XVA Analysis From the Balance Sheet

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Abstract

Since the great financial crisis of 2008–09, derivative dealers charge to their clients various add-ons, dubbed XVAs, meant to account for counterparty risk and its capital and funding implications. XVAs deeply affect the derivative pricing task by making it global, nonlinear, and entity dependent. But, before these technical implications, the fundamental points are to understand what deserves to be priced and what does not, and to establish, not only the pricing, but also the corresponding collateralization, accounting, and dividend policy of a bank.

As banks cannot replicate jump-to-default related cash flows, deals trigger wealth transfers from bank shareholders to bondholders and shareholders need to set capital at risk. On this basis, we devise a theory of XVAs, whereby the so-called contra-liabilities and cost of capital are sourced from bank clients at trade inceptions, on top of the fair valuation of counterparty risk, in order to compensate shareholders for wealth transfer and risk on their capital.

The resulting all-inclusive XVA formula, meant incrementally at every new deal, reads (CVA + FVA + KVA), where C sits for credit, F for funding, and where the KVA is a cost of capital risk premium. This formula corresponds to the cost of the possibility for the bank to go into run-off, while preserving a constant hurdle rate h, from any point in time onward if wished: It ensures to the bank shareholders a submartingale wealth process corresponding to the hurdle rate h, consistently between and throughout deals.

Keywords: Counterparty risk, market incompleteness, wealth transfer, cost of capital, credit valuation adjustment (CVA), funding valuation adjustment (FVA), capital valuation adjustment (KVA), balance sheet of a bank.

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Contents

Acronyms

Amounts on dedicated cash accounts of the bank:

CM Clean margin.

RC Reserve capital.

RM Risk margin.

SCR Shareholder capital at risk.

UC Uninvested capital.

Valuations:

CA Contra-assets valuation.

CL Contra-liabilities valuation.

CVA Credit valuation adjustment.

DVA Debt valuation adjustment.

FDA Funding debt adjustment.

FV Fair valuation of counterparty risk.

FVA Funding valuation adjustment.

KVA Capital valuation adjustment.

MtM Mark-to-market.

XVA Generic "X" valuation adjustment.

Also:

AE Accounting equity.

CCP Central counterparty.

CDS Credit default swap.

CET1 Core equity tier I capital.

CR Capital at risk.

EC Economic capital.

FRTB Fundamental review of the trading book.

FTP Funds transfer price.

SHC Shareholder capital (i.e. wealth).

1 Introduction

Since the great financial crisis of 2008–09, investment banks charge to their clients, in the form of rebates with respect to the counterparty-risk-free value of financial derivatives, various add-ons meant to account for counterparty risk and its capital and funding implications. In this paper the counterparty-risk-free value is dubbed mark-to-market (MtM) and the counterparty risk related add-ons are generically termed XVAs, where VA stands for valuation adjustment and X is a catch-all letter to be replaced by C for credit, D for debt, F for funding, M for margin, or K for capital.

XVAs deeply affect the derivative pricing task by making it global, nonlinear, and entity dependent. But, before these technical implications, the fundamental points are to:

1 Understand what deserves to be priced and what does not, and

2 Establish, not only the pricing, but also the corresponding collateralization, accounting, and dividend policy of the bank.

These are the **two objectives** of this paper. Its **contributions** are:

- The devising of a sustainable strategy for profits retention for a dealer bank, meant to prevent Ponzi kind of schemes, such as the one observed during the 2008–09 crisis, where always more trades are entered for the sole purpose of funding previously entered ones;

- An FTP (all-inclusive XVA add-on) meant as the cost of a "soft landing option" that allows the bank to go into run-off, i.e. lock its portfolio and let the balance sheet of the bank amortize in the future, while ensuring a constant target hurdle rate h to the shareholders, from any point in time onward if wished;

- An extension in continuous-time of the static notion of a "cost-of-capital pricing model in incomplete markets";

- A dynamic interpretation of the Solvency II risk margin formula;

- The connection between the XVAs and core equity tier I capital (CET1), the regulatory metric meant to represent the core financial strength of a bank;

- The connection between the XVAs and shareholder equity (i.e. shareholder wealth), which, under our cost-of-capital XVA approach, is shown to be a submartingale with drift coefficient corresponding to a constant hurdle rate h on shareholder capital at risk;

- An XVA theory based on a counterparty risk market incompleteness tenet, much more credible and realistic than the competing XVA replication paradigm;

- A theoretically sound but workable XVA approach that can be implemented in practice at the level of a real banking portfolio (as opposed to most theoretical pricing approaches in incomplete markets);

- The understanding that the rise of the XVA metrics reflects a shift of paradigm regarding the pricing and risk management of financial derivatives, from hedging to balance sheet optimization, and insights into related ways these metrics can be used in practice;

- A detailed comparison between the two competing XVA paradigms: replication, in most of the XVA literature, versus cost-of-capital in this paper;

- A refined understanding and formalization of the XVA related wealth transfer issues;

- An economical justification, through a notion of consistent valuations in a setup where two pricing models coexist, of the reduction of filtration methodology that is used for solving the XVA equations mathematically;

- XVA formulas that, accounting for the transfer of the residual value on the bank accounts from clients and shareholders to bondholders at the default of the bank, happen to be unilateral XVA formulas (even though the default of the bank is present as a key ingredient of our model), naturally in line with the regulatory requirement that capital should not diminish as an effect of the sole deterioration of the bank credit spread;

- A KVA formula, resulting from our approach, where the hurdle rate enters the discount factor applied to future capital at risk, which makes a crucial practical difference at the very large time horizon of a banking portfolio;

- An FVA formula, resulting from our approach, that is asymmetric and always nonnegative, in line with the pari passu requirement that there can be no shortfall but only windfall at the default of a bank;

- A self-consistent, self-contained XVA suite, where the input to KVA computations happens to be the output of the CVA and FVA model;

- A clarification of the connections between XVAs and the Modigliani-Miller invariance principle;

- A one-static XVA model with explicit formulas for all the quantities at hand, offering a concrete grasp on the related wealth transfer and risk premium issues.

1.1 Context

Coming after several papers on the valuation of defaultable assets in the 90's (see e.g. Duffie and Huang (1996)), Bielecki and Rutkowski (2002, Eq. (14.25) p. 448) yields the formula (CVA − DVA) for the fair valuation of bilateral counterparty risk on a swap, assuming risk-free funding. This formula, rediscovered and generalized by others since the 2008–09 financial crisis (cf. e.g. Brigo and Capponi (2010)), is symmetrical, i.e. it is the negative of the analogous quantity considered from the point

of view of the counterparty, consistent with the law of one price and the Modigliani and Miller (1958) theorem.

Around 2010, the materiality of the DVA windfall benefit of a bank at its own default time became the topic of intense debates in the quant and academic finance communities. At least, it seemed reasonable to admit that, if the own default risk of the bank was accounted for in the modeling, in the form of a DVA benefit, then the cost of funding (FVA) implication of this risk should be included as well, which would lead to the modified formula (CVA − DVA + FVA). See for instance Burgard and Kjaer (2011, 2013, 2017), Crépey (2015), Brigo and Pallavicini (2014), or Bichuch, Capponi, and Sturm (2018); see also Bielecki and Rutkowski (2015) for an abstract funding framework (without explicit reference to XVAs), generalizing Piterbarg (2010) to a nonlinear setup.

Then Hull and White (2012) objected that the FVA was only the compensator of another windfall benefit of the bank at its own default, corresponding to the non-reimbursement by the bank of its funding debt. Accounting for the corresponding "DVA2" (akin to the FDA in Albanese and Andersen (2014) or in the present paper) would bring back to the original fair valuation formula:

$$\text{CVA} - \text{DVA} + \text{FVA} - \text{FDA} = \text{CVA} - \text{DVA},$$

as FVA = FDA.

However (see Burgard and Kjaer (2013, end of Section 3.1) as well as Sections 5.3 and B.3 below), this argument assumes that the bank can perfectly hedge its own default, which, as a bank is an intrinsically leveraged entity, is not the case in practice. One can mention the related corporate finance notion of debt overhang in Myers (1977), by which a project valuable for the firm as a whole may be rejected by shareholders because the project is mainly valuable to bondholders. But, until recently, such considerations were hardly considered in the field of derivative pricing.

Related articles in this regard are Burgard and Kjaer (2011, 2013, 2017), Castagna (2012, 2013, 2014), Albanese and Andersen (2015, 2014), Albanese, Andersen, and Iabichino (2015), Andersen, Duffie, and Song (2017), and Green, Kenyon, and Dennis (2014). These papers are put further into perspective in Section 1.2. However, except for the last one, these papers only consider FVA (or its avatar MVA); they do not propose an approach to KVA.

1.2 Contents of the Paper

Our key premise, detailed in Section 2.1, is that counterparty risk entails two distinct but intertwined sources of market incompleteness:

- A bank cannot hedge its own jump-to-default exposure;

- A bank cannot perfectly hedge counterparty default losses.

In this paper we specify the banking XVA metrics that align derivative entry prices to shareholder interest, given this impossibility for a bank to replicate the jump-to-default

related cash flows. We develop a continuous-time "cost-of-capital" XVA approach consistent with the accounting standards set out in IFRS 4 Phase II (see International Financial Reporting Standards (2013)), inspired from the Swiss solvency test and Solvency II insurance regulatory frameworks (see Swiss Federal Office of Private Insurance (2006) and Committee of European Insurance and Occupational Pensions Supervisors (2010)), which so far has no analogue in the banking domain. Under this approach, the valuation CL of the so-called contra-liabilities and the cost of capital (KVA) are sourced from clients at trade inceptions, on top of the complete market valuation of counterparty risk, in order to compensate bank shareholders for wealth transfer and risk on their capital.

Regarding **objective 2** above, our XVA-inclusive trading strategies achieve a given hurdle rate to shareholders in the conservative limit case that no new trades occur. We devise a collateralization, accounting, and dividend policy of a dealer bank meant to preserve the possibility for the bank to go into run-off, i.e. lock its portfolio and let the balance sheet of the bank amortize in the future, while ensuring a constant target hurdle rate to the shareholders, from any point in time onward if wished. Such a "soft landing option" is key from a regulatory point of view, as it guarantees that the bank should not be tempted to go into snowball or Ponzi kind of schemes, such as the one observed during the 2008–09 financial crisis, where always more trades are entered for the sole purpose of funding previously entered ones.

Regarding **objective 1** above, we show that the XVA cost of the corresponding "soft landing option" is, by contrast with the complete market valuation $(\mathrm{CVA} - \mathrm{DVA})$ of counterparty risk:

$$\mathrm{CVA} + \mathrm{FVA} + \mathrm{KVA}, \tag{1}$$

computed unilaterally in a certain sense (even though we do crucially include the default of the bank itself in our modeling) and charged to clients on an incremental run-off basis at every new deal. Our KVA is a risk premium, in the line of the risk margin in the Solvency II insurance regulation (but devised in a consistent continuous-time framework): It devises entry prices which keep the position of a derivative market maker on an "efficient frontier" corresponding to a given return (constant hurdle rate h) on the shareholder capital at risk that is earmarked for coping with trading losses. Our cost-of-capital XVA strategy makes shareholder equity (i.e. wealth) a submartingale with drift corresponding to such a hurdle rate h, consistently between and throughout deals. Thus, we arrive to a sustainable strategy for profits retention, much like in the above-mentioned insurance regulation, but in a consistent continuous-time and banking framework.

The first ones to recast the XVA debate in the perspective of the balance sheet of the bank were Burgard and Kjaer (2011), but this was to conclude that adding an appropriately hedged derivative has no impact on the dealer's funding costs, notwithstanding the fact that the corresponding hedge is a violation of the pari passu rules that protect the junior creditors of a firm. Also relying on balance sheet models of a dealer bank, Castagna (2012, 2013, 2014) and Andersen, Duffie, and Song (2017)

end up with conflicting conclusions, namely that the FVA should, respectively should not, be included in the valuation of financial derivatives. Adding the KVA, but in a replication framework, Green, Kenyon, and Dennis (2014) conclude that both the FVA and the KVA should be included as add-ons in entry prices and as liabilities in the balance sheet.

By contrast, postulating that deals should guarantee to shareholders a certain hurdle rate h in an incomplete market setup, our approach implies that the FVA (and the MVA that is included into it in our notation in this paper) should be included as an add-on in entry prices and as a liability in the balance sheet; the KVA should be included as an add-on in entry prices but not as a liability in the balance sheet.

We view the counterparty risk market incompleteness tenet as much more credible and realistic than the competing XVA replication paradigm. Besides the divergent implications regarding inclusion or not of the different XVAs in entry prices and/or in the balance sheet, our approach also results in materially modified XVA formulas. In particular:

- Even though we do include the default of the bank itself in our modeling, our XVAs are, in a certain sense, unilateral. In particular, this makes our CVA immediately in line with the corresponding regulatory requirement that the reserve capital of a bank should not decrease uniquely because the credit risk of the bank worsens;

- The KVA that arises from our theory discounts future capital at risk projections at a rate including the hurdle rate h;

- Our XVA suite is self-contained and self-consistent: the main KVA input is the CVA and FVA desks trading loss process.

Generic pricing approaches in incomplete markets include risk-minimisation and minimal martingale measures, utility maximisation, utility indifference pricing, good-deal pricing, and probability distortions, among others (see e.g., respectively, Schweizer (2001), Rogers (2001), Carmona (2009), Cochrane and Saa-Requejo (2000), and Madan (2015)). We emphasize that, by contrast with most of these, as the companion paper Albanese, Caenazzo, and Crépey (2017) demonstrates, a dynamic cost-of-capital XVA approach is implementable at the scale of a real banking portfolio, including thousands of counterparties and hundreds of thousands of trades (provided optimized implementations and technologies are used).

1.3   Outline

Section 2 sets a financial stage where a bank is split across several trading desks with related cash flows detailed in Section 3. Consistency of valuation across the different trading desks is represented by a suitable martingale invariance principle exposed in Section 4. However, in our framework, valuation is not price, which entails an additional adjustment in the form of a cost-of-capital risk premium. Based on these principles,

Section 5 derives the XVA equations that apply to a static portfolio (or more precisely, a derivative portfolio held on a run-off basis). Section 6 revisits the corresponding approach at the trade incremental level, specifies the "soft landing" pricing, accounting, and dividend policy implied by our overall XVA approach, and establishes the corresponding submartingale property with hurdle rate h of shareholder wealth, consistenly between and throughout deals.

Sections 2 through 6 yield the conceptual backbone of the paper. Section 7 establishes various well-posedness and comparison results for the XVA equations of Section 5, specialized to a full-fledged continuous-time setup. Section 8 revisits the XVA equations when different choices are made on the fate of the residual reserve capital and risk margin in case of default of the bank. Section 9 discusses in the light of the outputs of this paper alternative XVA approaches that have been developed in the last years in the setup of what we therefore call the XVA benchmark model, namely a Black–Scholes model for an underlying market risk factor, combined with independent Poisson counterparties and bank defaults. Section 10 reviews our approach and discusses its main assumptions, results, and formulas.

Section A details the balance sheet perspective on the XVA metrics. Section B develops a stylized cost-of-capital XVA approach in a one-period static setup, yielding explicit XVA formulas and illustrating the XVA wealth transfer and risk premium issues.

1.4 Probabilistic Setup

We work in an abstract dynamic setup on a measurable space $(\Omega, \mathcal{A})$, endowed with a filtration with respect to which all the introduced processes are adapted, meant to cover discrete and continuous time in a common formalism (hence, in a first stage, we will not be able to distinguish between martingale and local martingale, etc.). This setup will be made more precise a first time in Section 4.1, specialized into a continuous-time model in Section 7, and simplified into a one-period model in Section B. All our notation is discounted at the risk-free rate. In other terms, we use the risk-free asset, assumed to exist, as a numéraire.

Remark 1.1 The core underlying issue here is the existence of a publicly observable, reference rate for the remuneration of cash collateral (cf. Section 2.1). In developed economies, this is provided by the overnight indexed swap (i.e. OIS) rate, which, whenever available, is together the best market proxy for a risk-free rate and the reference rate for the remuneration of cash collateral. ∎

To retrieve the corresponding "undiscounted" equations and formulas, just capitalize all cash flows and values (as well as the amounts on the different banking accounts, economic capital, etc.) at the risk-free (i.e. OIS) rate everywhere. All cumulative cash flow processes introduced below (accounting results, trading losses, hedging losses,..) start from 0 at time 0.

Unless explicitly specified, an amount paid (received) means effectively paid (received) if positive, but actually received (paid) if negative. A similar convention applies

to the notions of cost vs. benefit, loss vs. gain, etc.

2 Financial Setup

This section sets the financial stage. We consider a dealer bank, which is a market maker, engaged into derivative trading with clients.

A market maker cannot decide on asset selection: trades are proposed by clients and the market maker needs to stand ready to bid for a trade at a suitable price no matter what the trade is and when it arrives.

Bank clients are price taker corporates willing to accept a loss in a trade for the sake of receiving benefits acounting for their base business line. These benefits become apparent only once one includes their real investment portfolio, which cannot be done explicitly in a pricing model.

2.1 From Hedging to Netting, Capital, and Collateralization

Counterparty risk is related to cash flows or valuations linked to either counterparty default or the default of the bank itself.

The risk of financial loss as a consequence of client default is hard to replicate, because single name CDS instruments that could in principle be used for that purpose are illiquid and are typically written on bonds, not on swaps with rapidly varying value ("contingent credit swap exposures" to which counterparty risk exposures are tantamount).

The possibility for the bank of hedging its own default is even more questionable since, in order to hedge it, a bank would need to be able to freely trade its own debt. But banks are special firms that are intrinsically leveraged and cannot be transformed into a pure equity entity. There is also an argument of scale. It has been estimated that, if all European banks were to be required to have capital equal to a third of liabilities, then the total capitalization of banks would be greater than the total capitalization of the entire equity market as we know it today. Last, even if a bank was free to redeem all its debt, bank shareholders could not effectively monetise the hedging benefit, which would be hampered by bankruptcy costs. See Castagna and Fede (2013, Section 10.7) for a detailed discussion regarding this impossibility for a bank to hedge its own jump-to-default exposure, and cf. also Burgard and Kjaer (2013, Sections 3.1 and 3.2).

As counterparty risk is hard to hedge in practice, the management of counterparty risk is more about netting and collateralization than hedging, the residual risk being handled on reserve and capital bases, whether it is in a bilateral or in a centrally cleared trading setup (see e.g. Basel Committee on Banking Supervision (2015) and Official Journal of the European Union (2012)).

Netting means that default exposures are assessed at an aggregated portfolio (or netted set) level, after accounting for value compensation between the components of each netting set and for the transfer of property of the corresponding collateral. The exact specification of the netting sets depends on the trading setup of the bank

(bilateral, with netting by counterparty, and/or centrally cleared, with netting by asset class, in particular).

Collateral means cash or liquid assets that are posted to guarantee a netted set of transactions against the default of the posting party. Liquidation losses beyond the collateral posted by a defaulted party impact the capital of the shareholders within the bank. We assume cash only collateral, which stays the property of the posting party unless a default in a related netting set happens; posted collateral is remunerated at the risk-free (i.e. OIS, see Remark 1.1) rate. Collateral can be re-hypothecable, in the form of a variation margin (VM) that tracks the mark-to-market of a deal and can be re-used by the receiving party for its funding purposes. A variation-margined position is still subject to gap risk, which is the risk of slippage between the market value of a netting set and its variation margin during the liquidation period of a defaulted party. Hence, an additional layer of margin can be required, in the form of the so-called initial margin (IM). The initial margin is typically segregated and it is dynamically updated at a frequency that can be similar to the one used for variation margin. Under the current regulation, both forms of collateralization (VM and IM) are becoming mandatory, with some variants specific to bilateral versus centrally cleared transactions.

Remark 2.1 The industry terminology tends to distinguish an FVA, in the specific sense of the cost of funding the cash collateral for variation margin, from an MVA defined as the cost of funding segregated collateral posted as initial margin (see e.g. Albanese, Caenazzo, and Crépey (2017)). In this paper, we merge the two in an overall FVA meant in the broad sense of the cost of funding the derivative business of the bank.

If cash collateral happens to be remunerated at some basis with respect to the reference risk-free (OIS) rate (see Remark 1.1), then this entails further liquidity valuation adjustments. However, the corresponding bases and adjustments are typically much smaller than the other XVAs considered in this paper, hence we ignore them hereafter. ∎

2.2   One Bank, Several Desks, Different Stakeholders

A bank is a defaultable entity and has two different classes of stakeholders: shareholders and creditors.

More precisely, we model our bank as a composite entity split into ***shareholders*** and so-called ***bondholders***. Shareholders have the control of the bank and are solely responsible for investment decisions before bank default. At the bank default time τ, shareholders are wiped out. Bondholders represent the junior creditors of the bank, which have no decision power until the time of default, but are protected by pari-passu laws forbidding certain trades that would trigger wealth away from them to shareholders during the default resolution process of the bank. The bondholders have to cope with bankruptcy costs, which are outside the scope of an XVA model, that they need to face to liquidate the bank when it defaults. The trading cash flows received by the bank prior its default time τ go to bank shareholders, whereas the trading cash

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flows received by the bank during the default resolution period that starts at τ go to bondholders. Note that our setup is self-financed in the sense that, in order to make it a "closed system", we suppose that the shareholders and bondholders hold nothing outside the bank.

The bank also has senior creditors, represented in our framework by an external **funder** that can lend unsecured to the bank at some exogenously given risky funding spread.

Hereafter, in order to fix the mindset, we restrict ourselves to bilateral derivative portfolios. But we stress that centrally cleared derivatives can also be included along similar lines (in the post-crisis regulatory environment, bilateral exotic trades are typically hedged through cleared vanilla portfolios): see Remarks 2.3 and A.2.

The counterparty default losses and the (other than risk-free) funding expenditures of the bank can be viewed as synthetic derivative payoffs, which we call **contra-assets** (cf. Definition 3.5), that emerge, due to counterparty risk, at the aggregated level of the portfolio of the bank. We call **CA desk(s)** a mathematical entity regrouping the CVA desk and the FVA desk (the latter sometimes represented in practice by the ALM or the Treasury) of a bank. As will be detailed in the paper, the CA desk values the contra-assets and charges the corresponding CVA and FVA to the clients of the bank, deposits the corresponding payments in a **reserve capital account**, and then is exposed to the corresponding payoffs. As time proceeds, counterparty default losses and funding expenditures occur and are covered by the CA desk with the reserve capital account.

After the contracts have thus been filtered out of their counterparty risk and risky funding implications by the CA desk, the other trading desks of the bank, which we call **clean desks**, can ignore counterparty risk and focus on the management of the market risk of the contracts in their respective business lines. In particular, as we will formalize in Section 5.1, the re-hypothecable collateral provided by the Treasury of the bank to the clean desks funds their trading at the risk-free cost of collateral remuneration. Thus the clean desks do not need any cash account, the financing function of a cash account being already played by their risk-free remunerated collateral account, which we call the **clean margin account**.

Hence, in our setup, we, in fact, deal with two portfolios: the client portfolio, between the clients of the bank and the CA desk, and the so-called cleaned portfolio, between the CA desk and the clean desks. The corresponding promised cash flows are the same. But, as should be intuitively clear from the above and seen in detail below, counterparty risk only impacts the client portfolio.

Remark 2.2 Reflecting into the model the distinction (that exists for fact in banks) between the CA desks and the clean desks allows one to:

- Provide the economical background and motivation for the mathematical tools of reduction of filtration that will be introduced in the second part of the paper to solve our XVA equations;

- Explain the mechanism of "a fully collateralised market hedge" in the context of bilateral trade portfolios;

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- Provide direct derivations of the XVA equations, coming in contrast with most other XVA references in the literature, where the XVA equations are obtained in two steps, as the difference between a linear equation for the mark-to-market of the portfolio ignoring counterparty risk and another equation for the "risky value" of the portfolio;

- Yield a much clearer view on the contra-liability cash flows. For instance, the derivation (83) shows in an elementary static setup how, accounting for a fully collateralised market hedge, these cash flows are not a fiction or an abstract compensation of others, but actual cash flows that fall into the estate of the defaulted bank, increasing the realized recovery rate of bondholders. ∎

Remark 2.3 In the case of centrally cleared trading, the role of the clean desks is played by a central counterparty (CCP), which "clears the delta" of the bank, also providing, although by a different mechanism, a "fully collateralized market hedge" to the client portfolio of the bank (see Armenti and Crépey (2018)). ∎

Each (CA or clean) trading desk may also setup a related hedge, typically dynamic and possibly imperfect (especially in the case of the CA desk), which generates a (cumulative) hedging loss process intended to control the fluctuations of their respective trading loss process. Note that we are restricting ourselves to the most common situation of hedges that are either swapped or traded through a repo market, without upfront payment. The related hedging loss processes are meant inclusive of associated funding costs, assumed risk-free. See Example 4.1 below and cf. Crépey, Bielecki, and Brigo (2014, Section 4.2.1) for more details.

We emphasize that the CA desk is not in charge of the KVA, which under our approach is of a different nature and is treated separately by the management of the bank. Namely, on top of reserve capital, the so-called risk margin is sourced by the bank from the clients at each deal, deposited into a ***risk margin account***, and then gradually released as KVA payments into the shareholder dividend stream. The risk margin account also yields risk-free interest payments to the shareholders.

Another account contains the ***shareholder capital at risk*** earmarked by the bank to deal with exceptional trading losses (beyond the expected losses that are already accounted for by reserve capital). Last, there may be one more bank account with an additional (typically unknown) amount of ***uninvested capital***. These accounts also yields risk-free interest payments to the shareholders.

In practice, losses-and-earnings realization times are typically quarter ends for bank profits, released as dividends, versus recapitalization managerial decision times for losses. For simplicity in our model:

Assumption 2.1 (Mark-to-model) We write CM, RC, RM, SCR, and UC for the respective (discounted) amounts on the clean margin, reserve capital, risk margin, shareholder capital at risk, and uninvested capital accounts of the bank.

Except for UC, all these amounts are continuously, instantaneously reset to some (to-be-determined hereafter) theoretical target levels. ∎

As the amounts on all the accounts but one are marked-to-model through Assumption 2.1 and because our setup is self-financed, hence the last, residual amount UC plays the role of an "adjustment variable".

Definition 2.1 Shareholder capital is

$$SHC = SCR + UC; \tag{2}$$

The core equity tier I capital of the bank is

$$CET1 = RM + SCR + UC. \; \blacksquare \tag{3}$$

Remark 2.4 From a financial interpretation point of view, CET1 represents the financial strength of the bank when assessed from a regulatory (i.e. structural solvency) point of view (see Section A); SHC = SCR + UC represents ***shareholder wealth***, whereas ***bondholder wealth*** will be the value of the contra-liabilities (cf. Theorem 5.1). $\blacksquare$

The rise of the XVA metrics reflects a shift, regarding the pricing and risk management of financial derivatives, from a hedging paradigm to balance sheet optimization. In fact, the afore-described setup can be viewed as an abstraction of the bank balance sheet (or, vice versa, the latter can be viewed as a metaphorical representation of the former), displayed Figure 1 in Section A, where the horizontal subdivision (from bottom to top) embodies the functional distinction between clean desks, the CA desk, and the management of the bank, whereas the vertical subdivision has to do with the capitalistic distinction between shareholders and bondholders.

2.3   Default Model of the Bank

In the spirit of structural models of the default time of a firm, such as the Merton (1974) model, the core equity capital of a firm corresponds to its "distance to default" and the default itself would be modeled as the first time where $CET1 \leq 0$. However, a negative CET1 (via UC, specifically) is allowed in our setup, interpreted as recapitalisation (or equity dilution). Hence we never have structural default in our setup.

Instead we model the default of the bank as a totally unpredictable event at some exogenous time τ calibrated to the bank CDS curve, which we view as the most reliable and informative credit data regarding anticipations of markets participants about future recapitalization, government intervention, bail-in, and other bank failure resolution policies.

Assumption 2.2 At the bank default time τ, the property of the residual amount on the reserve capital and risk margin accounts is instantaneously transferred from the shareholders to the bondholders of the bank. $\blacksquare$

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Remark 2.5 The default of a bank does not necessarily mean insolvency, i.e. negative equity, but, more precisely, liquid assets less than short term liabilities (see e.g. Duffie (2010)). In fact, the legal definition of default is an unpaid coupon or cash flow, which is a liquidity event. For instance, at the time of its collapse in April 2008, Bear Stearns had billions of capital. ∎

At the bank default time, the property of any residual $SHC = SCR + UC$ amount flows back to shareholders. Note that, should the bank default instantaneously at time 0 right after the client portfolio has been setup and the corresponding RC and RM amounts have been sourced from client, having RC and/or RM (or part of them) flow back to shareholders would make it a "positive arbitrage" to their benefit. Conversely, having SCR (or part of it) go to bondholders would make it a "negative arbitrage" to shareholders. Shareholders declaring default at time 0 should be possible without contradiction (or "arbitrage") in the model. Hence, in our setup, the SCR amount is only "earmarked by the bank" and dedicated to absorb (exceptional) trading losses if they arise, but, in case of bank default, the residual SCR and a fortiori UC amounts flow back to the shareholders.

Remark 2.6 As the clean margin is collateral, its property is transferred from the clean margin poster (or lender) to the clean margin receiver (or borrower) at the liquidation time of the bank. In the case of the clean desks, this will exactly compensate the amount that they need to close their position (cf. Assumption 6.3). The CA desk nets its corresponding cash flow with the ones that it obtains from the liquidation of the client portfolio; the netted output flows into the so-called ***contra-liabilities*** of the bank, i.e. the realized recovery stream of the bondholders (see Definition 3.5), via the term $\mathcal{C}^\bullet$ in (13) below. ∎

3   Cash Flows

In this section, we describe the cash flows streams of the clean and the CA desks.

Assumption 3.1 The derivative portfolio of the bank is held on a run-off basis, i.e. set up at time 0 and such that no new unplanned trades[1] ever enter the portfolio in the future. ∎

In practice, unless precisely a bank goes into run-off, derivative portfolios are incremental and XVA models are especially required for computing an entry price (or counterparty risk related add-on) for every new trade. Assumption 3.1 will be relaxed accordingly in Section 6.

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[1]Other than the ones initially planned at time 0, noting that the latter may still include forward starting contracts or dynamic hedges.

The rules regarding the settlement of contracts at defaults yield the counterparty exposure (CVA and DVA originating) cash flows, denoted by $\mathcal{C}$: cf. Assumption 6.3 and see Lemmas 7.7 and B.1 for concrete examples below.

The risky funding cash flows[2] of the CA desk, which is in charge of the risky funding of the bank, are represented in the form of a process $\mathcal{F}$. In practice, $\mathcal{F}$ is determined by the collateralization and funding policy of the bank through a relevant self-financing condition: See e.g. Lemmas 7.6 and B.1 below.

Last, the (partial) hedging policy of the CA desk results in a (cumulative) CA desk hedging loss process $\mathcal{H}$.

3.1 Accounting Result Processes

Definition 3.1 We call accounting result of the CA desk, denoted by Γ^{ca}, the cumulative cash flow process triggered to the bank by the activity of the CA desks, i.e., in view of the above description:

$$\Gamma^{ca} = -(\text{RC} - \text{RC}_0 + \mathcal{C} + \mathcal{F} + \mathcal{H}). \quad \blacksquare \tag{4}$$

See (57) and (83) for illustrations in concrete set-ups (where RC = CA as per Proposition 5.2).

In particular, the term $\text{RC} - \text{RC}_0$ in (4) accounts for the resets of the RC amounts by the shareholders as per Assumption 2.1, beyond the initial contribution RC_0 provided by the clients of the bank at portfolio inception time 0.

We label the netting sets of the client portfolio by an index c (like 'contract', as, from an XVA viewpoint, any netted set is tantamount to a single contract). Given a netting set labeled by c, we denote by $\mathcal{P}^c$, CM^c, and $\mathcal{H}^c$ the related contractually promised cash flow stream, amount on the clean margin account (counted positively when received by the clean desks), and hedging loss. Hence, in particular,

$$\text{CM} = \sum_c \text{CM}^c, \tag{5}$$

where, here and henceforth, c in the summations ranges over all netting sets of the client portfolio.

Definition 3.2 We call accounting results of the clean desks, denoted by Γ^{cl}, the cumulative cash flow process triggered to the bank by the activity of the clean desks, i.e., in view of the above description:

$$\Gamma^{cl} = \sum_c (\text{CM}^c - \text{CM}_0^c + \mathcal{P}^c - \mathcal{H}^c)^{\tau_c^\delta}, \tag{6}$$

where the stopping time τ_c^δ denotes the liquidation time of the netting set c. $\blacksquare$

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[2]Other than the risk-free accrual of the reserve capital (cash) account of the CA desk, which is already included in our choice of the risk-free deposit asset as a numéraire.

In particular, the term $\text{CM}^c - \text{CM}_0^c$ in (6) accounts for the resets of the CM^c amounts between the CA and the clean desks as per Assumption 2.1, beyond the initial contribution CM_0^c provided by the clients of the bank at portfolio inception time 0.

3.2 Trading Loss Processes

For any process Y, the corresponding process stopped before the bank default time τ is denoted by Y° (whenever well-defined). We also define $Y^\bullet = Y^\circ - Y$, so that

$$Y = Y^\circ - Y^\bullet. \tag{7}$$

In order to avoid the ambiguity between "a process Y stopped before τ" in the respective meanings of $Y = Y^\circ$ or to denote the stopped process Y°, we (abusively) say that a process Y is "without jump at τ" if $Y = Y^\circ$, i.e. $Y^\bullet = 0$.

In particular, as the RC, RM, SCR, and UC amounts will only really matter before τ, we assume all these amounts without jump at τ, to avoid unnecessary use of the $\cdot^\circ$ notation. Moreover, as $\mathcal{C}^\circ$ is only made in practice of counterparty default losses of the bank (see e.g. Lemmas 7.7 and B.1):

Assumption 3.2 The counterparty exposure cash flow $\mathcal{C}$ is a finite variation process with nondecreasing component $\mathcal{C}^\circ$. ∎

In view of the distinction between shareholders and bondholders we must differentiate between cash flows from the point of view of the bank as a whole (shareholders and bondholders altogether), such as the processes Γ^{ca} and Γ^{cl}, and cash flows affecting the bank shareholders, corresponding in our terminology (assessed as loss after sign change) to the trading loss processes below. However, this is ignored by the clean desks, which do not incorporate the default of the bank in their modeling: the latter is only local, at the level of their particular business line. But this is then corrected by the CA desk, which has a modeling view at the level of the bank as a whole.

Accordingly, by trading loss L^{cl} of the (bank generated by the and from the point of view of the) clean desks (who ignore the default of the bank in their modeling), we just mean the negative of their accounting result process; by trading loss L^{ca} of the (bank shareholders generated by the) CA desk, we mean the negative of their accounting result process, stopped before τ for alignment with shareholder interest. Namely:

Definition 3.3 We call trading loss of the clean desks, of the CA desk, and of the bank, the respective processes

$$L^{cl} = -\Gamma^{cl}, \; L^{ca} = -(\Gamma^{ca})^\circ \text{, and } L = -(\Gamma^{cl} + \Gamma^{ca})^\circ = (L^{cl})^\circ + L^{ca}. \; ∎ \tag{8}$$

That is, in view of (4)–(6):

$$L^{cl} = -\sum_c (\text{CM}^c - \text{CM}_0 + \mathcal{P}^c - \mathcal{H}^c)^{\tau_c^\delta},$$
$$L^{ca} = \text{RC} - \text{RC}_0 + \mathcal{C}^\circ + \mathcal{F}^\circ + \mathcal{H}^\circ, \tag{9}$$
$$L = \text{RC} - \text{RC}_0 + \mathcal{C}^\circ + \mathcal{F}^\circ + \mathcal{H}^\circ + (L^{cl})^\circ.$$

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See (57) and (82) for illustrations in concrete set-ups (where $\mathrm{RC} = \mathrm{CA}$ as set after Proposition 5.2).

From the balance sheet perspective depicted on Figure 1 in Section A, as long as the bank is nondefault, the overall accounting result $\Gamma = \Gamma^{cl} + \Gamma^{ca}$ of the bank and its trading loss L correspond to the appreciation of the accounting equity and to the depreciation of the core equity of the bank (see (11) and Section A).

3.3 Dividend, Contra-Asset, and Contra-Liability Cash Flow Processes

The trading gains $(-L)$ of the bank (cf. Definition 3.3) continuously flow into the shareholder dividend stream. Also accounting for the risk margin payments, for the risk-free interest that is earned by the shareholders on the risk margin account, and for the default of the bank that stops the dividend stream before τ:

Definition 3.4 Shareholder (cumulative, discounted) dividends $\mathcal{D}$ are

$$\mathcal{D} = -(L + \mathrm{RM} - \mathrm{RM}_0). \ \blacksquare \tag{10}$$

We emphasize that, in our model, negative dividends are possible. They are interpreted as recapitalisation, i.e. equity dilution (cf. Section 2.3).

Lemma 3.1 *The core equity of the bank and shareholder wealth satisfy*

$$\mathrm{CET1} = \mathrm{CET1}_0 - L \tag{11}$$

$$\mathrm{SHC} = \mathrm{SHC}_0 + \mathcal{D}. \tag{12}$$

Proof. Recall that shareholders hold nothing outside the bank. Hence, as long as the bank is nondefault, the trading of the bank (i.e. the activity of the clean and the CA desks) requires from CET1 the amount L in (8), which can be analyzed as amounts $(\mathrm{RC} - \mathrm{RC}_0)$ and $-\sum_c (\mathrm{CM}^c - \mathrm{CM}_0^c)^{\tau_c^\delta}$ needed for resetting the reserve capital and clean margin amounts and further amounts $\mathcal{C} + \mathcal{F} + \mathcal{H}$ and $-\sum_c (\mathcal{P}^c - \mathcal{H}^c)^{\tau_c^\delta}$ needed for absorbing the CA and clean desks trading and hedging losses (the latter flowing between CET1 and clients, external funder and hedgers). This establishes the identity (11) before τ, and therefore also from τ onward as both sides are without jump at τ.

Recalling $\mathrm{SHC} = \mathrm{SCR} + \mathrm{UC} = \mathrm{CET1} - \mathrm{RM}$ (cf. (2) and (3)), we have by (11)

$$\mathrm{SHC} = \mathrm{CET1}_0 - L - \mathrm{RM} = \mathrm{CET1} - \mathrm{RM}_0 - (L + \mathrm{RM} - \mathrm{RM}_0),$$

which in view of (10) is (12). $\blacksquare$

Definition 3.5 We call contra-assets $(\mathcal{S})$ and contra-liabilities $(\mathcal{B})$ the cash flows triggered by the activity of the CA desk from the shareholders, respectively to the bondholders, i.e., also accounting for the related bank default cash flows RC_τ (cf. Assumption 2.2),

$$\begin{aligned}
\mathcal{S} &= \mathcal{C}^\circ + \mathcal{F}^\circ + \mathcal{H} + \mathbb{1}_{\{[\tau,+\infty)\}}\mathrm{RC}_\tau \\
\mathcal{B} &= \mathcal{C}^\bullet + \mathcal{F}^\bullet + \mathbb{1}_{\{[\tau,+\infty)\}}\mathrm{RC}_\tau. \ \blacksquare
\end{aligned} \tag{13}$$

Contra-assets, respectively contra-liabilities, draw their names from the fact that, from the point of view of the balance sheet of the bank depicted Figure 1 in Section A, their (to be defined) value processes will appear as special liabilities, respectively assets, that arise from the feedback of counterparty risk on financial payables (respectively receivables).

We emphasize that the purpose of our capital structure model of the bank (see Figure 1 in Section A) is not to model the default of the bank, like in a Merton (1974) model, as the point of negative equity (CET1 < 0), which in the case of a bank would be unrealistic (cf. Section 2.3). Instead, our aim is to put in a balance sheet perspective the contra-assets and contra-liabilities of a dealer bank, items which are not present in the Merton model and will play a key role in our analysis.

4 Pricing Principles

This section states our cost-of-capital pricing principles for the cash flows of Section 3.

4.1 Invariance Valuation Setup

We denote by T a finite and constant upper bound on the maturity of all claims in the portfolio, also including the time (such as two weeks) of liquidating the position between the bank or any of its clients. We write $\bar{\tau} = \tau \wedge T$, where τ is the default time of the bank

Consistent with the divergent views of the clean and the CA traders exposed before Definition 3.3, our setup involves not one, but actually two pricing models.

Assumption 4.1 Clean desks compute their model prices and hedging sensitivities using a filtration $\mathbb{F} = (\mathfrak{F}_t)_{t \geq 0}$ such that the bank default time τ is not an $\mathbb{F}$ stopping time. The CA desk uses a larger filtration $\mathbb{G} = (\mathfrak{G}_t)_{t \geq 0}$ such that τ is a $\mathbb{G}$ stopping time.

Any $\mathbb{G}$ semimartingale Y on $[0, \bar{\tau}]$ admits a unique $\mathbb{F}$ semimartingale Y' on $[0, T]$, called $\mathbb{F}$ reduction of Y, that coincides with Y before τ. Any $\mathbb{G}$ stopping time η admits an $\mathbb{F}$ stopping time η', called $\mathbb{F}$ reduction of η, such that

$$\eta \wedge \tau = \eta' \wedge \tau. \; \blacksquare \tag{14}$$

All cash flow and price processes are modeled as $\mathbb{G}$ or $\mathbb{F}$ semimartingales. The contractually promised derivative cash flow stream of the client portfolio is an $\mathbb{F}$ semimartingale. The counterparty exposure, risky funding, and hedging cash flows $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{H}$ are $\mathbb{G}$ semimartingales.

The representation of valuation by the clean and CA desks of the bank, as well as the consistency of valuation across their different perspectives, are encoded into the following:

Assumption 4.2 (Invariance valuation principle) The clean and XVA desks use not only different filtrations, but also potentially different pricing measures $\mathbb{P}$ on $\mathfrak{F}_T$ and $\mathbb{Q}$ on $\mathfrak{G}_T$, equivalent on $\mathcal{F}_T$.

The process L^{cl} is an $(\mathbb{F}, \mathbb{P})$ martingale on $[0, T]$. The process L^{ca} is a $(\mathbb{G}, \mathbb{Q})$ martingale on $[0, \bar{\tau}]$ without jump at τ.

Stopping before τ turns $(\mathbb{F}, \mathbb{P})$ martingales on $[0, T]$ into $(\mathbb{G}, \mathbb{Q})$ martingales on $[0, \bar{\tau}]$ without jump at τ. Conversely, the $\mathbb{F}$ reductions[3] of $(\mathbb{G}, \mathbb{Q})$ martingales on $[0, \bar{\tau}]$ without jump at τ are $(\mathbb{F}, \mathbb{P})$ martingales on $[0, T]$. ∎

Remark 4.1 We leave aside the question of knowing whether such framework could be related to a suitable notion of no arbitrage in a situation where two different pricing models (different filtrations, in particular) are involved. ∎

In continuous time (see Section 7.1), the probabilistic setup of Assumption 4.2 corresponds to the notions of invariance time τ and invariance probability measure $\mathbb{P}$ studied in Crépey and Song (2017b).

As we will see in Sections 5.1 and 5.3, the martingale conditions on the processes L^{cl} and L^{ca} in Assumption 4.2 embed the theoretical target levels for the levels CM and RC to be maintained by the bank on its clean margin and reserve capital accounts.

Remark 4.2 As a consequence of Assumption 4.2 and of (8), the trading loss L of the bank is a $(\mathbb{G}, \mathbb{Q})$ martingale on $[0, \bar{\tau}]$ without jump at τ; Its $\mathbb{F}$ reduction L' is an $(\mathbb{F}, \mathbb{P})$ martingale on $[0, T]$. ∎

4.2 Valuation of the Funding and Hedging Assets

Assumption 4.3 The hedging loss $\mathcal{H}^c$ related to the hedging of each netting set c by the clean desks, including the risk-free cost of setting the hedge, is an $(\mathbb{F}, \mathbb{P})$ martingale. ∎

The rationale is that clean desks hedging gains arise in practice as the stochastic integral of predictable hedging ratios against wealth processes of buy-and-hold strategies in available ($\mathbb{F}$ adapted, "clean") hedging assets. Recall from Section 2.2 that we are considering wealth processes inclusive of the associated funding costs, assumed risk-free, here. Such wealth processes are assumed to be $(\mathbb{F}, \mathbb{P})$ martingales, as are in turn stochastic integrals against them. As a consequence, each process $\mathcal{H}^c$ (of the corresponding discounted cash flows) is an $(\mathbb{F}, \mathbb{P})$ martingale.

Example 4.1 Assuming the hedge of a contract c of the cleaned portfolio implemented through a repo market on a (discounted) Black-Scholes stock S with volatility σ, then, supposing no dividends and risk-free repo funding of S:

$$d\mathcal{H}_t^c = -\zeta_t^c dS_t = -\zeta_t^c \sigma S_t dW_t, \tag{15}$$

where W is the $(\mathbb{F}, \mathbb{P})$ Brownian motion driving the stock and ζ^c is the corresponding hedging ratio (the instantaneous risk-free cost of funding the hedge is implicitly contained in (15) through our choice of the risk-free deposit asset as a numéraire). ∎

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[3]cf. Assumption 4.1.

Assumption 4.4 The hedging loss $\mathcal{H}$ of the CA desk, including the costs of setting the hedge, is a $(\mathbb{G}, \mathbb{Q})$ martingale without jump at τ (i.e. $\mathcal{H} = \mathcal{H}^\circ$). $\blacksquare$

The rationale is the same as for Assumption 4.3, this time with respect to $(\mathbb{G}, \mathbb{Q})$. The assumption $\mathcal{H} = \mathcal{H}^\circ$ is made for consistency with our premise that a bank cannot hedge its own jump-to-default exposure (cf. Section 2.1).

Assumption 4.5 The risky funding cash flow process $\mathcal{F}$ is a $(\mathbb{G}, \mathbb{Q})$ finite variation martingale with nondecreasing $\mathcal{F}^\circ$ component. $\blacksquare$

The rationale here is that risky funding cash flows arise as the stochastic integral of predictable funding ratios against wealth processes of buy-and-hold strategies into funding assets related to the default of the bank. These wealth processes are assumed to be $(\mathbb{G}, \mathbb{Q})$ martingales. Therefore the risky funding (discounted) costs handled by the CA desk accumulate into a $(\mathbb{G}, \mathbb{Q})$ martingale $\mathcal{F}$: See Lemmas 7.6 and B.1 for concrete illustration below.

The assumption that $\mathcal{F}^\circ$ is nondecreasing rules out models where the bank could invest (not only borrow) at its unsecured borrowing spread over risk-free, because, as a consequence on $\mathcal{F}^\bullet$ through the martingale condition on the process $\mathcal{F}$ as a whole, this would imply that the bank can hedge its own jump-to-default exposure. That is, to ban the latter, we assume only windfall at bank own default, no shortfall (cf. Sections 5.3 and 9.1).

The last parts in Assumptions 4.4 (i.e. $\mathcal{H} = \mathcal{H}^\circ$) and 4.5 also insure some kind of "orthogonality" between the risky funding martingale $\mathcal{F}$ and the CA hedging loss martingale $\mathcal{H}$, so that $\mathcal{F}$ and $\mathcal{H}$ are nonsubstitutable with each other (the bank cannot manipulate by using one for the other, see again Sections 5.3 and 9.1).

4.3 Capital at risk and Cost of Capital

Not only a bank cannot hedge its own jump-to-default: it cannot replicate its counterparty default losses either. Then the regulator comes and requires that capital be set at risk by the shareholders, which therefore require a risk premium. Capital at risk (CR) is the resource of the bank devoted to cope with losses beyond their expected levels, the latter being already taken care of by reserve capital (RC). Counterparty default losses, as also funding payments, are materialities for default if not paid. In contrast, risk margin payments are at the discretion of the bank management, hence they do not represent an actual liability to the bank. Accordingly (see Section 9.2 for discussion):

Assumption 4.6 The risk margin is loss-absorbing, hence part of capital at risk. $\blacksquare$

As a consequence, shareholder capital at risk (SCR) is only the difference between the capital at risk (CR) of the bank and the risk margin (RM), i.e.

$$\text{SCR} = \text{CR} - \text{RM}. \tag{16}$$

The broad axiom that follows will be turned into a tangible KVA equation in Section 5.4:

Assumption 4.7 (Constant hurdle rate principle) An exogenous and constant hurdle rate $h \geq 0$ prevails, in the sense that KVA payments are such that, at any time t,

"Shareholder instantaneous average return$_t$ = hurdle rate $h \times \text{SCR}_t$." ∎   (17)

Remark 4.3 In principle, the choice of h to be used in KVA computations is a managerial decision of the bank. In practice, the level of compensation required by shareholders on their capital at risk in a firm is driven by market considerations. Typically, investors in banks expect a hurdle rate of about 10% to 12%. In a real-life environment where banks compete for clients (as opposed to our setup where only one bank is considered), an endogenous and stochastic implied hurdle rate arises from the competition between banks (see Remark 5.4 and Section B.4). ∎

In our setup, the duality of perspective of the clean versus CA desks, on pricing as reflected by Assumptions 4.1 and 4.2, also applies to risk measurement.

Capital calculations are typically made "on a going concern", hence assuming that the bank is not in default. Instead, cost of capital calculations are made by the management in a model including the default of the bank.

Moreover, in the context of XVA computations that entail projections over decades, the main source of information is market prices of liquid instruments, which allow the bank to calibrate its pricing models (cf. Remark 6.2 below), and there is little of relevance that can be said about the historical probability measure.

Accordingly:

Assumption 4.8 Capital and cost of capital computations are made with respect to the respective filtrations $\mathbb{F}$ and $\mathbb{G}$.

The estimates $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{Q}}$ of the historical probability measure used by the bank in its respective economic capital and cost of capital computations are the pricing measures $\mathbb{P}$ and $\mathbb{Q}$. ∎

Any discrepancies between $\widehat{\mathbb{P}}$ and $\mathbb{P}$ or $\widehat{\mathbb{Q}}$ and $\mathbb{Q}$ are left to model risk, meant to be included in an AVA (additional valuation adjustment) in an FRTB terminology. Model risk in the XVA context is an important and widely open issue, which we leave for future research (see e.g. Glasserman and Yang (2018) for a tentative treatment in the case of the CVA).

We emphasize that we calibrate the pricing measures to derivative market prices, including the corresponding credit premia, and then we perform all our (including capital) calculations based on these. If we were able to estimate risk premia reliably over time horizons that, in the XVA context, can be as long as fifty years into the future, then we could have a more sophisticated, hybrid approaches, with historical measures distinct from the pricing measures, deduced from the latter by deduction of corresponding risk premia. In the absence of such a reliable methodology, we do all the computations under pricing measures. As, in particular, implied CDS spreads are typically larger than statistical estimates of default probabilities, we believe that

this approach is conservative. Moreover, the hurdle rate h can be interpreted as a risk aversion parameter of the bank shareholders and the KVA as a corresponding risk premium: see Section B.4.

5   Derivation of the XVA Equations

In this section, based on the cost-of-capital pricing principles of Assumptions 4.2 and 4.7, we derive the XVA equations that determine the theoretical target levels for the RC and RM amounts to be maintained by the bank on its reserve capital, and risk margin accounts. We also characterize the theoretical target level of the CM amount on the clean margin account.

 The well-posedness of the ensuing equations in continuous time will be established in Section 7. So far, all our XVA processes are only sought for in an abstract vector space $\mathcal{S}_2$ of $\mathbb{Q}$ (typically square) integrable $\mathbb{G}$ adapted processes defined until time $\bar{\tau} = \tau \wedge T$. We denote by $\mathcal{S}_2^\circ$ the corresponding subspace of processes Y without jump at τ and such that $Y_T = 0$ holds on $\{T < \tau\}$.

Definition 5.1 The conditional expectation with respect to $(\mathfrak{G}_t, \mathbb{Q})$ (respectively $(\mathfrak{F}_t, \mathbb{P})$) is denoted by $\mathbb{E}_t$ (respectively $\mathbb{E}'_t$), or simply by $\mathbb{E}$ (respectively $\mathbb{E}'$) if $t = 0$.

 Given a $\mathbb{G}$, respectively $\mathbb{F}$ adapted cumulative cash flow stream $\mathcal{X}$, we call $(\mathbb{G}, \mathbb{Q})$, respectively $(\mathbb{F}, \mathbb{P})$ value process of $\mathcal{X}$ the process X such that (assuming the integrability implied by the related identity in (18))

$$X_t = \mathbb{E}_t(\mathcal{X}_{\bar{\tau}} - \mathcal{X}_t), \, t \in [0, \bar{\tau}], \text{ respectively } X_t = \mathbb{E}'_t(\mathcal{X}_T - \mathcal{X}_t), \, t \in [0, T] \, . \blacksquare \qquad (18)$$

Remark 5.1 Put differently, the $(\mathbb{G}, \mathbb{Q})$ value process of a $\mathbb{G}$ adapted cash flow stream $\mathcal{X}$ is the $\mathbb{G}$ adapted process X on $[0, \bar{\tau}]$ such that $X_{\bar{\tau}} = 0$ and $(X + \mathcal{X})$ is a $(\mathbb{G}, \mathbb{Q})$ martingale on $[0, \bar{\tau}]$; The $(\mathbb{F}, \mathbb{P})$ value process of an $\mathbb{F}$ adapted cash flow stream $\mathcal{X}$ is the $\mathbb{F}$ adapted process X on $[0, T]$ such that $X_T = 0$ and $(X + \mathcal{X})$ is an $(\mathbb{F}, \mathbb{P})$ martingale on $[0, T]$. $\blacksquare$

 Note that our hypothesis underlying Assumption 4.5 (see the comments following it) that wealth processes of buy-and-hold strategies in the funding assets are $(\mathbb{G}, \mathbb{Q})$ martingales is equivalent to saying that the risky funding assets are priced by the $(\mathbb{G}, \mathbb{Q})$ valuation rule in Definition 5.1 (see e.g. (55)-(56) below); The hypothesis underlying Assumption 4.3 (see the comments following it) that wealth processes of buy-and-hold strategies in ($\mathbb{F}$ adapted, "clean") hedging assets, inclusive of risk-free funding costs, are $(\mathbb{F}, \mathbb{P})$ martingales, is equivalent to saying that the hedging assets are priced by the $(\mathbb{F}, \mathbb{P})$ valuation rule in Definition 5.1.

 As immediate consequences of Assumptions 4.4–4.5:

Remark 5.2 The processes $\mathcal{H} = \mathcal{H}^\circ$ and $\mathcal{F}$ have zero $(\mathbb{G}, \mathbb{Q})$ value. $\blacksquare$

5.1 Clean Margin and Mark-to-Market

Definition 5.2 By clean valuation of a contract (or portfolio of contracts) with $\mathbb{F}$ adapted contractually promised cash flow stream $\mathcal{D}$, we mean the $(\mathbb{F}, \mathbb{P})$ value process of $\mathcal{D}$. ∎

Note that, for simplicity, we only consider European derivatives.

By linearity of the conditional expectation that is embedded in the $(\mathbb{F}, \mathbb{P})$ valuation operator, clean valuation is additive over contracts and intrinsic to the contracts themselves. As it only entails the promised cash flows, it is independent of the involved parties and of their collateralization, funding, and hedging policies.

In the notion of mark-to-market of the (cleaned or client) portfolio, counterparty risk is ignored for the future, but only the contracts not yet liquidated are considered:

Definition 5.3 Given a netting set c of the client portfolio, we denote by P^c the related clean value process.

By mark-to-market (MtM) of the (cleaned or client) portfolio, we mean

$$\text{MtM} = \sum_c P^c \mathbb{1}_{[0, \tau_c^\delta]} \tag{19}$$

on $[0, T]$. ∎

Proposition 5.1 *If we set, for each netting set c,*

$$\text{CM}^c = P^c \mathbb{1}_{[0, \tau_c^\delta]}, \tag{20}$$

then the $(\mathbb{F}, \mathbb{P})$ martingale condition on L^{cl} in Assumption 4.2 is satisfied.

Proof. By Definition 5.2 and Remark 5.1, the process $(P^c + \mathcal{P}^c)$ is an $(\mathbb{F}, \mathbb{P})$ martingale, whereas $\mathcal{H}^c$ is an $(\mathbb{F}, \mathbb{P})$ martingale by Assumption 4.3. The conclusion then follows in view of the first line in (9). ∎

Hereafter we work under the clean margin specification (20). Hence, by plugging (20) into the first line in (9), we obtain

$$L^{cl} = -\sum_c (P^c - P_0^c + \mathcal{P}^c - \mathcal{H}^c)^{\tau_c^\delta}; \tag{21}$$

Moreover, by (5), (20), and (19),

$$\text{CM} = \text{MtM}. \tag{22}$$

5.2 Contra-assets and Contra-liabilities Valuation

Definition 5.4 We call CA, CVA (credit valuation adjustment), and FVA (funding valuation adjustment), the solutions to the following fixed-point problems, assumed well-posed in $\mathcal{S}_2^{\circ}$:[4] For $t \leq \bar{\tau}$,

$$\mathrm{CA}_t = \mathbb{E}_t\big(\mathcal{C}_{\bar{\tau}}^{\circ} + \mathcal{F}_{\bar{\tau}}^{\circ} - \mathcal{C}_t^{\circ} - \mathcal{F}_t^{\circ} + \mathbb{1}_{\{\tau < T\}}\mathrm{CA}_\tau\big), \tag{23}$$

$$\mathrm{CVA}_t = \mathbb{E}_t\big(\mathcal{C}_{\bar{\tau}}^{\circ} - \mathcal{C}_t^{\circ} + \mathbb{1}_{\{\tau < T\}}\mathrm{CVA}_\tau\big), \tag{24}$$

$$\mathrm{FVA}_t = \mathbb{E}_t\big(\mathcal{F}_{\bar{\tau}}^{\circ} - \mathcal{F}_t^{\circ} + \mathbb{1}_{\{\tau < T\}}\mathrm{FVA}_\tau\big). \tag{25}$$

We define

$$\mathrm{CL} = \mathrm{DVA} + \mathrm{FDA}, \tag{26}$$

where, denoting by $\mathrm{CVA}^{\mathrm{CL}}$ and $\mathrm{FVA}^{\mathrm{CL}}$ the respective $(\mathbb{G}, \mathbb{Q})$ value processes of bullet cash flows $\mathbb{1}_{\{\tau < T\}}\mathrm{CVA}_\tau$ and $\mathbb{1}_{\{\tau < T\}}\mathrm{FVA}_\tau$ at time $\bar{\tau}$:

- DVA (debt valuation adjustment) is the sum between the $(\mathbb{G}, \mathbb{Q})$ value process of $\mathcal{C}^{\bullet}$ and $\mathrm{CVA}^{\mathrm{CL}}$;

- FDA (funding debt adjustment) is the sum between the $(\mathbb{G}, \mathbb{Q})$ value process of $\mathcal{F}^{\bullet}$ and $\mathrm{FVA}^{\mathrm{CL}}$.

We call fair valuation of counterparty risk, denoted by FV, the $(\mathbb{G}, \mathbb{Q})$ value of $(\mathcal{C} + \mathcal{F} + \mathcal{H})$. ■

Lemma 5.1 *We have*

$$\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}, \tag{27}$$

$$\mathrm{FVA} = \mathrm{FDA}, \tag{28}$$

and

$$\mathrm{FV} = \mathrm{CA} - \mathrm{CL} = \mathrm{CVA} - \mathrm{DVA}, \tag{29}$$

which is also the $(\mathbb{G}, \mathbb{Q})$ value process of $\mathcal{C}$.

Proof. The assumed uniqueness of $\mathcal{S}_2^{\circ}$ solutions to the CVA, FVA, and CA fix-point equations yields (27). Since $\mathcal{F}$ and $\mathcal{H}$ are zero $(\mathbb{G}, \mathbb{Q})$ valued (cf. Remark 5.2), the remaining statements directly follow from Definition 3.5. ■

Proposition 5.2 *If*

$$\mathrm{RC} = \mathrm{CA}, \tag{30}$$

then the $(\mathbb{G}, \mathbb{Q})$ martingale condition on L^{ca} in Assumption 4.2 is satisfied.

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[4]See Section 7 for concrete well-posedness results.

Proof. Assuming (30), then, by the second line in (9) we have $L^{ca} = \text{CA} - \text{CA}_0 + \mathcal{C}^\circ + \mathcal{F}^\circ + \mathcal{H}$. In view of the definition of CA and of the $(\mathbb{G}, \mathbb{Q})$ martingale property of $\mathcal{H} = \mathcal{H}^\circ$ postulated in Assumption 4.4, this process L^{ca} is a $(\mathbb{G}, \mathbb{Q})$ martingale. ∎

Hereafter we work under the reserve capital specification (30)**.** Hence, in particular, observe that our CVA and FVA incorporate the impact of the default of the bank on shareholders as per Assumption 2.2, whereby the residual amounts $\mathbb{1}_{\{\tau < T\}}\text{CVA}_\tau$ and $\mathbb{1}_{\{\tau < T\}}\text{FVA}_\tau$, summing up to $\mathbb{1}_{\{\tau < T\}}\text{CA}_\tau = \mathbb{1}_{\{\tau < T\}}\text{RC}_\tau$ as per (30), are transferred from the reserve capital account to bank bondholders at time τ.

Regarding the contra-liabilities, our DVA and FDA include CVA^{CL} and FVA^{CL}, which are contra-liability components of the CVA and the FVA, valuing the above transfers.

5.3 Wealth Transfer Analysis

Let us now temporarily assume, for the sake of the argument, that the bank would be able to hedge its own jump-to-default risk by selling further on the market the contra-liability cash flow stream $\mathcal{B} = \mathcal{C}^\bullet + \mathcal{F}^\bullet + \mathbb{1}_{\{[\tau,+\infty)\}}\text{CA}_\tau$ (cf. (30) and (13)), cash flow stream assumed priced in the market by its $(\mathbb{G}, \mathbb{Q})$ value process CL (cf. the comments following Remark 5.1). Then, in order to reach the target amount RC = CA as per (30) on the cash (reserve capital) account of the CA desk, given the external contribution CL to RC already provided by the proceeds of the above hedge, it becomes enough for the CA desks to require from clients (at time 0) and shareholders (later on) the diminished amount

$$\text{CA} - \text{CL} = \text{FV} = \text{FV}_0 + (\text{FV} - \text{FV}_0),$$

where FV_0 is provided by clients at portfolio inception and $(\text{FV} - \text{FV}_0)$ by shareholders through reset of the RC account at later times. Note that the risky funding cash flow process $\mathcal{F}$ has not changed (assuming the funding policy of the bank unchanged), because the total amount on the reserve capital account of the CA desk is still the same as before, given by the CA process as per (30). In particular, due to the above hedge, the contra-liability process is now given by $\mathcal{B} - \mathcal{B} = 0$. See Section B.3 for the corresponding argument in a more explicit one-period static setup.

This observation qualifies the name of "fair valuation of counterparty risk" for the process FV, as follows: Would the bank be able to hedge its own default, then the price of counterparty risk given by FV would indeed be fair, as clients paying FV_0 to the CA desk at time 0 and shareholders resetting the reserve capital account by $(\text{FV} - \text{FV}_0)$ at later times would constantly suffice to let the shareholders cover their future counterparty default and funding costs in conditional expectation, whilst leaving the bondholders break even at τ.

But, due to the impossibility for a bank to sell credit protection on itself, a bank cannot hedge its own default and the amount required by the shareholders to cover their future counterparty default and funding costs in conditionally expected terms is in fact CA = FV + CL, even if this entails a "gift" from the clients (who provide

$\mathrm{CA_0 = FV_0 + CL_0}$) and shareholders (who provide $\mathrm{CA - CA_0 = FV - FV_0 + CL - CL_0}$) to the bondholders, of value $\mathrm{CA - FV = CL}$. Note, however, that this seemingly gift is not necessarily a "positive arbitrage" for the bondholders: the process CL can be interpreted as a market price of the bankruptcy costs, which are outside the scope of an XVA model, that the bondholders need to face in order to liquidate the bank when it defaults.

In conclusion, we interpret $\mathrm{CA - FV = CL}$ as the wealth transfer from shareholders (were it not for compensation into prices by the clients) to bondholders triggered by the derivative portfolio of the bank, due to the inability of the bank to hedge its own jump-to-default exposure.

5.4 Economic Capital and KVA

We recall that the value-at-risk of a random variable (loss) ℓ and its expected shortfall (also known as average value-at-risk), both at some level α ($\geq 50\%$), respectively denote the quantile of level α of ℓ, which we denote by $q_\alpha(\ell)$, and $(1-\alpha)^{-1} \int_\alpha^1 q_a(\ell) da$. The latter is well defined if ℓ is integrable: see e.g. Föllmer and Schied (2016, Section 4.4) for these and other basic properties.

Capital requirements are focused on the solvency issue, because it is when a regulated firm becomes insolvent that the regulator may choose to intervene, take over, or restructure a firm. Economic capital (EC) is the level of capital at risk that a regulator would like to see on an economic basis. Specifically, Basel II Pillar II defines economic capital as the 99% value-at-risk of the negative of the variation over a one-year period of core equity tier I capital (CET1), the regulatory metric that represents the core financial strength of a bank.

As CL is bondholder wealth that cannot be monetized by shareholders (see Remark 2.4 and the next-to-last paragraph in Section 5.3), it is not included in CET1 (cf. (3) and Section A). Now, by (11), CET1 depletions correspond in our setup to the trading loss process L of the bank. Moreover, the FRTB required a shift from 99% value-at-risk to 97.5% expected shortfall as the reference risk measure in capital calculations. Accordingly, under Assumption 4.8 and with L', the $\mathbb{F}$ reduction of L, for "L understood on a going concern basis":

Definition 5.5 The (discounted) economic capital of the bank at time t, EC_t, is the $(\mathfrak{F}_t, \widehat{\mathbb{P}} = \mathbb{P})$ conditional $\alpha = 97.5\%$ expected shortfall of $(L'_{t+1} - L'_t)$. ∎

Lemma 5.2 EC *is nonnegative.*

Proof. As seen in Remark 4.2, the process L' is an $(\mathbb{F}, \mathbb{P})$ martingale. The result then follows from the fact that the α expected shortfall of a centered random variable ℓ is nonnegative (as $\mathbb{E}(\ell \mathbb{1}_{\{\ell > \alpha\}}) \geq 0$). ∎

In view of (10) and (16), and since L is a $(\mathbb{G}, \widehat{\mathbb{Q}} = \mathbb{Q})$ martingale (cf. Remark 4.2 and Assumption 4.8), also accounting for the risk-free remuneration to shareholders of the risk margin account, our risk premium principle (17) translates in mathematical

terms into the following generic KVA equation in $\mathcal{S}_2^\circ$ for the theoretical target RM level:

$$(-\text{KVA}) \text{ has the } (\mathbb{G}, \mathbb{Q}) \text{ drift coefficient } h\,(\text{CR} - \text{KVA}). \tag{31}$$

The last step is to fully specify the capital at risk (CR) in (31). Toward this aim, assume that, for any tentative (nonnegative) capital at risk process C in a suitable space $\mathcal{L}_2$ of $\mathbb{Q}$ integrable processes containing both $\mathcal{S}_2$ and the process EC, the equation (cf. (31))

$$(-K) \text{ has the } (\mathbb{G}, \mathbb{Q}) \text{ drift coefficient } h(C - K) \tag{32}$$

defines a unique process $K = K(C)$ in $\mathcal{S}_2^\circ$. Then:

Definition 5.6 The set of admissible capital at risk processes is

$$\mathcal{C} = \{C \in \mathcal{L}_2; C \geq \max\big(K(C), \text{EC}\big)\}. \; \blacksquare \tag{33}$$

In (33), $C \geq \text{EC}$ is the risk acceptability condition and $C \geq K(C)$ expresses that the risk margin $K(C)$ that would correspond through (31) (cf. (17)) to a tentative capital at risk C is part of capital at risk, by Assumption 4.6.

Consistent with (32) and (33), with also minimality of (an admissible) capital at risk and its cost in mind[5]:

Definition 5.7 We set

$$\text{CR} = \max(\text{EC}, \text{KVA}), \tag{34}$$

for the supposedly unique process (theoretical target RM level) KVA in $\mathcal{S}_2^\circ$ such that

$$(-\text{KVA}) \text{ has the } (\mathbb{G}, \mathbb{Q}) \text{ drift coefficient } h\big(\max(\text{EC}, \text{KVA}) - \text{KVA}\big). \; \blacksquare \tag{35}$$

In particular, the (discounted) KVA is a $(\mathbb{G}, \widehat{\mathbb{Q}} = \mathbb{Q})$ supermartingale with drift

$$-h\big(\max(\text{EC}, \text{KVA}) - \text{KVA}\big) = -h\big(\text{EC} - \text{KVA}\big)^+ = -h\text{SCR}.$$

Remark 5.3 As a sanity check, let us temporarily assume, for the sake of the argument, perfect clean but also CA hedges, in the sense that both processes L^{cl} and L^{ca} (hence also L and L') would vanish. That is, in view of (21) and (9), (30),

$$\mathcal{H}^c = P^c - P_0^c + \mathcal{P}^c, \text{ for each netting set c, and } \mathcal{H} = -\big(\text{CA} - \text{CA}_0 + \mathcal{C}^\circ + \mathcal{F}^\circ\big)$$

(assuming such $\mathcal{H}^c$ and $\mathcal{H}$ would be achievable hedging loss processes). Then EC vanishes, by Definition 5.5, and KVA $= 0$ obviously solves (35) in $\mathcal{S}_2^\circ$: There is no unhedged risk in this case, hence no capital at risk and no KVA risk premium. $\blacksquare$

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[5]Minimality to be formally established in the continuous-time setup of Section 7, along with the well-posedness of (35) in the first place, as Theorem 7.1 and Lemma 7.4 below.

Remark 5.4 The KVA formula (48) that will result from (35) in a full-fledged continuous time setup (see also Section B.4 in a static setup) can be used either in the direct mode, for computing the KVA corresponding to a given target hurdle rate h, or in the reverse-engineering mode, like the Black–Scholes model with volatility, for defining the "implied hurdle rate" associated with the actual RM level on the risk margin account of the bank.

Cost of capital proxies have always been used to estimate return on equity (ROE). Whether it is used in the direct or in the implied mode, the KVA is a refinement, dynamic and fine-tuned for derivative portfolios, but the base concept of ROE is far older than even the CVA.

Although, in the current state of the market, the KVA and even the MVA (which is included in the FVA in this paper) are rarely passed into entry prices, they are actively used for collateral and capital optimization purposes and detection of good/bad trade opportunities: This kind of balance sheet optimization is very active in top tier banks at the moment.

In any case, we stress that, beyond the notions of KVA level or of a corresponding implied hurdle rate, as the axiomatic introduction (17) emphasized, our KVA concept is a sustainable dividend release strategy in the first place. ∎

5.5 Shareholder Balance Conditions

Summing up our conclusions in this section:

Theorem 5.1 *Under a cost-of-capital XVA strategy applied to a portfolio held on a run-off basis, the following shareholder balance conditions hold at all times:*

$$\text{CM} = \text{MtM, RC} = \text{CVA} + \text{FVA, RM} = \text{KVA}, \tag{36}$$

where $\text{CA} = \text{CVA} + \text{FVA}$ *and KVA are given by the equations (23)–(25) and (35) (all assumed well posed in a suitable vector space* $\mathcal{S}_2^\circ$ *of* $\mathbb{Q}$ *integrable processes without jump at* τ*), while* MtM *is given by (19).*

The processes $\text{CA} = \text{CVA} + \text{FVA}$ *and* $\text{CL} = \text{DVA} + \text{FDA}$ *are the respective* $(\mathbb{G}, \mathbb{Q})$ *values of the contra-asset and contra-liability processes* $\mathcal{S}$ *and* $\mathcal{B}$ *in (13).*

Shareholder wealth $\text{SHC} = \text{SCR} + \text{UC}$ *satisfies*

$$\text{SHC} = \text{SHC}_0 - (L + \text{KVA} - \text{KVA}_0), \tag{37}$$

which is a $(\mathbb{G}, \mathbb{Q})$ *submartingale with drift coefficient* $h\text{SCR}$*, without jump at* τ *(the trading loss process* $L = \text{CET1}_0 - \text{CET1}$ *in (37) is a* $(\mathbb{G}, \mathbb{Q})$ *martingale).*

Proof. The shareholder balance conditions (36) are the synthesis of Proposition 5.1 and the ensuing specification (20) for CM^c, Proposition 5.2 and the ensuing specification (30) for RC, and of our RM specification in the form of a KVA as per (35). All the subsequent statements in the theorem until (37) included then follow from (10) and (13). The last part of the theorem follows from the facts that L is a $(\mathbb{G}, \mathbb{Q})$ martingale while KVA is a $(\mathbb{G}, \mathbb{Q})$ supermartingale with drift coefficient $(-h\text{SCR})$ (see the comment

following Definition 5.7), both without jump at τ (regarding L , see (11) and Remark 4.2). ∎

One can also note that, equivalently to the $(\mathbb{G}, \mathbb{Q})$ submartingale property without jump at τ of SHC in Theorem 5.1, the $\mathbb{F}$ reduction SHC$'$ of SHC is an $(\mathbb{F}, \mathbb{P})$ submartingale with drift coefficient hSCR$'$ on $[0, T]$.

5.6 Connection With the Modigliani-Miller Theory

The Modigliani-Miller invariance result, with Modigliani and Miller (1958) as a seminal reference, consists in various facets of a broad statement that the funding and capital structure policies of a firm are irrelevant to the profitability of its investment decisions. Modigliani-Miller (MM) irrelevance, as we put it for brevity hereafter, was initially seen as a pure arbitrage result. However, it was later understood that there may be market incompleteness issues with it. So quoting Duffie and Sharer (1986, page 9), "generically, shareholders find the span of incomplete markets a binding constraint [...] shareholders are not indifferent to the financial policy of the firm if it can change the span of markets (which is typically the case in incomplete markets)"; or Gottardi (1995, page 197): "When there are derivative securities and markets are incomplete the financial decisions of the firm have generally real effects".

A situation where shareholders may "find the span of incomplete markets a binding constraint" is when market completion is legally forbidden. This may seem a narrow situation but it is precisely the XVA case, which is also at the crossing between market incompleteness and the presence of derivatives pointed out above as the MM non irrelevance case in Gottardi (1995). Specifically, the contra-assets and contra-liabilities that emerge endogenously from the impact of counterparty risk on the derivative portfolio of a bank cannot be "undone" by shareholders, because jump-to-default risk cannot be replicated by a bank.

As a consequence, MM irrelevance is expected to break down in the XVA setup. In fact, as will be directly visible on the trade incremental FTP (pricing) formula (38), cost of funding and cost of capital are material to banks and need be reflected in entry prices for ensuring shareholder indifference to the trades.

For further related discussion, see Andersen, Duffie, and Song (2017, Section VII.A).

6 Incremental XVA Approach, Funds Transfer Price, and Cost of Run-Off Interpretation

So far we worked under Assumption 3.1 that the derivative portfolio of the bank is held on a run-off basis, i.e. set up at time 0 and assuming that no new unplanned trades will ever enter the portfolio in the future. Then the balance conditions (36) hold at all times.

But, in practice derivative portfolios are incremental. Yet, given the derivative porfolio of a bank at time 0, the considerations of the previous sections can be applied

to the version of the portfolio that **would be** run-off by the bank from time 0 onward. In this perspective, the above notions of MtM and XVAs can therefore be interpreted as the appropriate amounts to be maintained by the bank on its clean margin, reserve capital, and risk margin accounts (cf. (36)), so that the bank **could go** into run-off, while staying in line with shareholder interest, i.e. preserving the hurdle rate h, if wished. We refer to this possibility as the "soft landing option" for the bank. Moreover, since we relied on a dynamic analysis, such a soft landing option, for a bank respecting the balance conditions (36), is granted not only from time 0 onward, but from any future time onward, as long as there are no new deals in the portfolio.

Now, when a new deal shows up at some time $\theta > 0$, it has three impacts: it modifies the mark-to-market of the portfolio (cf. Section 5.1), it triggers a wealth transfer from clients and shareholders to bondholders (cf. Section 5.2-5.3), and it alters the risk profile of the portfolio (cf. Section 5.4). Hence the shareholder balance conditions are impaired by the deal, unless somebody fills in the missing amounts on the clean margin, reserve capital, and risk margin accounts to restore them. Since shareholders have the control of the bank before bank default, it is natural to postulate that, for a new deal to occur, shareholders should be indifferent to it, in the following sense:

Assumption 6.1 (Preservation of the hurdle rate throughout a new deal) The entry price of a new deal should be set at a level allowing the shareholder balance conditions (36) (with constant and unchanged hurdle rate h, in particular) to be preserved throughout the deal, without modification of the shareholder equity[6] SHC = SCR + UC. ∎

Denoting by $\Delta\cdot$, for any portfolio related process, the difference between the time θ value of this process for the run-off version of the portfolio with and without the new deal, we propose the following pricing and accounting scheme in case a new contract with clean valuation P is concluded at time θ (deal obviously assumed not part of a netting set already involved in a liquidation procedure, hence $\Delta\text{MtM} = P$):

- The clean desks pay P to the client; If $P > 0$, then an amount P is added by the CA desk on the clean margin account, otherwise an amount $(-P)$ is added to this account by the clean desks;

- The CA desk charges to the client an amount ΔCA and adds it on[7] the reserve capital account;

- The management of the bank charges the amount ΔKVA to the client and adds it on[8] the risk margin account.

Proposition 6.1 *The above-described trade incremental pricing and accounting scheme satisfies Assumption 6.1. The ensuing all-inclusive XVA add-on passed by the bank to*

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[6]i.e. shareholder wealth, see Remark 2.4.
[7]i.e. removes $|\Delta\text{CA}|$ from, if $\Delta\text{CA} < 0$.
[8]i.e. removes $|\Delta\text{KVA}|$ from, if $\Delta\text{KVA} < 0$.

*the client of the deal, called funds transfer price (*FTP*), amounts to*

$$\begin{aligned} \text{FTP} &= \Delta\text{CA} + \Delta\text{KVA} = \Delta\text{CVA} + \Delta\text{FVA} + \Delta\text{KVA} \\ &= \Delta\text{FV} + \Delta\text{CL} + \Delta\text{KVA}, \end{aligned} \tag{38}$$

which is interpreted as the rebate to deduct from the clean valuation P of the contract in order to preserve the hurdle rate h of bank shareholders, without altering their wealth $\text{SHC} = \text{SCR} + \text{UC}$, throughout the deal.

Proof. By construction of the above FTP scheme, the soft landing option with constant hurdle rate h, i.e. the balance conditions (36), are preserved throughout the deal. Moreover, it is the client that provides all the required amounts on the clean margin, reserve capital and risk margin accounts of the bank, hence $\Delta\text{SHC} = 0$.

The ensuing overall client price is $\text{MtM} - \Delta\text{CA} - \Delta\text{KVA} = \text{MtM} - \text{FTP}$ with FTP as per (38), in view of the identities (27) and (29) applied to the portfolios with and without the new deal. The market incompleteness interpretation follows from the analyses of Section 5.3 and Remark 5.3. ∎

Hereafter we stick to the FTP scheme considered in Proposition 6.1.

In view of the FTP formula (38), the only XVAs that ultimately matter in entry prices are the CVA, the FVA, and the KVA. Obviously, the endowment (legacy portfolio of the bank) has a key impact on the FTP of a new trade. It may very well happen that a new deal is risk-reducing with respect to the endowment, in which case FTP < 0, i.e. the overall XVA-inclusive price charged by the bank to the client would be $P - \text{FTP} > P$ (subject of course to the commercial attitude adopted by the bank under such circumstance).

Remark 6.1 In an asymmetric setup with a price maker and a price taker, the price maker passes his costs to the price taker. For transactions between dealers, it is possible that one is the price maker and the other one is the price taker. It is also possible that a transaction triggers gains for the shareholders of both entities. The detailed consideration of these dynamics would lead to an understanding of the drivers to economical equilibrium in a situation where multiple dealers are present (as opposed to our setup where only one bank is considered). ∎

Remark 6.2 Market quotes used for model calibration are typically prices provided by the clean desks of different banks at time 0. Hence, in view of the first bullet point in Proposition 6.1, clean valuation is the relevant notion of valuation for model calibration purposes. ∎

6.1   Cost-of-Capital XVA Rolling Strategy

We assume the following cost-of-capital XVA strategy:

- between time 0 and the next deal, as if no new deal should ever enter the portfolio, as per the balance conditions (36),

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- the way described before Proposition 6.1 at the ($\mathbb{G}$ stopping) time θ (whenever finite) of the next deal.

The processes L and KVA of Section 5, as well as the amounts on the different banking accounts, are extended from $[0, \bar{\tau}]$ to $[0, +\infty) \cap [0, \tau]$ by constancy from time T onward.

Lemma 6.1 *Under the above cost-of-capital XVA strategy, assuming the processes L and KVA do not jump at time θ, shareholder equity (i.e. wealth) SHC is a $(\mathbb{G}, \mathbb{Q})$ submartingale on $[0, +\infty) \cap [0, \tau \wedge \theta]$, with drift coefficient hSCR.*

Proof. Assuming the new trade at time θ handled by the trade incremental policy of Proposition 6.1 after the balance conditions (36) have been held before θ, then, by Theorem 5.1, (37) holds on $[0, +\infty) \cap [0, \tau] \cap [0, \theta)$. Moreover, by invariance of SHC at the new deal and having assumed L and KVA do not jump at θ, (37) still holds at θ itself. This establishes (37) on $[0, +\infty) \cap [0, \tau \wedge \theta]$. The result then follows from the submartingale property that holds on $[0, +\infty)$ regarding the process $\text{SHC}_0 - (L + \text{KVA} - \text{KVA}_0)$ in (37), by the last part in Theorem 5.1. ∎

In view of the nature of the processes L and KVA, which are respectively martingales and solutions to BSDEs, the technical assumption that these processes do not jump at time θ in Lemma 6.1 holds naturally if θ is a predictable stopping time in a continuous time setup. This is a reasonable assumption regarding the time at which a financial contract is concluded. It was actually already implicitly assumed regarding the fixed time 0 at which the portfolio of the bank is supposed to have been setup in the first place.

The cost-of-capital XVA strategy described before Lemma 6.1 can be iterated at every new trade. We call this approach the ***cost-of-capital XVA rolling strategy***. By an iterated application of Lemma 6.1 at every new trade, we obtain the following:

Theorem 6.1 *Under the cost-of-capital XVA rolling strategy, assuming the trading loss L and KVA processes relative to each successive portfolio do not jump at the time where the following deal is concluded, the shareholder equity (i.e. wealth) SHC is a $(\mathbb{G}, \mathbb{Q})$ submartingale on $[0, +\infty) \cap [0, \tau]$ with drift coefficient hSCR.* ∎

As already noted about Lemma 6.1, the technical assumption about "the processes L and KVA relative to each successive portfolio" is not an issue in Theorem 6.1. Also note that Lemma 6.1 and Theorem 6.1 are in fact equally valid with $(\mathbb{F}, \mathbb{P})$ instead of $(\mathbb{G}, \mathbb{Q})$ (cf. Remark 4.2 regarding L), replacing all the processes involved by $\mathbb{F}$ reductions and τ by $+\infty$.

Hence, the application of the FTP (38) as a rebate on the clean valuation of every new deal, associated with the clean collateralization, accounting, and dividend policies preserving the balance conditions (36) between deals, ensure to the bank the possibility of "soft landing" its portfolio, with a constant hurdle rate h, from any point in time onward if wished. Thus we arrive to a sustainable strategy for profits retention, which is already the key principle behind Solvency II.

Note that, without the KVA (i.e. for $h = 0$), the (discounted) shareholder wealth process would only be a risk-neutral martingale, which could only be acceptable to shareholders without risk aversion (cf. Section B.4).

6.2 Fine-Tuning of the Assumptions

In view of Proposition 6.1, XVA computations, whether they regard a new trade, which is then treated on an incremental run-off basis, or targeting the preservation of the shareholder balance conditions (36) between deals, only entail portfolios that would be held on a run-off basis. Hence, **in the following we may and do restrict attention to the case of a portfolio held on a run-off basis**.

Moreover, in view of the above soft landing interpretation, it is natural to perform the XVA computations under the following assumption, in line with a run-off procedure where market risk is first hedged out by the bank, but we conservatively assume no XVA hedge, and the portfolio is then let to amortize until its final maturity:

Assumption 6.2 We assume a perfect clean hedge by the clean traders, i.e. L^{cl} constant, which can be taken as zero as trading loss processes only matter through their fluctuations, and we assume no CA hedge, i.e. $\mathcal{H} = 0$. ∎

As it then immediately follows from (4), (9), (12), and (30), we thus have

$$
\begin{aligned}
&\Gamma^{ca} = -(\mathrm{CA} - \mathrm{CA}_0 + \mathcal{C} + \mathcal{F}), \\
&L = L^{ca} = \mathrm{CA} - \mathrm{CA}_0 + \mathcal{C}^\circ + \mathcal{F}^\circ, \\
&\mathcal{D} = -(\mathcal{C}^\circ + \mathcal{F}^\circ + \mathrm{CA} - \mathrm{CA}_0 + \mathrm{KVA} - \mathrm{KVA}_0).
\end{aligned}
\tag{39}
$$

In particular, the process L, which is the input to the capital and KVA computations (cf. (35) and Definition 5.5), happens to be an output of the $\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}$ computations.

Moreover, the rules regarding the settlement of contracts at defaults are that:

Assumption 6.3 On the client portfolio side, at the time a party (a client or the bank itself) defaults, the property of the collateral posted on each involved netting set is transferred to the collateral receiver. Moreover, at the liquidation time of each netting set c:

- any positive value due by a nondefaulted party on this netting set is paid in full to its bondholders,

- any positive value due by a defaulted counterparty on this netting set is only paid up to some exogenously specified recovery rate,

where value is here understood on a counterparty risk free basis as P^c, net of the corresponding (already transferred) client collateral, but inclusive of all the promised contractual cash flows unpaid during the liquidation period;

On the cleaned portfolio side, at each liquidation time, the property of the corresponding clean margin amount $\mathrm{CM}^c = P^c$ is transferred from the clean margin paying

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party to the clean margin receiving party (i.e. from the CA desk to the clean desks in case P^c at liquidation time is positive and vice versa in case P^c at liquidation time is negative). ∎

In concrete setups, taking the difference between the client and cleaned portfolio cash flows as per the two bullet points in Assumption 6.3 yields the counterparty exposure cash flows $\mathcal{C}$. Hence, a major underlying to the counterparty exposure $\mathcal{C}$ and, in turn, risky funding processes $\mathcal{F}$ is the MtM process: See e.g. Lemmas 7.6–7.7 and B.1.

These connections make the MtM, CA, and KVA equations, hence the derivative portfolio pricing problem as a whole, a self-contained problem under a cost-of-capital XVA approach.

Clean valuation that is involved in MtM computations is not the main topic of the paper. Hence we focus on the XVA equations in the sequel.

As a final note, in practice, the equity capital of the bank (capital at risk CR and uninvested capital UC beyond it) can be used by the bank for its funding purposes (provided shareholders agree with it, in exchange of some risk-free remuneration). As developed in Crépey, Élie, Sabbagh, and Song (2018), this induces an intertwining of capital at risk (CR) and the FVA. Instead, for simplicity in the sequel:

Assumption 6.4 The bank does not use its equity capital (CR + UC) for funding purposes. ∎

7   XVA Equations Well-Posedness and Comparison Results

In Section 5, we formulated the CVA, FVA, and KVA equations as fix-point problems (24), (25), and (35) in an abstract space $\mathcal{S}_2^\circ$ of (square) integrable processes without jump at the bank default time τ. In this section, we establish concrete well-posedness and comparison results for these problems in the continuous-time setup. See also Section B for a stylized cost-of-capital XVA approach in a one-period static setup.

7.1   Continuous-Time Setup

In continuous time, Assumptions 4.1 and 4.2, with filtrations $\mathbb{F}$ and $\mathbb{G}$ satisfying the usual conditions, martingale understood as local martingale, and existence of $\mathbb{F}$ semi-martingale reductions reinforced[9] into existence of an "$\mathbb{F}$ predictable reduction" on $[0, T]$ coinciding until τ (included) with any $\mathbb{G}$ predictable process, mean that τ is an invariance time as per Crépey and Song (2017b), with so called invariance probability measure $\mathbb{P}$. This covers the mainstream immersion setup (but not only, see e.g. Crépey and Song (2017a)), where $(\mathbb{F}, \mathbb{Q})$ local martingales are $(\mathbb{G}, \mathbb{Q})$ local martingales without jump at τ, in which case τ is an invariance time with $\mathbb{P} = \mathbb{Q}$.

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[9]As shown in Song (2016).

We recall the following result, where $J = \mathbb{1}_{[0,\tau)}$ denotes the survival indicator process of τ:

Theorem 3.5 in Crépey and Song (2017b) *Assuming that an $\mathbb{F}$ predictable reduction of any $\mathbb{G}$ predictable process exists and that $\mathsf{S}_T = \mathbb{Q}(\tau > T | \mathfrak{F}_T) > 0$, then*

$$
\begin{array}{c}
\textit{any } \mathbb{G} \textit{ optional process admits a unique } \mathbb{F} \textit{ optional} \\
\textit{reduction coinciding with it before } \tau \textit{ on } [0, T].
\end{array} \tag{40}
$$

If, moreover, τ has a $(\mathbb{G}, \mathbb{Q})$ intensity process $\gamma = \gamma J_-$ such that $e^{\int_0^\tau \gamma_s ds}$ is $\mathbb{Q}$ integrable, then the existence on $(\Omega, \mathcal{A})$ and uniqueness on $(\Omega, \mathfrak{F}_T)$ of an invariance probability measure $\mathbb{P}$ hold. ∎

Hereafter we work in the sequel under the corresponding specialization of Assumptions 4.1 and 4.2 and we assume that γ is $\mathbb{F}$ predictable, without loss of generality by $\mathbb{F}$ reduction.

As can be established by section theorem, for any $\mathbb{G}$ progressive Lebesgue integrand X such that the $\mathbb{G}$ predictable projection ${}^p X$ exists,[10] the indistinguishable equality $\int_0^\cdot {}^p X_s ds = \int_0^\cdot X_s ds$ holds. As a consequence, one can actually consider the $\mathbb{F}$ reduction X' of any $\mathbb{G}$ progressive Lebesgue integrand X (even if this means replacing X by ${}^p X$).

All the value, XVA, and price processes are now continuous-time processes modeled as $\mathbb{G}$ (or "clean" $\mathbb{F}$) semimartingales (in a càdlàg version).

In continuous time, the process Y° in (7) is defined, for any left-limited process Y, by

$$
Y^\circ = Y^{\tau-} = JY + (1-J)Y_{\tau-}.
$$

The CVA, FVA, and KVA fix-point problems (24), (25), and (35) are then a matter of backward stochastic differential equations (BSDEs) involving the following spaces of processes (recall $\mathbb{E}$ and $\mathbb{E}'$ denote $\mathbb{Q}$ and $\mathbb{P}$ expectations):

- $\mathcal{S}_2$, the space of càdlàg $\mathbb{G}$ adapted processes Y over $[0, \bar{\tau}]$ such that, denoting $Y_t^* = \sup_{s \in [0,t]} |Y_s|$:

$$
\mathbb{E}\left[Y_0^2 + \int_0^T e^{\int_0^s \gamma_u du} \mathbb{1}_{\{s < \tau\}} d(|Y_s^*|^2) \right] = \mathbb{E}'\left[\sup_{t \in [0,T]} |Y_t'|^2 \right] < \infty, \tag{41}
$$

  where the equality follows from Lemma 5.1 in Crépey, Élie, Sabbagh, and Song (2018);

- $\mathcal{S}_2^\circ$, the subspace of the processes in $\mathcal{S}_2$ such that Y is without jump at τ and $Y_T = 0$ holds on $\{T < \tau\}$;

---

[10] For which σ integrability of X valued at any stopping time, e.g. X bounded or càdlàg, is enough.

- $\mathcal{L}_2$, the space of $\mathbb{G}$ progressive processes X over $[0, T]$ such that

$$\mathbb{E}\Big[\int_0^T e^{\int_0^s \gamma_u du} \mathbb{1}_{\{s < \tau\}} X_s^2 ds \Big] = \mathbb{E}'\Big[\int_0^T |X_t'|^p dt \Big] < +\infty, \tag{42}$$

where the equality follows from Lemma 5.1 in Crépey, Élie, Sabbagh, and Song (2018);

- $\mathcal{S}_2'$ and $\mathcal{L}_2'$, the respective spaces of $\mathbb{F}$ adapted càdlàg and $\mathbb{F}$ progressive processes Y' and X' over $[0, T]$ such that $Y_T' = 0$ and that make the corresponding norm finite in the right-hand side of (41) or (42).

Finally, we postulate a classical weak martingale representation setup a la Jacod (1980).

7.2 KVA in the Case of a Default-Free Bank

Note that the primary raison d'etre for the KVA is the default of bank clients, as opposed to the default of the bank itself (which on the other hand is the key of the contra-liabilities wealth transfer issue, cf. Section 5.3). In this section we temporarily suppose the bank default free, i.e., formally,

$$\text{``}\tau = +\infty, \ (\mathbb{F}, \mathbb{P}) = (\mathbb{G}, \mathbb{Q}).\text{''}$$

The results are then extended to the case of a defaultable bank in Section 7.3.

At that stage we use the "$\prime$" notation, not in the sense of $\mathbb{F}$ reduction (as $\mathbb{F} = \mathbb{G}$), but simply in order to distinguish the equations in this part, where $\mathbb{F} = \mathbb{G}$, from the ones in Section 7.3, where $\mathbb{F} \neq \mathbb{G}$ (the present data will then be interpreted as the $\mathbb{F}$ reductions of the corresponding data in Section 7.3).

Given $C' \geq \text{EC} \geq 0$ representing a putative capital at risk process for the bank, we consider the following BSDEs (cf. (32) and (35) with τ set to ∞):

$$K_t' = \mathbb{E}_t' \int_t^T h\big(C_s' - K_s'\big) ds, \ t \in [0, T] \tag{43}$$

$$\text{KVA}_t' = \mathbb{E}_t' \int_t^T \big(h(\text{EC}_s - \text{KVA}_s')^+ ds, \ t \in [0, T], \tag{44}$$

to be solved for respective processes K' and KVA$'$.

Lemma 7.1   • *If C' is in $\mathcal{L}_2'$, then the the K' BSDE (43) is well posed in $\mathcal{S}_2'$, with $\mathcal{S}_2'$ solution*

$$K_t' = h\mathbb{E}_t' \int_t^T e^{-h(s-t)} C_s' ds, \ t \in [0, T] \tag{45}$$

(which is nonnegative, like C');

- If EC *is in* $\mathcal{L}'_2$, *then the* KVA' *BSDE* (44) *is well posed in* $\mathcal{S}'_2$,

 where well-posedness here and later includes square integrable existence, uniqueness, and comparison[11].

Proof. The KVA' BSDE (44) has a Lipschitz coefficient

$$k_t(y) = h\big(\mathrm{EC}_t - y\big)^+, \, y \in \mathbb{R}. \tag{46}$$

Therefore, by standard results (see e.g. Becherer (2006) or Kruse and Popier (2016)), the BSDE (44) is well-posed in $\mathcal{S}'_2$ if EC is in $\mathcal{L}'_2$, where well-posedness includes existence, uniqueness, and comparison (note that comparison is not an issue in our setup, even with jumps, where our coefficient k "only depends on y", cf. e.g. Kruse and Popier (2016, Assumption (H3') and Proposition 4)). Even simpler considerations prove the statements pertaining to the linear BSDE (43). Finally, (45) obviously solves (43). ∎

To emphasize the dependence on C', we henceforth denote by $K' = K'(C')$ the solution (45) to the linear BSDE (43). Assuming that EC is in $\mathcal{L}'_2$, we define the set of admissible capital at risk processes as (cf. (33))

$$\mathcal{C}' = \{C' \in \mathcal{L}'_2; C' \geq \max\big(\mathrm{EC}, K'(C')\big)\}$$

and we set (cf. (34))

$$\mathrm{CR}' = \max(\mathrm{EC}, \mathrm{KVA}'), \tag{47}$$

where KVA' is the $\mathcal{S}'_2$ solution to (44).

Lemma 7.2 *Assuming that* EC *is in* $\mathcal{L}'_2$, *the* $\mathcal{S}'_2$ *solution* KVA' *to* (44) *solves the linear BSDE* (43) *for the implicit data* $C' = \mathrm{CR}'$ *in* (47), *i.e. we have* $\mathrm{KVA}' = K'(\mathrm{CR}')$, *that is,*

$$\mathrm{KVA}'_t = h\mathbb{E}'_t \int_t^T e^{-h(s-t)} \max(\mathrm{EC}_s, \mathrm{KVA}'_s) ds, \, t \in [0, T] \tag{48}$$

(which is nonnegative, like EC*).*

Proof. The process KVA' is in $\mathcal{S}'_2$ and, by virtue of (44) and (47), we have, for $t \in [0, T]$,

$$\begin{aligned}
\mathrm{KVA}'_t &= \mathbb{E}'_t \int_t^T \Big(h\big(\mathrm{EC}_s - \mathrm{KVA}'_s\big) + ds \\
&= \mathbb{E}'_t \int_t^T h\Big(\mathrm{CR}'_s - \mathrm{KVA}'_s\Big) ds.
\end{aligned} \tag{49}$$

Hence, the process KVA' solves the linear BSDE (43) for $C' = \mathrm{CR}'$ in $\mathcal{L}'_2$. The identity $\mathrm{KVA}' = K'(\mathrm{CR}')$ follows by uniqueness of an $\mathcal{S}'_2$ solution to the linear BSDE (43) established in Lemma 7.1. Equation (48) then follows by an application of (45) with $C' = \mathrm{CR}'$ as per (47). ∎

---

[11] "Ordered data implies ordered solutions", see e.g. Kruse and Popier (2016, Proposition 4).

Remark 7.1 The KVA formula (48) appears as a continuous-time analog of the risk margin formula under the Swiss solvency test cost of capital methodology. see Swiss Federal Office of Private Insurance (2006, Section 6, middle of page 86 and top of page 88). ∎

Proposition 7.1 *Under the assumptions of Lemma 7.2, we have:*
(i) $\text{CR}' = \min \mathcal{C}', \text{KVA}' = \min_{C' \in \mathcal{C}'} K'(C')$;
(ii) *The process* KVA' *is nondecreasing in the hurdle rate* h.

Proof. **(i)** We saw in Lemma 7.2 that $\text{KVA}' = K'(\text{CR}')$, hence

$$\text{CR}' = \max(\text{EC}, \text{KVA}') = \max\big(\text{EC}, K'(\text{CR}')\big),$$

therefore $\text{CR}' \in \mathcal{C}'$. Moreover, for any $C' \in \mathcal{C}'$, we have (cf. (46)):

$$k_t(K'_t(C')) = h\big(\text{EC}_t - K'_t(C')\big)^+ \leq h(C'_t - K'_t(C')).$$

Hence, the coefficient of the KVA BSDE (44) never exceeds the coefficient of the linear BSDE (43) when both coefficients are evaluated at the solution $K'_t(C')$ of (43). Since these are BSDEs with equal (null) terminal condition, the BSDE comparison principle applied to the BSDEs (43) and (44) yields $\text{KVA}' \leq K'(C')$. Consequently, $\text{KVA}' = \min_{C' \in \mathcal{C}'} K(C')$ and, for any $C' \in \mathcal{C}'$,

$$C' \geq \max(\text{EC}, K'(C')) \geq \max(\text{EC}, \text{KVA}') = \text{CR}'.$$

Hence $\text{CR}' = \min \mathcal{C}'$.
(ii) By Lemma 5.2, EC is nonnegative. Hence, in view of (46), the coefficient $k_t(y)$ of the KVA' BSDE (44) is nondecreasing in the parameter h. So is therefore the $\mathcal{S}'_2$ solution KVA' to (44), by the BSDE comparison theorem of Kruse and Popier (2016, Proposition 4) applied to the BSDE (44) for different values of the parameter h. ∎

7.3 KVA in the Case of a Defaultable Bank

In the case of a defaultable bank, specializing to the continuous-time setup of Section 7.1 the abstract equations of Section 5.4, "·'" now denoting $\mathbb{F}$ reduction (predictable, optional, or progressive, as applicable):

Lemma 7.3 *For any* $C \in \mathcal{L}_2$, *we have* $C' \in \mathcal{L}'_2$, *and the* $(\mathbb{G}, \mathbb{Q})$ *BSDE (32) for a process* $K = K(C)$ *in* $\mathcal{S}^\circ_2$ *is equivalent, via* $\mathbb{F}$ *reduction, to the* $(\mathbb{F}, \mathbb{P})$ *BSDE (43) for a process* $K'(C')$ *in* $\mathcal{S}'_2$. *Assuming* EC *in* $\mathcal{L}'_2$, *the* $(\mathbb{G}, \mathbb{Q})$ *KVA BSDE (35) in* $\mathcal{S}^\circ_2$ *is equivalent, via* $\mathbb{F}$ *reduction, to the* $(\mathbb{F}, \mathbb{P})$ *KVA' BSDE (44) in* $\mathcal{S}'_2$.

Proof. We only prove the stated equivalence between (35) and (44), as the one between (32) and (43) can be established similarly. First we show the equivalences in the sense of local martingale solutions between the differential formulations of (35) and (44), i.e.

$$\begin{aligned} & \text{KVA}_T = 0 \text{ on } \{T < \tau\} \text{ and, for } t \in (0, \bar{\tau}], \\ & d\text{KVA}^{\tau-}_t = -h\big(\text{EC}_t - \text{KVA}_t\big)^+ dt + d\nu_t, \end{aligned} \tag{50}$$

where ν denotes some $(\mathbb{G}, \mathbb{Q})$ local martingale stopped before τ, respectively

$$\mathrm{KVA}'_T = 0 \text{ and, for } t \in (0, T],$$
$$d\mathrm{KVA}'_t = -h\big(\mathrm{EC}_t - \mathrm{KVA}'_t\big)^+ dt + d\mu_t, \tag{51}$$

for some $(\mathbb{F}, \mathbb{P})$ local martingale μ.

By definition of $\mathbb{F}$ reductions (see Section 7.1), the terminal condition in (51) obviously implies the one in (50). Conversely, taking the $\mathcal{F}_T$ conditional expectation of the terminal condition in (50) yields

$$0 = \mathbb{E}[\mathrm{KVA}_T \mathbb{1}_{\{T < \tau\}} | \mathcal{F}_T] = \mathbb{E}[\mathrm{KVA}'_T \mathbb{1}_{\{T < \tau\}} | \mathcal{F}_T] = \mathrm{KVA}'_T \mathsf{S}_T,$$

hence $\mathrm{KVA}'_T = 0$ (as by assumption $\mathsf{S}_T > 0$, see above (40)), which is the terminal condition in (51).

The martingale condition in (51) implies the one in (50), by stopping before τ and application to $\nu = \mu^{\tau-}$ of the second part in Assumption 4.2 (specialized to continuous time local martingales as explained in the beginning of Section 7.1). Conversely, assuming the martingale condition in (50) implies, by application to $\mu = \nu'$ of the third part in Assumption 4.2, that $(\mathrm{KVA}', \mu = \nu')$ satisfies the martingale condition in (51) on $[0, \bar{\tau}]$, hence on $[0, T]$, by (40).

Given this equivalence between (50) and (51), i.e. between (35) and (44), in the sense of local martingale solutions, if KVA solves the $(\mathbb{G}, \mathbb{Q})$ BSDE (35) in $\mathcal{S}_2^\circ$, hence in the sense of local martingale solutions, then KVA$'$ solves (44) in the sense of local martingale solutions and therefore in $\mathcal{S}'_2$, by virtue of the identity (41). Conversely, if KVA$'$ solves (44) in $\mathcal{S}'_2$, then KVA solves the $(\mathbb{G}, \mathbb{Q})$ BSDE (35) in the sense of local martingale solutions and therefore in $\mathcal{S}_2^\circ$, by (41). ∎

Lemma 7.4   • If $C \in \mathcal{L}_2$, hence $C' \in \mathcal{L}'_2$ (cf. the identity (42)), then the $(\mathbb{G}, \mathbb{Q})$ linear BSDE (32) for $K = K(C)$ is well posed in $\mathcal{S}_2^\circ$ and the $\mathbb{F}$ optional reduction K' of its $\mathcal{S}_2^\circ$ solution K is the $\mathcal{S}'_2$ solution to the continuous-time version (43) of (32);

• If EC is in $\mathcal{L}'_2$, then the $(\mathbb{G}, \mathbb{Q})$ KVA BSDE (35) is well posed in $\mathcal{S}_2^\circ$ and the $\mathbb{F}$ optional reduction KVA$'$ of its $\mathcal{S}_2^\circ$ solution KVA is the $\mathcal{S}'_2$ solution to the continuous-time version (44) of (35).

Proof. This follows from Lemma 7.1 through the equivalences of Lemma 7.3. ∎

This allows us to prove the minimality result announced in Section 5.4 according to which, in the present continuous-time setup, CR and KVA as per Definition 5.7 (are well-defined, by Lemma 7.4, and) provide the minimal admissible capital at risk process, with the cheapest ensuing cost of capital (for the fixed and given hurdle rate h), the latter being nondecreasing in the hurdle rate h:

Theorem 7.1 *Assuming that* EC *is in* $\mathcal{L}'_2$, *using the notation of Section 5.4 specified by Lemma 7.4:*
(i) *We have* $\mathrm{CR} = \min \mathcal{C}, \mathrm{KVA} = \min_{C \in \mathcal{C}} K(C)$;
(ii) *The process* KVA *is nondecreasing in h.*

Proof. This follows by application of Proposition 7.1 via Lemma 7.4. ∎

7.4 CVA and FVA

We now consider the CVA and the FVA.

Note that the nondecreasing cash flows $\mathcal{C}^\circ$ and $\mathcal{F}^\circ$ (by Assumptions 3.2 and 4.5) can be assumed to be $\mathbb{F}$ (rather than simply $\mathbb{G}$) adapted, without loss of generality by $\mathbb{F}$ reduction (and recall Assumption 6.2 set $\mathcal{H} = 0$). Moreover, regarding the funding cash flows, whenever the CVA process is already well defined in $\mathcal{S}_2^\circ$ so that the quest for a theoretical target $\mathrm{RM} = \mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}$ level (cf. (36) and (27)) reduces to the one for the FVA component, we postulate a more specific

$$d\mathcal{F}_t^\circ = f_t(\mathrm{FVA}_t)dt \text{ until } \tau, \tag{52}$$

for some random function $f = f_t(y)$ measurable with respect to the product of the $\mathbb{F}$ predictable σ field by the Borel σ field on $\mathbb{R}$.

Lemma 7.5 *For* $\mathcal{C}$ *and* $\mathcal{F}$ *thus specialized, the* CVA *and* FVA *equations* (24) *and* (25) *in* $\mathcal{S}_2^\circ$ *are equivalent to the following equations in* $\mathcal{S}'_2$:

$$\mathrm{CVA}'_t = \mathbb{E}'_t(\mathcal{C}_T^\circ - \mathcal{C}_t^\circ),\ t \in [0, T], \tag{53}$$

and (provided CVA′ *defined by* (53) *is in* $\mathcal{S}'_2$)

$$\mathrm{FVA}'_t = \mathbb{E}'_t \int_t^T f_s(\mathrm{FVA}'_s)ds,\ t \in [0, T]. \tag{54}$$

Proof. Analogous to the proof of Lemma 7.3, hence omitted. ∎

Remark 7.2 A structure (52) for $\mathcal{F}$ is a slight departure from the abstract setup postulated since Section 3.1, where $\mathcal{F}$ had been introduced as an exogenous process (implicitly, at least). But, in applications, the dependence (52) of $\mathcal{F}$ on the FVA is only semi-linear (see e.g. (59) and (90)). Provided the ensuing FVA fixed-point problem is well-posed, for which sufficient conditions will be given in Proposition 7.2 based on Lemma 7.5, one can readily check, by revisiting all the above, that this dependence does not affect any of the qualitative conclusions drawn in the previous sections of the paper. ∎

The remainder of this section is only illustrative example. We refer the reader to Albanese et al. (2017) and Armenti and Crépey (2018) for more developed applications, involving in particular initial margin (cf. Remark 2.1). Let

$$U_0 = 1 \text{ and } dU_t = \lambda_t U_t dt + (1 - R)U_{t-} dJ_t = U_{t-}\big(\lambda_t dt + (1 - R)\, dJ_t\big), \ 0 \le t \le \bar{\tau}, \quad (55)$$

represent the (discounted) risky funding asset supposed to be used by the bank for its unsecured borrowing purposes, for some exogenous and constant recovery rate $R \in [0, 1]$.

Remark 7.3 Such a funding asset is based on idealized "instantaneous digital CDSs" that we assume available for notational simplicity here; in practice one could use rolling CDSs as per Bielecki, Jeanblanc, and Rutkowski (2008) (see the concluding Remark to their Section 2.2.3).

Note that the bank can only be short in U, as pari passu rules forbid to sell default protection on oneself. ∎

The $(\mathbb{G}, \mathbb{Q})$ martingale condition that applies to U under the $(\mathbb{G}, \mathbb{Q})$ valuation rule in Definition 5.1 (cf. the comments following Remark 5.1) implies that $\lambda = (1 - R)\gamma$, where γ is the $(\mathbb{G}, \mathbb{Q})$ default intensity of the bank and R its anticipated recovery rate. Hence

$$\lambda_t dt + (1 - R)\, dJ_t = (1 - R)d\mu_t, \quad (56)$$

where $d\mu_t = \gamma dt + dJ_t$ is the $(\mathbb{G}, \mathbb{Q})$ compensated jump-to-default martingale of the bank.

We assume all re-hypothecable collateral and we denote by D a $\mathbb{G}$ optional process representing the difference between the collateral MtM posted by the CA desk to the clean desks (cf. (22)) and the collateral received by the CA desk from the clients.

Lemma 7.6 *For $0 \le t \le \bar{\tau}$:*

$$\begin{aligned}
d\mathcal{F}_t &= (1 - R)(D_{t-} - \mathrm{CA}_{t-})^+ d\mu_t, \\
d\Gamma_t^{ca} &= -(d\mathrm{CA}_t + d\mathcal{C}_t) - (1 - R)(D_{t-} - \mathrm{CA}_{t-})^+ d\mu_t, \\
dL_t &= dL_t^{ca} = d\mathrm{CA}_t + d\mathcal{C}_t^\circ + \lambda_t(D_t - \mathrm{CA}_t)^+ dt.
\end{aligned} \quad (57)$$

Assuming the CVA equation (24) already well-posed in $\mathcal{S}_2^\circ$, we have

$$\begin{aligned}
d\mathcal{F}_t^\circ &= \lambda_t(D_t - \mathrm{CVA}_t' - \mathrm{FVA}_t)^+ dt, \\
d\mathcal{F}_t^\bullet &= (1 - R)(D_{t-} - \mathrm{CVA}_{t-} - \mathrm{FVA}_{t-})^+(-dJ_t).
\end{aligned} \quad (58)$$

In particular, $\mathcal{F}^\circ$ is of the form (52), with

$$f.(y) = \lambda(D - \mathrm{CVA}' - y)^+. \quad (59)$$

Proof. Under Assumption 6.4 that equity capital is not used for funding purposes by the bank, the funding strategy of the CA desk reduces to a splitting of the amount RC_t on the reserve capital account as

$$
\begin{aligned}
\mathrm{RC}_t = \quad & \underbrace{D_t}_{\text{Posted collateral remunerated risk-free}} \\
+ \quad & \underbrace{(\mathrm{RC}_t - D_t)^+}_{\text{Cash invested at the risk-free rate}} \\
- \quad & \underbrace{(\mathrm{RC}_t - D_t)^-}_{\text{Cash unsecurely funded}} \\
= \quad & \underbrace{\left(D_t + (\mathrm{RC}_t - D_t)^+\right)}_{\text{Amount } \xi_t \text{ invested at the risk-free rate}} \quad - \quad \underbrace{(\mathrm{RC}_t - D_t)^-}_{\text{Unsecurely funded as } \eta_t U_t}
\end{aligned}
\tag{60}
$$

(all in discounted amounts). Given our current use of the risk-free asset as numéraire, a standard continuous-time self-financing equation expressing the conservation of cash flows yields[12]

$$
d\left(\xi_t - \eta_t U_t\right) = -\eta_{t-}\, dU_t = -(1-R)\eta_{t-}U_{t-}\, d\mu_t = -(1-R)(D_{t-} - \mathrm{CA}_{t-})^+ d\mu_t,\, , \ 0 \le t \le \bar{\tau}.
$$

by the second balance condition $\mathrm{RC} = \mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}$ (cf. (36) and (27)). This yields the first line in (57), which is equivalent to (58) (assuming the CVA already well-defined in $\mathcal{S}_2^\circ$). An application of the first line in (39) then yields the second line in (57), from which the third line (under Assumption 6.2 that $L^{cl} = 0$) follows by Definition 3.3 (also noting that CA does not jump at τ). ∎

In what follows we further assume that the bank portfolio involves a single client with default time denoted by τ_1, that $\mathbb{Q}(\tau_1 = \tau) = 0$, that the liquidation of a defaulted party is instantaneous, and that no contractual cash flows are promised at the exact times τ and τ_1. We refer the reader to Albanese et al. (2017) and Armenti and Crépey (2018) for the relaxation of all these assumptions, in respective bilateral and centrally cleared setups.

Let J and J^1, respectively R and R_1, denote the survival indicator processes and the assumed recovery rates of the bank and its client toward each other. In this case, D is of the form $J^1 Q$, where Q is the difference between the clean valuation P of the client portfolio and the amount VM of variation margin (re-hypothecable collateral) to be transferred[13] between the client and the CA desk in case of a default.

Lemma 7.7 *For $0 \le t \le \bar{\tau}$,*

$$
\begin{aligned}
d\mathcal{C}_t^\circ &= (1 - R_1)Q_{\tau_1}^+(-dJ_t^1), \\
d\mathcal{C}_t^\bullet &= \mathbb{1}_{\{\tau \le \tau_1\}}(1 - R)Q_\tau^-(-dJ_t).
\end{aligned}
\tag{61}
$$

---

[12]A left-limit in time is required in η because U jumps at time τ, so that the process η, which is defined through (60) as $\frac{(\mathrm{RC}-D)^-}{U}$, is not predictable.

[13]Property-wise, having already been posted as a loan by the client to the CA desk (if positive, or by the CA desk to the client otherwise).

Proof. Before the defaults of the bank or its client, the contractual cash flows from the client to the CA desk exactly compensate the contractual cash flows from the clean desk to the CA desk, so that there are no contributions to the process $\mathcal{C}$. Because of this, and since liquidations are instantaneous, it is enough to focus on the contributions to $\mathcal{C}$ at time $\tau \wedge \tau_1$. By symmetry (noting that $\mathbb{1}_{\{\tau_1 \leq \tau\}}$ is implicitly present in the first line of (61) via the restriction $0 \leq t \leq \bar{\tau}$), it is enough to prove the first line in (61). Let $\epsilon = Q_{\tau_1}^+$. By Assumption 6.3, if the counterparty defaults at $\tau_1 < \tau$, then (as $Q = P - \mathrm{VM}$ and having excluded the possibility of contractual cash flows at times τ or τ_1):

- On the client portfolio side, the CA desk receives

$$\mathrm{VM}_{\tau_1} + R_1 Q_{\tau_1}^+ - Q_{\tau_1}^- = \mathbb{1}_{\epsilon=0} P_{\tau_1} + \mathbb{1}_{\epsilon>0}(\mathrm{VM}_{\tau_1} + R_1\epsilon);$$

- The cleaned portfolio between the CA desk and the clean desk of the bank is unwound, resulting in a settlement of P_{τ_1} from the CA desk to the clean desks (transfer of property of the corresponding amount on the CA account).

Combining both cash flows, the CA desk loss amounts to

$$P_{\tau_1} - \big(\mathbb{1}_{\epsilon=0} P_{\tau_1} + \mathbb{1}_{\epsilon>0}(\mathrm{VM}_{\tau_1} + R_1\epsilon)\big) = \mathbb{1}_{\epsilon>0}(P_{\tau_1} - \mathrm{VM}_{\tau_1} - R_1\epsilon) = (1 - R_1)\epsilon,$$

which shows the first line in (61). ∎

Note that, by $\mathbb{F}$ reduction, we may and do assume that τ_1 is an $\mathbb{F}$ stopping time and that Q is an $\mathbb{F}$ optional process.

Proposition 7.2 *Assuming that* CVA$'$ *defined by* (63) *is in* $\mathcal{S}_2'$, *and that the processes* λ *and* $\lambda J^1 Q^+$ *are in* $\mathcal{L}_2'$, *then the* FVA$'$ *equation* (54) *is well-posed in* $\mathcal{S}_2'$, *the CA, CVA, and FVA equations* (23), (24), *and* (25) *are well-posed in* $\mathcal{S}_2^\circ$, *and we have, for* $0 \leq t \leq \bar{\tau}$:

$$\mathrm{CVA}_t = (\mathrm{CVA}')_t^{\tau-} \ \text{ and } \ \mathrm{FVA}_t = (\mathrm{FVA}')_t^{\tau-}, \ \text{where, for } 0 \leq t \leq T: \tag{62}$$

$$\mathrm{CVA}'_t = \mathbb{E}'_t\big[\mathbb{1}_{\{t<\tau_1<T\}}(1 - R_1)Q_{\tau_1}^+\big], \tag{63}$$

$$\mathrm{FVA}'_t = \mathbb{E}'_t\int_t^T \lambda_s(J_s^1 Q_s - \mathrm{CVA}'_s - \mathrm{FVA}'_s)^+ ds; \tag{64}$$

$$\mathrm{CL}_t = \underbrace{\overbrace{\mathbb{E}_t\big[\mathbb{1}_{\{t<\tau\leq\tau_1\wedge T\}}(1 - R)Q_\tau^-\big]}^{\mathrm{FTDDVA}_t} + \overbrace{\mathbb{E}_t\big[\mathbb{1}_{\{t<\tau<T\}}\mathrm{CVA}_\tau\big]}^{\mathrm{CVA}_t^{\mathrm{CL}}}}_{\mathrm{DVA}_t}$$

$$+ \underbrace{\overbrace{\mathbb{E}_t\big[\mathbb{1}_{\{t<\tau<T\}}(J_{\tau-}^1 Q_{\tau-} - \mathrm{CA}_{\tau-})^+\big]}^{\mathrm{FTDFVA}_t} + \overbrace{\mathbb{E}_t\big[\beta_t^{-1}\beta_\tau \mathbb{1}_{\{t<\tau<T\}}\mathrm{FVA}_\tau\big]}^{\mathrm{FVA}_t^{\mathrm{CL}}}}_{\mathrm{FDA}_t};$$

$$\tag{65}$$

$$\mathrm{FV}_t = \underbrace{\mathbb{E}_t\big[\mathbb{1}_{\{t<\tau_1\leq\tau\wedge T\}}(1-R_1)Q_{\tau_1}^+\big]}_{\mathrm{FTDCVA}_t} - \underbrace{\mathbb{E}_t\big[\mathbb{1}_{\{t<\tau\leq\tau_1\wedge T\}}(1-R)Q_{\tau}^-\big]}_{\mathrm{FTDDVA}_t}; \qquad (66)$$

$$dL_t = (1-R_1)Q_{\tau_1}^+(-dJ_t^1) + d\mathrm{CVA}_t \qquad\qquad (67)$$
$$+ \lambda_t(J_t^1 Q_t - \mathrm{CVA}_t - \mathrm{FVA}_t)^+ dt + d\mathrm{FVA}_t.$$

Proof. Under the assumptions of the proposition, the $(\mathbb{F},\mathbb{P})$ FVA$'$ BSDE (54), for f as per (59) with $D = J^1 Q$ there, is a Lipschitz coefficient BSDE well-posed in $\mathcal{S}_2'$, by standard results (see e.g. Kruse and Popier (2016, Sect. 4) or Becherer (2006)). In view of Lemma 7.5, this proves the CVA and FVA, hence CA = CVA + FVA, related statements. The CL and FV formulas (65) and (66) readily follow from (26), (58) and (61), for CL, and, for FV, from (61) and the last part in Lemma 5.1. The dynamics (67) for L are obtained by plugging the first line in (61) into the last line in (57). ∎

In the contra-liability or fair valuation formulas (65) and (66), "FTD" in FTDCVA or FTDDVA stands for "first to-default" CVA and DVA, which only value the related cash flows until the bank default τ.

The FV formula (66) is symmetrical between the bank and its client, hence consistent with the so-called "law of one price". But the corresponding notion of fair valuation is forgetful of the wealth transfers from clients and shareholders to bondholders that arise due to the incompleteness of counterparty risk (cf. Sections 5.3 and 5.6).

Remark 7.4 In the simple example above, without joint defaults in particular, we have

$$\mathrm{FTDCVA} = \mathrm{CVA} - \mathrm{CVA}^{\mathrm{CL}}, \ \mathrm{FTDDVA} = \mathrm{DVA} - \mathrm{CVA}^{\mathrm{CL}}. \qquad (68)$$

See Albanese et al. (2017) for detail differences that arise in more general cases with joint defaults: with FTDCVA and FTDDVA defined as in (66), the identity FV = FTDCVA − FTDDVA (= CVA − DVA, by the second identity in (29)) remains valid in general, but the term by term identification (68) is no longer exactly true. ∎

8 Unilateral Versus Bilateral XVAs

Even though our setup includes the default of the bank itself (which is the essence of the contra-liabilities wealth transfer issue), we end up with unilateral CVA, FVA, and KVA formulas (63), (64), and (48) pricing the related cash flows until the final maturity T of the portfolio (as opposed to $\bar{\tau} = \tau \wedge T$)—under a reduced filtration (and possibly changed pricing measure), but without any bank default intensity discounting.

Recall from (38) that the only XVAs that ultimately matter in entry prices are the CVA, the FVA, and the KVA. Unilateral XVA costs to be accounted for in entry prices is indeed what follows from our analysis of the wealth transfers involved and from the mathematical analysis of the resulting fix-point equations. However, this also

makes the corresponding XVAs more expensive than the bilateral XVAs that appear in most of the related literature (see for instance Burgard and Kjaer (2017, Section 4.4)).

A unilateral CVA is actually required for being in line with the regulatory requirement that reserve capital should not diminish as an effect of the sole deterioration of the bank credit spread (see Albanese and Andersen (2014, Section 3.1)). But a bilateral FVA already satisfies this regulatory monotonicity requirement (essentially, as, when the bank credit spread deteriorates, the shortest duration of a bilateral FVA is compensated by the higher funding spread). The KVA is not directly concerned by this requirement but indirectly it is, through the related monotonicity condition on capital at risk $\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA})$.

In this section we revisit the XVA equations when different choices are made on the fate of the residual risk margin and reserve capital in case of default of the bank (cf. Assumption 2.2 and Definition 3.5).

We denote by $\mathcal{S}_2^{\bullet}$ the subspace of the processes in $\mathcal{S}_2$ with a null terminal condition at $\bar{\tau}$.

8.1 From Unilateral to Bilateral KVA

A bilateral KVA would follow by deciding that, upon bank default, as an exception to Assumption 2.2, the residual risk margin flows back into equity capital and not to bondholders.

In the thus-modified setup, we obtain the following modified KVA equation in $\mathcal{S}_2^{\bullet}$:

$$(-\mathrm{KVA}) \text{ has the } (\mathbb{G}, \mathbb{Q}) \text{ drift coefficient } h\big(\max(\mathrm{EC}, \mathrm{KVA}) - \mathrm{KVA}\big), \qquad (69)$$

instead of our original KVA equation (35) in $\mathcal{S}_2^{\circ}$. That is, in the continuous-time setup of Sections 7.2–7.3:

$$\mathrm{KVA}_t = h\mathbb{E}_t \int_t^{\bar{\tau}} e^{-h(t-s)} \max(\mathrm{EC}_s, \mathrm{KVA}_s) ds, \ t \in [0, \bar{\tau}]. \qquad (70)$$

Or, in an an equivalent $(\mathbb{F}, \mathbb{P})$ formulation obtained via the identity (41): $\mathrm{KVA} = (\mathrm{KVA}')^{\tau}$ on $[0, \bar{\tau}]$, where

$$\mathrm{KVA}_t' = h\mathbb{E}_t' \int_t^{T} e^{-\int_t^s (h+\gamma_u) du} \max(\mathrm{EC}_s, \mathrm{KVA}_s') ds, \ t \in [0, T]. \qquad (71)$$

By comparison with our original KVA' equations (47)–(48), note in particular the bank default intensity discounting in (71).

8.2 From Unilateral to Bilateral FVA

A bilateral FVA, which already satisfies the regulatory monotonicity requirement on the related reserve capital, might be advocated as follows (similar developments would apply to the MVA if we had it explicitly in our model, see Remark 2.1).

Assume, for the sake of the argument, that the portfolio of the defaulted bank with clients is unwound with counterparty risk-free counterparties, called novators. The residual amount of CVA reserve capital is required by the novators to deal with the residual counteparty risk on the deals. But the residual amount of FVA reserve capital is useless to the (counterparty risk-free) novators. In view of this, one could disentangle the CA desk into a CVA desk and an FVA desk, each endowed with their own reserve capital account, and decide that, upon bank default, as an exception to Assumption 2.2, the residual FVA capital reserve flows back into equity capital and not to bondholders.

Revisiting the CVA and FVA derivations of Sections 5.3 and 7.4 in the thus-modified setup, the CVA equations do not change, but we obtain the following modified FVA equation in $\mathcal{S}_2^\bullet$:

$$\mathrm{FVA}_t = \mathbb{E}_t\big(\mathcal{F}_{\bar{\tau}}^\circ - \mathcal{F}_t^\circ\big),\, t \leq \bar{\tau}, \tag{72}$$

instead of the FVA equation (25) in $\mathcal{S}_2^\circ$. That is, in the continuous-time setup of Section 7.4:

$$\mathrm{FVA}_t = \mathbb{E}_t \int_t^{\bar{\tau}} f_s(\mathrm{FVA}_s)ds,\, t \in [0, \bar{\tau}]. \tag{73}$$

Or, equivalently: $\mathrm{FVA} = (\mathrm{FVA}')^\tau$ on $[0, \bar{\tau}]$, where

$$\mathrm{FVA}'_t = \mathbb{E}'_t \int_t^T e^{-\int_t^s \gamma_u du} f_s(\mathrm{FVA}'_s)ds,\, t \in [0, T]. \tag{74}$$

Note again the bank default intensity discounting in (74).

8.3 A Bank with Four Floors

The considerations of Section 8.2 point out to the vision of a bank with four floors:[14] clean desks are the bottom floor. On top of it sits the CVA floor, which obtains risk-free funding from the FVA floor above it, just like the clean desks, and is only in charge of filtering out counterparty risk from client deals. Conversely, the FVA (third) floor is just in charge of the funding of the bank, obtaining risky funding from the outside of the bank and distributing risk-free funding, in the form of collateral provided at an risk-free cost, to the CVA and clean desks for funding their trading.

The CVA and FVA desks are now on different floors: They have their own and separate reserve capital account, hedge, and trading loss process, whereas they were sharing each of these in the earlier sections of the paper. Clean desks continue to be served by the floors above them with a cleaned portfolio worth MtM (in the sense of Definition 5.3). Each of the three trading floors generates a martingale trading loss process:

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[14]It would be even five if the MVA was disentangled from a FVA then meant in the strict sense of the cost of funding variation margin, cf. Remark 2.1.

- For the clean floor, an $(\mathbb{F}, \mathbb{P})$ martingale that, stopped before τ, is a $(\mathbb{G}, \mathbb{Q})$ martingale;

- For the CVA floor, a $(\mathbb{G}, \mathbb{Q})$ martingale without jump at τ;

- For the FVA floor, a $(\mathbb{G}, \mathbb{Q})$ martingale jumping to 0 at time τ, this jump to 0 reflecting the deviation from Assumption 2.2 considered in Section 8.2.

The upper (fourth) floor is the management in charge of the KVA payments, i.e. of the dividend distribution policy of the bank.

This view is applied to bilateral portfolios (with a bilateral KVA as per Section 8.1) in Albanese et al. (2017). A more simplistic take made in Armenti and Crépey (2018), where the focus is less mathematical and more on the CCP feature of the setup there, is to view all the XVA equations as endowed with a null terminal condition at $\bar{\tau}$. This allows avoiding any enlargement of filtration technicality, as all equations can then be solved directly in $\mathbb{G}$. However, it results in a bilateral CVA (such as $\mathrm{CVA} - \mathrm{CVA}^{\mathrm{CL}}$, or FTDCVA in the context of Proposition 7.2) that misses the $\mathrm{CVA}^{\mathrm{CL}}$ wealth transfer from clients and shareholders to bondholders. Likewise, a bilateral KVA misses the fact that the residual RM_τ falls into the estate of the bondholders at time τ (if $< T$).

9 Comparison with Other Approaches in the XVA Benchmark Model

To conclude the main body of this paper, we compare our findings with alternative approaches in the literature that have been developed in the last years in what we therefore call the XVA benchmark model, namely a Black–Scholes model S for an underlying market risk factor, in conjunction with independent Poisson counterparties and bank defaults: See, not exhaustively, the Burgard and Kjaer (2011, 2013, 2017) CVA and FVA approach, referred to as the BK approach below, the Green et al. (2014) KVA approach, referred to as the GK approach, Bichuch et al. (2018), and Crépey et al. (2014, Section 4.6).[15]

A general comment in this regard is that using a Black–Scholes replication framework as an XVA toy model can be quite misleading. The view developed in the present paper is that XVAs are mainly about market incompleteness, and therefore fall under a logic orthogonal to Black–Scholes. From this perspective, promoting a Black–Scholes replication approach in the XVA context is a bit comparable to the mispractice developed during the credit derivative pre-2008 crisis era, when the notion of "Gaussian copula implied correlation" of a CDO tranche was presented as a relative of the Black–Scholes implied volatility of an option, whereas the Gaussian copula model is a purely static device not supported by a sound hedging basis.

Furthermore it is interesting to note that Burgard and Kjaer (2011, 2013, 2017), although availing themselves of a replication pricing framework and blaming risk-neutral

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[15]In journal form Crépey (2015, Part II, Section 5).

approaches outside the realm of replication (see the first paragraph in their 2013 paper), end-up doing what they call semi-replication, which is nothing but a form of risk-neutral pricing without (exact) replication.

Actually, in this XVA field, even the restriction to a Markov setup (beyond Black–Scholes or replication) is not necessarily innocuous, as we will see in the next-to-last paragraph of Section 9.1.

What follows provides more detailed comparisons between these approaches and the one of the present paper.

9.1   CVA and FVA: Comparison with the BK Approach

Burgard and Kjaer (2011, 2013, 2017) repeatedly (and rightfully) say that only pre-default cash flows matter to shareholders. For instance, quoting the first paragraph in the second reference:

> "Some authors have considered cases where the post-default cash flows on the funding leg are disregarded but not the ones on the derivative. But it is not clear why some post default cashflows should be disregarded but not others",

to which we subscribe fully and refer to as their *first principle* below.

The introduction of their classical "(funding) strategy I : semi-replication with no shortfall at own default" (see e.g. (Burgard and Kjaer 2013, Section 3.2)) seems to be in line with the idea, which we also agree with (see Assumption 4.5 and the comment following it) and refer to as their *second principle*, that a shortfall of the bank at its own default (i.e. "CL < 0") does not make sense and should be excluded from a model.

However, being rigorous with their *first principle* above implies that the valuation jump of the portfolio at the own default of the bank should be disregarded in the shareholder cash flow stream, as included in our Assumption 2.2 (cf. also Definition 3.5). But their computations, stated in terms of $(d\widehat{V} + d\bar{\bar{\Pi}})$ in Burgard and Kjaer (2013, equation (9)) or $(d\widehat{V}^\alpha + d\Pi)$ in Burgard and Kjaer (2017, equation (3.5)), include this cash flow.

For illustration, in order to be able to restrict attention on the CVA for simplicity, let us assume, for the sake of the argument, that the bank, although risky, can both fund itself and invest at the risk-free rate.[16] This corresponds to the limiting case where $\mathcal{F} = 0$ in Assumption 4.5. We put ourselves in the framework of Section 7.4 where Proposition 7.2 was derived, specialized further to a BK setup with volatility σ of a stock S underlying a (single) contract with payoff $\phi(S_T)$ sold by the client to the bank. Taking the difference between the CVA$'$ PDE (cf. (63) specialized to the present BK setup) and the Black–Scholes PDE for the clean valuation of the contract shows that the bank pre-default CVA-deducted value of the contract, $\widehat{V}'_t := P_t - \text{CVA}'_t$, can be represented in

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[16]For instance because the bank is highly capitalized and, notwithstanding Assumption 6.4, can in fact use its capital for funding its trading.

functional form as $\widehat{V}'(t, S_t, J_t^1)$, where the function $\Pi(t, S) = \widehat{V}'(t, S, J^1 = 1)$ satisfies the following pricing equation:

$$\Pi(T, S) = \phi(S) \text{ and, for } t < T,$$
$$(\partial_t + \frac{1}{2}\sigma^2 S^2 \partial_{S^2}^2)\Pi + \gamma_1(R_1 P^+ - P^-) - \gamma_1\Pi = 0. \tag{75}$$

Here γ_1 is the default intensity of the counterparty, assumed constant in BK. Now, (75) is nothing but the equation (10) for $\widehat{V}$ in Burgard and Kjaer (2013) or (3.8) for $\widehat{V}^\alpha$ in Burgard and Kjaer (2017), but for a default intensity γ of the bank, denoted by λ_B there, formally set equal to 0.

Hence, in this BK setup, the CVA-deducted value of the option truly disregarding all cash flows from time τ onward, including the jump in valuation at time τ, is **not** given by the solution $\widehat{V}$ to equation (10) in Burgard and Kjaer (2013) or $\widehat{V}^\alpha$ to equation (3.8) in Burgard and Kjaer (2017), **but** by $\widehat{V} = (\widehat{V}')^{\tau-}$, where $\widehat{V}'$ satisfies the formal analog of these equations **with intensity of the bank set equal to 0**.

Incidentally, the Itô derivation of the portfolio XVA-deducted value process in Burgard and Kjaer (2013, Eq. (9)) relies on the implicit assumption that this value process $\widehat{V}$ is in the first place a (regular enough) function of the postulated risk factors (for Itô's formula to apply), which is not justified in their paper. For instance, in the simplified (CVA only) BK setup above, it is only the process CVA$'$ that is obviously Markov in t, S, and the nondefault indicator process J^1 of the bank client, and then it only holds that CVA $= (\text{CVA}')^{\tau-}$.

As a consequence of the above pitfall regarding $\widehat{V}$, reviewing the funding strategies in Burgard and Kjaer (2017, Section 4), in the special case with $s_B = 0$ there of a pure CVA setup:

- Their strategy III, claimed to imply a unilateral CVA as per Albanese and Andersen (2014) (i.e. (62)–(63) in the present paper), does in fact not: Accounting for the transfer of the residual reserve capital from shareholders to bondholders at the bank default time τ (cf. Assumption 2.2 and Definition 3.5), the funding strategy that implies a unilateral CVA is simply the obvious one (having assumed $s_B = 0$), i.e. funding and investing at the risk-free rate;

- Their respective strategies I and II, not only do not imply the claimed XVA formulas (as all the BK equations forget the above-mentioned wealth transfer), but they also breach the second part in Assumption 4.5, which may lead to a violation of their **second principle** of no bank shortfall at its default, through a negative FVA = FDA (unless in their notation $V \geq 0$, respectively $\widehat{V} \geq 0$, i.e. in our notation above $P \geq 0$, respectively $\Pi \geq 0$);

- Their replication strategy is not practical, as admitted at the end of Burgard and Kjaer (2013, Section 3.1), which is the motivation for their other (but partly flawed as seen) strategies.

9.2 KVA: Comparison With the GK Approach

Despite what the "valuation adjustment" terminology fallaciously induces one to believe, in our view, the KVA is not part of the value of the derivative portfolio, but a risk premium in incomplete counterparty risk markets: Under our incomplete market approach, risk margin payments are meant to remunerate the risk of unhedged trading losses. Hence, including the KVA to contra-assets (which contribute to the trading loss process of the bank) would induce a circularity that we think is illogical.

In Green et al. (2014) the KVA is instead treated as a liability in a replication framework. The KVA is also treated as a liability in some theoretical actuarial literature, under the name of risk margin or market-value margin MVM there (see Salzmann and Wüthrich (2010, Section 4.4)). Viewing the KVA as a liability results in a non loss-absorbing risk margin, hence CR = SCR = EC (as opposed to (16) in our setup), and therefore hEC instead of $h(\text{EC} - \text{KVA})^+$ in the KVA equation (35). This implies no discounting at the hurdle rate h in the (already risk-free discounted) KVA formula (48) (where KVA$'$ and KVA coincide before τ), and therefore a "much higher" KVA (considering the very long time horizon T).

Moreover, if the KVA is viewed as a liability, forward starting one-year-ahead fluctuations of the KVA must be simulated for economic capital calculation. This makes it intractable numerically, unless one switches from economic capital to regulatory capital in the KVA equation. Using regulatory instead of economic capital is then motivated by practical considerations but is less self-consistent. It loses the connection, established from a balance sheet perspective in our approach, whereby the KVA input is the CA desk loss process $L = L^{ca}$ as per (39).

10 Conclusion

To conclude this paper we summarize our dynamic cost-of-capital XVA strategy.

We root our XVA approach in a balance sheet perspective, depicted Figure 1 in Section A, which is key in identifying the economic meaning of the XVA accounting terms as well as their connections with the core equity CET1 of the bank and shareholder wealth SHC. Due to counterparty risk incompleteness, the derivative portfolio of the bank triggers a wealth transfer from clients and shareholders to bondholders, for which shareholders can only be compensated by a corresponding add-on to entry prices.

Moreover, shareholders bear the trading risk of the bank. Our KVA is devised as the cost of a corresponding dividend policy which would be sustainable even in the limit case of a portfolio held on a run-off basis, with no new trades ever entered in the future. Conceived in the spirit of a portfolio optimization tool for a derivative market maker in incomplete counterparty risk markets, the KVA in this sense is a risk premium tuned to keep the shareholders on an "efficient frontier" such that

"Shareholder instantaneous average return$_t$ = hurdle rate $h \times \text{SCR}_t$"

(cf. (17)), where SCR is shareholder capital at risk. Our KVA is genuinely dynamic, resulting in a cost-of-capital pricing framework in continuous time. It is a risk premium taking as input data the marked-to-model counterparty defaults and risky funding (i.e. contra-assets) loss process L. Thanks to this connection, the equation for contra-assets CA valuation and the KVA equation, i.e. the XVA equations as a whole, are a self-contained, self-consistent problem. As risk margin is retained earnings meant to be released to the bank shareholders, the KVA in our sense does not belong to the balance sheet as a liability.

The **main hypotheses** characterizing our cost-of-capital XVA modeling approach are Assumptions 4.2 and 4.7, which respectively define valuation and risk premium at the portfolio level, and Assumption 6.1, which conveys the ensuing pricing rule at the trade incremental level by postulating the constancy of the hurdle rate throughout each new trade.
The main results of the paper are:

- Proposition 5.2, which identifies the contra-asset CA = CVA + FVA value process of the bank as the counterparty risk deduction to apply to the mark-to-market of a run-off portfolio in order to ensure the martingality of the core equity of the bank (CET1 process, cf. (11)); the value CL of the contra-liabilities is interpreted in Section 5.3 as the wealth transfer triggered by the derivative portfolio from clients and shareholders to bondholders, due to the inability of the bank to hedge its own jump-to-default exposure;

- Theorem 7.1, which identifies the cheapest admissible capital at risk process and the ensuing KVA (cost of capital) pricing correction ensuring to shareholders a positive hurdle rate h on their capital at risk, in the case of run-off portfolio;

- Proposition 6.1, which identifies the funds transfer price (FTP), i.e. the all-inclusive counterparty risk add-on preserving the hurdle rate of bank shareholders throughout a new trade, as the incremental (CVA + FVA + KVA) of the trade;

- The companion Theorems 5.1 and 6.1, concluding that under our XVA approach, shareholder equity (i.e. wealth) SHC is a submartingale with drift corresponding a positive hurdle rate h on shareholder capital at risk, consistently between and throughout deals.

Our **CVA, FVA, and KVA formulas** (63), (64), and (48) convey the general message (even if the first two are specific to the illustrative setup of Section 7.4), discussed in detail in Section 8, that, although we include the default of the bank itself in our modeling, the XVA formulas accounting for all the wealth transfers involved are unilateral, pricing the related cash flows until the final maturity T of the portfolio –under a reduced filtration (and possibly changed probability measure) ignoring the default of the bank itself, but without bank default intensity discounting.

A XVAs Accounting Interpretation

With XVAs, the pricing and risk management of financial derivatives evolve from a hedging paradigm to balance sheet optimization. This section details the balance sheet perspective on the XVA metrics.

From an accounting perspective, the wealth of a firm is measured by its so-called accounting equity.

Definition A.1 The accounting equity (AE) of a (dealer) bank is

$$AE = CL + \text{Other Assets} - CA - \text{Other Liabilities}, \tag{76}$$

where the contra-assets (CA) and the contra-liabilities (CL) have been formally introduced in Definition 3.5, and:

- In "Other Assets", one places the mark-to-market MtM^+ of the portfolio receivables and the collateral CM^- posted by the clean desks (cf. (22)); "Other Assets" also include reserve capital (RC), risk margin (RM), shareholder capital at risk (SCR), and an additional (typically unknown) amount of uninvested capital (UC);

- In "Other Liabilities", one places the mark-to-market MtM^- of the portfolio payables and the collateral CM^+ received by the clean desks. ∎

The above decompositions are reflected in the bank balance sheet Figure 1, where:

- The mark-to-market of the portfolio payables and receivables, as well as the corresponding collateral (clean margin CM), are shown in dashed boxes at the bottom, because their role vanishes under Assumption 6.2, as the clean desks then generate no trading losses (or profit, i.e. $L^{cl} = 0$);

- Contra-liabilities at the top of the figure are shown in dotted boxes because, as explained in Section 5.3, they can only be an actual benefit to bank bondholders, upon bank default, whereas it is the interest of shareholders that matters regarding the determination of derivative entry prices and the related accounting and dividend policies of the bank, as long as the bank is nondefault;

- The arrows in the left column represent trading losses of the CA desk in "normal years 1 to 39" and in an "exceptional year 40" with full depletion (i.e. refill via UC, under our mark-to-model Assumption 2.1) of RC, RM, and SCR (the numberings yr1 to yr40 are fictitious yearly scenarios in line with the 97.5% expected shortfall of the losses that is used for defining the notion of economic capital which underlies our definition of capital at risk);

- The arrows in the right column symbolize the average depreciation in time of contra-assets between deals.
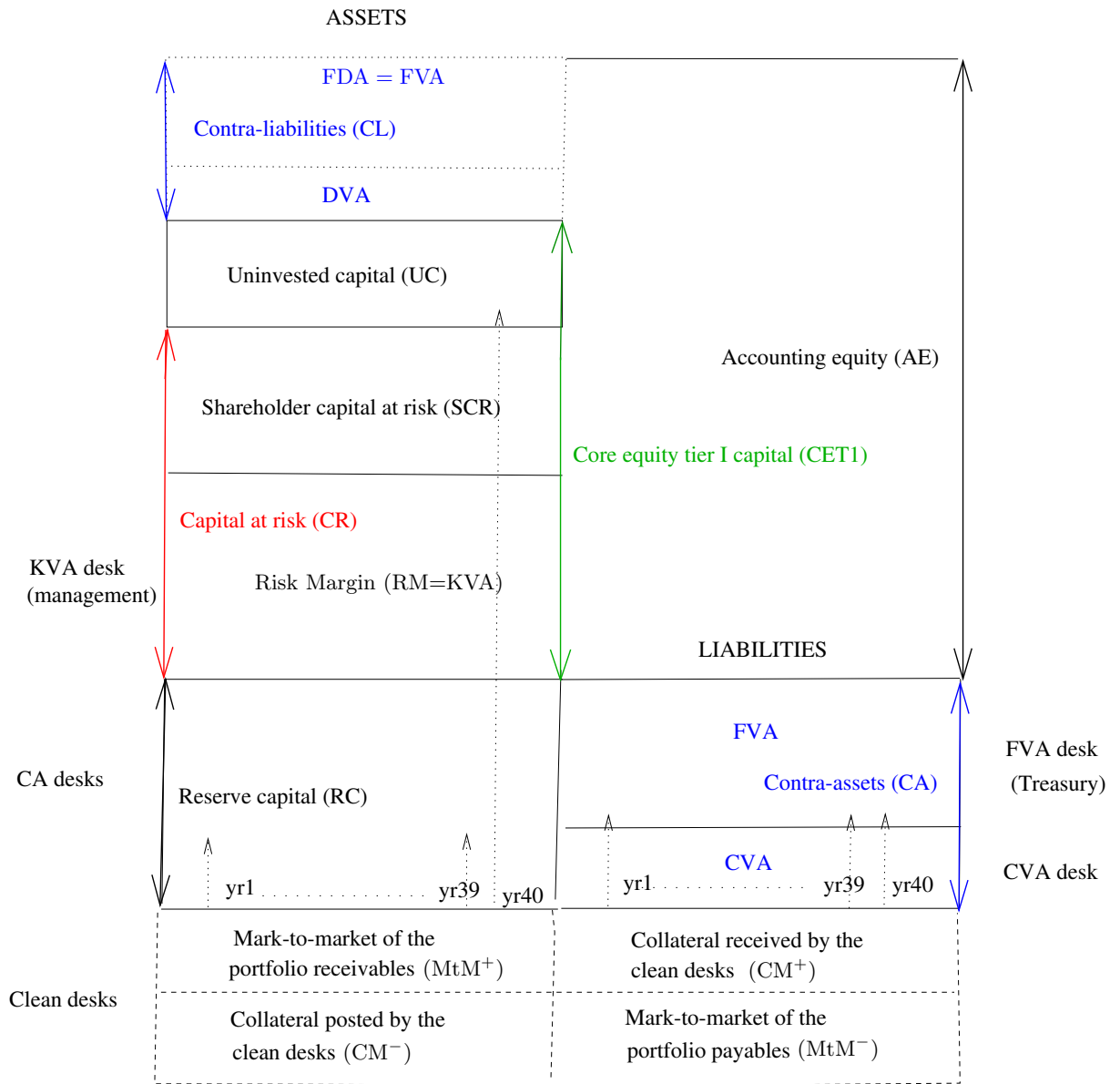
Figure 1: Balance sheet of the bank. Shareholder wealth is $SHC = SCR + UC$.

In particular, via the balance conditions (36) that hold in our (or in any marked-to-model) setup,

$$AE = RM + SCR + UC + CL. \tag{77}$$

We emphasize that, since our risk margin is loss-absorbing (part of capital at risk), we do not put its theoretical KVA target value as a liability (or contra-asset) on the balance sheet (cf. Sections 4.3 and 9.2), which explains the presence of RM = KVA in AE.

Remark A.1 Figure 1 depicts the balance sheet of a "three floor" bank, as considered in most of the paper. To render the situation of a "four floor" bank as per Sections 8.2–8.3, the CA floor would need to be disentangled into a CVA floor and an FVA floor, each endowed with their own reserve capital account (and hedge). ∎

Remark A.2 Although we wrote in Section 2.2 that we restricted ourselves to bilateral trade portfolios to fix the mindset, the conceptual picture of the balance-sheet a dealer bank provided by Figure 1 is general and also applies to a bank engaged into both bilateral and centrally cleared transactions. The involvement of a bank in centrally cleared transactions can be either direct, as clearing member of CCPs (as typical for major banks), or/and indirect, via external CCP accounts. A detailed consideration of the balance sheet of a bank also involved into centrally cleared derivatives only requires a more stratified "mille-feuille" of items, with, in particular, an overall risk margin account that needs be split into one risk margin account dedicated to each CCP membership and a residual risk margin account for the bilateral transactions: cf. Remark 2.3 and, for more detail, see Armenti and Crépey (2018). ∎

Core equity tier I capital is the regulatory metric meant to represent the core financial strength of a bank when assessed from a structural solvency perspective. Broadly speaking, it corresponds to the wealth of the shareholders within the bank. Since the contra-liabilities of a dealer bank do not benefit to its shareholders:

Definition A.2 (CET1) The core equity tier I capital (CET1) of a dealer bank is given by

$$CET1 = AE - CL. \tag{78}$$

Hence, by (77),

$$CET1 = CR + UC = RM + SCR + UC, \tag{79}$$

which is (3).

However, in our setup where the risk margin is only gradually released to the shareholders through the KVA payments, shareholder wealth strictly speaking only corresponds to shareholder equity SHC = SCR + UC = CET1 − RM (see Remark 2.4).

B XVA Analysis in a One-Period Setup

In this section, we apply a cost-of-capital XVA approach to an uncollateralized portfolio made of a single deal $\mathcal{P}$ (promised random variable) between a client and the bank without prior endowment and with fully invested initial equity CR, in an elementary one-period (one year) setup.

The bank and client are both default prone with zero recovery to each other. The bank also has zero recovery to its external funder. The bank wants to charge to its client an add-on (or obtain from its client a rebate, depending on whether the bank is seller or buyer), denoted by CA, accounting for its expected counterparty default losses and funding expenditures, as well as a KVA risk premium sized by its unhedged trading loss over the year. The all-inclusive XVA add-on to the entry price for the deal, which we call funds transfer price (FTP), follows as

$$\text{FTP} = \underbrace{\text{CA}}_{\text{Expected costs}} + \underbrace{\text{KVA}}_{\text{Risk premium}}. \tag{80}$$

A one-period setup is too narrow for practical XVA purposes. However, the key concept of XVAs as pricing adjustments aligning the mark-to-market of a deal (or portfolio) to the interest of bank shareholders remains (cf. Proposition 6.1). Moreover, this setup gives useful insights, as it is then possible to derive stylized formulas for all the quantities at hand and for the particularly transparent view that it offers on the related wealth transfer and risk premium issues.

B.1 One-Period XVA Setup

In a one-period setup, there are no filtrations involved and it is enough to work with, instead of processes throughout the paper, random variables, which correspond to the increment of these processes over the single time step in the model. Accordingly, in this part, all the processes $Y = \mathcal{C}$, $\mathcal{F}$ (recall Assumption 6.2 set $\mathcal{H} = 0$) and the related processes Y° and $Y^\bullet$ in Sections 4–5, as also all the trading loss processes, are identified with random variables on the (one and only, in this context) probability space $(\Omega, \mathcal{A}, \mathbb{Q})$. Hence, we may and do dismiss the $\cdot'$ notation everywhere (cf. Assumption 4.1).

Since values, prices, and XVAs only matter at time 0 in a one-period setup, we identify all the XVA processes, as well as the clean valuation process P of the portfolio, with their values at time 0. We assume that all the prices that are due are instantaneously paid at time 0. We assume likewise that the KVA paid by the client at time 0 is immediately transferred by the bank management to the shareholders. Hence there is no need and purpose for reserve capital and risk margin bank accounts, so that we may and do assume these to be zero.

There is no residual value on the (non-existing) reserve capital account when the bank defaults. Hence CVA^{CL} and FVA^{CL} are redefined as zero.

Definition 5.5 just corresponds to defining EC as the 97.5% expected shortfall of the bank trading loss L. As L is centered, $\text{EC} \geq 0$ (cf. Lemma 5.2). Besides, as there

is no risk margin account, hence $\mathrm{CR} = \mathrm{SCR}$, so that the minimal admissible capital at risk (cf. Section 5.4) is simply $\mathrm{CR} = \mathrm{EC}$.

We denote by J and J_1 the survival indicators (random variables) of the bank and client at time 1, with default probability of the bank $\mathbb{Q}(J = 0) = \gamma$ and no joint default for simplicity, i.e $\mathbb{Q}(J = J_1 = 0) = 0$. The terms Y° and $Y^\bullet$ in the decomposition (7) are meant as

$$\text{Random variables } Y^\circ \text{ of the form } JU \text{ and } Y^\bullet \text{ of the form } (1 - J)V, \qquad (81)$$

for some random variables U and V.

B.2 One-Period XVA Formulas

Lemma B.1 *We have*

$$
\begin{aligned}
\mathcal{C}^\circ &= (1 - J_1)\mathcal{P}^+, \ \mathcal{F}^\circ = \gamma(P - \mathrm{CA})^+ \\
\mathcal{C}^\bullet &= (1 - J)\mathcal{P}^-, \ \mathcal{F}^\bullet = (1 - J)(P - \mathrm{CA})^+ \\
\Gamma^{ca} &= -\left(\mathcal{C}^\circ + \mathcal{F}^\circ - \mathrm{CA}\right) + \mathcal{C}^\bullet + \mathcal{F}^\bullet \\
L &= L^{ca} = \mathcal{C}^\circ + \mathcal{F}^\circ - \mathrm{CA}.
\end{aligned}
\qquad (82)
$$

Proof. For the deal to occur, the CA desk needs borrow $(P - \mathrm{CA})^+$ unsecured or invest $(P - \mathrm{CA})^-$ risk-free. Having assumed zero recovery to the external funder, unsecured borrowing is priced $\gamma \times$ the amount borrowed by the bank (cf. the comments following Remark 5.1), so that at time 0 the CA desk must pay

$$\gamma(P - \mathrm{CA})^+$$

for the risky funding of the bank.

At time 1 (cf. Assumption 6.3):

- On the client portfolio side:

  - If the bank is not in default (i.e. $J = 1$), then the CA desk closes the position with the client while receiving $\mathcal{P}$ from its client if the latter is not in default (i.e. $J_1 = 1$), whereas the bank pays $\mathcal{P}^-$ to its client if the latter is in default (i.e. $J_1 = 0$). In addition, the CA desk reimburses its funding debt $(P - \mathrm{CA})^+$ or receives back the amount $(P - \mathrm{CA})^-$ it had lent at time 0.
  - If the bank is in default (i.e. $J = 0$), then the CA desk receives back $\mathcal{P}^+$ on the derivative as well as the amount $(P - \mathrm{CA})^-$ it had lent at time 0.

- On the cleaned portfolio side, the CA desk delivers a cash flow $(\mathcal{P} - P)$ to the clean desks, whatever the default status of the parties at hand.

Collecting all cash flows, the result Γ^{ca} of the CA desk over the year is:

$$
\begin{aligned}
\Gamma^{ca} = {} & -\gamma(P - \mathrm{CA})^+ \\
& + J\big(J_1\mathcal{P} - (1 - J_1)\mathcal{P}^- - (P - \mathrm{CA})^+ + (P - \mathrm{CA})^-\big) \\
& + (1 - J)\big(\mathcal{P}^+ + (P - \mathrm{CA})^-\big) - (\mathcal{P} - P) \\
= {} & -\big(\underbrace{(1 - J_1)\mathcal{P}^+}_{\mathcal{C}^\circ} + \underbrace{\gamma(P - \mathrm{CA})^+}_{\mathcal{F}^\circ} - \mathrm{CA}\big) \\
& + \underbrace{(1 - J)\mathcal{P}^-}_{\mathcal{C}^\bullet} + \underbrace{(1 - J)(P - \mathrm{CA})^+}_{\mathcal{F}^\bullet}
\end{aligned}
\tag{83}
$$

(as easily checked for each of the three possible values of the pair (J, J_1)), where the identification of the different terms as $\mathcal{C}^\circ$, $\mathcal{F}^\circ$, $\mathcal{C}^\bullet$, and $\mathcal{F}^\bullet$ follows from their financial interpretation. In view of (8) and of the interpretation (81) of the decomposition (7) in a one-period setup, (83), which is the next-to-last line in (82), also implies the last line (recalling Assumption 6.2 set $L^{cl} = 0$). These results are also consistent with (39), remembering that Assumption 6.2 set $\mathcal{H} = 0$ and that, in the present one-period setup, CA corresponds to the time 0 value or, equivalently, to "the negative of the first increment" of the CA process in (39)). ∎

Observe that $\mathcal{F}^\circ = \gamma(P - \mathrm{CA})^+$ is deterministic in our one-period setup.

Remark B.1 Dropping Assumption 6.4 while conservatively assuming $\mathrm{UC} = 0$ would lead to the same expressions for the various quantities in (82), except for $(P - \mathrm{CA})$ replaced by $(P - \mathrm{CA} - \mathrm{EC})$ in the funding cash flows. ∎

Remark B.2 The derivation (83) implicitly allows for negative equity (that arises whenever $L > \mathrm{CR}$), which is interpreted as recapitalization. In a variant of the model excluding recapitalization, the default of the bank would be modeled in a structural fashion as $L = \mathrm{EC}$ $(= \mathrm{CR})$, where

$$
L = L^{ca} = (\mathcal{C}^\circ + \mathcal{F}^\circ - \mathrm{CA}) \wedge \mathrm{EC},
\tag{84}
$$

and we would have

$$
\begin{aligned}
\Gamma^{ca} &= (\mathrm{EC} - L)^+ + \mathbb{1}_{\{\mathrm{EC} < L\}}(\mathcal{P}^- + (P - \mathrm{CA})^+) - \mathrm{EC}) \\
&= \mathbb{1}_{\{\mathrm{EC} > L\}}(-L) + \mathbb{1}_{\{\mathrm{EC} \le L\}}(\mathcal{P}^- + (P - \mathrm{CA})^+ - \mathrm{EC}),
\end{aligned}
$$

with L as per (84) (compare with the two last lines in (82)). In this paper we consider a model with recapitalization for the reasons explained in Section 2.3. ∎

Proposition B.1 *We have:*

$$
\mathrm{CVA} = \mathbb{E}\mathcal{C}^\circ = \mathbb{E}[(1 - J_1)\mathcal{P}^+], \ \mathrm{DVA} = \mathbb{E}\mathcal{C}^\bullet = \mathbb{E}[(1 - J)\mathcal{P}^-]
\tag{85}
$$

$$
\mathrm{FVA} = \mathrm{FDA} = \mathbb{E}\mathcal{F}^\circ = \mathbb{E}\mathcal{F}^\bullet = \frac{\gamma}{1 + \gamma}(P - \mathrm{CVA})^+
\tag{86}
$$

$$
\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}, \ \mathrm{CL} = \mathrm{DVA} + \mathrm{FVA}
\tag{87}
$$

$$
\mathrm{KVA} = h\mathrm{EC}.
\tag{88}
$$

Proof. In line with Definition 3.5, but for CVA^{CL} and FVA^{CL} redefined as zero in a one-period setup, we have

$$\text{CVA} = \mathbb{E}\mathcal{C}^\circ, \ \text{DVA} = \mathbb{E}\mathcal{C}^\bullet$$
$$\text{FVA} = \text{FDA} = \mathbb{E}\mathcal{F}^\circ = \mathbb{E}\mathcal{F}^\bullet, \tag{89}$$

where the involved cash flows are given by Lemma B.1. But, since $\mathcal{F}^\circ = \gamma(P - \text{CA})^+$ where $\text{CA} = \text{CVA} + \text{FVA}$ (cf. (27)), the formula $\text{FVA} = \mathbb{E}\mathcal{F}^\circ$ in the above is in fact a semi-linear equation

$$\text{FVA} = \gamma(P - \text{CVA} - \text{FVA})^+. \tag{90}$$

However (cf. Remark 7.2), as γ (a probability) is nonnegative, this equation has the unique solution given by the right-hand side in (86).

In our one-period setup with, there is no risk margin account and we have $\text{CR} = \text{SCR} = \text{CR}$. Hence (17) reduces to (88). ∎

Through the last line in (82), constancy of $\mathcal{F}^\circ$, and cash-invariance of the expected shortfall risk measure, one can also note that EC reduces to the 97.5% expected shortfall of $\mathcal{C}^\circ$, net of the CVA.

The value $\text{FDA} = \mathbb{E}[(1 - J)(P - \text{CA})^+]$ of the default funding cash flow $\mathcal{F}^\bullet$ equals the FVA cost $\gamma(P - \text{CA})^+$ for the bank of funding its position. But the FVA and the FDA do not impact the same economic agent. Namely, the FVA hits bank shareholders, whereas the FDA benefits bondholders. Hence the net effect of funding is not nil to shareholders, but reduces to an FVA cost.

In the static setup, our hurdle rate h is nothing but the return on equity (ROE).

B.3 Monetizing the Contra-Liabilities?

Let us now assume, for the sake of the argument, that the bank would be able to hedge its own jump-to-default exposure through a further deal, corresponding to a payment by the bank to a third party of

$$\mathcal{C}^\bullet + \mathcal{F}^\bullet = (1 - J)(\mathcal{P}^- + (P - \text{CA})^+)$$

at time 1, in exchange of a time 0 fee fairly valued as $\text{CL} = \mathbb{E}(\mathcal{C}^\bullet + \mathcal{F}^\bullet) = \text{DVA} + \text{FVA}$ (cf. Proposition B.1). As CL has already been received through the hedge fee in order to preserve a trading loss $L^{ca} = L$ as per (82), it suffices to the CA desk to charge to the client a diminished rebate $\text{CA} - \text{CL} = \text{FV}$. The amount that needs by borrowed by the CA desk for implementing this strategy is still $\mathcal{F}^\circ$ as before. The client is better off by the amount $\text{CA} - \text{FV} = \text{CL}$. The shareholders are indifferent in expected counterparty default and funding expenses term. But the bondholders are completely wiped out (because of the hedge their realized recovery is now 0). By comparison with (83), the CL originating cash flow $\mathcal{C}^\bullet + \mathcal{F}^\bullet = (1 - J)(\mathcal{P}^- + (P - \text{CA})^+)$ has been hedged out and monetized by the shareholders as an amount CL received by the bank at time 0.

Hence, if the bank was able to hedge its own jump-to-default in the above sense, then, in order to satisfy its shareholders in expectation, it would be enough for the bank to charge to its client the compressed add-on FV = CVA − DVA. However, the bank would still charge to its client the KVA add-on hEC as risk compensation for the non flat loss-and-profit L triggered by the trading of the bank—unless the bank would also hedge its client default (on top of its own), e.g. through a CDS contract assumed available on the client, in order to cancel out L, in which case the KVA would in turn vanish.

In conclusion, FV is interpreted as the fair valuation of counterparty risk when market completeness and no pari passu trading restrictions are assumed. By contrast, under market incompleteness and pari passu trading restrictions, the FTP aligning the deal to shareholder interest (in the sense of a given hurdle rate h) is (80), which can be decomposed as (cf. (38))

$$
\begin{aligned}
\text{FTP} = \underbrace{\text{CVA} + \text{FVA}}_{\text{Expected costs CA}} &+ \underbrace{\text{KVA}}_{\text{Risk premium}} \\
= \underbrace{\text{CVA} - \text{DVA}}_{\text{Fair valuation FV}} &+ \underbrace{\text{DVA} + \text{FDA}}_{\text{Wealth transfer CL}} + \underbrace{\text{KVA}}_{\text{Risk premium}},
\end{aligned}
\tag{91}
$$

where CVA and FVA are given by (85) and where the random variable L used to size the economic capital in the KVA formula (88) is the bank trading loss L as per (82).

B.4   KVA Risk Premium and Indifference Pricing Interpretations

The CA component of the FTP corresponds to the expected costs for the CA desk of concluding the deal, making the trading loss $L = \mathcal{C}^\circ + \mathcal{F}^\circ - \text{CA} = \mathcal{C}^\circ - \text{CVA}$ of the bank centered. On top of these expected costs, the bank charges a suitable risk margin (RM). Assume the bank shareholders endowed with a (standard) utility function U on $\mathbb{R}$ such that $U(0) = 0$. Under a canonical indifference pricing framework (see e.g. Carmona (2009)), RM is defined through the following equation:

$$
\mathbb{E}U(\text{RM} - L) = \mathbb{E}U(0) = 0,
\tag{92}
$$

the expected utility of the bank without the deal. The corresponding value of RM is interpreted as the minimal admissible risk margin for the deal to occur, seen from the bank perspective.

Taking for concreteness $U(-\ell) = \frac{1 - e^{\rho\ell}}{\rho}$, for some fixed risk aversion parameter ρ, (92) yields $\text{RM} = (\frac{1}{\rho}) \ln \mathbb{E}e^{\rho L}$. In the limiting case where the shareholder risk aversion parameter $\rho \to 0$ and $\mathbb{E}U(-L) = -\mathbb{E}(L) = 0$, then $\text{RM} = 0$.

Comparing with the expression (88) for the KVA, with EC there sized as the $\alpha = 97.5\%$ expected shortfall of L as suggested in Section B.1, which we denote below by $\mathbb{ES}(L)$, we conclude that the implied KVA and hurdle rate corresponding to pricing the deal by exponential utility would be

$$
\text{KVA} = \rho^{-1} \ln \mathbb{E}e^{\rho L}, \; h = \frac{\rho^{-1} \ln \mathbb{E}e^{\rho L}}{\mathbb{ES}(L)}.
\tag{93}
$$

This makes apparent that the hurdle rate h in our KVA setup plays the role of a risk aversion parameter, like ρ in an exponential utility framework. It shows that ending up with a centered trading loss L, i.e. setting the historical probability measure equal to the pricing probability measure in our model (cf. Assumption 4.8), does by no means imply that our setup is "risk-neutral" in the sense that shareholders would have no aversion to risk: It is only so when h is equal to 0 (i.e. $\rho = 0$) in the above. A positive value of h encodes a positive risk aversion of the bank shareholders. In fact, there are two layers of risk premium in our model, the first one which is included in the choice of the pricing measure by the traders of the bank and the second one through the KVA that is charged by the management of the bank.

Under a canonical pricing by indifference framework backed to a standard utility function, the indifference price has a competitive game (or optimization) interpretation. Assume the bank is competing for the client with other banks. Then the bank can only win the deal is if its own indifference price, i.e. the minimal admissible price from its own perspective, is the smallest indifference price of all the competing banks: depending on the detail of the bargaining process taking place between the client and the banks, the effective price at which the transaction will settle can be any intermediate price between the indifference price of the winning bank and the one immediately next above it (assuming both of them less than the reservation price of the client). In the limit of a continuum of competing banks with a continuum of indifference prices, whenever a bank makes a deal, this can only be at its own indifference price.

Our stylized indifference pricing model of a KVA defined by a constant hurdle rate h exogenizes (by comparison with the "endogenous KVA" (93)) the impact on pricing of the competition between banks. It does so in a way that generalizes smoothly in continuous time, as required to deal with a real derivative banking portfolio. It provides a refined notion of return on equity (ROE) for derivative portfolios, where an "explicit" optimization approach would be impractical.

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