Bilateral Counterparty Risk under Funding Constraints – Part I: Pricing.

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Abstract

This and the follow-up paper deal with the valuation and hedging of bilateral counterparty risk on OTC derivatives. Our study is done in a multiple-curve setup reflecting the various funding constraints (or costs) involved, allowing one to investigate the question of interaction between bilateral counterparty risk and funding.

The first task is to define a suitable notion of no arbitrage price in the presence of various funding costs. This is the object of this paper, where we develop an "additive, multiple curve" extension of the classical "multiplicative (discounted), one curve" risk-neutral pricing approach. We derive the dynamic hedging interpretation of such an "additive risk-neutral" price, starting by consistency with pricing by replication in the case of a complete market. This is illustrated by a completely solved example building over previous work by Burgard and Kjaer.

Keywords: Counterparty Risk, Funding Costs, Nonlinear Pricing and Hedging, Arbitrage, Backward Stochastic Differential Equation (BSDE).

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1 Introduction

Counterparty risk is the risk of either party defaulting in an OTC derivative contract (or portfolio of contracts). This is the native form of credit risk, which affects any OTC transaction between two parties. Early treatments of counterparty risk can be found for instance in Duffie and Huang (1996) or Bielecki and Rutkowski (2002, Chapter 14). See also the collection of papers in Pykhtin (2005), and see Gregory (2009) or Cesari et al. (2010) for practically oriented presentations in book form. The interest in counterparty risk, along with counterparty risk itself, has exploded since the credit crisis of 2007-09, when it was realized that the resilience of a bank to a major financial turmoil, is largely determined by its ability to properly value and hedge this risk. With the sovereign debt crisis the issue is more topical than ever.

We shall deal in this paper with valuation and hedging of a generic contract, to be understood in practice as a portfolio of OTC derivatives, between two defaultable counterparties. These two parties, which will be referred to as "the bank" and "the investor", are tied by a legal agreement, the Credit Support Annex (CSA), prescribing the collateralization scheme and the close-out cash-flow in case of default of either party. The aim of a CSA is to mitigate counterparty risk. Collateral means cash or various possible eligible securities posted through margin calls as default guarantee by the two parties. The CSA close-out cash-flow is the terminal cash-flow, including the accumulated collateral at that time, to occur in case of default of either party.

A counterparty risk related issue, especially when dealing with bilateral counterparty risk, is a proper accounting for the costs and benefits of funding one's position into the contract. From the perspective of say the bank (and symmetrically so for the investor), this lets a third party enter the scene, namely the funder of the position of the bank. This gives rise to another close-out cash-flow, from the external funder to the bank, at time of the bank's default (coming as a funding benefit of the bank upon her own default). Interaction between the pricing, the hedging and the funding problems, has become a major topic of concern for practitioners, reflected for instance in Piterbarg (2010), Morini and Prampolini (2010), (Burgard and Kjaer 2011a; Burgard and Kjaer 2011b) or Castagna (2011).

The first task is to define a suitable notion on no arbitrage price in the presence of various funding costs. This is the object of this paper, where we develop an "additive, multiple curve" extension of the classical "multiplicative (discounted), one curve" risk-neutral pricing approach.

1.1 Outline of the Paper

In Section 2, we define our market model of the contract between the bank and the investor, its hedging assets and its funding assets. In Section 3 we characterize the hedging error arising from a given pricing and hedging scheme, accounting in particular for the funding cash-flows. Given potential nonlinearities in the funding cash-flows, it is not possible to get rid of funding costs through discounting as in a classical one-curve setup. In Section 4, the cash-flows are priced instead under an "additive, flat" extension of the classical "multiplicative, discounted" risk-neutral assumption. We also derive the dynamic hedging interpretation of this "additive risk-neutral" price. Section 5 provides an illustrative example, discussing in the additive martingale pricing perspective of this paper the situation in a complete Black-Scholes market which is considered in (Burgard and Kjaer 2011a;

Burgard and Kjaer 2011b).

The take-away message of this paper is that for properly valuing and hedging (bilateral in particular) counterparty risk in a multiple-curve setup reflecting the presence of various funding costs, it is necessary to focus on a party of interest, say the bank, and to consider the "system" consisting of the bank, the investor and the funder of the bank. One must also have a clear view of the three equally important pillars of the bank's position consisting of the contract itself, its hedging portfolio and its funding portfolio (as opposed to getting rid of the funding component of the position by risk-free discounting in a classical one-curve setup).

2 Market Model

This Section specifies a model of a contract (generic CSA portfolio of OTC derivatives with time horizon T between the bank and the investor), its hedging assets and its funding assets. Let $(\Omega, \mathcal{G}_T, \mathcal{G})$ stand for a filtered space with a finite horizon T, which is used throughout the paper for describing the evolution of a financial market model. The filtration $\mathcal{G}$ as well as any other filtration in this or the follow-up paper, are assumed to satisfy the usual conditions. All random variables are $\mathcal{G}_T$-measurable. All random times are $[0,T] \cup \{\infty\}$-valued $\mathcal{G}$-stopping times. All processes are defined over $[0,T]$ and $\mathcal{G}$-adapted. We endow the measurable space $(\Omega, \mathcal{G}_T)$ with a probability measure $\mathbb{P}$, which is fixed throughout the paper, and will later be interpreted as a martingale pricing measure in some sense. We assume in particular that $\mathbb{P}$ is equivalent to the historical probability measure $\widehat{\mathbb{P}}$ over $(\Omega, \mathcal{G}_T)$. We denote by $\mathbb{E}_t$ the conditional expectation given $\mathcal{G}_t$. All cash-flows that appear in the paper are assumed to be $\mathbb{P}$-integrable. By default, all price and value processes (including the collateral process Γ) are assumed to be semimartingales, and all semimartingales (including finite variation processes) are taken in a càdlàg version; "martingale" simply means local martingale; all inequalities between random quantities are to be understood $d\mathbb{P}$-almost surely or $dt \otimes d\mathbb{P}$-almost everywhere, as suitable.

We denote by θ and $\overline{\theta}$ the default times of the bank and the investor, in the sense of the times at which promised dividends and margin calls, cease to be paid by a distressed party. We assume that θ and $\overline{\theta}$ cannot occur jointly at fixed (constant) times, which is for instance satisfied in all intensity models of credit risk. Note that this does not preclude the possibility of θ and $\overline{\theta}$ jointly occurring at some stopping time, as it is for instance the case in a Marshall Olkin model of two default times (see Bielecki et al. (2012) or Morini (2011)). We denote

$$\tau = \theta \wedge \overline{\theta}\,, \;\; \bar{\tau} = \tau \wedge T,$$

where $\bar{\tau}$ represents the effective time horizon of our problem, since there will be no cash-flows after it.

Remark 2.1 (Unilateral or Bilateral Counterparty Risk?) In principle the possibility of one's own default should be accounted for by a suitable correction, actually standing as a benefit, to the value of the contract. This benefit is the so-called Debt Valuation Adjustment (DVA), see for instance Brigo and Capponi (2010). There is a debate among practitioners however regarding the relevance of accounting for one's own credit risk as a benefit through bilateral counterparty risk valuation. The point is that since selling protection on oneself is hardly doable in practice (and typically illegal), it is not really possible to hedge one's own credit risk. It is recent news that Goldman hedges DVA risk through peers. But while this may take partly care of spread hedging, it does not hedge the jump to default risk. To hedge jump to default risk one may try to repurchase one's own bond (see Section 5 for an example), but this is not practical in general. Given that hedging bilateral counterparty risk is difficult, the principle of risk-neutral valuation of bilateral counterparty risk is thus questionable.

But the practical justification for using a model of bilateral counterparty risk is that unilateral valuation of counterparty risk induces a significant gap between the CVAs computed by the two parties. This implies that it is difficult to agree on a CSA on the basis of unilateral counterparty risk valuations.

If in the end one does not want to account for bilateral counterparty risk, one simply considers a model of unilateral counterparty risk, which corresponds in our formalism to letting $\theta = \infty$ everywhere below (for unilateral counterparty risk from the perspective of the bank).

We let D represent the clean or promised cumulative dividend process of the contract, assumed to be of finite variation. A promised dividend dD_t is only effectively paid if none of the parties defaulted by time t, resulting in the effective dividend process $dC_t = \mathbb{1}_{t<\tau} dD_t$.

In order to mitigate counterparty risk, the contract is collateralized. Collateral consists of cash or various possible eligible securities posted through CSA regulated margin calls as default guarantee by the two parties. We model collateral by means of an algebraic margin amount Γ_τ passing from the bank to the investor at time $\tau < T$. So, before τ, a positive Γ_t represents an amount "lent" by the bank to the investor (and remunerated as such by the investor), but devoted to become the property of the investor in case of default of either party at time τ (if $< T$). Symmetrically, before τ, a positive $(-\Gamma_t)$ represents an amount "lent" by the investor to the bank (and remunerated as such by the bank), but devoted to become the property of the bank in case of default of either party at time $\tau < T$.

There is also a CSA close-out cash-flow $\mathbb{1}_{\tau<T} R^i$ from the bank to the investor at time of default $\tau < T$, in which R^i is a $\mathcal{G}_\tau$-measurable random variable which will be specified in the follow-up paper.

We shall focus on the bank shortening the contract to the investor under the rules of a given CSA, and setting up a related hedge. By the bank shortening the contract to the investor we mean that all the cash-flows of the contract are paid by the bank. This is conventional however since promised cash-flows are algebraic. For instance $\Delta D_t = \pm 1$ means a bullet cash-flow of ± 1 "from" the bank to the investor at time t.

We call external funder (or funder for short) a generic third-party[1] insuring funding of the position of the bank. External here stands in opposition to the internal sources of funding provided to the bank via the remuneration of the swapped component of her hedge, or via the remuneration of the margin amount (see Subsections 2.1 and 2.2).

Remark 2.2 For simplicity we assume the external funder to be default-free. Note that this hinders a direct application of our framework to the study of systemic risk.

In the context of this paper where the focus is on counterparty risk, recoveries upon default are more conveniently excluded from dividends and accounted for separately as boundary conditions. We shall thus distinguish two categories of related cash-flows:

[1] Possibly composed in practice of several entities and/or devices.

- Dividends, in the sense of all pre-default cash-flows involving the bank, decomposing into:
  - Counterparty clean or promised contract dividends;
  - Gains on the hedging instruments before time τ;
  - The dt-cost/benefit of funding the position/investing into it;
    * This includes in particular the remuneration of the collateral;
- Close-out cash-flows, meaning cash-flows at the default time τ (if $< T$), consisting of:
  - The CSA close-out cash-flow, or recovery on the contract paid by the bank to the investor upon default of either party;
    * This includes in particular the delivery of the collateral;
  - A close-out funding cash flow from the funder to the bank in case of a default of the bank.

Remark 2.3 Apart from the promised dividends of the contract and the remuneration of the collateral, which are exchanged between the two parties, all other cash-flows differ between them. This induces an asymmetry between the parties, to the consequence that the value of the contract will not be the same from their perspectives. This is why we need to focus on a given party, the one conventionally called "the bank" in this paper. Of course symmetrical considerations apply to "the investor", but with non-symmetrical hedging positions and funding conditions.

The fact that the economic value (in a sense of cost of hedging as we shall see below) of a deal is different to the two parties, poses the practical problem of agreeing on a price between them. Asymmetry in economic value is as factual as for instance market incompleteness in a real world. One thus has to live with it and find trading agreements in this context (still better knowing it than ignoring it). One might claim that asymmetry is only on the funding side, not on the (bilateral) counterparty risk side, so that "maybe at first order" the asymmetry is not so big. This would need to be checked numerically however. Moreover the point of this paper is that counterparty risk and funding are tied together. So even the view that symmetry is at least there counterparty risk side, is a simplistic one.

2.1 Hedging Assets

Let $\mathcal{P}$ denote the $\mathbb{R}^d$-valued semimartingale price process of a family of hedging assets, and let $\mathcal{C}$ stand for the corresponding $\mathbb{R}^d$-valued cumulative effective dividend process. The finite variation dividend process $\mathcal{C}$ represents all the cash-flows that are paid to a holder of a buy-and-hold position in the hedging assets until time $\bar{\tau}$.

An hedging asset can be traded either in swapped form, at no upfront payment, or (at least for a physical asset as opposed to a natively swapped primary asset, see below) directly on a primary market. Hedging assets traded in swapped form include (counterparty risk clean) CDS on the two parties which are typically used for hedging the counterparty jump-to-default exposure of the contract. Note that a fixed CDS (of a given contractual spread in particular) cannot be traded dynamically in the market. Indeed, only freshly emitted CDS can be entered into, at no cost and at the related fair market spread, at a given time. What is used in practice for hedging corresponds to the concept of a rolling CDS, formally introduced in Bielecki et al. (2008),[2] which is essentially a self-financing trading strategy in market CDS. So, much like as in futures contracts, the value of a rolling CDS is null at any point in time, yet due to the trading gains of the strategy, the related cumulative value process is not zero. The case of hedging assets traded in swapped form also covers the situation of a physical (as opposed to natively swapped) hedging asset traded via a repo market.

We assume in this paper that every hedging asset can be traded in swapped form, either as a natively swapped instrument rolled over time, or, for a physical asset, via a corresponding repo market. In mathematical terms, trading the hedging asset with price $\mathcal{P}^i_t$ in swapped form effectively means than one uses, instead of the original (physical or fixed swap) asset, a synthetic asset with price process $\mathcal{S}^i_t = 0$ and gain process given by

$$d\mathcal{P}^i_t - \big(r_t\mathcal{P}^i_t + c^i_t\big)dt + d\mathcal{C}^i_t, \tag{2.1}$$

where:

- In case of a physical primary asset traded via a repo market, the basis c^i corresponds to the so-called repo basis; the meaning of all other terms in (2.1) is clear;
- In case of a natively swapped asset rolled over time, the different terms in (2.1) are to be understood as[3]

$$d\mathcal{P}^i_t = d\bar{\mathcal{P}}^{i,t_0}_t\,|_{t_0=t}\ ,\ \ \mathcal{P}^i_t = \bar{\mathcal{P}}^{i,t}_t = \mathcal{S}^i_t = 0\,,\ \ c^i = 0\,,\ \ d\mathcal{C}^i_t = d\bar{\mathcal{C}}^{i,t_0}_t\,|_{t_0=t} \tag{2.2}$$

  where $(\bar{\mathcal{P}}^{i,t_0}_t)_{t\geq t_0}$ is the price process at time t of the corresponding fixed (as opposed to rolled) swap emitted at time $t_0 \leq t$, with dividend process $(\bar{\mathcal{C}}^{i,t_0}_t)_{t\geq t_0}$.

All the dt-funding costs in this paper are expressed in terms of a basis, like a repo basis c^i in (2.1), to the risk-free rate r_t.

2.2 Funding Assets

This Subsection provides a comprehensive specification of funding cash-flows. The corresponding notion of a self-financing trading strategy will be derived in Subsection 3.1. A general formulation of the pricing and hedging problem under abstract funding constraints will then be given in Subsection 3.2.

We assume that the bank can lend money to (respectively borrow money from) its external funder at an excess cost over the risk-free rate r_t determined by a funding credit and/or liquidity basis λ (respectively $\bar{\lambda}$). In case the bank is indebted to its (default-free) funder at time $\tau = \theta < T$, the bank will in principle not be in a position to reimburse the totality of its external debt, but only a fraction $\mathfrak{r}$ of it, where a $[0,1]$-valued $\mathcal{G}_\theta$-measurable random variable $\mathfrak{r}$ represents the recovery rate of the bank towards its external funder. This results as we shall see below in a close-out funding cash-flow, proportional to $(1-\mathfrak{r})$, from the external funder to the bank in case $\tau = \theta < T$. This cash-flow corresponds to the funding side of "the bank benefiting from her own default". In case $\mathfrak{r} < 1$ the bank defaults at time θ not only on its commitments in the contract with regard to the investor, but also on its related funding debt. The case $\mathfrak{r} = 1$ can be seen as a partial default in which at time θ the bank only defaults on its contractual commitments with regard to the investor, but

[2] See also the related concept of Floating Rate CDS in Brigo (2005).

[3] See Bielecki et al. (2008) and Bielecki et al. (2011) for more details.

not on its funding debt with respect to its funder. It can be used for modeling the situation of a bank in a global net lender position, so that it actually does not need any external lender (in case cash is needed for funding its position, the bank simply uses its own cash). Also note that in case of unilateral counterparty risk (from the perspective of the bank, so $\theta = \infty$ almost surely) the value of $\mathfrak{r}$ is immaterial; by convention in this case one shall let $\mathfrak{r} = 1$ almost surely.

Regarding the collateral, we restrict ourselves for simplicity to collateral posted as cash. We follow the most common CSA covenant under which the party getting the collateral can use it in its trading, as opposed to a covenant where collateral is segregated by a third party in order to avoid the so-called re-hypothecation risk (see Bielecki and Crépey (2011)). Specific CSA rates $r_t + b_t$ and $r_t + \bar{b}_t$, where b and $\bar{b}$ stand for related bases, are then typically used to remunerate the collateral owned by either party. This results in a dt-remuneration of the margin amount which is worth

$$(r_t + b_t)\Gamma_t^+ dt - (r_t + \bar{b}_t)\Gamma_t^- dt = r_t\Gamma_t dt + \big(b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^-\big)dt$$

to the bank, and the opposite to the investor.

Regarding funding of the hedging instruments, we suppose that a hedging position is either entirely swapped, or funded in totality by the external lender, and that this choice is given and fixed once for all at time 0 for every hedging instrument. We let a superscript ^{s} refer to the subset of the hedging instruments traded in swapped form, and $^{\bar{s}}$ refer to the subset, complement of ^{s}, of (physical) hedging instruments which are traded directly on a primary market (and are therefore funded together with the contract by the external funder).

Remark 2.4 In the above specification it is assumed for simplicity that the funding cost is not deal specific but always the same, distinguishing only the direction of the flow (funding or investment). A different model of treasury, more on the micro scale, would allocate different funding to deals with different characteristics.

In order to account for the above funding specification in a classical formalism of self-financing trading strategies, we introduce the following funding assets on $[0, \bar{\tau}]$ (with all initial conditions set to one):

- Two collateral funding assets, B^0 and $\bar{B}^0$, evolving as

$$dB_t^0 = (r_t + b_t)B_t^0 dt\,,\;\; d\bar{B}_t^0 = (r_t + \bar{b}_t)\bar{B}_t^0 dt, \tag{2.3}$$

  dedicated to the funding of the positive and the negative part of the margin account,

- Two external funding assets, B^f and $\bar{B}^f$, evolving as

$$dB_t^f = (r_t + \lambda_t)B_t^f dt\,,\;\; d\bar{B}_t^f = (r_t + \bar{\lambda}_t)\bar{B}_t^f dt - (1 - \mathfrak{r})\bar{B}_{t-}^f \delta_\theta(dt) \tag{2.4}$$

  where the symbol δ denotes a Dirac measure; these are the investing and funding assets of the bank by its external lender.

Remark 2.5 By funding assets we mean riskless, finite variation assets which are used for funding a position. In the classical one-curve setup with risk-free rate r_t, there is only one funding asset, the so-called savings account, growing at rate r_t. The savings account is thus the inverse of the risk-free discount factor $\beta_t = e^{-\int_0^t r_s ds}$. In this paper, we do not postulate the existence of the savings account. The risk-free rate r_t simply corresponds to the time-value of money, and one can only think of β_t^{-1} as a "fictitious" savings account. What we have instead is the coexistence of various funding assets with different growth rates in the economy. This raises the question of arbitrage that might result from trading between these rates. These can simply reflect different levels of credit-riskiness, so that a related arbitrage opportunity is only a pre-default view, disregarding losses-upon-defaults. Even without credit risk, different funding rates may consistently coexist in an economy, reflecting trading constraints, or, in other words, liquidity funding costs. The rationale here is that a given funding rate may be only accessible for a definite notional and for a specific purpose, so that funding arbitrage strategies are either not possible, or not sought for by the parties. An example that arises in the context of counterparty risk is that of the collateral, in which the two parties must have a contractually prescribed amount ($\Gamma_t^{\pm}$ below) at any point in time. The question of the co-existence of no-arbitrage with several funding assets is work in progress in Bielecki et al. (2011).

3 Trading Strategies

The task of the bank shortening the contract to the investor, consists in devising a price and a dynamic hedging portfolio for the contract sold to the investor, whilst getting funded by its external lender. In this Section we characterize the hedging error arising from a given pricing and hedging scheme, detailing in particular the funding cash-flows.

A hedge process is defined as a left-continuous (see Remark 3.1) and locally bounded, $\mathbb{R}^d$-valued row-vector process ζ over $[0, \bar{\tau}]$, representing the number of units of the hedging assets which are held in the hedging portfolio. By price-and-hedge of the contract for the bank shortening it to the investor, we mean any pair-process $(\bar{\Pi}, \zeta)$ over $[0, \bar{\tau}]$, where ζ is a hedge process and $\bar{\Pi}$ is an $\mathbb{R}$-valued semimartingale such that $\bar{\Pi}_{\bar{\tau}} = \mathbb{1}_{\tau < T} R^i$ (the CSA close-out cash-flow). Accordingly, by hedging error process of the price-and-hedge $(\bar{\Pi}, \zeta)$, we mean $\varrho = \bar{\Pi} - \bar{\mathcal{W}}$, where $\bar{\mathcal{W}}$ is the value process of the collateralization, hedging and funding portfolio. Replication corresponds to $\varrho_{\bar{\tau}} = 0$ almost surely.

3.1 Self-Financing Condition

The position of the bank being funded as described in Subsection 2.2, one has for $t \in [0, \bar{\tau}]$,

$$\bar{\mathcal{W}}_t = \left(\Gamma_t^+ - \Gamma_t^-\right) + \left(\zeta_t^s \mathcal{S}_t^s + \zeta_t^{\bar{s}} \mathcal{P}_t^{\bar{s}}\right) + \left((\bar{\mathcal{W}}_t - \Gamma_t - \zeta_t^{\bar{s}} \mathcal{P}_t^{\bar{s}})^+ - (\bar{\mathcal{W}}_t - \Gamma_t - \zeta_t^{\bar{s}} \mathcal{P}_t^{\bar{s}})^-\right) \quad (3.1)$$

where the three terms in the right-hand side correspond to the amounts respectively invested as collateral, into the hedging assets (swapped and non swapped components ζ^s and $\zeta^{\bar{s}}$, see the explanations surrounding Equation (2.1)) and into the external funding assets. Note $\mathcal{S}_t = 0$, so a hedging instrument traded in swapped form does not directly contribute to the value $\bar{\mathcal{W}}_t$. However it contributes as we shall see below to the dynamics of $\bar{\mathcal{W}}_t$, via the related gain process in (2.1). Equivalently to (3.1), one can put in a more formal notation

$$\bar{\mathcal{W}}_t = \eta_t^0 B_t^0 + \bar{\eta}_t^0 \bar{B}_t^0 + \zeta_t^s \mathcal{S}_t^s + \zeta_t^{\bar{s}} \mathcal{P}_t^{\bar{s}} + \eta_t^f B_t^f + \bar{\eta}_t^f \bar{B}_t^f \quad (3.2)$$

with

$$\eta_t^0 = \frac{\Gamma_t^+}{B_t^0}, \quad \bar{\eta}_t^0 = -\frac{\Gamma_t^-}{\bar{B}_t^0}, \quad \eta_t^f = \frac{(\bar{\mathcal{W}}_t - \Gamma_t - \zeta_t^{\bar{s}} \mathcal{P}_t^{\bar{s}})^+}{B_t^f}, \quad \bar{\eta}_t^f = -\frac{(\bar{\mathcal{W}}_t - \Gamma_t - \zeta_t^{\bar{s}} \mathcal{P}_t^{\bar{s}})^-}{\bar{B}_t^f}$$

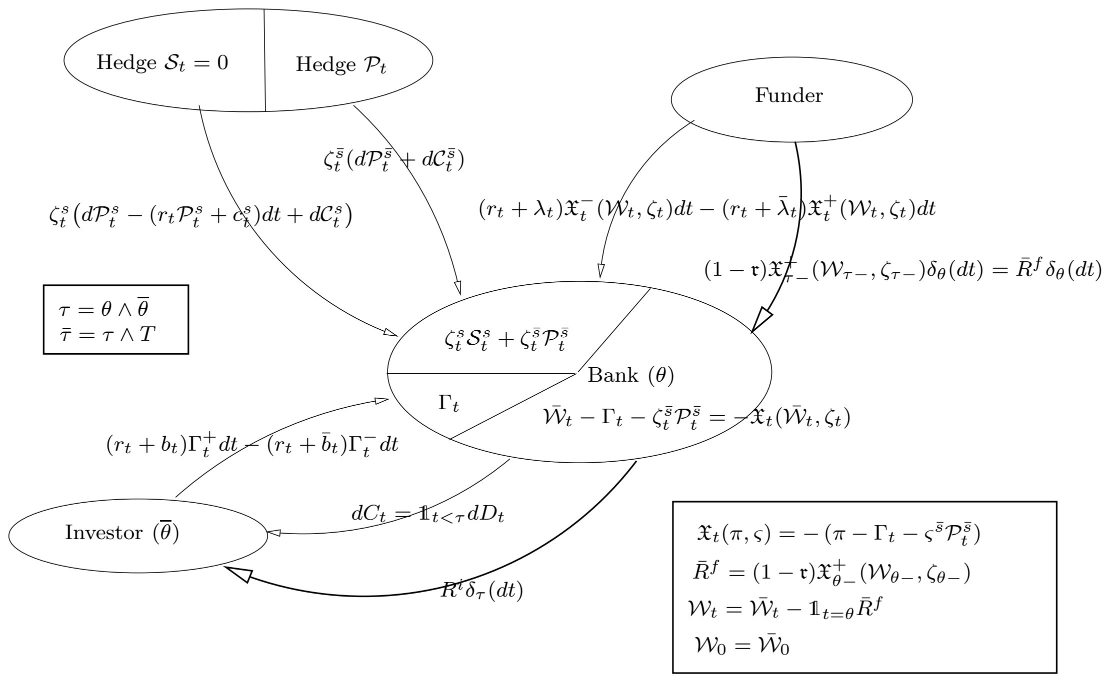

Figure 1: Cash-flows of the bank over $[0, \bar{\tau}]$.

Following a standard terminology, we then say in view of (3.2) that a price-and-hedge $(\bar{\Pi}, \zeta)$ of the bank is self-financing if and only if $\bar{\mathcal{W}}_0 = \bar{\Pi}_0$ and for $t \in [0, \bar{\tau}]$

$$\begin{aligned} d\bar{\mathcal{W}}_t = &-dC_t + \eta_t^0 dB_t^0 + \bar{\eta}_t^0 d\bar{B}_t^0 + \\ &\zeta_t^s\big(d\mathcal{P}_t^s - (r_t\mathcal{P}_t^s + c_t^s)dt + d\mathcal{C}_t^s\big) + \zeta_t^{\bar{s}}(d\mathcal{P}_t^{\bar{s}} + d\mathcal{C}_t^{\bar{s}}) + \eta_t^f dB_t^f + \bar{\eta}_{t-}^f d\bar{B}_t^f \end{aligned} \tag{3.3}$$

where the "minus" in $\bar{\eta}_{t-}^f$ is needed[4] because $\bar{B}_t^f$ jumps at time θ (and process $\bar{\eta}^f$ is not predictable).

Remark 3.1 Being able to take a left-limit in $\bar{\eta}^f$ is the reason why we restrict ourselves to left-continuous hedges ζ, as opposed to predictable hedges in general in the hedging literature. This restriction does not harm in practice (see for example the hedges found in Section 5 of this paper or in Section 4 of the follow-up paper).

Figure 1 provides a graphical representation of all the cash-flows over $[0, \bar{\tau}]$. We denote for every real number π and $\mathbb{R}^d$-valued row-vector ς

$$\begin{aligned} &\mathfrak{X}_t(\pi, \varsigma) = -\big(\pi - \Gamma_t - \varsigma^{\bar{s}}\mathcal{P}_t^{\bar{s}}\big) \\ &f_t(\pi, \varsigma) = b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^- + \lambda_t\big(\pi - \Gamma_t - \varsigma^{\bar{s}}\mathcal{P}_t^{\bar{s}}\big)^+ - \bar{\lambda}_t\big(\pi - \Gamma_t - \varsigma^{\bar{s}}\mathcal{P}_t^{\bar{s}}\big)^- - \varsigma^s c_t^s \end{aligned} \tag{3.4}$$

where $\mathfrak{X}_{t-}(\bar{\mathcal{W}}_{t-}, \zeta_{t-})$ will be interpreted as the (algebraic) debt of the bank towards its external funder at time t, and $f_t(\bar{\mathcal{W}}_t, \zeta_t)$ as the dt-excess-funding-benefit of the bank. Let finally for $t \in [0, \bar{\tau}]$

$$\Pi_t^* = \bar{\Pi}_t - \mathbb{1}_{t \geq \theta}\bar{R}^f \,, \;\; \mathcal{W}_t = \bar{\mathcal{W}}_t - \mathbb{1}_{t \geq \theta}\bar{R}^f \tag{3.5}$$

[4]We thank Marek Rutkowski for pointing this out as well as for a significant clarification of this section of the paper.

where $\bar{R}^f := (1-\mathfrak{r})\mathfrak{X}^+_{\theta-}(\mathcal{W}_{\theta-},\zeta_{\theta-})$ will appear in the last line of (3.7) as the close-out cash-flow from the external funder to the bank at time $\tau=\theta<T$.

Proposition 3.1 *Under the funding specification of Subsection 2.2, a price-and-hedge* $(\bar{\Pi},\zeta)$ *is self-financing if and only if* $\mathcal{W}_0=\Pi^*_0(=\bar{\Pi}_0)$ *and for* $t\in[0,\bar{\tau}]$

$$d\mathcal{W}_t=-dC_t+\big(r_t\mathcal{W}_t+f_t(\mathcal{W}_t,\zeta_t)\big)dt+\zeta_t(d\mathcal{P}_t-r_t\mathcal{P}_tdt+d\mathcal{C}_t). \tag{3.6}$$

Proof. Plugging (2.3)-(2.4) into (3.3) and using also the current specification of the funding policy regarding hedging assets, yields that the strategy is self-financing if and only if for $t\in[0,\bar{\tau}]$

$$\begin{aligned}
d\bar{\mathcal{W}}_t=&-dC_t+(r_t+b_t)\Gamma^+_tdt-(r_t+\bar{b}_t)\Gamma^-_tdt+\zeta_t(d\mathcal{P}_t+d\mathcal{C}_t)-\zeta^s\left(r_t\mathcal{P}^s_t+c^s_t\right)dt\\
&+(r_t+\lambda_t)\big(\bar{\mathcal{W}}_t-\Gamma_t-\zeta^{\bar{s}}_t\mathcal{P}^{\bar{s}}_t\big)^+dt-(r_t+\bar{\lambda}_t)\big(\bar{\mathcal{W}}_t-\Gamma_t-\zeta^{\bar{s}}_t\mathcal{P}^{\bar{s}}_t\big)^-dt\\
&-\bar{\eta}^f_{\tau-}(1-\mathfrak{r})\bar{B}^f_{\tau-}\delta_\theta(dt)\\
=&-dC_t+r_t(\bar{\mathcal{W}}_t-\zeta_t\mathcal{P}_t)dt+\zeta_t(d\mathcal{P}_t+d\mathcal{C}_t)+b_t\Gamma^+_tdt-\bar{b}_t\Gamma^-_tdt-\zeta^sc^s_tdt\\
&+\lambda_t\big(\bar{\mathcal{W}}_t-\Gamma_t-\zeta^{\bar{s}}_t\mathcal{P}^{\bar{s}}_t\big)^+dt-\bar{\lambda}_t\big(\bar{\mathcal{W}}_t-\Gamma_t-\zeta^{\bar{s}}_t\mathcal{P}^{\bar{s}}_t\big)^-dt\\
&+(1-\mathfrak{r})\left(\bar{\mathcal{W}}_{\tau-}-\Gamma_{\tau-}-\zeta^{\bar{s}}_{\tau-}\mathcal{P}^{\bar{s}}_{\tau-}\right)^-\delta_\theta(dt)\\
=&-dC_t+r_t\bar{\mathcal{W}}_tdt+\zeta_t(d\mathcal{P}_t-r_t\mathcal{P}_tdt+d\mathcal{C}_t)+f_t(\bar{\mathcal{W}}_t,\zeta_t)dt+\bar{R}^f\delta_\theta(dt).
\end{aligned} \tag{3.7}$$

□

3.2 General Price-and-Hedge

As illustrated in Subsection 2.2, the exact nature of the funding cash-flows depends on the specification of a funding policy defined in terms of related funding assets. For the sake of clarity and generality one shall work henceforth with the following abstract definition of a (self-financing) price-and-hedge, in which the funding component of the hedging portfolio only shows up through an abstract dt-excess-benefit-funding coefficient $f_t(\pi,\varsigma)$ and a funding close-out cash-flow $(1-\mathfrak{r})\mathfrak{X}^+_t(\pi,\varsigma)$, without explicit reference to specific funding assets.[5] In this expression, a $\mathcal{G}_\theta$-measurable random variable $\mathfrak{r}$ represents as before a recovery rate of the bank towards an external funder (assumed risk-free), the case $\mathfrak{r}=1$ still covering by convention the situation of unilateral counterparty risk with $\theta=\infty$; $\mathfrak{X}$ represents an abstract debt function of the bank to its external funder.

The following definition of a general price-and-hedge is put in the form of a forward-backward stochastic differential equation (FBSDE, see Ma and Yong (2007)) in $(\mathcal{W},\Pi^*,\zeta,\varrho)$, where ϱ equals $\Pi^*-\mathcal{W}$ (the hedging error) in view of the second line in (3.8). What solving this FBSDE would mean is solving the related control problem, that is finding a general price-and-hedge $(\bar{\Pi},\zeta)$ such that the related process ϱ has "nice" properties in terms of arbitrage (typically: ϱ being a martingale under some equivalent probability measure) and replication (typically: ϱ being small in some appropriate norm). This would be a fairly non-standard FBSDE however, accounting in particular for the "forward-backward initial condition" $\mathcal{W}_0=\Pi^*_0$ of $\mathcal{W}$ (see Horst et al. (2011) for related technical issues). But we shall not try to solve the problem in this form, rather introducing soon a more tractable backward stochastic differential equation (BSDE).

[5] There are also the internal sources of funding, measured by the data $b,\bar{b}$ and c, provided to the bank by the remuneration of the collateral and of the swapped component of her hedge.

Definition 3.1 (General Price-and-Hedge) Let semimartingales $\mathcal{W}, \Pi^*$, ϱ and a hedge ζ form a quadruplet $(\mathcal{W}, \Pi^*, \zeta, \varrho)$ satisfying the initial conditions $\mathcal{W}_0 = \Pi^*_0$, $\varrho_0 = 0$, and such that for $t \in [0, \bar{\tau}]$

$$\begin{aligned} d\mathcal{W}_t &= -dC_t + \big(r_t\mathcal{W}_t + f_t(\mathcal{W}_t, \zeta_t)\big)dt + \zeta_t(d\mathcal{P}_t - r_t\mathcal{P}_t dt + d\mathcal{C}_t) \\ d\Pi^*_t &= -dC_t + \big(r_t\mathcal{W}_t + f_t(\mathcal{W}_t, \zeta_t)\big)dt + \zeta_t(d\mathcal{P}_t - r_t\mathcal{P}_t dt + d\mathcal{C}_t) + d\varrho_t \end{aligned} \tag{3.8}$$

along with a terminal condition $\Pi^*_{\bar{\tau}} = \mathbb{1}_{\tau<T}\bar{R}$ where

$$\bar{R} = R^i - \mathbb{1}_{\tau=\theta}\bar{R}^f \tag{3.9}$$

in which $\bar{R}^f := (1-\mathfrak{r})\mathfrak{X}^+_{\tau-}(\mathcal{W}_{\tau-}, \zeta_{\tau-})$.

One then calls general price-and-hedge with hedging error ϱ, the pair-process $(\bar{\Pi}, \zeta)$ where for $t \in [0, \bar{\tau}]$

$$\bar{\Pi}_t := \Pi^*_t + \mathbb{1}_{t\geq\theta}\bar{R}^f.$$

We say that $(\bar{\Pi}, \zeta)$ is a replicating strategy if $\varrho_{\bar{\tau}} = 0$ almost surely.

Observe that $\bar{R}$ represents the total close-out cash-flow delivered by the bank at time $\tau < T$ (CSA close-out cash flow R^i paid to the investor minus close-out funding cash-flow $\mathbb{1}_{\tau=\theta}\bar{R}^f$ got from the external funder). Also note that under the funding specification of Subsection 2.2, Definition 3.1 is consistent with the developments of Subsection 3.1. In the abstract Definition 3.1 we focus on processes Π^* and $\mathcal{W}$ rather on $\bar{\Pi}$ and $\bar{\mathcal{W}}$ that concurrently showed-up in the specific setup of Subsections 2.2 and 3.1, because Π^* (actually, ultimately Π to be introduced in Definition 4.1 below) and $\mathcal{W}$ will be more convenient mathematically. By a slight abuse of terminology we call $\mathcal{W}$ the value of the hedging portfolio. To be precise, it is process $\bar{\mathcal{W}}$ which corresponds to what should be called exactly the value of the collateralization, hedging and funding portfolio.

Also observe that in case $\mathfrak{r} = 1$ and $f = c = 0$ (classical one-curve setup without excess funding costs), one recovers the usual notion of a self-financing hedging strategy with related wealth process $\mathcal{W}$. The funding close-out cash-flow $\bar{R}^f$ and the funding bases f and c can thus be interpreted as the corrections to a classical one-curve setup.

4 Martingale Pricing Approach

In this Section we deal with pricing of the contract shortened by the bank to the investor, under the funding conditions of the bank defined by the external recovery rate $\mathfrak{r}$ and the funding bases f and c. Note that given nonlinearities in the funding (unless $\mathfrak{r} = 1$ and the funding coefficient f is linear), it will not be possible to get rid of the funding costs in the pricing through discount factors as in a linear one-curve setup.[6] Cash-flows will be priced instead under an "additive, flat" extension of the classical "multiplicative, discounted" risk-neutral assumption. We also derive the dynamic hedging interpretation of such an additive risk-neutral price, starting by consistency with pricing by replication in the case of a complete market.

[6] Unless one resorts to an implicit discount factor depending on the value of the contract.

4.1 Primary Market

Let $\mathcal{M}$ denote the gain process of all hedging instruments traded in swapped form. So, in view of (2.1): $\mathcal{M}_0 = 0$ and for $t \in [0, \bar{\tau}]$

$$d\mathcal{M}_t = d\mathcal{P}_t - \big(r_t\mathcal{P}_t + c_t\big)dt + d\mathcal{C}_t. \tag{4.1}$$

Our standing probability measure $\mathbb{P}$ is henceforth interpreted as a risk-neutral pricing measure on the primary market of hedging instruments traded in swapped form, in the sense that

Assumption 4.1 The primary gain process $\mathcal{M}$ is an $\mathbb{R}^d$-valued $\mathbb{P}$-martingale.

By arbitrage, we mean a self-financing strategy with a related gain at time $\bar{\tau}$ which is almost surely non-negative, and which is positive with a positive probability (under the historical or any equivalent probability measure). Since the pricing measure $\mathbb{P}$ is equivalent to the historical probability measure $\widehat{\mathbb{P}}$, Assumption 4.1 precludes arbitrage opportunities on the primary market of the hedging assets traded in swapped form.

Remark 4.1 Expression (4.1) for the primary gain process $\mathcal{M}$ (and similar expressions related to valuation of the contract below) involves no discounting at some “reference rate”. This is why we call this approach an “additive” version of the “multiplicative” risk-neutral assumption which is more commonly used through the language of discounting at the risk-free rate r_t in the one-curve literature. For instance, in a Black-Scholes model on a stock S (with constant risk-free rate r and nil repo basis on S for notational simplicity), the “additive” martingale $\mathcal{M}$ and the usual one-curve “multiplicative” martingale write respectively

$$\mathcal{M}_t = \mathcal{M}_t^+ := \int_0^t dS_s - \int_0^t rS_s ds\,,\ \ \mathcal{M}_t^\times = e^{-rt}S_t$$

so $d\mathcal{M}_t^\times = e^{-rt}d\mathcal{M}_t^+$. In a one-curve setup more generally, an additive and a multiplicative approach are equivalent, and the multiplicative one is actually more convenient, since it allows one to get rid of the funding issue, “absorbed” in the discounting at the risk-free rate. In a multiple-curve setup however, the funding issue (intrinsically a nonlinear one in a bilateral counterparty risk setup, unless $\mathfrak{r} = 1$, via the positive part $\mathfrak{X}^+$ in the external close-out cash-flow) has to be accounted for explicitly, “additively”.

Under Assumption 4.1 it is convenient to rewrite (3.6) in martingale form as

$$d\mathcal{W}_t = \big(r_t\mathcal{W}_t + g_t(\mathcal{W}_t, \zeta_t)\big)dt - dC_t + \zeta_t d\mathcal{M}_t \tag{4.2}$$

where for $\pi \in \mathbb{R}$ and $\varsigma \in \mathbb{R}^d$

$$g_t(\pi, \varsigma) = f_t(\pi, \varsigma) + \varsigma c_t. \tag{4.3}$$

Example 4.1 Under the funding specification of Subsection 2.2, one gets in view of the expression of f in (3.4) that

$$g_t(\pi, \varsigma) = b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^- + \lambda_t\big(\pi - \Gamma_t - \varsigma^{\bar{s}}\mathcal{P}_{t-}^{\bar{s}}\big)^+ - \bar{\lambda}_t\big(\pi - \Gamma_t - \varsigma^{\bar{s}}\mathcal{P}_{t-}^{\bar{s}}\big)^- + \varsigma^{\bar{s}}c_t^{\bar{s}} \tag{4.4}$$

which depends on ς through $\varsigma^{\bar{s}}$, the position in the hedging assets funded together with the contract by the external lender.

Remark 4.2 In the case of a physical (as opposed to a natively swapped) primary asset, the coefficient c^i corresponds to a repo basis, and one might wonder why in Example 4.1 the repo rates eventually present in g are actually those of the hedging instruments which are not traded in swapped form. An interpretation is that in case of a hedging instrument traded in swapped form, the opportunity of getting it funded at the excess cost c^i is exploited, whereas for a hedging instrument not traded in swapped form this opportunity is not, creating an (algebraic) "loss of income" which should be reflected in the final "pricing formula", and therefore in the coefficient g of the corresponding pricing equation as we shall see in Definition 4.1 below.

From the BSDE point of view, a particularly simple situation will be the one where

$$\mathfrak{X}_t^+(\pi,\varsigma) = \mathfrak{X}_t^+(\pi)\,,\quad g_t(\pi,\varsigma) = g_t(\pi). \tag{4.5}$$

We call this the fully swapped hedge case in reference to its financial interpretation under the funding specification of Subsection 2.2. Otherwise we shall talk of an externally funded hedge.

4.2 $\mathbb{P}$-Price-and-Hedge BSDE

The class of general price-and-hedges introduced in Definition 3.1 is too large for practical purposes. This leads us to introduce the following more restrictive definition. Given a hedge ζ and a semimartingale Π, we denote $R = R^i - \mathbb{1}_{\tau=\theta} R^f$, in which

$$R^f := (1-\mathfrak{r})\mathfrak{X}_{\tau-}^+(\Pi_{\tau-},\zeta_{\tau-}). \tag{4.6}$$

Let us stress that R implicitly depends on $(\Pi_{\tau-},\zeta_{\tau-})$ in this notation.

Definition 4.1 ($\mathbb{P}$-price-and-hedge) Let a pair (Π,ζ) made of a semimartingale Π and a hedge ζ satisfy the following BSDE on $[0,\bar{\tau}]$:

$$\begin{aligned} &\Pi_{\bar{\tau}} = \mathbb{1}_{\tau<T} R \text{ and for } t\in[0,\bar{\tau}] : \\ &d\Pi_t + dC_t - \big(r_t\Pi_t + g_t(\Pi_t,\zeta_t)\big)dt = d\nu_t \end{aligned} \tag{4.7}$$

for some martingale ν null at time 0. Letting for $t\in[0,\bar{\tau}]$

$$\bar{\Pi}_t := \Pi_t + \mathbb{1}_{t\geq\theta} R^f,$$

process $(\bar{\Pi},\zeta)$ is then said to be a $\mathbb{P}$-price-and-hedge. The related cost process is the martingale ε defined by $\varepsilon_0 = 0$ and for $t\in[0,\bar{\tau}]$

$$d\varepsilon_t = d\nu_t - \zeta_t d\mathcal{M}_t. \tag{4.8}$$

Equivalently to the BSDE (4.7) in differential form, one can write in integral form, for $t\in[0,\bar{\tau}]$ (recall $\beta_t = e^{-\int_0^t r_s ds}$)

$$\beta_t\Pi_t = \mathbb{E}_t\Big(\int_t^{\bar{\tau}} \beta_s dC_s \;-\; \int_t^{\bar{\tau}} \beta_s g_s(\Pi_s,\zeta_s)ds + \beta_{\bar{\tau}}\mathbb{1}_{\tau<T}R\Big). \tag{4.9}$$

The reader is referred to El Karoui et al. (1997) for a general reference about BSDEs in finance, and to Example 1.1 therein as a basic example of use of BSDEs in connection with funding constraints (different borrowing and lending rates).

The $\mathbb{P}$-price-and-hedge BSDE (4.7) is made non-standard by the random terminal time $\bar{\tau}$, the dependence of the terminal condition R in $(\Pi_{\tau-}, \zeta_{\tau-})$, the contract effective dividend term dC_t, and the fact that it is not driven by an explicit set of fundamental martingales like Brownian motions and/or compensated Poisson measures. In this last respect, the representation (4.8) rather suggests that this BSDE will be solved with respect to the "primary martingale" $\mathcal{M}$, up to a (typically orthogonal) martingale ε.

Remark 4.3 (BSDEs) An analysis of counterparty risk under funding constraints led us to the $\mathbb{P}$-price-and-hedge BSDE (4.7), as a simplification of the general price-and-hedge FBSDE of Subsection 3.2. BSDE modeling should not be really seen as a choice here, but rather as an output of this paper. This is mainly due to the intrinsically nonlinear feature of the funding issue under bilateral counterparty risk (via the positive part $\mathfrak{X}^+$ in the external close-out cash-flow unless $\mathfrak{r} = 1$, see Remark 4.1). As observed recurrently in the mathematical finance literature, BSDEs and FBSDEs emerge naturally in the context of nonlinear pricing problems. In this the reader can be referred to the literatures on large investors, insiders and pricing impact, see for instance Cvitanic and Ma (1996). Regarding counterparty risk, another good point with BSDEs is that since they are a "nonlinear pricing tool," they can as easily deal with recursive features which will appear in the follow-up paper like a CVA close-out valuation process Q given not exogenously, but in terms of the counterparty risky price (or CVA) that one is looking for – a practically important issue as recently pointed out in Brigo and Morini (2010).

Again, the $\mathbb{P}$-price-and-hedge BSDE (4.7) is rather non standard at first sight. However we shall see in the follow-up paper that it can ultimately be reduced to a rather classical pre-default CVA BSDE. This will also raise interesting questions from the point of view of BSDEs per se. Firstly, the solution of the general price-and-hedge FBSDE that we got before simplification into the more tractable BSDE (4.7), is an open problem to the best of our knowledge. Secondly, via path-dependent collateralization, this provides an important field of motivation and application for the theory of time-delayed BSDEs (see Remark 3.3 in the follow-up paper). Thirdly, via the large-size and high-dimensional features of the CVA computational pricing problem, this provides an important field of motivation and application for numerical BSDEs.

About numerics, it's interesting to note that a BSDE modeling approach is consistent with the American Monte Carlo technology which is advocated for practical CVA computations (without explicit references to BSDEs) in the book by Cesari et al. (2010). The regression scheme which is used in Brigo and Pallavicini (2008) for computing the "clean price P" (in the terminology and notation of our follow-up paper) at all points of a simulated grid, also has a flavor of numerical BSDEs.

By construction, a $\mathbb{P}$-price-and-hedge $(\bar{\Pi}, \zeta)$ is a general price-and-hedge in the sense of Definition 3.1. In Subsection 4.3 we shall comment upon a $\mathbb{P}$-price-and-hedge from the points of view of arbitrage, hedging and computational tractability. For this one first needs to derive the equations for the wealth $\mathcal{W}$ of the corresponding hedging portfolio, and for the corresponding hedging error ϱ. This is the object of the following

Lemma 4.1 *Given a $\mathbb{P}$-price-and-hedge $(\bar{\Pi}, \zeta)$ and the related process Π, let a process $\mathcal{W}$ be defined by the first line in (3.8), starting from the initial condition $\mathcal{W}_0 = \Pi_0$; let then a process Π^* be defined by, for $t \in [0, \bar{\tau}]$*

$$\Pi^*_t = \bar{\Pi}_t - \mathbb{1}_{\tau=\theta} \bar{R}^f$$

where $\bar{R}^f := (1-\mathfrak{r})\mathfrak{X}^+_{\tau-}(\mathcal{W}_{\tau-}, \zeta_{\tau-})$*. Then* $(\bar{\Pi}, \zeta)$ *is a general price-and-hedge, with wealth* $\mathcal{W}$ *of the hedging portfolio such that for* $t \in [0, \bar{\tau}]$

$$\Big(\beta_t \Pi_t - \int_0^t \beta_s g_s(\Pi_s, \zeta_s) ds\Big) - \Big(\beta_t \mathcal{W}_t - \int_0^t \beta_s g_s(\mathcal{W}_s, \zeta_s) ds\Big) = \int_0^t \beta_s d\varepsilon_s \tag{4.10}$$

and with hedging error $\varrho = \Pi^* - \mathcal{W}$ *such that for* $t \in [0, \bar{\tau}]$

$$\begin{aligned} d\varrho_t &= d\varepsilon_t + \big(r_t \varrho_t + g(\Pi_t, \zeta_t) - g(\mathcal{W}_t, \zeta_t)\big)dt \; - \; \mathbb{1}_{\tau=\theta}(1-\mathfrak{r})(\bar{R}^f - R^f)\delta_\tau(dt) \\ &= d\varepsilon_t + \big(r_t \varrho_t + g(\Pi_t, \zeta_t) - g(\mathcal{W}_t, \zeta_t)\big)dt \\ &\qquad - \; \mathbb{1}_{\tau=\theta}(1-\mathfrak{r})\Big(\mathfrak{X}^+_{\tau-}(\mathcal{W}_{\tau-}, \zeta_{\tau-}) - \mathfrak{X}^+_{\tau-}(\Pi_{\tau-}, \zeta_{\tau-})\Big)\delta_\tau(dt) \end{aligned} \tag{4.11}$$

(and $\varrho_0 = \varepsilon_0 = 0$*).*

Proof. Identity (4.10) immediately follows from (4.2), (4.7) and (4.8) (plus the fact that $\mathcal{W}_0 = \Pi_0$). Rewritten in term of the hedging error $\varrho = \Pi^* - \mathcal{W}$, Equation (4.2) for the value $\mathcal{W}$ of the hedging portfolio of $(\bar{\Pi}, \zeta)$ yields that for $t \in [0, \bar{\tau}]$

$$d\Pi^*_t = \big(r_t \mathcal{W}_t + g_t(\mathcal{W}_t, \zeta_t)\big)dt - dC_t + \zeta_t d\mathcal{M}_t + d\varrho_t. \tag{4.12}$$

Besides, the equation part (second line) in the $\mathbb{P}$-price-and-hedge BSDE (4.7) can be written in terms of the cost $d\varepsilon_t = d\nu_t - \zeta_t d\mathcal{M}_t$ of (4.8) as

$$d\Pi_t = \big(r_t \Pi_t + g_t(\Pi_t, \zeta_t)\big)dt - dC_t + \zeta_t d\mathcal{M}_t + d\varepsilon_t. \tag{4.13}$$

Since $\Pi_t - \Pi^*_t = \mathbb{1}_{t=\theta}(\bar{R}^f - R^f)$, substracting (4.13) from (4.12) yields (4.11). □

4.3 Arbitrage, Replication and Computational Issues

Assume first that it is possible to find a $\mathbb{P}$-price-and-hedge process $(\bar{\Pi}, \zeta)$ with a vanishing cost process $\varepsilon = 0$, and second that for this $(\bar{\Pi}, \zeta)$ and the related process Π, uniqueness holds for the following forward SDE in Y: $Y_0 = \Pi_0$ and for $t \in [0, \bar{\tau}]$,

$$d(\beta_t Y_t) - \beta_t g_t(Y_t, \zeta_t)dt = d(\beta_t \Pi_t) - \beta_t g_t(\Pi_t, \zeta_t)dt.$$

Via the BSDE machinery (see for instance El Karoui et al. (1997)), the first assumption is typically met by application of a martingale representation property (whenever available), whereas the second assumption is a technical requirement guaranteeing that Π and $\mathcal{W}$ coincide if they solve the same forward SDE. Under these assumptions, one gets by (4.10) with $\varepsilon = 0$ therein that $\mathcal{W} = \Pi$. It follows that $R^f = \bar{R}^f$, and therefore by (4.11) that $\varrho = \varepsilon = 0$. In this case the $\mathbb{P}$-price-and-hedge process $(\bar{\Pi}, \zeta)$ is thus a replicating strategy.

We refer the reader to Section 5, expanding over Burgard and Kjaer (2011b), for a practical example of replication. Since replication ultimately relies on a martingale representation property, it typically holds (or not) not only for a particular contract, but for any financial derivative. We shall thus refer to this case henceforth as the "complete market" case.

In a more general, "incomplete" market, the cost ε of a $\mathbb{P}$-price-and-hedge $(\bar{\Pi}, \zeta)$, and in turn its hedging error ϱ, can only be reduced up to a level "proportional" to the "degree of incompleteness" of the primary market. The bank shortening the contract to the investor can only partially hedge its position, ending-up with a non-vanishing hedging error $\varrho_{\bar{\tau}}$.

Remark 4.4 (Arbitrage) In a complete market or if $\mathfrak{r} = 1$ (under unilateral counterparty risk in particular), the Dirac-driven term vanishes in (4.11). Under suitable conditions, one can then change the measure $\mathbb{P}$ into an equivalent measure $\mathbb{Q}$ such that the hedging error ϱ is a $\mathbb{Q}$-martingale. This excludes that $\varrho_{\bar{\tau}}$ could be non-negative almost surely and positive with positive probability. In conclusion a $\mathbb{P}$-price-and-hedge $(\bar{\Pi}, \zeta)$ cannot be an arbitrage in this case. On the opposite, in an incomplete market with moreover $\mathfrak{r} < 1$, a $\mathbb{P}$-price-and-hedge $(\bar{\Pi}, \zeta)$ is, in principle, arbitrable.

A non-arbitrable strategy would be a general price-and-hedge $(\bar{\Pi}, \zeta)$ such that the quadruplet $(\mathcal{W}, \Pi^*, \zeta, \varrho)$ in Definition 3.1 solves the related FBSDE, in the sense in particular that the hedging error ϱ would be a martingale under some equivalent probability measure. However in an incomplete market and with moreover $\mathfrak{r} < 1$ this FBSDE seems intractable (again, see Horst et al. (2011) for related technical issues). The $\mathbb{P}$-price-and-hedge BSDE can be viewed as a simplified version of this theoretical FBSDE. The price to pay for this simplification is that it opens the door to an arbitrage (unless a market is complete or $\mathfrak{r} = 1$, in which cases the $\mathbb{P}$-price-and-hedge BSDE and the above FBSDE are essentially equivalent). However we believe that this arbitrage is quite theoretical (the corresponding "free lunch" seems quite difficult to lock in).

In view of the above arbitrage, hedging and computational considerations, we restrict ourselves to $\mathbb{P}$-price-and-hedges in the sequel. For brevity we write henceforth in this and the follow-up paper "a price-and-hedge (Π, ζ)" when the related pair-process $(\bar{\Pi}, \zeta)$ is a $\mathbb{P}$-price-and-hedge. By price related to a hedge process ζ, we mean any process Π such that (Π, ζ) is a price-and-hedge (solves the BSDE (4.7)). Also in the sequel we simply call (4.7) the price BSDE, as opposed to CVA BSDEs to appear in the follow-up paper.

Note that this shift of terminology is rather immaterial since nobody cares about the price of the contract at time $\bar{\tau}$ (as soon as $\bar{\Pi}_{\bar{\tau}} = \mathbb{1}_{\tau<T} R^i$, for consistency with our definition of the hedging error ϱ as $\bar{\mathcal{W}} - \bar{\Pi}$). What matters in practice is the price for $t < \bar{\tau}$, in which case $\Pi_t = \bar{\Pi}_t = \Pi^*_t$.

Remark 4.5 (Symmetries) Similarly to the funding benefit coefficient g and the external funding recovery rate $\mathfrak{r}$ of the bank, one can introduce a funding cost coefficient $\overline{g}$ and an external funding recovery rate $\bar{\mathfrak{r}}$ for the investor. In the very case where $\mathfrak{r} = \bar{\mathfrak{r}} = 1$ and $g = \overline{g} = g(\pi)$, all cash-flows are symmetric from the point of view of the two parties. In this case the seller price of the bank will agree with the buyer price of the investor. An example of symmetric funding costs is the setup of Fujii and Takahashi (2011), where the funding close-out cash-flows are not represented (so implicitly $\mathfrak{r} = \bar{\mathfrak{r}} = 1$) and excess funding costs reduce to collateral bases b and $\bar{b}$ in the sense of our Subsection 2.2. Since collateral remuneration cash-flows are between the two parties of the contract (they do not involve external entities), collateral bases does not break the symmetry in our sense.[7]

A contrario it is worth emphasizing that as soon as $\mathfrak{r}$ or $\bar{\mathfrak{r}} < 1$, or the g-coefficients depend on ς, or they don't but $g \neq \overline{g}$, funding induces an asymmetry between the two parties, resulting in a short bank price of the contract, different from its long investor price (and in turn different CVAs in the follow-up paper). See Remark 2.3 for the practical "violation of money conservation" issue.

Another notable specification, corresponding to the setup of Piterbarg (2010), is the linear case where $\mathfrak{r} = 1$ and $g = g(\pi)$ is linear. The bank has then a common buyer and

[7] Fujii and Takahashi (2011) consider in their paper a different notion of symmetry, which may be broken even in their setup.

seller price. Under the funding specification of Subsection 2.2, the linear case corresponds to $\mathfrak{r} = 1$, $b = \bar{b}$ and $\lambda = \bar{\lambda}$.

Finally the one-curve setup corresponds to the case where $\mathfrak{r} = 1$ and all the bases are equal to 0, so $g = c = 0$. The only funding rate[8] in the economy is then the risk-free interest rate r.

5 Example

This Section illustrates our approach by applying it to a Black-Scholes case considered in (Burgard and Kjaer 2011a; Burgard and Kjaer 2011b). Note however that Burgard and Kjaer disregard defaultability of the bank regarding her funding debt to her external funder. So their setup corresponds to the special case $\mathfrak{r} = 1$. As soon as one wants to deal with bilateral counterparty risk, defaultability of the bank regarding her funding debt is an important issue. We thus treat the general case where $\mathfrak{r} \leq 1$.

5.1 Setup

We consider an European option with payoff $\phi(S_T)$ on a Black-Scholes stock S. The option is sold by the bank to the investor at time 0. Both parties are defaultable but they cannot default simultaneously. The option position is hedged by the bank with the stock S and zero-recovery risky bonds B and I issued by the bank itself and by the investor. Repo markets (with nil repo bases for notational simplicity) are assumed to exist for S, B and I. Assuming a constant risk-free rate r, the gain process $\mathcal{M}$ of a buy-and-hold position into the hedging assets traded in swapped form writes as follows

$$d\mathcal{M}_t = \begin{pmatrix} dS_t - rS_t dt \\ dB_t - rB_t dt \\ dI_t - rI_t dt \end{pmatrix}.$$

Consistently with the martingale requirement of Assumption 4.1 on $\mathcal{M}$, one assumes the following model for (S, B, I):

$$\begin{cases} dS_t - rS_t dt = \sigma S_t dW_t \\ dB_t - rB_t dt = B_{t-}\left(dJ_t^\theta + \gamma dt\right) \\ dI_t - rI_t dt = I_{t-}\left(dJ_t^{\overline{\theta}} + \overline{\gamma} dt\right) \end{cases} \tag{5.1}$$

where W_t is a $\mathbb{P}$-Brownian motion, and $J_t^\theta = \mathbb{1}_{t<\theta}$ and $J_t^{\overline{\theta}} = \mathbb{1}_{t<\overline{\theta}}$ are the non-default indicator processes of the bank and the investor, with constant $\mathbb{P}$-default intensities γ and $\overline{\gamma}$. However only S and I are assumed to be traded by the bank on their repo market. As for B, the bank funds it, together with its option position, by an external funder, at a constant external borrowing basis $\bar{\lambda}$ over the risk-free rate r (and her external lending rate is simply r). Moreover there is no collateralization ($\Gamma = 0$). In this case, (3.4) and (4.4) yield

$$\mathfrak{X}_t(\pi, \varsigma) = -\left(\pi - \varsigma^B B_t\right), \quad g_t(\pi, \varsigma) = -\bar{\lambda}\left(\pi - \varsigma^B B_t\right)^-. \tag{5.2}$$

Note that in this case of a non fully swapped hedge, $\mathfrak{X}$ and g depend on $\varsigma = (\varsigma^S, \varsigma^B, \varsigma^I)^\mathsf{T}$ (so (4.5) does not hold), through the hedging position ς^B of the bank in her own bond.

[8] Assuming the existence of a riskless asset with growth rate r.

Finally one assumes a CSA close-out cash-flow of the form (consistently with the general CSA close-out framework of the follow-up paper)

$$R^i = \mathbb{1}_{\tau=\theta}\big(\rho\chi^+ - \chi^-\big) - \mathbb{1}_{\tau=\overline{\theta}}\big(\overline{\rho}\chi^- - \chi^+\big) \tag{5.3}$$

where $\chi = Q(\tau, S_\tau)$ for a CSA close-out pricing function Q, and where ρ and $\overline{\rho}$ denote constant recovery rates of the bank and the investor to each other. Letting $\zeta = (\zeta^S, \zeta^B, \zeta^I)$, the price BSDE (4.7) writes as follows:

$$\begin{aligned} &\Pi_{\bar{\tau}} = \mathbb{1}_{\tau<T} R \text{ and for } t \in [0, \bar{\tau}] : \\ &d\Pi_t + \mathbb{1}_{T<\tau}\delta_T(dt)\phi(S_T) = \big(r\Pi_t - \bar{\lambda}\big(\Pi_t - \zeta_t^B B_t\big)^-\big)dt + \zeta_t d\mathcal{M}_t + d\varepsilon_t \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} R &= R^i - \mathbb{1}_{\tau=\theta} R^f = R^i - \mathbb{1}_{\tau=\theta}(1-\mathfrak{r})\mathfrak{X}^+_{\tau-}(\Pi_{\tau-}, \zeta_{\tau-}) \\ &= \mathbb{1}_{\tau=\theta}\big(\rho\chi^+ - \chi^- - (1-\mathfrak{r})\left(\Pi_{\tau-} - \zeta^B_{\tau-} B_{\tau-}\right)^-\big) - \mathbb{1}_{\tau=\overline{\theta}}\big(\overline{\rho}\chi^- - \chi^+\big) \end{aligned} \tag{5.5}$$

for some recovery rate $\mathfrak{r}$ (assumed constant) of the bank toward her external funder.

5.2 Analysis of a Solution

In this simple, complete market case (note there are three independent sources of randomness W, θ, $\overline{\theta}$ in the model and three hedging assets S, B and I, plus an external funding source), a solution (Π, ζ) to the price BSDE can be guessed intuitively. Moreover this will be a solution with vanishing cost process $\varepsilon = 0$, in other words a replication strategy for the bank selling the option to the investor.

The Markovian structure of the problem leads us to seek for a solution (Π, ζ) to (5.4) such that for $t \in [0, \bar{\tau}]$

$$\begin{aligned} &\Pi_t + \mathbb{1}_{t=T<\tau}\phi(S_T) = u(t, S_t, J_t^\theta, J_t^{\overline{\theta}}) \\ &\zeta_t = \delta(t, S_t) \end{aligned} \tag{5.6}$$

for suitable pricing and "delta" functions u and $\delta = (\delta^S, \delta^B, \delta^I)$. We denote $\widetilde{u}(t, S) = u(t, S, 1, 1)$, $\widetilde{B}^c(t) = B_0^c e^{(r+\gamma_c)t}$ and $\widetilde{B}^b(t) = B_0^b e^{(r+\gamma_b)t}$. Equation (5.5) writes as follows

$$\begin{aligned} R = \mathbb{1}_{\tau=\theta}\Big(&\rho Q(\tau, S_\tau)^+ - Q(\tau, S_\tau)^- - (1-\mathfrak{r})\left(\widetilde{u}(\tau, S_\tau) - \delta^B(\tau, S_\tau)\widetilde{B}(\tau)\right)^-\Big) \\ &- \mathbb{1}_{\tau=\overline{\theta}}\Big(\overline{\rho} Q(\tau, S_\tau)^- - Q(\tau, S_\tau)^+\Big). \end{aligned}$$

One then has in view of the first lines in (5.4) and (5.6), for every $(t, S) \in [0, T] \times (0, \infty)$

$$\begin{aligned} &u(t, S, 0, 1) = \rho Q(t, S)^+ - Q(t, S)^- - (1-\mathfrak{r})\left(\widetilde{u}(t, S) - \delta^B(t, S)\widetilde{B}(t)\right)^- \\ &u(t, S, 1, 0) = Q(t, S)^+ - \overline{\rho} Q(t, S)^-. \end{aligned} \tag{5.7}$$

Besides, on one hand, the first line of (5.6) and the second line of (5.4) (assuming $\varepsilon = 0$ therein) yield that for $t \in [0, \bar{\tau}]$

$$du(t, S_t, J_t^\theta, J_t^{\overline{\theta}}) = \Big(r\widetilde{u}(t, S_t) - \bar{\lambda}\big(\widetilde{u}(t, S_t) - \delta^B(t, S_t)\widetilde{B}(t)\big)^-\Big)dt + \delta(t, S_t)d\mathcal{M}_t. \tag{5.8}$$

On the other hand, assuming $\widetilde{u}$ of class $\mathcal{C}^{1,2}$, one has in view of the model dynamics for $(S_t, J^\theta_t, J^{\overline{\theta}}_t)$ the following Itô formula over $[0, \bar{\tau}]$:

$$du(t, S_t, J^\theta_t, J^{\overline{\theta}}_t) = \Big(\partial_t \widetilde{u}(t, S_t) + \mathcal{A}^{bs}\widetilde{u}(t, S_t)\Big)\, dt + \partial_S \widetilde{u}(t, S_t)\sigma S_t\, dW_t \\ - (u(t, S_t, 0, 1) - \widetilde{u}(t, S_t))\, dJ^\theta_t - (u(t, S_t, 1, 0) - \widetilde{u}(t, S_t))\, dJ^{\overline{\theta}}_t$$

where $\mathcal{A}^{bs} = rS\partial_S + \frac{\sigma^2 S^2}{2}\partial^2_{S^2}$ is the Black-Scholes generator, and where the right-hand side can be rewritten as

$$\begin{aligned}&\Big(\partial_t \widetilde{u}(t, S_t) + \mathcal{A}^{bs}\widetilde{u}(t, S_t) + \gamma\big(u(t, S_t, 0, 1) - \widetilde{u}(t, S_t)\big) + \overline{\gamma}\big(u(t, S_t, 1, 0) - \widetilde{u}(t, S_t)\big)\Big) dt \\ &\quad + \partial_S \widetilde{u}(t, S_t)\sigma S_t\, dW_t \\ &\quad - (u(t, S_t, 0, 1) - \widetilde{u}(t, S_t)) \left(dJ^\theta_t + \gamma dt\right) \\ &\quad - (u(t, S_t, 1, 0) - \widetilde{u}(t, S_t)) \left(dJ^{\overline{\theta}}_t + \overline{\gamma} dt\right).\end{aligned} \tag{5.9}$$

Equating the martingale terms of the right-hand side in (5.8) and of (5.9), and using also (5.7), denoting additionally $\widetilde{I}(t) = I_0 e^{(r+\overline{\gamma})t}$, one obtains that for every $(t, S) \in [0, T] \times (0, \infty)$:

$$\begin{aligned}-\delta^B(t, S)\widetilde{B}(t) &= \rho Q(t, S)^+ - Q(t, S)^- - (1 - \mathfrak{r}) \left(\widetilde{u}(t, S) - \delta^B(t, S)\widetilde{B}(t)\right)^- - \widetilde{u}(t, S) \\ -\delta^I(t, S)\widetilde{I}(t) &= Q(t, S)^+ - \overline{\rho}Q(t, S)^- - \widetilde{u}(t, S) \\ \delta^S(t, S) &= \partial_S \widetilde{u}(t, S).\end{aligned} \tag{5.10}$$

The first line of (5.10) is equivalent to

$$\big(\widetilde{u}(t, S) - \delta^B(t, S)\widetilde{B}(t)\big)^+ - \mathfrak{r}\big(\widetilde{u}(t, S) - \delta^B(t, S)\widetilde{B}(t)\big)^- = \rho Q(t, S)^+ - Q(t, S)^- \tag{5.11}$$

which reduces to (assuming $\mathfrak{r} > 0$)

$$-\delta^B(t, S)\widetilde{B}(t) = \begin{cases} \rho Q(t, S) - \widetilde{u}(t, S), & Q(t, S) \geq 0 \\ \frac{1}{\mathfrak{r}}Q(t, S) - \widetilde{u}(t, S), & Q(t, S) \leq 0. \end{cases} \tag{5.12}$$

This together with the two last lines of (5.10) explicitly yields $\delta(t, S)$ in terms of $\widetilde{u}(t, S)$. In particular (5.11) yields that

$$\big(\widetilde{u}(t, S) - \delta^B(t, S)\widetilde{B}(t)\big)^- = \frac{1}{\mathfrak{r}}Q(t, S)^-. \tag{5.13}$$

Equating now the dt-terms of the right-hand side in (5.8) and of (5.9), and accounting also for (5.7), (5.13) and for the terminal payoff $\phi(S_T)$ in case $T < \tau$, one obtains that the pre-default pricing function $\widetilde{u}(t, S)$ should satisfy the following pricing equation over $[0, T] \times (0, \infty)$:

$$\begin{cases} \widetilde{u}(T, S) = \phi(S)\,, \ S \in (0, \infty) \\ \left(\partial_t + \mathcal{A}^{bs}\right) \widetilde{u}(t, S) + \widetilde{g}(t, S, \widetilde{u}(t, S)) = 0\,, \ t < T, S \in (0, \infty) \end{cases} \tag{5.14}$$

with for every real u

$$\widetilde{g}\,(t, S, u) = \gamma\Big(\rho Q(t, S)^+ - Q(t, S)^- - (1 - \mathfrak{r})\frac{1}{\mathfrak{r}}Q(t, S)^- - u\Big) \\ + \overline{\gamma}\Big(Q(t, S)^+ - \overline{\rho}Q(t, S)^- - u\Big) - ru + \bar{\lambda}\frac{1}{\mathfrak{r}}Q(t, S)^-.$$

Or equivalently in terms of $\widetilde{\lambda} = \bar{\lambda} - (1-\mathfrak{r})\gamma$ (the related "liquidity basis", see Example 3.1 in the follow-up paper) and $\widetilde{r} = r + \gamma + \overline{\gamma}$:

$$\begin{aligned}\widetilde{g}\,(t,S,u) = \frac{\widetilde{\lambda}}{\mathfrak{r}}Q(t,S)^- + \gamma\Big(\rho Q(t,S)^+ - Q(t,S)^-\Big)\\ + \overline{\gamma}\Big(Q(t,S)^+ - \overline{\rho}Q(t,S)^-\Big) - \widetilde{r}u.\end{aligned} \tag{5.15}$$

This achieves the analysis of a solution, assumed to exist of the form (5.6) and with null cost process ε, to the price BSDE (5.4). Conversely, the linear PDE (5.14) is known to have a unique classical solution $\widetilde{u}(t,S)$ under mild conditions on the coefficients. This function $\widetilde{u}(t,S)$ and the function $\delta(t,S)$ associated to it via (5.10) and (5.12), yield by reverse-engineering in the above computations a solution of the price BSDE (5.4) with null cost process ε. One finally obtains a solution in the sense of replication to the pricing and hedging problem of the bank, accounting also for the defaultability of the latter regarding her funding debt, for any $0 < \mathfrak{r} \leq 1$.

Remark 5.1 **(i)** Practicality of the solution only holds if $\delta^B \geq 0$, corresponding to the bank repurchasing her own bond. Otherwise $\delta^B \leq 0$ would mean that the bank should issue more bond for hedging her CVA, which is not practical (see Burgard and Kjaer).
(ii) The case $\mathfrak{r} = 0$ can be dealt with likewise provided $Q \geq 0$, otherwise (5.11) has no solution and replicability does not hold.

5.2.1 CVA

Letting $v(t,S)$ denote the Black-Scholes pricing function of the option (price clean of counterparty risk and excess funding costs), and defining the pre-default CVA function $\widetilde{w} = v - \widetilde{u}$, the following pre-default CVA pricing equation follows from (5.14)-(5.15):

$$\begin{cases}\widetilde{w}(T,S) = 0\,,\ S \in (0,\infty)\\ \Big(\partial_t + \mathcal{A}^{bs}\Big)\,\widetilde{w}(t,S) + \widetilde{h}(t,S,\widetilde{w}(t,S)) = 0\,,\ t < T, S \in (0,\infty)\end{cases} \tag{5.16}$$

with for every real w

$$\begin{aligned}\widetilde{h}\,(t,S,w) = -\frac{\widetilde{\lambda}}{\mathfrak{r}}Q(t,S)^- + (\gamma+\overline{\gamma})(v(t,S) - Q(t,S)) + \gamma(1-\rho)Q(t,S)^+\\ - \overline{\gamma}(1-\overline{\rho})Q(t,S)^- - \widetilde{r}w.\end{aligned} \tag{5.17}$$

5.2.2 CSA Close-Out Pricing Schemes

It is implicitly understood above that a CSA close-out valuation scheme Q is an exogenous process, as in the standard clean CSA close-out pricing scheme $Q(t,S) = v(t,S)$ (in this latter case the $(\gamma+\overline{\gamma})(v(t,S) - Q(t,S))$-term, a replacement benefit/cost of the bank in the financial interpretation, vanishes in (5.17)).

However one can also see by reverse-engineering in the above computations that it is possible to deal likewise with the so-called pre-default CSA close-out pricing scheme $Q(t,S) = \widetilde{u}(t,S)$. In this "recursive" case where the data (in principle) Q depends on the solution $\widetilde{u}$, the pre-default price and CVA PDEs become semilinear, via the resulting nonlinear dependence of $\widetilde{g}$ or $\widetilde{h}$ in their third argument u or w. This implies some viscosity (instead of classical) solution technicalities but it does essentially not change the flow of arguments.

Note finally that the solution by replication that we were able to obtain in the complete market, Black-Scholes example of this Section, was only possible due to many detail features of the setup like, for instance, the use of zero-recovery bonds as hedging instruments. With non-zero recovery bonds, even though one has three hedging assets S, B and I in regard of three independent sources of randomness W, θ, $\overline{\theta}$, the nonlinearity (unless $\mathfrak{r} = 1$) of the funding close-out cash-flow (we take a positive part in $\mathfrak{X}^+$ in (5.5)) makes the replication equations nonlinear, and therefore non trivial to solve (even if three equations in three unknowns, see for instance the case made in the end of Remark 5.1(ii)).

5.3 Comparison with the results of Burgard and Kjaer

In the special case where $\mathfrak{r} = 1$, the results of Burgard and Kjaer coincide with the ones we just derived (with in this case $\widetilde{\lambda} = \bar{\lambda}$ representing a liquidity basis, since for $\mathfrak{r} = 1$, $\bar{\lambda}$ should not incorporate any credit spread). There is a caveat however. Burgard and Kjaer implicitly disregard defaultability of the bank regarding her funding debt (they do not have any close-out funding cash-flow in their setup, which is tantamount to letting $\mathfrak{r} = 1$ in our notation). But in the interpretation of their results, they dwell upon a case where despite of the bank being practically risk-free with regard to her funding debt, the external borrowing basis $\bar{\lambda}$ (that they denote s_F) would be of the form $(1-\rho)\gamma$. However $\bar{\lambda} = (1-\rho)\gamma$ implicitly refers to a case (actually a sensible one) where $\mathfrak{r}(=\rho) < 1$, whereas in their case $\mathfrak{r}$ is always (implicitly) equal to one. As a consequence, in this case, the simplifications that they find in the coefficient $\widetilde{h}$ in (5.17) (for $Q = v$, see Subsubsection 5.2.2), and the conclusions that they draw regarding the appropriate internal organization of the bank for managing counterparty risk and funding costs, may not be relevant (in this situation $s_F = (1-\rho)\gamma$ and $Q = v$ that they consider among others). More precisely, in Equation (5.14)-(5.15) above, the coefficient of the first $Q(t, S)^-$-term that appears in the expression of $\widetilde{g}$ in (5.15) is $\frac{\widetilde{\lambda}}{\mathfrak{r}}$,[9] as opposed to s_F in Burgard and Kjaer. This $\frac{\widetilde{\lambda}}{\mathfrak{r}}$ has no special relation with $\gamma(1-\rho)$. So no simplification occurs between the $-\frac{\widetilde{\lambda}}{\mathfrak{r}}Q(t, S)^-$-term and the $\gamma(1-\rho)Q(t, S)^+$-term[10] in $\widetilde{h}$ in (5.17).

Perspectives

In conclusion of this paper, in the presence of nonlinear funding costs, a martingale pricing approach is already useful in the context of a complete market model, allowing us to streamline the analysis of Burgard and Kjaer in the previous Section. However the corresponding computations were quite setup-dependent, and a general study at the level of the price BSDE (5.4) needs to be conducted, as soon in particular as one leaves the realm of complete markets.

Observe in the above example that passing from a price BSDE to a CVA BSDE allowed one to get rid of coupon payments (the option payoff in the example) that blur the picture in the price BSDE. One also learned from this example that the valuation problem is essentially a pre-default one, so that a reduced-form approach should be a fruitful way to go.

These avenues of research will be explored systematically in the follow-up paper, which

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[9] which indeed coincides with $\bar{\lambda}$ alias s_F in Burgard and Kjaer in case $\mathfrak{r} = 1$, but in this case only.

[10] corresponding to the FCA- and the DVA-terms in Burgard and Kjaer (2011a).

will see the emergence of the CVA not only as a very important financial issue, but also as a valuable tool in the mathematical analysis of counterparty risk under funding constraint.

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