-absorbing, hence part of capital at risk. ∎

As a consequence, $\mathrm{CR} \geq \mathrm{KVA}$ and *shareholder* capital at risk (SCR) is only the difference between the capital at risk (CR) of the bank and the risk margin (KVA), i.e.

$$\mathrm{SCR} = \mathrm{CR} - \mathrm{KVA}. \tag{18}$$

Specifically:

Definition 3.3 We set

$$\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA}), \tag{19}$$

where KVA is the shareholder value process of $\int_0^{\cdot} h\mathrm{SCR}_s ds$, i.e.

$$\mathrm{KVA}_t = \mathbb{E}_t\Big[\int_t^{\tau} h\mathrm{SCR}_s ds + \mathrm{KVA}_{\tau-}\Big],\ t < \tau, \tag{20}$$

and KVA is killed at $\tau \wedge T$. ∎

The process $\mathrm{KVA}^{\tau-}$ is then a supermartingale with drift coefficient

$$-h\mathrm{SCR} = -h\big(\max(\mathrm{EC}, \mathrm{KVA}) - \mathrm{KVA}\big) = -h\big(\mathrm{EC} - \mathrm{KVA}\big)^+, \tag{21}$$

by (18)-(19).

As more directly visible on the following differential formulation of (20) (cf. (3)):

$$\begin{aligned}
&\mathrm{KVA}_T^{\tau-} = 0 \text{ on } \{T < \tau\} \text{ and, for } t \leq \tau \wedge T,\\
&d\mathrm{KVA}_t^{\tau-} = -hd\mathrm{SCR}_t + d\nu_t,\\
&\text{for some martingale } \nu,
\end{aligned} \tag{22}$$

the KVA corresponds financially to the amount to be maintained by the bank on its risk margin account in order to be in a position to deliver to its shareholders, dynamically into the future, a hurdle rate h on their capital at risk (SCR). Moreover the amount on the risk margin account should land off at $\mathrm{KVA}_T = 0$ on $\{T < \tau\}$, as ending up in the negative would mean an insufficient risk margin for ensuring the hurdle rate h to the shareholders, whereas ending up in the positive at $T < \tau$ would mean that the bank is unnecessarily expensive to its clients.

Remark 3.6 In theory, the choice of a target hurdle rate h is a managerial decision of the bank. In practice, the level of compensation required by shareholders on their capital at risk in a firm is driven by market considerations. Typically, investors in banks expect a hurdle rate of the order of 10%. In a real-life environment where banks compete for clients (as opposed to our setup where only one bank is considered), an endogenous and stochastic implied hurdle rate arises from the competition between banks (cf. Section 7.2). ∎

We will see in Proposition 5.1 and Theorem 5.2 that CR as per (19) is in fact the minimal and cheapest capital at risk process C satisfying the risk admissibility condition $C \geq \mathrm{EC}$ and consistent with the target hurdle rate h on shareholder capital at risk.

Shareholder dividends (shareholder trading gains and KVA risk margin payments) accumulate as $-(L^{\tau-} + \mathrm{KVA}^{\tau-} - \mathrm{KVA}_0)$. We emphasize that negative dividends are possible in our model. They are interpreted as recapitalization.

As an immediate consequence of the respective martingale and supermartingale properties of $L^{\tau-}$ and $\mathrm{KVA}^{\tau-}$:

Corollary 3.1 *Shareholder dividends accumulate into a submartingale (stopped before τ), with drift coefficient hSCR.* ∎

4 Technical Setup

This section yields a technical specification of the above abstract setup, in which the XVA equations are shown to be well posed in Section 5.

4.1 Reduction of Filtration Setup

In addition to the full (financial) model filtration $\mathbb{G} = (\mathfrak{G}_t)_{t\in\mathbb{R}_+}$, on $(\Omega, \mathfrak{A})$, we introduce a smaller (technical) filtration $\mathbb{F} = (\mathfrak{F}_t)_{t\in\mathbb{R}_+}$, such that the bank default time τ is a "$\mathbb{G}$ but not $\mathbb{F}$" stopping time.

Assumption 4.1 For any $\mathbb{G}$ semimartingale Y on $[0, \tau \wedge T]$, there exists a unique $\mathbb{F}$ semimartingale Y' on $[0, T]$, called the $\mathbb{F}$ reduction of Y, that coincides with Y before τ. ∎

In particular, any $\mathbb{G}$ stopping time θ admits an $\mathbb{F}$ stopping time θ', called $\mathbb{F}$ reduction of θ, such that $\theta \wedge \tau = \theta' \wedge \tau$.

Assumption 4.2 There exists a probability measure $\mathbb{P}$ on $\mathfrak{F}_T$, equivalent to the restriction of $\mathbb{Q}$ to $\mathfrak{F}_T$, such that stopping before τ turns $(\mathbb{F}, \mathbb{P})$ local martingales on $[0, T]$ into $(\mathbb{G}, \mathbb{Q})$ local martingales on $[0, \tau \wedge T]$ (stopped before τ); Conversely, the $\mathbb{F}$ reductions of $(\mathbb{G}, \mathbb{Q})$ local martingales on $[0, \tau \wedge T]$ without jump at τ are $(\mathbb{F}, \mathbb{P})$ local martingales on $[0, T]$. ∎

Remark 4.1 The case where $\mathbb{P} \neq \mathbb{Q}$ corresponds to situations of hard wrong way risk (strong adverse dependence, see e.g. Crépey and Song (2016) and Crépey and Song (2017a)) between the defaults of the bank and a client, or between the default of the bank and its portfolio exposure with a client. A regulator should consider such situations with caution, as they include the cases of a client which is an affiliate of the bank itself; or of the bank and a client trading CDS contracts on an affiliate of the bank itself, or on a reference entity susceptible to default at the same time as the bank. In other words, even if we are not restricted to it mathematically, our base case is the "immersion" case where $\mathbb{P} = \mathbb{Q}$ (hard wrong-way risk should mainly regard the dependence between client defaults and portfolio exposures). ∎

Unless explicitly mentioned, probabilistic statements still refer to the stochastic basis $(\mathbb{G}, \mathbb{Q})$.

Assumptions 4.1 and 4.2, with filtrations $\mathbb{F}$ and $\mathbb{G}$ satisfying the usual conditions and existence of $\mathbb{F}$ semimartingale reductions reinforced[1] into existence of an "$\mathbb{F}$ predictable reduction" on $[0, T]$ coinciding until τ (included) with any $\mathbb{G}$ predictable

[1]As shown in Song (2016).

process, mean that τ is an invariance time as per Crépey and Song (2017b) and Crépey and Song (2018), with so called invariance probability measure $\mathbb{P}$. We recall the following results (with $J = \mathbb{1}_{[\![0,\tau[\![}$):

Lemma 2.3, Theorem 3.5, and Section 4.2 in Crépey and Song (2017b) *Assuming that an $\mathbb{F}$ predictable reduction of any $\mathbb{G}$ predictable process exists and that $\mathsf{S}_T = \mathbb{Q}(\tau > T \,|\, \mathfrak{F}_T) > 0$, then:*

$$\begin{aligned}&\textit{Any } \mathbb{G} \textit{ optional process admits a unique } \mathbb{F} \textit{ optional}\\ &\textit{reduction coinciding with it before } \tau \textit{ on } [0,T].\end{aligned} \tag{23}$$

If, moreover, τ has a $(\mathbb{G},\mathbb{Q})$ intensity process $\gamma = \gamma J_-$ such that $e^{\int_0^\tau \gamma_s ds}$ is $\mathbb{Q}$ integrable, then the existence on $(\Omega,\mathfrak{A})$ and uniqueness on $(\Omega,\mathfrak{F}_T)$ of an invariance probability measure $\mathbb{P}$ hold; on $\mathfrak{F}_T$, $\mathbb{P}$ coincides with the bank survival probability measure associated with $\mathbb{Q}$, i.e.[2] *the measure with $(\mathbb{G},\mathbb{Q})$ density process $Je^{\int_0^{\cdot} \gamma_s ds}$.* ∎

Hereafter we work under the corresponding specialization of Assumptions 4.1 and 4.2.

The conditional expectation with respect to $(\mathfrak{G}_t,\mathbb{Q})$ (respectively $(\mathfrak{F}_t,\mathbb{P})$) is denoted by $\mathbb{E}_t$ (respectively $\mathbb{E}'_t$), or simply by $\mathbb{E}$ (respectively $\mathbb{E}'$) if $t=0$.

Definition 4.1 Given an $\mathbb{F}$ optional and $\mathbb{P}$ integrable process $\mathcal{X}$ stopped at T, we call clean value process of $\mathcal{X}$ the $\mathbb{F}$ adapted process X vanishing on $[T,+\infty)$ and such that

$$X_t = \mathbb{E}'_t(\mathcal{X}_T - \mathcal{X}_t), t \le T. \blacksquare \tag{24}$$

As can be established by section theorem, for any $\mathbb{G}$ progressive Lebesgue integrand X such that the $\mathbb{G}$ predictable projection ${}^p(-X)$ exists,[3] the indistinguishable equality $\int_0^{\cdot} {}^pX_s ds = \int_0^{\cdot} X_s ds$ holds. As a consequence, one can actually consider the $\mathbb{F}$ reduction X' of any $\mathbb{G}$ progressive Lebesgue integrand X (even if this means replacing X by ${}^pX$). We will need the following spaces of processes:

- $\mathcal{S}_2$, the space of càdlàg $\mathbb{G}$ adapted processes Y over $[0,\tau\wedge T]$ without jump at time τ such that, denoting $Y^*_t = \sup_{s\in[0,t]}|Y_s|$:

$$\mathbb{E}\Big[Y_0^2 + \int_0^T J_s e^{\int_0^s \gamma_u du} d(Y^*)^2_s\Big] = \mathbb{E}'\Big[\sup_{t\in[0,T]} (Y')^2_t\Big] < \infty, \tag{25}$$

  where the equality follows from the identity (9.3) in Crépey and Song (2018); Note that, for $Y\in\mathcal{S}_2$,

$$\mathbb{E}\Big[\sup_{t\in[0,\tau\wedge T]} Y_t^2\Big] \le \mathbb{E}\Big[Y_0^2 + \int_0^T J_s\, e^{\int_0^s \gamma_u du} d\big(Y^*\big)^2_s\Big] < \infty; \tag{26}$$

[2] cf. Schönbucher (2004), Collin-Dufresne et al. (2004), and see also Crépey and Song (2018, Section A) for more details about it.

[3] For which σ integrability of X valued at any stopping time, e.g. X bounded or càdlàg, is enough.

- $\mathcal{L}_2$, the space of $\mathbb{G}$ progressive processes X over $[0,T]$ such that

$$\mathbb{E}\big[\int_0^{\tau\wedge T} e^{\int_0^s \gamma_u du} X_s^2 ds\big] = \mathbb{E}'\big[\int_0^T (X'_s)^2 ds\big] < +\infty, \tag{27}$$

  where the equality follows from the identity (9.4) in Crépey and Song (2018);

- $\mathcal{S}'_2$ and $\mathcal{L}'_2$, the respective spaces of $\mathbb{F}$ adapted càdlàg and $\mathbb{F}$ progressive processes Y' and X' over $[0,T]$ that make the corresponding squared norm finite in the right-hand side of (25) or (27).

Note that, in view of the above properties:

The $\mathbb{F}$ optional reduction operator is an isometry from $\mathcal{S}_2$ onto $\mathcal{S}'_2$, with stopping before τ as the reciprocal operator. (28)

Finally, we postulate a standard weak martingale representation setup, driven by a multivariate Brownian motion and an integer valued random measure (see e.g. Crépey et al. (2019, Section 2.2)).

4.2 Trading Cash Flows

We now specify the trading cash flows: the contractually promised cash flows $\mathcal{P}$ (which, via Assumption 3.4, also determine the hedging cash flows $\mathcal{H}$), the counterparty credit cash flows $\mathcal{C}$, and the risky funding cash flows $\mathcal{F}$.

The client (assumed all bilateral) portfolio of the bank is partitioned into netting sets of contracts which are jointly collateralized and liquidated upon clients or bank default. Given each netting set c of the client portfolio, we denote by:

- $\mathcal{P}^c$, its contractually promised cash flows;

- τ_c and R_c, the default time and recovery rate of the client corresponding to the netting set c, whereas τ and R are the analogous data regarding the bank itself;

- $\tau_c^\delta \geq \tau_c$ and $\tau^\delta \geq \tau$, the end of the so called close-out periods of the related client and of the bank itself, so that the effective liquidation of the netting set c happens at time $\tau_c^\delta \wedge \tau^\delta$;

- P^c, the clean value process of the netting set, i.e. of $(P^c)'$ (assumed $\mathbb{P}$ integrable), so that, by Definition 4.1,

$$P_t^c = \mathbb{E}'_t\big((\mathcal{P}^c)'_T - (\mathcal{P}^c)'_t\big),\ t \leq T. \tag{29}$$

Note that, by linearity, (29) is the sum over the netting set c of the analogous quantity pertaining to each individual deal in c, called clean valuation of the deal (recall that we restrict ourselves to European derivatives).

The rules regarding the settlement of contracts following defaults are that:

Assumption 4.3 At the time a party (a client or the bank itself) defaults, the property of the collateral posted on each involved netting set is transferred to the collateral receiver. Moreover, at the liquidation time of a netting set c:

- any positive value due by a nondefaulted party on this netting set is paid in full,
- any positive value due by a defaulted counterparty on this netting set is only paid up to the corresponding (assumed exogenously specified) recovery rate,

where value is here understood on a clean valuation basis as P^c (cf. (29)), net of the corresponding (already transferred) client collateral, but inclusive of all the promised contractual cash flows unpaid during the liquidation period;

In addition, during the liquidation period, the CA desks pay to the clean desks all the unpaid contractual cash flows and, at liquidation time, the property of an amount P^c on the clean margin amount is transferred from the CA desks to the clean desks. ∎

One is then in the abstract setup of the previous sections (cf. the proof of Lemma 5.3 for a detailed derivation), for

$$
\begin{aligned}
\mathcal{P} &= \sum_c \Big((\mathcal{P}^c)^{\tau_c^\delta \wedge \tau^\delta} + \mathbb{1}_{[\tau_c^\delta \wedge \tau^\delta, \infty)} P^c_{\tau_c^\delta \wedge \tau^\delta} \Big) \text{ and} \\
\mathcal{C} &= \sum_{c; \tau_c \le \tau^\delta} (1 - R_c) \big(P^c_{\tau_c^\delta \wedge \tau^\delta} + \mathcal{P}^c_{\tau_c^\delta \wedge \tau^\delta} - \mathcal{P}^c_{(\tau_c \wedge \tau)-} - \Gamma^c_{(\tau_c \wedge \tau)-} \big)^+ \mathbb{1}_{[\tau_c^\delta \wedge \tau^\delta, \infty)} \\
&\qquad - (1 - R) \sum_{c; \tau \le \tau_c^\delta} \big(P^c_{\tau^\delta \wedge \tau_c^\delta} + \mathcal{P}^c_{\tau^\delta \wedge \tau_c^\delta} - \mathcal{P}^c_{(\tau \wedge \tau_c)-} - \bar{\Gamma}^c_{(\tau \wedge \tau_c)-} \big)^- \mathbb{1}_{[\tau^\delta \wedge \tau_c^\delta, \infty)},
\end{aligned} \tag{30}
$$

where Γ^c and $\bar{\Gamma}^c$ are the collateral amounts received and posted by the bank in relation with the netting set c (cash amounts assumed stopped before the first-default time of the involved parties).

The risky funding cash flows $\mathcal{F}$ depend on the actual nature (re-hypothecable variation margin and/or segregated initial margin) of the collateral amounts Γ^c and $\bar{\Gamma}^c$ and on the funding policy of the bank: see Lemma 5.2 for a basic variation margin illustration and see Albanese et al. (2017) for richer specifications, also involving initial margin.

In any case, $\mathcal{P}$ and $\mathcal{H}$ are additive over individual trades, but $\mathcal{C}$ is only additive over netting sets, and $\mathcal{F}$ only over (at least as large) funding sets (re-hypotecable collateral is actually aggregatable throughout the whole bank).

4.3 Economic Capital

We recall that the value-at-risk of a random variable (loss) ℓ and its expected shortfall, both at some level $\alpha \in (\frac{1}{2}, 1)$, respectively denote the left quantile of level α of ℓ, which we denote by $q^\alpha(\ell)$, and $(1-\alpha)^{-1} \int_\alpha^1 q^a(\ell) da$ (see e.g. Föllmer and Schied (2016, Section 4.4)). As is well known, the expected shortfall operator is $(1-\alpha)^{-1}$ Lipschitz from the space of integrable losses ℓ to $\mathbb{R}$, and to $\mathbb{R}_+$ when restricted to centered losses ℓ.

Capital requirements are focused on the solvency issue, because it is when a regulated firm becomes insolvent that the regulator may choose to intervene, take over, or restructure a firm. Specifically, Basel II Pillar II defines economic capital as the $\alpha = 99\%$ value-at-risk of the negative of the variation over a one-year period of core equity tier I capital (CET1). Moreover, the Fundamental review of the trading book required a shift from 99% value-at-risk to 97.5% expected shortfall as the reference risk measure in capital calculations.

In our setup, before bank default, CET1 depletions correspond to the shareholder trading loss process $L^{\tau-}$ (see Albanese et al. (2019, Proposition 4.1) for more detail about it). In addition, economic capital calculations are typically made by a bank "on a going concern", hence assuming that the bank itself does not default. Accordingly, under Assumption 3.5 and with L' seen under $(\mathbb{F}, \mathbb{P})$ for "losses of the bank assessed on a going concern basis" (cf. the last sentence in the paragraph following (23)):

Definition 4.2 The economic capital of the bank at time t, EC_t, is the $(\mathfrak{F}_t, \mathbb{P})$ conditional expected shortfall of the random variable $(L'_{(t+1)\wedge T} - L'_t)$ (assumed $\mathbb{P}$ integrable) of level $\alpha = 97.5\%$, killed at τ. ∎

Remark 3.4 and the converse part in Assumption 4.2 imply that the process L' is an $(\mathbb{F}, \mathbb{P})$ local martingale. Assuming its $\mathbb{P}$ integrability is not a practical restriction as, in concrete setups such as the one of Proposition 5.4, $L^{\tau-}$ and L' are even square integrable $(\mathbb{G}, \mathbb{Q})$ and $(\mathbb{F}, \mathbb{P})$ martingales.

Since the expected shortfall of a centered random variable is nonnegative:

Remark 4.2 EC is nonnegative. ∎

Remark 4.3 In practice, capital at risk (CR) can be used by the bank for its funding purposes. As developed in Crépey, Élie, Sabbagh, and Song (2019), this induces an interference of CR with $\mathcal{F}$, hence an intertwining of the FVA and the KVA. Instead, for simplicity hereafter, we assume that the bank does not use capital at risk (CR) for funding purposes. ∎

5 XVA Equations Well-Posedness and Comparison Results

In this section, we show well-posedness results for the CVA, FVA, and KVA equations (whereas the MtM process is characterized in (41)). We also establish a KVA optimality principle.

5.1 Shareholder Valuation and Clean Valuation

Recall that the shareholder value equation (2), for a process Y vanishing on $[T, +\infty)$ if $T < \tau$, is equivalent to the BSDE (3) for $Y^{\tau-}$. This is in particular the case for the MtM, CVA, and FVA equations (9), (10), and (11). In the case of the KVA equation

(20), the drift in the equation also depends on the KVA itself. To formally include this case (and also, as we shall see below, certain FVA specifications), we extend the notion of shareholder value to cash flows including a component, depending on Y itself, of the form

$$\int_0^{\cdot} J_t j_t(Y_t)dt, \tag{31}$$

for some random function $j = j_t(y)$ measurable with respect to the product of the $\mathbb{F}$ predictable σ field by the Borel σ field on $\mathbb{R}$. We thus consider the following shareholder value equation (which generalizes (3)):

$$Y_t^{\tau-} = \mathbb{E}_t\big(\mathcal{Y}_{\tau-}^{\tau-} - \mathcal{Y}_t^{\tau-} + \int_t^{\tau\wedge T} j_s(Y_s)ds + Y_\tau^{\tau-}\big),\ t \le \tau \wedge T, \tag{32}$$

respectively the following clean value equation for Y' (cf. (24)):

$$Y_t' = \mathbb{E}_t'\big(\mathcal{Y}_T' - \mathcal{Y}_t' + \int_t^T j_s(Y_s')ds\big), t \le T \tag{33}$$

(and Y' vanishes on $[T, +\infty)$).

Definition 5.1 By $\mathcal{S}_2$ solution (respectively $\mathcal{S}_2'$ solution) to the $(\mathbb{G},\mathbb{Q})$ BSDE (32) for $Y^{\tau-}$ (respectively the $(\mathbb{F},\mathbb{P})$ BSDE (33) for Y'), we mean any $(\mathbb{G},\mathbb{Q})$ semimartingale solution $Y^{\tau-}$ in $\mathcal{S}_2$ to (32) with $(Y + \mathcal{Y} + \int_0^{\cdot} j_s(Y_s)ds)^{\tau-}$ in $\mathcal{S}_2$ (respectively any $(\mathbb{F},\mathbb{P})$ semimartingale solution Y' in $\mathcal{S}_2'$ to (33) with $(Y' + \mathcal{Y}' + \int_0^{\cdot} j_s(Y_s')ds)$ in $\mathcal{S}_2'$). By this equation in $\mathcal{S}_2$ (respectively $\mathcal{S}_2'$), we mean this equation considered in terms of $\mathcal{S}_2$ (respectively $\mathcal{S}_2'$) solutions. By well-posedness of this equation in $\mathcal{S}_2$ (respectively $\mathcal{S}_2'$), we mean existence and uniqueness of an $\mathcal{S}_2$ (respectively $\mathcal{S}_2'$) solution. ∎

Theorem 5.1 *The shareholder value equation* (32) *in $\mathcal{S}_2$ for $Y^{\tau-}$ is equivalent, through the bijection* (28)*, to the clean value equation* (33) *in $\mathcal{S}_2'$ for Y'.*

In the case where $\mathcal{Y}'$ is in $\mathcal{S}_2$, if the random function $z \mapsto j_t(z - \mathcal{Y}_t')$ is Lipschitz in the real z and such that $j_\cdot(-\mathcal{Y}_\cdot') \in \mathcal{L}_2'$, then the clean value equation (33) *for Y' is well-posed in $\mathcal{S}_2'$, and so is the shareholder value equation* (32) *in $\mathcal{S}_2$ for $Y^{\tau-}$.*

Proof. To alleviate the notation, we show the stated equivalence in the base case $j = 0$, i.e. the one between (3) and

$$Y_t' = \mathbb{E}_t'(\mathcal{Y}_T' - \mathcal{Y}_t'), t \le T. \tag{34}$$

First we show an equivalence between the following differential forms of (3) and (34):

$$\begin{aligned}&Y_T^{\tau-} = 0 \text{ on } \{T < \tau\} \text{ and, for } t \le \tau \wedge T,\\ &dY_t^{\tau-} = -d\mathcal{Y}_t^{\tau-} + d\nu_t,\\ &\text{for some } (\mathbb{G},\mathbb{Q}) \text{ martingale } \nu \text{ in } \mathcal{S}_2,\end{aligned} \tag{35}$$

respectively

$$\begin{aligned}
&Y'_T = 0 \text{ and, for } t \le T,\\
&dY'_t = -d\mathcal{Y}'_t + d\mu_t, \qquad (36)\\
&\text{for some } (\mathbb{F},\mathbb{P}) \text{ martingale } \mu \text{ in } \mathcal{S}'_2.
\end{aligned}$$

By definition of $\mathbb{F}$ reductions, the terminal condition in (36) obviously implies the one in (35). Conversely, taking the $\mathcal{F}_T$ conditional expectation of the terminal condition in (35) yields

$$0 = \mathbb{E}[Y_T^{\tau-}\mathbb{1}_{\{T<\tau\}}|\mathcal{F}_T] = \mathbb{E}[Y'_T\mathbb{1}_{\{T<\tau\}}|\mathcal{F}_T] = Y'_T\mathsf{S}_T,$$

hence $Y'_T = 0$ (as by assumption $\mathsf{S}_T > 0$, see above (23)), which is the terminal condition in (36).

For $Y^{\tau-}$ in $\mathcal{S}_2$, the martingale condition in (36) implies the one in (35), by stopping before τ and application to $\nu = \mu^{\tau-}$ of (28) and of the first part in Assumption 4.2. Conversely, the martingale condition in (35) implies that $(Y', \mu = \nu')$ satisfies the second line in (36) on $[0, \tau \wedge T]$, hence on $[0, T]$, by (23). Moreover, by application of the second part in Assumption 4.2 and of (28), $\mu = \nu'$ is an $(\mathbb{F},\mathbb{P})$ martingale in $\mathcal{S}'_2$.

Summarizing, if $Y^{\tau-}, \nu$ in $\mathcal{S}_2$ solve (35), then $Y', \mu = \nu'$ in $\mathcal{S}'_2$ solve (36); Conversely, if Y', μ in $\mathcal{S}'_2$ solve (36), then $Y^{\tau-} = (Y')^{\tau-}, \nu = \mu^{\tau-}$ in $\mathcal{S}_2$ solve (35).

Now, if $Y^{\tau-}$ is an $\mathcal{S}_2$ solution to (20), then $Y^{\tau-}, \nu$ in $\mathcal{S}_2$ solve (35) (for some ν), hence $Y', \mu = \nu'$ in $\mathcal{S}'_2$ solve (36), therefore Y' is an $\mathcal{S}'_2$ solution to (40); Conversely, if Y' is an $\mathcal{S}'_2$ solution to (40), then Y', μ in $\mathcal{S}'_2$ solve (36) (for some μ), hence $Y^{\tau-} = (Y')^{\tau-}, \nu = \mu^{\tau-}$ in $\mathcal{S}_2$ solve (35), thus $Y^{\tau-}$ is an $\mathcal{S}_2$ solution to (20) (noting that $\nu \in \mathcal{S}_2$ is $\mathbb{Q}$ square integrable over $[0, \tau \wedge T]$, by (26)).

This shows the first part of the theorem. Under the additional assumptions made in the second part, the well-posedness in $\mathcal{S}'_2$ of the clean value equation (33) follows from standard results (see e.g. Kruse and Popier (2016)) applied to the $(\mathbb{F},\mathbb{P})$ BSDE for $Z' = Y' + \mathcal{Y}'$, i.e. the $(\mathbb{F},\mathbb{P})$ BSDE with terminal condition $\mathcal{Y}'_T$ and coefficient $z \mapsto j(z - Y')$. The well-posedness of the shareholder value equation (32) in $\mathcal{S}_2$ for $Y^{\tau-}$ then follows from the first part of the theorem. ∎

In particular, the equations (9), (10), (11), and (20) for $\mathrm{MtM}, \mathrm{CVA}, \mathrm{FVA}$, and KVA are respectively equivalent to the following more explicit formulations: For $t \le T$,

$$\mathrm{MtM}'_t = \mathbb{E}'_t\big(\mathcal{P}'_T - \mathcal{P}'_t\big), \tag{37}$$

$$\mathrm{CVA}'_t = \mathbb{E}'_t\big(\mathcal{C}'_T - \mathcal{C}'_t\big), \tag{38}$$

$$\mathrm{FVA}'_t = \mathbb{E}'_t\big(\mathcal{F}'_T - \mathcal{F}'_t\big), \tag{39}$$

$$\mathrm{KVA}'_t = \mathbb{E}'_t \int_t^T h(\mathrm{EC}'_s - \mathrm{KVA}'_s)^+ ds \tag{40}$$

(recalling (21)).

Regarding MtM, we then have, by specification (30) of $\mathcal{P}$, (37), and (9),

$$\mathrm{MtM}' = \sum_c P^c \mathbb{1}_{[0,(\tau_c^\delta)')},\ \mathrm{MtM}^{\tau-} = \big(\sum_c P^c \mathbb{1}_{[0,\tau_c^\delta \wedge \tau^\delta)}\big)^{\tau-}, \tag{41}$$

whereas, from τ onward, MtM proceeds from $\mathcal{P}$ by the second part in (7).

Clean valuations at the individual trade level that are involved in MtM computations (see the observation made after (29)) are of course standard (at least, in the base case where $\mathbb{P} = \mathbb{Q}$). Hence we focus on the XVA equations in the sequel.

5.2 KVA in the Case of a Default-Free Bank

In this section we temporarily suppose the bank default free, i.e., formally,

$$\text{“}\tau = +\infty,\ (\mathbb{F}, \mathbb{P}) = (\mathbb{G}, \mathbb{Q}).\text{”}$$

The results are then extended to the case of a defaultable bank in Section 5.3.

At that stage we use the “$\cdot'$” notation, not in the sense of $\mathbb{F}$ reduction (as $\mathbb{F} = \mathbb{G}$), but simply in order to distinguish the equations in this part, where $\mathbb{F} = \mathbb{G}$, from the ones in Section 5.3, where $\mathbb{F} \neq \mathbb{G}$ (the present data will then be interpreted as the $\mathbb{F}$ reductions of the corresponding data in Section 5.3).

Given $C' \geq \mathrm{EC}' \geq 0$ representing a putative capital at risk process for the bank, we consider the auxiliary BSDE,

$$K'_t = \mathbb{E}'_t \int_t^T h\big(C'_s - K'_s\big) ds,\ t \leq T, \tag{42}$$

with the same interpretation as the KVA (cf. (22) and the following comment), but relative to any putative capital at risk process C' (and simplified to the present setup of a risk-free bank).

Lemma 5.1 • *If C' is in $\mathcal{L}'_2$, then the equation* (42) *for K' has for unique $\mathcal{S}'_2$ solution*

$$K'_t = h\mathbb{E}'_t \int_t^T e^{-h(s-t)} C'_s ds,\ t \in [0, T] \tag{43}$$

(which is nonnegative, like C');

- *If L' is in $\mathcal{L}'_2$, then* EC' *is in $\mathcal{L}'_2$ and the* KVA' *equation* (40) *has a unique $\mathcal{S}'_2$ solution.*

Proof. If L' is in $\mathcal{L}'_2$, then EC' is in $\mathcal{L}'_2$, by Definition 4.2 and $(1-\alpha)^{-1}$ Lipschitz property of the (also conditional) expected shortfall operator recalled in the beginning of Section 4.3. Moreover, the KVA' BSDE (40) has a Lipschitz coefficient

$$k_t(y) = h\big(\mathrm{EC}'_t - y\big)^+,\ y \in \mathbb{R}. \tag{44}$$

Hence the result regarding KVA' follows from the second part in Theorem 5.1 (applied with $\mathcal{Y} = 0$).

Even simpler considerations prove the analogous result pertaining to the linear BSDE (42). Finally, the $\mathcal{S}'_2$ solution K' to (43) solves (42). ∎

To emphasize the dependence on C', we henceforth denote by $K' = K'(C')$ the solution (43) to the linear BSDE (42). Assuming that L' is in $\mathcal{L}_2'$, we define the set of admissible capital at risk processes as

$$\mathrm{Adm}' = \{C' \in \mathcal{L}_2'; C' \geq \max\big(\mathrm{EC}', K'(C')\big)\}, \tag{45}$$

where $C' \geq \mathrm{EC}'$ is the risk acceptability condition, while $C' \geq K'(C')$ expresses that the risk margin $K'(C')$, which would correspond through the constant hurdle rate h to the tentative capital at risk process C' (cf. the comment regarding the KVA itself made after (22)), is part of capital at risk, by Assumption 3.6.

Let (cf. (19))

$$\mathrm{CR}' = \max(\mathrm{EC}', \mathrm{KVA}'), \tag{46}$$

where KVA' is the $\mathcal{S}_2'$ solution to (40).

Corollary 5.1 *Assuming that L' is in $\mathcal{L}_2'$, the $\mathcal{S}_2'$ solution* KVA' *to* (40) *solves the linear BSDE* (42) *corresponding to the implicit data $C' = \mathrm{CR}'$ as per* (46)*, i.e. we have* $\mathrm{KVA}' = K'(\mathrm{CR}')$, *that is,*

$$\mathrm{KVA}'_t = h\mathbb{E}'_t \int_t^T e^{-h(s-t)} \max(\mathrm{EC}'_s, \mathrm{KVA}'_s) ds,\ t \in [0,T] \tag{47}$$

(which is nonnegative, as already seen for EC *in Remark 4.2).*

Proof. The process KVA' is in $\mathcal{S}_2'$ with martingale part in $\mathcal{S}_2'$ and, by virtue of (40), we have, for $t \in [0,T]$,

$$\mathrm{KVA}'_t = \mathbb{E}'_t \int_t^T h\big(\mathrm{EC}'_s - \mathrm{KVA}'_s\big)^+ ds = \mathbb{E}'_t \int_t^T h\Big(\mathrm{CR}'_s - \mathrm{KVA}'_s\Big) ds, \tag{48}$$

by (46). Hence, the process KVA' solves the linear BSDE (42) corresponding to $C' = \mathrm{CR}' \in \mathcal{L}_2'$. The identity $\mathrm{KVA}' = K'(\mathrm{CR}')$ follows by uniqueness of an $\mathcal{S}_2'$ solution to the linear BSDE (42) as seen in Lemma 5.1. Equation (47) then follows by an application of (43) with $C' = \mathrm{CR}'$ as per (46). ∎

Remark 5.1 The KVA formula (47) appears as a continuous-time analog of the risk margin formula under the Swiss solvency test cost of capital methodology: See Swiss Federal Office of Private Insurance (2006, Section 6, middle of page 86 and top of page 88). ∎

We are now in a position to establish the minimality result announced after Remark 3.6.

Proposition 5.1 *Assuming that L' is in $\mathcal{L}_2'$, we have:*
(i) $\mathrm{CR}' = \min \mathrm{Adm}', \mathrm{KVA}' = \min_{C' \in \mathrm{Adm}'} K'(C')$;
(ii) *The process* KVA' *is nondecreasing in the hurdle rate h.*

Proof. **(i)** We saw in Corollary 5.1 that $\mathrm{KVA}' = K'(\mathrm{CR}')$, hence

$$\mathrm{CR}' = \max(\mathrm{EC}', \mathrm{KVA}') = \max\big(\mathrm{EC}', K'(\mathrm{CR}')\big),$$

therefore $\mathrm{CR}' \in \mathrm{Adm}'$. Moreover, for any $C' \in \mathrm{Adm}'$, we have (cf. (44)):

$$k_t(K'_t(C')) = h\big(\mathrm{EC}'_t - K'_t(C')\big)^+ \leq h(C'_t - K'_t(C')).$$

Hence, the coefficient of the KVA' BSDE (40) never exceeds the coefficient of the linear BSDE (42) when both coefficients are evaluated at the solution $K'_t(C')$ of (42). Since these are BSDEs with equal (null) terminal condition, the BSDE comparison principle of Proposition 4 in Kruse and Popier (2016)[4] applied to the BSDEs (42) and (40) yields $\mathrm{KVA}' \leq K'(C')$. Consequently, $\mathrm{KVA}' = \min_{C' \in \mathrm{Adm}'} K'(C')$ and, for any $C' \in \mathrm{Adm}'$,

$$C' \geq \max(\mathrm{EC}', K'(C')) \geq \max(\mathrm{EC}', \mathrm{KVA}') = \mathrm{CR}'.$$

Hence $\mathrm{CR}' = \min \mathrm{Adm}'$.
(ii) The coefficient (44) of the KVA' BSDE (40) is nondecreasing in the parameter h. So is therefore the $\mathcal{S}'_2$ solution KVA' to (40), by the BSDE comparison theorem of Kruse and Popier (2016, Proposition 4) applied to the BSDE (40) for different values of the parameter h. ∎

5.3 KVA in the Case of a Defaultable Bank

In the case of a defaultable bank, "$\cdot'$" now denoting $\mathbb{F}$ reduction (predictable, optional, or progressive, as applicable), we have by applications of the first part in Theorem 5.1 (with $\mathcal{Y} = 0$ there):

Proposition 5.2 *The equation*

$$K^{\tau-}_t = \mathbb{E}_t\Big(\int_t^{\tau\wedge T} h\big(C_s - K_s\big)ds + \mathbb{1}_{\{\tau \leq T\}} K^{\tau-}_\tau\Big),\ t \leq \tau \wedge T \tag{49}$$

in $\mathcal{S}_2$ for $K^{\tau-}$ is equivalent, through the bijection (28), *to the equation* (42) *in $\mathcal{S}'_2$ for K' .*

The equation (20) *in $\mathcal{S}_2$ for* $\mathrm{KVA}^{\tau-}$ [5] *is equivalent, through the bijection* (28), *to the equation* (40) *in $\mathcal{S}'_2$ for* KVA'. ∎

Hence, given Lemma 5.1 :

Corollary 5.2
- *If $C' \in \mathcal{L}'_2$, then the K equation* (49) *is well posed in $\mathcal{S}_2$ and the $\mathbb{F}$ optional reduction K' of its $\mathcal{S}_2$ solution K is the $\mathcal{S}'_2$ solution to* (42);
- *If L' is in $\mathcal{L}'_2$, then the* $\mathrm{KVA}^{\tau-}$ *equation* (20) *is well posed in $\mathcal{S}_2$ and the $\mathbb{F}$ optional reduction* KVA' *of its $\mathcal{S}_2$ solution* $\mathrm{KVA}^{\tau-}$ *is the $\mathcal{S}'_2$ solution to* (40). ∎

[4] Note that jumps are not an issue for comparison in our setup, where the coefficient k "only depends on y"; cf. Kruse and Popier (2016, Assumption (H3')).
[5] See (3) and the following comment.

In the case of a defaultable bank, writing $K = K(C)$ for the $\mathcal{S}_2$ solution to (49), the set of admissible capital at risk processes is defined by (cf. (45) and the following comments)

$$\mathrm{Adm} = \{C \in \mathcal{L}_2; C \geq \max\big(\mathrm{EC}, K(C)\big)\}. \tag{50}$$

Theorem 5.2 *Assuming that L' is in $\mathcal{L}'_2$:*
(i) *We have* $\mathrm{CR} = \min \mathrm{Adm}, \mathrm{KVA}^{\tau-} = \min_{C\in\mathrm{Adm}} K(C)$;
(ii) *The process* KVA *is nondecreasing in h.*

***Proof*.** This follows by application of Proposition 5.1 via Corollary 5.2. ∎

5.4 CVA and FVA

We now consider the CVA and the FVA equations (10) and (11) and their reduced forms (38) and (39).

Regarding the funding cash flows, we postulate (as typical regarding variation margin funding expenses, cf. (57) for a concrete example below)

$$d\mathcal{F}^{\tau-}_t = J_t f_t(\mathrm{FVA}^{\tau-}_t)dt, \tag{51}$$

for some random function $f = f_t(y)$ measurable with respect to the product of the $\mathbb{F}$ predictable σ field by the Borel σ field on $\mathbb{R}$.

Proposition 5.3 *The equation* (10) *in $\mathcal{S}_2$ for* $\mathrm{CVA}^{\tau-}$ *is equivalent to the* CVA' *formula* (38)*, which, if $\mathcal{C}'_T$ is $\mathbb{P}$ square integrable, yields a well defined process in $\mathcal{S}'_2$.*

Then, for $\mathcal{F}^{\tau-}$ as per (51)*, the equation* (11) *in $\mathcal{S}_2$ for* $\mathrm{FVA}^{\tau-}$ *is equivalent to the following equation in $\mathcal{S}'_2$ (cf.* (39)*):*

$$\mathrm{FVA}'_t = \mathbb{E}'_t \int_t^T f_s(\mathrm{FVA}'_s)ds,\ t \leq T. \tag{52}$$

Assuming f Lipschitz in y and $f_\cdot(0)$ in $\mathcal{L}'_2$, this equation is well posed in $\mathcal{S}'_2$ and the equation (11) *for* $\mathrm{FVA}^{\tau-}$ *is therefore well-posed in $\mathcal{S}_2$.*

***Proof*.** By two successive applications of Theorem 5.1 (with $j = 0$ in the CVA case and $\mathcal{Y} = 0$ in the FVA case). ∎

Remark 5.2 A structure (51) for $\mathcal{F}^{\tau-}$ as per (31) is a slight departure, as per Section 5.1, from the abstract setup postulated Section 3.1, where $\mathcal{F}$ had been introduced as an exogenous process. But, provided the ensuing $\mathrm{FVA}^{\tau-}$ equation is well-posed, for which sufficient conditions are given in Proposition 5.3 (see also Proposition 5.4 for a more explicit result in a concrete setup), one can readily check, by revisiting all the above, that this dependence does not affect any of the qualitative conclusions drawn in the previous sections of the paper. ∎

5.5 Example

Let

$$U_0 = 1 \text{ and } dU_t = \lambda_t U_t dt + (1-R)U_{t-} dJ_t = U_{t-}\big(\lambda_t dt + (1-R)\, dJ_t\big),\ t \le \tau \wedge T, \quad (53)$$

represent the (risk-free discounted) risky funding asset supposed to be used by the bank for its unsecured borrowing purposes, for some exogenous and constant recovery rate $R \in [0,1]$. Note that the bank can only be short in U, assuming the bank is not allowed to sell default protection on itself.

The martingale condition that applies to U (cf. the comments preceding Assumption 3.2) implies that $\lambda = (1-R)\gamma$, where γ is the default intensity of the bank and R its recovery rate toward its external funder (see the second paragraph of Section 2). Hence

$$\lambda_t dt + (1-R)\, dJ_t = (1-R) d\mu_t, \quad (54)$$

where $d\mu_t = \gamma dt + dJ_t$ is the compensated jump-to-default martingale of the bank.

We assume all re-hypothecable collateral (i.e. no initial margin) and we denote by D an optional process representing the difference between the collateral MtM posted by the CA desks to the clean desks and the collateral received by the CA desks from the clients.

Lemma 5.2 *For $t \le T$, we have*

$$d\mathcal{F}_t = (1-R)(D_{t-} - \mathrm{CA}_{t-})^+ d\mu_t, \quad (55)$$

i.e.

$$\begin{aligned} d\mathcal{F}^{\tau-}_t &= J_t \lambda_t (D_t - \mathrm{CVA}_t - \mathrm{FVA}_t)^+ dt, \\ d({}^{\tau-}(-\mathcal{F}))_t &= (1-R)(D_{t-} - \mathrm{CVA}_{t-} - \mathrm{FVA}_{t-})^+ (-dJ_t). \end{aligned} \quad (56)$$

If $\mathcal{C}'_T$ is $\mathbb{P}$ square integrable, then $\mathcal{F}^{\tau-}$ is of the form (51) *with*

$$f_\cdot(y) = \lambda'(D' - \mathrm{CVA}' - y)^+, \quad (57)$$

where CVA' *is defined by* (38)*.*

Proof. Assuming that capital at risk is not used by the bank for its funding purposes (cf. Remark 4.3), the funding strategy of the CA desks reduces to a splitting of the amount CA_t on the reserve capital account as

$$\begin{aligned} \mathrm{CA}_t = &\underbrace{D_t}_{\text{Posted collateral remunerated at the risk-free rate}} \\ &+ \underbrace{(\mathrm{CA}_t - D_t)^+}_{\text{Cash invested at the risk-free rate}} \\ &- \underbrace{(\mathrm{CA}_t - D_t)^-}_{\text{Cash unsecurely funded}} \\ = &\underbrace{\big(D_t + (\mathrm{CA}_t - D_t)^+\big)}_{=:\ \xi_t,\ \text{invested at the risk-free rate}} - \underbrace{(\mathrm{CA}_t - D_t)^-}_{=:\ \eta_t U_t,\ \text{unsecurely funded}} \end{aligned} \quad (58)$$

(all risk-free discounted amounts). Given our use of the risk-free asset as numéraire, a standard self-financing equation yields[6]

$$
\begin{aligned}
d\left(\xi_t - \eta_t U_t\right) &= -\eta_{t-} dU_t = -(1-R)\eta_{t-} U_{t-} d\mu_t \\
&= -(1-R)(D_{t-} - \mathrm{CA}_{t-})^+ d\mu_t,\ t \le \tau \wedge T.
\end{aligned}
$$

As $\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}$, this yields (55), i.e. (56). If $\mathcal{C}'_T$ is $\mathbb{P}$ square integrable, then (57) follows by the first part in Proposition 5.3. ∎

In what follows we further assume that the bank portfolio involves a single client with default time denoted by τ_1, that $\mathbb{Q}(\tau_1 = \tau) = 0$, that the liquidation of a defaulted party is instantaneous, and that no contractual cash flows are promised at the exact times τ and τ_1.

Let J and J^1, respectively R and R_1, denote the survival indicator processes and the assumed recovery rates of the bank and its client toward each other; we also assume that the bank has identical recovery rates toward its client and its external funder. In this case, D is of the form $J^1 Q$, where Q is the difference between the clean valuation P of the client portfolio and the amount VM of variation margin (re-hypothecable collateral) to be transferred[7] between the client and the CA desks in case of a default of the client or the bank.

The following result is in line with the generic specification of $\mathcal{C}$ in (30). For illustration we provide a detailed derivation from Assumption 4.3.

Lemma 5.3 *For $t \le T$,*

$$
\begin{aligned}
d\mathcal{C}^\star_t &= \mathbb{1}_{\{\tau_1 \le \tau\}}(1-R_1)Q^+_{\tau_1}(-dJ^1_t), \\
d({}^{\tau-}(-\mathcal{C}))_t &= \mathbb{1}_{\{\tau \le \tau_1\}}(1-R)Q^-_{\tau}(-dJ_t).
\end{aligned}
\tag{59}
$$

Proof. Before the defaults of the bank or its client, the contractual cash flows are delivered as promised, hence there are no contributions to the process $\mathcal{C}$. Because of this, and since liquidations are instantaneous, it is enough to focus on the contributions to $\mathcal{C}$ at time $\tau \wedge \tau_1$. By symmetry, it is enough to prove the first line in (59). Let $\epsilon = Q^+_{\tau_1}$. By Assumption 4.3, if the counterparty defaults at $\tau_1 < \tau$, then (as $Q = P - \mathrm{VM}$ and having excluded the possibility of contractual cash flows at times τ or τ_1):

- On the client portfolio side, the CA desks receive

$$
\mathrm{VM}_{\tau_1} + R_1 Q^+_{\tau_1} - Q^-_{\tau_1} = \mathbb{1}_{\epsilon=0} P_{\tau_1} + \mathbb{1}_{\epsilon>0}(\mathrm{VM}_{\tau_1} + R_1 Q_{\tau_1});
$$

- The property of the amount P_{τ_1} on the clean margin account is transferred from the CA desks to the clean desks.

[6] A left-limit in time is required in η because U jumps at time τ, so that the process η, which is defined through (58) as $\frac{(\mathrm{CA}-D)^-}{U}$, is not predictable.

[7] Property-wise, having already been posted as a loan by the client to the CA desks (if positive, or by the CA desks to the client otherwise), cf. Assumption 4.3.

Combining both cash flows, the loss of the CA desks triggered by the default of the client amounts to

$$P_{\tau_1} - \big(\mathbb{1}_{\epsilon=0}P_{\tau_1} + \mathbb{1}_{\epsilon>0}(\mathrm{VM}_{\tau_1} + R_1 Q_{\tau_1})\big) = \mathbb{1}_{\epsilon>0}(P_{\tau_1} - \mathrm{VM}_{\tau_1} - R_1 Q_{\tau_1}) = (1-R_1)Q^+_{\tau_1},$$

which shows the first line in (59). ∎

Proposition 5.4 *If $\mathbb{1}_{\{\tau_1'<T\}}(Q'_{\tau_1'})^+$ is $\mathbb{P}$ square integrable and $\lambda' J^1(Q')^+$ is in $\mathcal{L}_2'$, then the* FVA′ *equation* (52) *is well-posed in $\mathcal{S}_2'$, the* CVA *and* FVA *equations* (10)–(11) *are well-posed in $\mathcal{S}_2$ and we have, for $t \le T$:*

$$\begin{aligned}
&\mathrm{CVA}'_t = \mathbb{E}'_t\big[\mathbb{1}_{\{t<\tau_1'<T\}}(1-R_1)(Q'_{\tau_1'})^+\big], && (60)\\
&\mathrm{FVA}'_t = \mathbb{E}'_t\int_t^T \lambda'_s((J^1)'_s Q'_s - \mathrm{CVA}'_s - \mathrm{FVA}'_s)^+ ds, && (61)\\
&\mathrm{DVA}_t = \mathbb{E}_t\big[\mathbb{1}_{\{t<\tau\le\tau_1\wedge T\}}(1-R)Q^-_\tau\big] + \mathbb{E}_t\big[\mathbb{1}_{\{t<\tau\le T\}}\mathrm{CVA}_{\tau-}\big], && (62)\\
&\mathrm{FDA}_t = \mathbb{E}_t\big[\mathbb{1}_{\{t<\tau\le T\}}(J^1_{\tau-}Q_{\tau-} - \mathrm{CA}_{\tau-})^+\big] + \mathbb{E}_t\big[\beta_t^{-1}\beta_\tau\mathbb{1}_{\{t<\tau\le T\}}\mathrm{FVA}_{\tau-}\big], && (63)\\
&\mathrm{FV}_t = \mathbb{E}_t\big[\mathbb{1}_{\{t<\tau_1\le\tau\wedge T\}}(1-R_1)Q^+_{\tau_1}\big] - \mathbb{E}_t\big[\mathbb{1}_{\{t<\tau\le\tau_1\wedge T\}}(1-R)Q^-_\tau\big], && (64)\\
&dL'_t = (1-R_1)(Q'_{\tau_1'})^+(-d(J^1)'_t) + d\mathrm{CVA}'_t && (65)\\
&\qquad\quad + \lambda'_t(J^1)'_t(Q'_t - \mathrm{CVA}'_t - \mathrm{FVA}'_t)^+ dt + d\mathrm{FVA}'_t.
\end{aligned}$$

Moreover, L' is in $\mathcal{S}_2'$ and the ensuing KVA *implications of Corollary 5.2 are in force, in particular the* KVA′ *formula* (47) *holds.*

Proof. The CVA and FVA related statements are obtained by application of Proposition 5.3. The DVA, FDA, and FV formulas readily follow from Definition 3.2, (59), and (56). The dynamics (65) for L' are obtained by plugging the first lines of (59) and (56) into (12) and then taking $\mathbb{F}$ reductions of all the data. Finally L' belongs to $\mathcal{S}_2'$ as the sum (modulo a constant) between the $(\mathbb{F},\mathbb{P})$ optional projection of $\mathbb{1}_{\{\tau_1'<T\}}(Q'_{\tau_1'})^+$ (assumed $\mathbb{P}$ square integrable) and the $(\mathbb{F},\mathbb{P})$ martingale part of FVA′. ∎

The FV formula (64) is symmetrical between the bank and its client, hence consistent with the so-called "law of one price". But, as detailed in Section 3.5, the corresponding notion of fair valuation is only a theoretical cost of replication formula, forgetful of the wealth transfers from clients and shareholders to bondholders that arise due to the incompleteness of counterparty risk.

6 Comparison with the XVA Replication Theory

In this section, we compare our method with alternative approaches in the literature that have been developed in the last years in what we therefore call the XVA benchmark model, namely a Black–Scholes model S for an underlying market risk factor, in conjunction with independent Poisson counterparties and bank defaults: See, non-limitatively, the Burgard and Kjaer (2011, 2013, 2017) CVA and FVA approach, referred to as the BK approach below, the Green et al. (2014) KVA approach, referred to as the GK approach, Bichuch, Capponi, and Sturm (2018), or Crépey, Bielecki, and Brigo (2014, Section 4.6).[8]

As a general comment, using a Black–Scholes replication framework as an XVA toy model can be misleading. The view developed in the present paper is that XVAs are mainly about market incompleteness, and therefore fall under a logic orthogonal to Black–Scholes. From this perspective, promoting a Black–Scholes replication approach in the XVA context is a bit comparable to the mispractice developed during the credit derivative pre-2008 crisis era, when the notion of "Gaussian copula implied correlation" of a CDO tranche was presented as a relative of the Black–Scholes implied volatility of an option, whereas the Gaussian copula model is a purely static device not supported by a sound hedging basis. Actually, in a complete setup, the all-inclusive XVA should just be FV, i.e. the difference between the CVA and the DVA (see Section 3.5).

As a matter of fact, Burgard and Kjaer (2011, 2013, 2017), although availing themselves of a replication pricing framework and blaming risk-neutral approaches outside the realm of replication (see the first paragraph in their 2013 paper), end-up doing what they call semi-replication, which is nothing but a form of risk-neutral pricing without (exact) replication.

In the XVA field, even the restriction to a Markov setup (beyond Black–Scholes or replication) is not necessarily innocuous, as we will see in the next-to-last paragraph of Section 6.1.

What follows provides more detailed comparisons between these approaches and the one of the present paper.

6.1 CVA and FVA: Comparison with the Burgard and Kjaer Approach

Burgard and Kjaer (2011, 2013, 2017) repeatedly (and rightfully) say that only pre-default cash flows matter to shareholders. For instance, quoting the first paragraph in the second reference:

> "Some authors have considered cases where the post-default cash flows on the funding leg are disregarded but not the ones on the derivative. But it is not clear why some post default cashflows should be disregarded but not others",

to which we subscribe fully and refer to as their ***first principle***.

The introduction of their classical "(funding) strategy I : semi-replication with no shortfall at own default" (see e.g. (Burgard and Kjaer 2013, Section 3.2)) seems to be in line with the idea, which we also agree with (see Assumption 3.2 and the comment following it) and refer to as their ***second principle***, that a shortfall of the bank at its own default does not make sense and should be excluded from a model (which should

[8] In journal form Crépey (2015, Part II, Section 5).

imply that DVA ≥ 0, FDA ≥ 0, and no admissible hedge causes a shortfall at bank default that would allow the bank to manipulate the latter inequalities, see Section 3.5).

However, being rigorous with their first principle implies that the valuation jump of the portfolio at the own default of the bank should be disregarded in the shareholder cash flow stream. But their computations, stated in terms of $(d\widehat{V} + d\bar{\Pi})$ in Burgard and Kjaer (2013, equation (9)) or $(d\widehat{V}^{\alpha} + d\Pi)$ in Burgard and Kjaer (2017, equation (3.5)), include this cash flow (which in our setup would be tantamount to ignoring the last terms in equations (9)–(11), cf. Remark 2.1).

For illustration, in order to be able to restrict attention on the CVA for simplicity, let us assume, for the sake of the argument, that the bank, although risky (with default intensity $\gamma > 0$), can both fund itself and invest at the risk-free rate.[9] This corresponds to the limiting case where $\mathcal{F} = 0$ in Assumption 3.2. We put ourselves in the framework of Section 5.4 where Proposition 5.4 was derived, specialized further to a BK setup with volatility σ of a stock S underlying a (single) contract with payoff $\phi(S_T)$ sold by the client to the bank. Taking the difference between the CVA$'$ PDE (cf. (60) specialized to the present BK setup) and the Black–Scholes PDE for the clean valuation of the contract shows that the bank pre-default CVA-deducted value of the contract, $\widehat{V}'_t := P'_t - \mathrm{CVA}'_t$, can be represented in functional form as $\widehat{V}'(t, S_t, J^1_t)$, where the function $\Pi(t, S) = \widehat{V}'(t, S, J^1 = 1)$ satisfies the following pricing equation:

$$
\begin{aligned}
&\Pi(T, S) = \phi(S) \text{ and, for } t < T,\\
&(\partial_t + \frac{1}{2}\sigma^2 S^2 \partial^2_{S^2})\Pi(t, S) + \gamma_1\big(R_1 P^+ - P^-\big)(t, S) - \gamma_1 \Pi(t, S) = 0.
\end{aligned}
\tag{66}
$$

Here γ_1 is the default intensity of the counterparty, assumed constant in BK, and $P(t, S)$ is the Black-Scholes price of the option. Observe that the (positive) default intensity γ of the bank does not appear in (66). In fact, (66) is nothing but the equation (10) for $\widehat{V}$ in Burgard and Kjaer (2013) or (3.8) for $\widehat{V}^{\alpha}$ in Burgard and Kjaer (2017), but for a default intensity of the bank, denoted by λ_B there, formally set equal to 0.

Hence, in a BK pure CVA setup, the CVA-deducted value of the option truly disregarding all cash flows from time τ onward, including the jump in valuation at time τ, is ***not*** given by the solution $\widehat{V}$ to equation (10) in Burgard and Kjaer (2013) or $\widehat{V}^{\alpha}$ to equation (3.8) in Burgard and Kjaer (2017), ***but*** by $\widehat{V} = (\widehat{V}')^{\tau-}$, where $\widehat{V}'$ satisfies the formal analog of these equations ***with intensity of the bank set equal to 0***.

In the plain (counterparty risk and funding) BK setup, a strict application of their two principles (which is nothing but taking a bank shareholder-centric view given the impossiblity for a bank of hedging its own jump-to-default) should lead to a unilateral CVA and to a unilateral (and asymetric) FVA, where the credit riskiness of the bank only shows up throuh a funding spread driving the FVA.

Incidentally, the Itô derivation of the portfolio XVA-deducted value process in Burgard and Kjaer (2013, Eq. (9)) relies on the implicit assumption that this value

[9] For instance because the bank is highly capitalized and can in fact use its capital at risk for funding its trading (cf. Remark 4.3), or simply assuming exogenously a recovery rate of the bank R equal to 1.

process $\widehat{V}$ is in the first place a (regular enough) function of the postulated risk factors (for Itô's formula to apply), which is not justified in their paper. For instance, in the simplified (CVA only) BK setup above, CVA'_t is a measurable function of t, S_t, and J^1_t, and then it only holds that $\mathrm{CVA}^{\tau-} = (\mathrm{CVA}')^{\tau-}$.

As a consequence of the above pitfall regarding $\widehat{V}$, reviewing the funding strategies in Burgard and Kjaer (2017, Section 4), but accounting for the transfer of the residual reserve capital from shareholders to bondholders at the bank default time τ (cf. the last part of Assumption 2.2):

- Their strategy III would no longer imply a unilateral CVA as per Albanese and Andersen (2014) (i.e. (60) in the present paper). Considering for instance the special case with $s_B = 0$ there of a pure CVA setup, the funding strategy that implies a unilateral CVA is simply the obvious one (having assumed $s_B = 0$), i.e. funding and investing at the risk-free rate;

- Their respective strategies I and II would not imply the claimed XVA formulas. These strategies also breach the last part in Assumption 3.2, which may lead to a violation of their second principle of no bank shortfall at its default, through a negative FVA = FDA (unless in their notation $V \geq 0$, respectively $\widehat{V} \geq 0$, i.e. in our notation above $P \geq 0$, respectively $\Pi \geq 0$);

- Their replication strategy is not practical, as stated at the end of Burgard and Kjaer (2013, Section 3.1), which is the motivation for their other strategies.

6.2 KVA: Comparison With the Green and Kenyon Approach

Despite what the "valuation adjustment" terminology induces one to believe, our KVA is not part of the value of the derivative portfolio, but a risk premium in incomplete counterparty risk markets: our risk margin (i.e. KVA) payments are meant to remunerate the risk of unhedged trading losses. Hence, including the KVA to contra-assets, which contribute to the trading losses of the bank, would be illogical (and induce a circularity).

In Green et al. (2014) the KVA is instead treated as a liability in a replication framework. The KVA is also treated as a liability in some theoretical actuarial literature, under the name of market-value margin (MVM, see Salzmann and Wüthrich (2010, Section 4.4)). Viewing the KVA as a liability results in a non loss-absorbing risk margin, hence $\mathrm{CR} = \mathrm{SCR}$ (as opposed to (18) in our setup), and therefore $h\mathrm{EC}$ instead of $h\mathrm{SCR} = h(\mathrm{EC} - \mathrm{KVA})^+$ as Lebesgue integrand in the KVA equation (20) (cf. (21)). This implies no discounting at the hurdle rate h in the (already risk-free discounted) KVA formula (47) (recall KVA' and KVA coincide before τ).

Moreover, if the KVA is viewed as a liability, forward starting one-year-ahead fluctuations of the KVA must be simulated for economic capital calculation. This makes it intractable numerically, unless one switches from economic capital to a (scriptural) regulatory capital specification in the KVA equation. But using regulatory instead of economic capital is less self-consistent: It loses the connection, established from the balance sheet in our approach (see Albanese et al. (2019, Proposition 4.1) for more detail about it), whereby the KVA input is the shareholder trading loss $L^{\tau-}$ as per (12).

A potential drawback of our approach with respect to using (scriptural) regulatory capital specification is enhanced model risk. More generally, model risk in the XVA context is an important and widely open issue, which we leave for further research.

7 Conclusion

Under a cost-of-capital, economical capital based, XVA approach, the input to economic capital and KVA computations, i.e. the process $L^{\tau-}$ (cf. Definition 4.2 and (18)–(20)), is an output of the CVA and FVA computations (cf. (12)). Moreover, in view of (30), key inputs to the counterparty credit cash flows $\mathcal{C}$, and, in turn, to the risky funding cash flows $\mathcal{F}$, are the clean value processes P^c, as well as the collateralization schemes that underlie Γ^c and $\bar{\Gamma}^c$. Hence, MtM computations flow into CVA computations, which in turn flow into FVA computations, which all flow into KVA computations. These connections make the MtM, CA = CVA + FVA, and KVA equations, thus the derivative pricing problem as a whole, a self-contained and self-consistent problem.

7.1 Unilateral Versus Bilateral XVAs

Even though our setup crucially includes the default of the bank itself (which is the essence of the contra-liabilities wealth transfer issue), we end up with unilateral (and nonnegative) portfolio-wide CVA, FVA, and KVA formulas (38)–(40) pricing the related cash flows until the final maturity T of the portfolio (as opposed to $\tau \wedge T$), and without any bank default intensity discounting—under the reduced filtration $\mathbb{F}$ and possibly changed pricing measure $\mathbb{P}$, but, as discussed in Remark 4.1, the base case is $\mathbb{P} = \mathbb{Q}$.

Unilateral and nonnegative portfolio-wide XVA costs to be accounted for in entry prices is indeed what follows from our analysis of all wealth transfers involved. This makes the corresponding XVAs more conservative than the bilateral (and sometimes negative) XVAs that appear in most of the related literature.

A unilateral CVA is actually required for being in line with the regulatory requirement that reserve capital should not diminish as an effect of the sole deterioration of the bank credit spread (see Albanese and Andersen (2014, Section 3.1)).

A bilateral KVA would be obtained instead of a unilateral one by deciding that, upon bank default, notwithstanding the last part of Assumption 2.2, the residual risk margin flows back into equity capital and not to bondholders. Likewise, disentangling the CA desks into a CVA desk and an FVA desk, each endowed with their own reserve capital account, a bilateral FVA would follow from asserting that, upon bank default, the residual reserve capital of the FVA desk flows back into equity capital and not to bondholders: See e.g. Albanese et al. (2017, Proposition 4.2). But the corresponding violations of Assumption 2.2 induce “shareholder arbitrage”, in the sense of a riskless profit to shareholders in the case where the bank would default instantaneously at time

0, right after the client portfolio has been setup and the corresponding reserve capital and risk margin amounts have been sourced from the clients.

7.2 Cost-of-Capital XVA Approach: Theory and Practice

The KVA formula (47) can be used either in the direct mode, for computing the KVA corresponding to a given target hurdle rate h, or in the reverse-engineering mode, like the Black–Scholes model with volatility, for defining the "implied hurdle rate" associated with the actual amount on the risk margin account of the bank. Cost of capital proxies have always been used to estimate return-on-equity. Whether it is used in the direct or in the implied mode, the KVA is a refinement, dynamic and fine-tuned for derivative portfolios, but the base concept is far older than even the CVA.

In the current state of the market, even when they are computed, the KVA and even the MVA (which is included in the FVA in this paper) are not necessarily passed into entry prices. But they are strategically used for collateral and capital optimization purposes and detection of good/bad trade opportunities. This kind of balance sheet optimization is very active in top tier banks at the moment.

In the post-crisis regulatory environment, bilateral exotic trades are typically hedged by vanilla portfolios that are cleared through central counterparties. In this paper, we considered a dealer bank involved into bilateral derivative portfolios with clients and we refer the reader to Albanese et al. (2017) for numerical applications at the scale of real-life bilateral trade portfolios; see also Abbas-Turki, Diallo, and Crépey (2018) for a focus on the embedded nested Monte Carlo issues. Moreover, the abstract analysis of Section 2–3 can be adapted to the case of centrally cleared derivative portfolios, which is done in Armenti and Crépey (2019). A cost-of-capital XVA approach, thus extended, can then be applied to the situation of a bank involved into an arbitrary combination of bilateral and centrally cleared derivative portfolios.

References

Abbas-Turki, L., B. Diallo, and S. Crépey (2018). XVA principles, nested Monte Carlo strategies, and GPU optimizations. *International Journal of Theoretical and Applied Finance 21*, 1850030.

Albanese, C. and L. Andersen (2014). Accounting for OTC derivatives: Funding adjustments and the re-hypothecation option. Working paper available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2482955.

Albanese, C. and L. Andersen (2015). FVA: What's wrong, and how to fix it. *Risk Magazine*, September 54–56.

Albanese, C., L. Andersen, and S. Iabichino (2015). FVA: Accounting and risk management. *Risk Magazine*, February 64–68. Long preprint version available as ssrn.2517301.

Albanese, C., S. Caenazzo, and S. Crépey (2017). Credit, funding, margin, and capital valuation adjustments for bilateral portfolios. *Probability, Uncertainty and Quantitative Risk 2*(7), 26 pages.

Albanese, C., S. Crépey, R. Hoskinson, and B. Saadeddine (2019). XVA analysis from the balance sheet. Working paper available on https://math.maths.univ-evry.fr/crepey.

Andersen, L., D. Duffie, and Y. Song (2019). Funding value adjustments. *Journal of Finance 74*(1), 145–192.

Armenti, Y. and S. Crépey (2019). XVA Metrics for CCP optimisation. Working paper available on https://math.maths.univ-evry.fr/crepey.

Bichuch, M., A. Capponi, and S. Sturm (2018). Arbitrage-free XVA. *Mathematical Finance 28*(2), 582–620.

Burgard, C. and M. Kjaer (2011). In the balance. *Risk Magazine*, October 72–75.

Burgard, C. and M. Kjaer (2013). Funding costs, funding strategies. *Risk Magazine*, December 82–87. Preprint version available at https://ssrn.com/abstract=2027195.

Burgard, C. and M. Kjaer (2017). Derivatives funding, netting and accounting. *Risk Magazine*, March 100–104. Preprint version available at https://ssrn.com/abstract=2534011.

Castagna, A. (2012). Yes, FVA is a cost for derivatives desks: A note on Is FVA a cost for derivatives desks? by Prof. Hull and Prof. White. *Available at: SSRN*.

Castagna, A. (2013). Funding valuation adjustment (FVA) and theory of the firm: A theoretical justification of the inclusion of funding costs in the evaluation of financial contracts. *Available at: SSRN*.

Castagna, A. (2014). Towards a theory of internal valuation and transfer pricing of products in a bank: Funding, credit risk and economic capital. Available at http://ssrn.com/abstract=2392772.

Collin-Dufresne, P., R. Goldstein, and J. Hugonnier (2004). A general formula for valuing defaultable securities. *Econometrica 72*(5), 1377–1407.

Crépey, S. (2015). Bilateral counterparty risk under funding constraints. Part I: Pricing, followed by Part II: CVA. *Mathematical Finance 25*(1), 1–22 and 23–50. First published online on 12 December 2012.

Crépey, S., T. R. Bielecki, and D. Brigo (2014). *Counterparty Risk and Funding: A Tale of Two Puzzles.* Chapman & Hall/CRC Financial Mathematics Series.

Crépey, S., R. Élie, W. Sabbagh, and S. Song (2019). When capital is a funding source: The XVA Anticipated BSDEs. Working paper available on https://math.maths.univ-evry.fr/crepey.

Crépey, S. and S. Song (2016). Counterparty risk and funding: Immersion and beyond. *Finance and Stochastics 20*(4), 901–930.

Crépey, S. and S. Song (2017a). Invariance properties in the dynamic Gaussian copula model. *ESAIM: Proceedings and Surveys 56*, 22–41. Preprint on https://math.maths.univ-evry.fr/crepey.

Crépey, S. and S. Song (2017b). Invariance times. *The Annals of Probability 45*(6B), 4632–4674.

Crépey, S. and S. Song (2018). Invariance times transfer properties. Working paper available on https://math.maths.univ-evry.fr/crepey.

Föllmer, H. and A. Schied (2016). *Stochastic Finance: An Introduction in Discrete Time* (4th ed.). De Gruyter Graduate.

Green, A., C. Kenyon, and C. Dennis (2014). KVA: capital valuation adjustment by replication. *Risk Magazine*, December 82–87. Preprint version "KVA: capital valuation adjustment" available at ssrn.2400324.

Kruse, T. and A. Popier (2016). BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration. *Stochastics: An International Journal of Probability and Stochastic Processes 88*(4), 491–539.

Salzmann, R. and M. Wüthrich (2010). Cost-of-capital margin for a general insurance liability runoff. *ASTIN Bulletin 40*(2), 415–451.

Schönbucher, P. (2004). A measure of survival. *Risk Magazine 17*(8), 79–85.

Song, S. (2016). Local martingale deflators for asset processes stopped at a default time $s^{\mathfrak{t}}$ or just before $s^{\mathfrak{t}-}$. arXiv:1405.4474v4.

Swiss Federal Office of Private Insurance (2006). Technical document on the Swiss solvency test. https://www.finma.ch/FinmaArchiv/bpv/download/e/SST_techDok_061002_E_wo_Li_20070118.pdf.