When Capital is a Funding Source: The XVA Anticipated BSDEs

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Abstract

Economic capital (EC) can be used as a funding source by banks, at a risk-free cost instead of the additional credit spread of the bank in the case of unsecured borrowing. This results in a significant reduction of funding costs and an FVA (funding valuation adjustment) ignoring it would be grossly overestimated.

Mathematically the intertwining of EC and FVA leads to an anticipated BSDE (ABSDE) for the FVA, with coefficient entailing a conditional risk measure of the one-year-ahead increment of the martingale part of the FVA itself. Accounting further for the KVA (capital valuation adjustment) component of economic capital, with the ensuing feedback condition that EC must be greater than KVA, yields a system of ABSDEs for the FVA and the KVA processes considered simultaneously.

In this paper we show that the ensuing (FVA, KVA) system of ABSDEs is well-posed and we establish the convergence of a Picard approximation scheme. This is first done for a bank without debt. In the realistic case of a defaultable bank, the resulting ABSDEs, which are stopped before the default of the bank, are solved by reduction to a reference filtration.

Keywords: Funding valuation adjustment (FVA), capital valuation adjustment (KVA), economic capital, anticipated BSDE (ABSDE).

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1 Introduction

XVAs, where VA stands for valuation adjustment and X is a catch-all letter to be replaced by C for credit, F for funding or K for capital (!), denote various pricing adjustments applied to financial derivatives since the crisis, in order to account for counterparty risk and its capital and funding implications.

In Albanese and Crépey (2017), the XVA equations are shown to be well-posed in the base case where economic capital is not used by the bank for its funding purposes. But, in reality, economic capital can be used for funding the so-called variation margin (cf. the parameter ϕ representing "the fraction of capital used for funding" in Green, Kenyon, and Dennis (2014)), which may cause a material FVA reduction: See Figure 1, drawn from Albanese, Caenazzo, and Crépey (2017) (cf. Section 5.2 there for every detail about it), which compares, on a real-life banking portfolio, the CDS curve of a bank and its FVA blended curve. The corresponding term structures of economic capital and KVA are shown on Figure Figure 2.

The FVA blended curve is the funding curve, which, whenever applied to an FVA computation neglecting the impact of economic capital, gives rise to the same term structure for the FVA as the calculation carried out with the CDS curve of the bank, but accounting for the economic capital as a funding source. This blended curve is often inferred by consensus estimates based on the Markit XVA data service. However, here it is computed from the ground up, based on the equations and Picard iterations of the present paper.

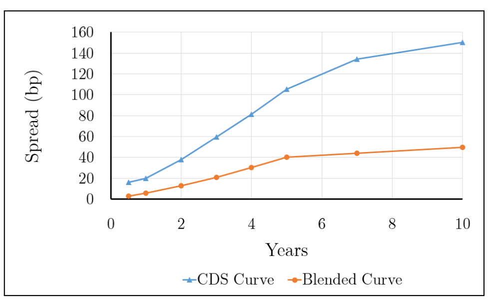

Figure 1: FVA blended funding curve of a bank compared with the bank CDS curve.

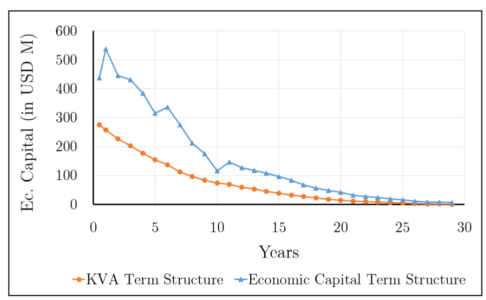

Figure 2: Term structure of economic capital compared with the term structure of KVA.

So, in this paper, we study the XVA equations in the realistic case where economic capital is deemed fungible as a source of funding for variation margin. In order to derive the well posedness of the XVA equations in such framework, our approach relies on an extension of the class of so-called anticipated backward stochastic differential equations (ABSDEs), introduced by Peng and Yang (2009). Specifically, the nature of the dynamics driving the XVA equations leads to the consideration of monotone coefficient ABSDEs, whose anticipated part consists in a conditional risk measure (expected shortfall) of the martingale increment of the solution on a future time period.

The paper is outlined as follows. Section 2 sets the mathematical stage. Section 3 recalls the XVA results of Albanese and Crépey (2017) in the base case ignoring the use of economic capital for funding purposes. Sections 4, 5, and 6 extend these results to the respective cases where the bank uses a stylized economic capital (ignoring the KVA) as a funding source for variation margin, where the KVA is included, and where the defaultability of the bank is accounted for. In each case, the well-posedness and noticeable properties of the corresponding (A)BSDEs are derived.

2 Mathematical Framework

Throughout the paper, we denote by:

- $(\Omega, \mathcal{A}, \mathbb{G}, \mathbb{Q})$, a filtered probability space, with complete and right continuous filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$;
- $\mathbb{E}$ and $\mathbb{E}_t$, the corresponding expectation and conditional expectation;
- $m(S)$ (with $m(S)_0 = S_0$), the canonical Doob-Meyer local martingale component of a special semimartingale S (all under $(\mathbb{G}, \mathbb{Q})$);
- $\mathcal{P}$, the $\mathbb{G}$ predictable sigma-field;
- $\mathcal{B}(\cdot)$, the Borel σ field on the underlying (topological) space;
- $|\cdot|$, an Euclidean norm in the dimension of its argument (which may change with the context);
- C, a positive constant, the value of which may change from line to line.

As can be established by section theorem, for any progressive Lebesgue integrand X such that the predictable projection ${}^pX$ exists,[1] the indistinguishable equality $\int_0^\cdot {}^pX_s ds = \int_0^\cdot X_s ds$ holds. As a consequence, we only consider predictable Lebesgue integrands in the paper (even if this means replacing X by ${}^pX$). In particular, we consider:

- r and λ, $\mathcal{P}$ measurable risk-free interest rate and bank funding spread processes, assumed **bounded from below** and **bounded**, respectively;
- $\beta = e^{-\int_0^\cdot r_t dt}$, the risk-free discount factor.

Until Sect. 6, we consider the hypothetical case of a default-free bank. In this case, the bank borrowing spread λ is interpreted as a liquidity spread. However, in reality, banks are defaultable and λ is, essentially, an instantaneous credit spread process, which is typically proxied by the CDS curve of the bank (see Sect. 6.1). Assuming λ bounded is therefore no issue.

By contrast, in the context of XVA computations, interest rates are typically modeled as stochastic (unbounded above but possibly negative since the crisis) processes, which grounds our assumption on r. ∎

[1] For which σ integrability of X valued at any stopping time, e.g. X bounded or càdlàg, is enough.

2.1 Martingale Representation Setup

We denote by W a $(\mathbb{G},\mathbb{Q})$ standard d variate Brownian motion, by E an Euclidean space, and by π a σ finite measure on $(E,\mathcal{B}(E))$ such that $\int_E (1\wedge|e|^2)\pi(de) < \infty$. We consider a $\mathbb{G}$ optional integer-valued random measure on $([0,T]\times E, \mathcal{B}([0,T])\otimes\mathcal{B}(E))$, with $\mathbb{Q}$ compensator measure $\eta(t,e)\pi(de)dt$ and compensated martingale measure $M(dt,de)$, for some $\widehat{\mathcal{P}}$ measurable, nonnegative and bounded function η, where $\widehat{\mathcal{P}} = \mathcal{P}\otimes\mathcal{B}(E)$. Stochastic integrals of $\mathcal{P}$ measurable processes against semimartingales, vector stochastic integrals extensions of the latter, and stochastic integrals of $\widehat{\mathcal{P}}$ measurable random functions with respect to jump measures or their compensations, are all meant in the sense of Jacod (1979). We introduce the following spaces:

- $\mathcal{S}_2$, the space of real valued $\mathbb{G}$ adapted càdlàg processes Y such that

$$\|Y\|^2_{\mathcal{S}_2} = \mathbb{E}\big[\sup_{0\le t\le T} Y_t^2\big] < +\infty;$$

- $\mathcal{H}_1$, the space of real valued $\mathbb{G}$ progressive processes X such that

$$\|X\|_{\mathcal{H}_1} = \mathbb{E}\big[\int_0^T |X_t|dt\big] < +\infty.$$

- $\mathcal{H}_2$, the space of $\mathbb{R}^d$ valued $\mathcal{P}$ measurable processes Z such that

$$\|Z\|^2_{\mathcal{H}_2} = \mathbb{E}\big[\int_0^T |Z_t|^2 dt\big] < +\infty;$$

- $\mathcal{L}_0$, the space of $\mathcal{B}(E)$ measurable functions endowed with the topology of convergence in measure induced by π, and we write, for any time t and $u\in\mathcal{L}_0$,

$$|u|_t = \left(\int_E |u(e)|^2\eta(t,e)\pi(de)\right)^{1/2};$$

- $\widehat{\mathcal{H}}_2$, the space of $\widehat{\mathcal{P}}$ measurable random functions U such that

$$\|U\|^2_{\widehat{\mathcal{H}}_2} = \mathbb{E}\Big[\int_0^T\int_E U_t^2(e)\eta(t,e)\pi(de)dt\Big] = \mathbb{E}\Big[\int_0^T |U_t|_t^2 dt\Big] < \infty.$$

We define $\|\cdot\|_2$ on $\mathcal{S}_2\times\mathcal{H}_2\times\widehat{\mathcal{H}}_2$ by

$$\|(Y,Z,U)\|_2^2 = \|Y\|^2_{\mathcal{S}_2} + \|Z\|^2_{\mathcal{H}_2} + \|U\|^2_{\widehat{\mathcal{H}}_2}.$$

We assume that every $(\mathbb{G},\mathbb{Q})$ square integrable martingale null at time 0 has a representation of the form

$$\int_0^t Z_s dW_s + \int_0^t\int_E U_s(e)M(ds,de), \tag{1}$$

for suitable integrable integrands $Z \in \mathcal{H}_2$ and $U \in \widehat{\mathcal{H}}_2$, uniquely defined modulo $d[W,W]$ and $\eta(t,e)\pi(de)dt$ negligible sets, respectively.

In particular, under this **square integrable martingale representation property**, one can readily check that all the results in Kruse and Popier (2016, Sect. 4),[2] which are derived in the special case of a Poisson random measure, are still valid in the above setup. Hence we use these results freely in the sequel.

Until Sect. 6, we call **square integrable solution** to any XVA BSDE[3] a special semimartingale solution XVA in $\mathcal{S}_2$ with $m(\mathrm{XVA})$ in $\mathcal{S}_2$.

3 XVA Framework

This section reviews the XVA framework of Albanese and Crépey (2017). For other XVA frameworks, see, for instance, Brigo and Pallavicini (2014), Bichuch, Capponi, and Sturm (2017) (without KVA), or, with a KVA meant as an additional contra-asset like the CVA and the FVA (as opposed to a risk premium in our case), Green, Kenyon, and Dennis (2014) or Elouerkhaoui (2016).

We consider a derivative portfolio with final maturity T between a bank and a client. In order to focus on counterparty risk and XVA analysis, we assume that the bank has setup with other banks[4] a fully collateralized back-to-back market hedge of its client portfolio. But we conservatively assume no XVA hedge. Hence only the counterparty risk related cash flows remain. Until until Sect. 6, we assume that **only the client is default prone**. The defaultability of the bank itself is added in Sect. 6.

The credit valuation adjustment (CVA) exposure of the bank to the default of its client is modeled as $Q_t\delta(dt)$, where δ is a Dirac measure at the default time of the client and where Q is some $\mathbb{G}$ optional nonnegative loss process of the bank given the client default.

Collateral set against counterparty risk includes variation margin, which tracks the mark-to-market of the portfolio and is typically rehypothecable, and initial margin set as a cushion against gap risk, i.e. the risk of slippage between the mark-to-market of the portfolio and its variation margin during the liquidation period that follows a default.

Now, the bank can use capital as variation margin (only, as opposed to initial margin, for reasons explained Section 3.2 in Albanese, Caenazzo, and Crépey (2017)). As economic capital can only be used for funding variation margin and because this feature is our main focus in this paper, we ignore initial margin for simplicity and focus on cash and rehypothecable variation margin. Initial margin could be added without harm as in Albanese, Caenazzo, and Crépey (2017). We denote by P the collateral (variation margin) needs of the bank ignoring the funding source provided by the different bank accounts also available for that purpose.

[2] At least, the part of their results derived under square integrable assumptions, which we only need in this paper.

[3] This notion will only be actually used for the FVA and KVA BSDEs.

[4] e.g. through a CCP, see Armenti and Crépey (2017).

Example 3.1 Let the mark-to-market (MtM) process denote the risk-free value process of the portfolio, i.e. the conditional expectation of the future contractual cash flows D promised by the client to the bank, ignoring counterparty risk and risky funding costs. Let VM denote the variation margin exchanged between the client and the bank, counted positively when received by the bank. In line with our assumption of a perfectly collateralized back to back hedge, we also suppose that the bank posts MtM as variation margin on its back to back hedge. Let R denote the recovery rate of the client in case it defaults. Then, assuming instantaneous liquidation of the client default, we have

$$P = (\mathrm{MtM} - \mathrm{VM}) \text{ killed at the default time of the client,}$$

$$Q = (1 - R)\big(\mathrm{MtM} + D - D_{-} - \mathrm{VM}\big)^{+}.$$

Note that the jump process $(D - D_{-})$ of the contractual cash flows contributes to the CVA exposure Q of the bank, because it fails to be paid by the client if the latter defaults. ∎

Remark 3.1 Everything in the paper can be readily extended to a bank engaged into bilateral trade portfolios with several clients, as considered in Albanese, Caenazzo, and Crépey (2017), by summing the $Q\delta$ and P processes over the different clients of the bank in all equations. ∎

We assume that

- The bank can invest at the risk-free rate r but can only obtain unsecured funding at the shifted rate $(r + \lambda)$;
- Posted collateral is remunerated at the risk-free rate by the receiving party.

3.1 Contra-Assets

The contra-asset (CA) value process corresponds to the conditionally expected future counterparty default and risky funding losses, i.e. $\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}$, computed under some (calibrated) risk-neutral pricing measure $\mathbb{Q}$. The corresponding amount, assessed incrementally on a run-off basis[5] at every new deal, is charged by the CVA desk and the Treasury of the bank to its client and put into the so-called reserve capital (RC) account in order to cope with these expected losses. From an accounting perspective, the CVA and the FVA represent special liabilities of the bank, meant to come as equity deductions, hence their name of contra-assets.

We assume that all the losses and earnings of the bank are marked to the model and realized in real time. In particular, the RC account is reset to its theoretical CA value at all times, any discrepancy between them being instantaneously released into the realized trading loss process L of the bank. Note that reserve capital can also be used as collateral. Accounting for it and since $\mathrm{RC} = \mathrm{CA}$, the funding (or investing, if negative) needs of the bank are reduced from P to $(P - \mathrm{CA})$.

[5] i.e. assuming only static portfolios, in line with the fact that a bank is a market maker that cannot anticipate future trades, see Section 6 in Albanese and Crépey (2017).

3.2 Economic Capital and Cost of Capital

By regulation, economic capital (EC) needs to be earmarked by the bank to cope with exceptional losses, beyond the expected losses already accounted for by reserve capital. This entails another charge for the bank, which is the cost, called KVA (capital valuation adjustment), of remunerating shareholders at some hurdle rate for their capital at risk. We assume a constant hurdle rate h, hence

$$\mathrm{KVA}_t = h\mathbb{E}_t \int_t^T e^{-\int_t^s (r_u+h)du}\mathrm{EC}_s ds, \quad 0 \le t \le T \tag{2}$$

(cf. the equation (46) in Albanese and Crépey (2017)). The corresponding cost, assessed incrementally on a run-off basis at every new deal, is sourced from the client and put into the so-called risk margin (RM) account. Under a continuous reset assumption on all bank accounts, any discrepancy between the RM account and its theoretical KVA value is instantaneously released into the shareholder dividend stream. Hence RM is equal to the KVA at all times. Moreover the risk margin is itself loss-absorbing, so that it is part of economic capital Hence,

$$\mathrm{KVA} = \mathrm{RM} \le \mathrm{EC}. \tag{3}$$

As economic capital is fungible with variation margin, the funding needs of the bank are reduced further from $(P - \mathrm{CA})$ to $(P - \mathrm{CA} - \mathrm{EC})$.

3.3 XVA Base Equations Ignoring the Fungibility of Economic Capital and Variation Margin

We assume that $\int_0^T \beta_s Q_s \delta(ds)$ is $\mathbb{Q}$ integrable and we define

$$\mathrm{CVA}_t = \mathbb{E}_t\Big[\int_t^T \beta_t^{-1}\beta_s Q_s \delta(ds)\Big],\ t \in [0,T], \tag{4}$$

which is nonnegative (as $Q \ge 0$).

In what follows we ignore the fungibility of economic capital as a funding source, so that the funding needs of the bank are seen as $(P - \mathrm{CA})$. Then, rephrasing the above qualitative descriptions in mathematical terms (but for funding needs of the bank simply seen as $(P - \mathrm{CA})$), the trading loss process L of the bank satisfies

$$\begin{aligned}
&L_0 = z \text{ (the accrued trading loss of the bank at the initial time 0) and, for } t \in (0,T] \\
&dL_t = \underbrace{Q_t\delta(dt)}_{\text{loss in case of client default}} + \underbrace{\Big((r_t+\lambda_t)\big(P_t - \mathrm{CA}_t\big)^+ - r_t\big(P_t - \mathrm{CA}_t\big)^-\Big)dt}_{\text{portfolio funding costs/benefits}} \\
&\qquad \underbrace{-r_t P_t dt}_{\text{remuneration of the collateral}} + \underbrace{d\mathrm{CA}_t}_{\text{appreciation of the liability CA of the bank}} \\
&= d\mathrm{CA}_t + Q_t\delta(dt) + \Big(\lambda_t\big(P_t - \mathrm{CA}_t\big)^+ - r_t\mathrm{CA}_t\Big)dt.
\end{aligned} \tag{5}$$

A no-arbitrage risk-neutral martingale condition on the trading loss L of the bank (cf. Albanese and Crépey (2017, Section 4.2)), together with the terminal condition $\mathrm{CA}_T = 0$, lead to the following equation for the CA process:

$$\mathrm{CA}_t = \underbrace{\mathbb{E}_t\Big[\int_t^T \beta_t^{-1}\beta_s Q_s \delta(ds)\Big]}_{\mathrm{CVA}_t} + \underbrace{\mathbb{E}_t\Big[\int_t^T \beta_t^{-1}\beta_s\lambda_s\big(P_s - \mathrm{CA}_s\big)^+ ds\Big]}_{\mathrm{FVA}_t},\ t\in[0,T]. \tag{6}$$

Since the CVA in (6) is an exogenous process, this equation for the CA process is equivalent to letting CA = CVA + FVA, for an FVA process that would be defined through the following BSDE:

$$\beta_t \mathrm{FVA}_t = \mathbb{E}_t \int_t^T \beta_s\lambda_s\Big(P_s - \mathrm{CVA}_s - \mathrm{FVA}_s\Big)^+ ds,\ 0\le t\le T. \tag{7}$$

Proposition 3.1 *Assuming that the processes r and $((P-\mathrm{CVA})^+)^2$ are in $\mathcal{H}_1$,*[6] *then the* FVA *BSDE* (7) *admits a unique square integrable solution.*

Proof. In terms of the coefficient

$$f:(t,y)\mapsto \lambda_t\Big(P_t - \mathrm{CVA}_t - y\Big)^+ - r_t y,\ y\in\mathbb{R}, \tag{8}$$

a direct application of Ito's formula induces that the FVA BSDE (7) rewrites as

$$\mathrm{FVA}_t = \mathbb{E}_t\int_t^T f(s,\mathrm{FVA}_s)ds,\ 0\le t\le T. \tag{9}$$

For any $t\in[0,T]$ and $y,y'\in\mathbb{R}$, we have

$$\begin{aligned}\big(f(t,y)-f(t,y')\big)(y-y') &= -r_t(y-y')^2\\ &\quad + \lambda_t(y-y')\Big(\big(P_t-\mathrm{CVA}_t-y\big)^+ - \big(P_t-\mathrm{CVA}_t-y'\big)^+\Big)\\ &\le C(y-y')^2,\end{aligned}$$

as r and λ are bounded from below. Hence the BSDE coefficient f satisfies the so-called monotonicity condition. Moreover, we have

$$\begin{aligned}&f(\cdot,0)=\lambda\big(P-\mathrm{CVA}\big)^+\\ &|f(\cdot,y)-f(\cdot,0)| = |\lambda\big(P-\mathrm{CVA}-y\big)^+ - ry - \lambda(P-\mathrm{CVA})^+| \le (|\lambda|+|r|)|y|.\end{aligned}$$

Since r and $((P-\mathrm{CVA})^+)^2$ are in $\mathcal{H}_1$, hence so are $\sup_{|y|\le c}|f(\cdot,y)-f(\cdot,0)|$ (for any $c>0$) and $(f(\cdot,0))^2$. Therefore the result follows by application of the general filtration BSDE results of Kruse and Popier (2016, Sect. 4)[7]. ∎

[6] Note that a sufficient condition for $((P-\mathrm{CVA})^+)^2\in\mathcal{H}_1$ is simply $(P^+)^2\in\mathcal{H}_1$, since $\mathrm{CVA}\ge 0$, .

[7] cf. also Kruse and Popier (2017), noting that we only use the part of their results derived under square integrable solutions anyway.

Remark 3.2 The FVA BSDE (9) also satisfies a comparison principle (see Kruse and Popier (2016, Sect. 4)).

As λ is bounded, (6), considered as an equation for (βFVA), is a Lipschitz BSDE. However, we want to deal with the FVA process directly, as opposed to working with (βFVA), which, in Markov numerical setups, would oblige to introduce an additional factor process to account for the path-dependence formally induced by β.

As visible from the above proof, Proposition 3.1 can be extended to λ bounded from below (only), for the modified integrability conditions

$$r,\ \lambda, \text{ and } (\lambda(P - \mathrm{CVA})^+)^2 \text{ in } \mathcal{H}_1. \blacksquare$$

4 Using Economic Capital as a Funding Source

We assume that a risk measure of its future losses L is dynamically earmarked by the bank as economic capital (EC). A stylized EC specification is

$$\mathrm{ES}_t\Big(\int_t^{t+1} \beta_t^{-1}\beta_s dL_s\Big), \tag{10}$$

where $\mathrm{ES}_t(\ell)$ denotes the $\mathcal{G}_t$ conditional expected shortfall of some level α ($\geq 50\%$, e.g. $\alpha = 97.5\%$) of a $\mathcal{G}_T$ measurable, $\mathbb{Q}$ integrable random variable (loss) ℓ, i.e.

$$\mathrm{ES}_t(\ell) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_t^a(\ell)da \in \mathbb{R}, \tag{11}$$

where $\mathrm{VaR}_t^a(\ell)$ denotes the $\mathcal{G}_t$ conditional quantile (value-at-risk) of level a of ℓ. Note that we will only deal with centered loss variables ℓ, hence $\mathrm{ES}_t(\ell) \geq 0$ (having assumed $\alpha \geq 50\%$).

Lemma 4.1 *For any $\mathcal{G}_T$ measurable, $\mathbb{Q}$ integrable random variables ℓ and ℓ', we have, for $0 \leq t \leq T$,*

$$|\mathrm{ES}_t(\ell) - \mathrm{ES}_t(\ell')| \leq \sup\left\{\mathbb{E}_t^{\mathbb{R}}[|\ell - \ell'|]\ ,\ \frac{d\mathbb{R}}{d\mathbb{Q}}_{|\mathcal{G}_T}\Big/\frac{d\mathbb{R}}{d\mathbb{Q}}_{|\mathcal{G}_t} \leq \frac{1}{1-\alpha}\right\} \ \leq\ \frac{1}{1-\alpha}\mathbb{E}_t[|\ell - \ell'|].$$

Proof. This follows from the conditional version of the following classical representation (cf. e.g. Artzner, Delbean, Eber, and Heath (1999)):

$$\mathrm{ES}_0(\ell) \ =\ \sup\left\{\mathbb{E}_0^{\mathbb{R}}[\ell]\ ;\ \frac{d\mathbb{R}}{d\mathbb{Q}}_{|\mathcal{G}_T} \leq \frac{1}{1-\alpha}\right\}, \tag{12}$$

where $\mathbb{E}^{\mathbb{R}}$ denotes the expectation under $\mathbb{R}$. $\blacksquare$

As explained in Sect. 3.2, on top of reserve capital, economic capital can also be used by the bank for its funding purposes, provided the bank pays to its shareholders the risk-free rate on EC that they would gain by depositing it otherwise. Under the stylized specification (10) for economic capital, the funding needs of the bank are then reduced from $(P_t - \mathrm{CA}_t)$ in (5) to $(P_t - \mathrm{CA}_t - \mathrm{ES}(\int_t^{t+1} \beta_t^{-1}\beta_s dL_s))$. Accordingly, the trading loss equation (5) and the CA equation (6) of the bank become, respectively,

$$
\begin{aligned}
&L_0 = z \text{ (the accrued trading loss of the bank at the initial time 0) and, for } t \in (0,T]\\
&dL_t = Q_t\delta(dt)\\
&+ \underbrace{r_t\mathrm{ES}_t\Big(\int_t^{t+1} \beta_t^{-1}\beta_s dL_s\Big)dt}_{\text{remuneration to shareholders for the EC funding source}}\\
&+ \underbrace{\Big((r_t+\lambda_t)\big(P_t - \mathrm{CA}_t - \mathrm{ES}_t\big(\int_t^{t+1} \beta_t^{-1}\beta_s dL_s\big)\big)^+ - r_t\big(P_t - \mathrm{CA}_t - \mathrm{ES}_t\big(\int_t^{t+1} \beta_t^{-1}\beta_s dL_s\big)\big)^-\Big)dt}_{\text{portfolio funding costs/benefits}}\\
&- r_tP_tdt + d\mathrm{CA}_t\\
&= d\mathrm{CA}_t + Q_t\delta(dt) + \Big(\lambda_t\big(P_t - \mathrm{CA}_t - \mathrm{ES}_t\big(\int_t^{t+1} \beta_t^{-1}\beta_s dL_s\big)\big)^+ - r_t\mathrm{CA}_t\Big)dt,
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
\mathrm{CA}_t = &\underbrace{\mathbb{E}_t\Big[\int_t^T \beta_t^{-1}\beta_s Q_s\delta(ds)\Big]}_{\mathrm{CVA}_t}\\
&+ \underbrace{\mathbb{E}_t\Big[\int_t^T \beta_t^{-1}\beta_s\lambda_s\big(P_s - \mathrm{CA}_s - \mathrm{ES}_s\big(\int_s^{s+1} \beta_s^{-1}\beta_u dL_u\big)\big)^+ ds\Big]}_{\mathrm{FVA}_t}, \; 0 \le t \le T.
\end{aligned}
\tag{14}
$$

Instead of an exogenous CA process in (6) feeding the dynamics (5) for L, we now face an FBSDE made of a forward SDE (13) for L coupled with a backward SDE (14) for the CA process. However, the system (13)–(14) can be decoupled as follows. We introduce the martingale component μ of the CVA such that

$$
\beta_t d\mu_t = d\mathbb{E}_t\Big[\int_0^T \beta_s Q_s\delta(ds)\Big]. \tag{15}
$$

Recalling that $m(\cdot)$ denotes the canonical Doob–Meyer local martingale component of a special semimartingale, we consider the following BSDE for a special semimartingale FVA

$$
\begin{aligned}
\mathrm{FVA}_t = \mathbb{E}_t \int_t^T \beta_t^{-1}\beta_s\lambda_s\Big(&P_s - \mathrm{CVA}_s - \mathrm{FVA}_s\\
&- \mathrm{ES}_s\Big[\int_s^{s+1} \beta_s^{-1}\beta_u\big(d\mu_u + dm(\mathrm{FVA})_u\big)\Big]\Big)^+ ds, \; 0 \le t \le T.
\end{aligned}
\tag{16}
$$

As the $\mathrm{ES}_s[\dots]$ term entails the conditional law of the one-year-ahead increments of $m(\mathrm{FVA})$, the FVA BSDE (16) is an anticipated BSDE (ABSDE) in the line of Peng and Yang (2009).

Lemma 4.2 *Given the* CVA *process exogenously defined by* (6) *under* $\mathbb{Q}$ *integrability of* $\int_0^T \beta_s Q_s \delta(ds)$, *the FBSDE* (13)–(14) *for* CA *and* L *is equivalent to the FVA ABSDE* (16) *through the following correspondence:*

$$\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA},\ dL_t = d\mu_t + dm(\mathrm{FVA})_t. \tag{17}$$

Specifically (all solutions here being meant in the sense of semimartingale solutions):

- *If processes* L *and* CA *solve the FBSDE* (13)–(14), *then the process* FVA *defined through the first identity in* (17) *solves the ABSDE* (16).
- *If a process* FVA *solves the ABSDE* (16), *then the processes* L *and* CA *defined through* (17) *solve the FBSDE* (13)–(14).

***Proof*.** Given processes L and CA solving the FBSDE (13)–(14), we show that the FVA process defined through the first identity in (17), i.e. by the second line in (14), solves the ABSDE (16). All we need to show is that, for such L, CA and FVA processes, the second identity holds in (17). By (13)–(14), we have, for $0 \le t \le T$,

$$\begin{aligned}
\beta_t dL_t &= d(\beta_t \mathrm{CA}_t) + \beta_t Q_t \delta(dt) + \beta_t \lambda_t \big(P_t - \mathrm{CA}_t - \mathrm{ES}_t(\int_t^{t+1} \beta_t^{-1}\beta_s dL_s)\big)^+ dt \\
&= d\mathbb{E}_t \int_t^T \beta_s Q_s \delta(ds) \\
&\qquad + d\mathbb{E}_t \Big[\int_t^T \beta_s \lambda_s \big(P_s - \mathrm{CA}_s - \mathrm{ES}_s(\int_s^{s+1} \beta_s^{-1}\beta_u dL_u)\big)^+ ds \Big] \\
&\qquad + \beta_t Q_t \delta(dt) + \beta_t \lambda_t \big(P_t - \mathrm{CA}_t - \mathrm{ES}_t(\int_t^{t+1} \beta_t^{-1}\beta_s dL_s)\big)^+ dt \\
&= \beta_t (d\mu_t + dm(\mathrm{FVA})_t),
\end{aligned} \tag{18}$$

by definitions of the CVA in (6), (15) of μ and of the FVA in the second line of (14). Hence the second identity in (17) follows.

Conversely, given a process FVA solving the ABSDE (16), in order to show that the processes L and CA defined through (17) solve the FBSDE (13)–(14), all we need to show is that the process L satisfies (13). By (16) and the second part in (17),

$$dm(\beta \mathrm{FVA})_t = d\mathbb{E}_t \int_0^T \beta_s \lambda_s \Big(P_s - \mathrm{ES}_s(\int_s^{s+1} \beta_s^{-1} \beta_u dL_u) - \mathrm{CVA}_s - \mathrm{FVA}_s\Big)^+ ds.$$

Going through the computations (18) in the reverse direction shows that L satisfies the first identity there, which is equivalent to (13). ∎

4.1 ABSDE Setup

The anticipated dependence of the forthcoming ABSDEs will be encompassed through a real valued, $\mathcal{P}\otimes\mathcal{B}(\mathcal{H}_2)\otimes\mathcal{B}(\widehat{\mathcal{H}}_2)$ measurable map ρ, satisfying the following:

Assumption 4.1 *There exists a constant $c_\rho > 0$ such that, for any $t \in [0,T]$ and $(Z,U),(Z',U') \in \mathcal{H}_2 \times \widehat{\mathcal{H}}_2$,*

$$|\rho_t(Z,U) - \rho_t(Z',U')| \leq \\ c_\rho\Big(\mathbb{E}_t \int_t^{(t+1)\wedge T} \big(|Z_s - Z'_s|^2 + |U_s - U'_s|_s^2\big)ds\Big)^{\frac{1}{2}}. \quad (19)$$

∎

We also consider a $\mathcal{G}_T$ measurable terminal condition ξ and a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathcal{L}_0)\otimes\mathcal{B}(\mathbb{R})$ function f, such that, abbreviating $\rho_t(0,0)$ into $\rho_t(0)$ and $f(t,0,0,0,\rho_t(0))$ into $f(t,0)$):

Assumption 4.2 *(i) For every arguments t,z,u,ϱ, the mapping $y \mapsto f(t,y,z,u,\varrho)$ is continuous. Moreover there exists a constant c_m such that*

$$(f(t,y,z,u,\varrho) - f(t,y',z,u,\varrho))(y-y') \leq c_m(y-y')^2 \quad (20)$$

holds uniformly;

(ii) There exists a constant $c_f > 0$ such that, for any arguments t, y, ϱ, ϱ', z, z', u, u',

$$|f(t,y,z,u,\varrho) - f(t,y,z',u',\varrho')| \leq c_f(|z-z'| + |u-u'|_t + |\varrho - \varrho'|;$$

(iii) The mappings $\sup_{|y|\leq c}|f(\cdot,y,0,0,\rho_t(0)) - f(\cdot,0)|$ (for any constant $c > 0$) and $(f(\cdot,0))^2$ are in $\mathcal{H}_1$;

(iv) $\mathbb{E}[|\xi|^2] < +\infty$. ∎

In particular, the coefficient f is monotone in y (cf. (20)).

We consider the following ABSDE (cf. Peng and Yang (2009)) with data ρ, ξ, and f, to be solved for a $\mathbb{G}$ adapted process Y, a $\mathcal{P}^{\otimes d}$ measurable process Z, and a $\widehat{\mathcal{P}}$ measurable function U such that (the three integrals that appear in (21) are well defined[8] and)

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s, U_s, \rho_s(Z,U)\right) ds \\ - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) M(ds,de),\ 0 \leq t \leq T \quad (21)$$

holds $d\mathbb{Q}\otimes dt$ almost everywhere. A solution (Y,Z,U) to (21) is dubbed **square integrable** if it is in $\mathcal{S}_2 \times \mathcal{H}_2 \times \widehat{\mathcal{H}}_2$.

[8] In which case, the second line in (21) defines a local martingale.

Remark 4.1 If (Y, Z, U) solves (21), then

$$(\bar{Y}, \bar{Z}, \bar{U}) = (e^{c_m \cdot} Y., e^{c_m \cdot} Z., e^{c_m \cdot} U.)$$

satisfies

$$\begin{aligned}\bar{Y}_t &= e^{c_m T}\xi + \int_t^T e^{c_m s} f\left(s, e^{-c_m s}\bar{Y}_s, e^{-c_m s}\bar{Z}_s, e^{-c_m s}\bar{U}_s, \rho_s(e^{-c_m \cdot}\bar{Z}., e^{-c_m \cdot}\bar{U}.)\right) ds \\ &\quad - \int_t^T \int_E \bar{U}_s(e) M(ds, de) - \int_t^T \bar{Z}_s dW_s,\ 0 \le t \le T.\end{aligned}$$

Hence $(\bar{Y}, \bar{Z}, \bar{U})$ solves an ABSDE with modified data that satisfy Assumptions 4.1 and 4.2 with $c_m = 0$ in (20). Moreover,

$$\|(Y, Z, U)\|_2 < +\infty \Longleftrightarrow \|(\bar{Y}, \bar{Z}, \bar{U})\|_2 < +\infty.$$

Hence we may and do suppose that $c_m = 0$ in what follows, without loss of generality. ∎

4.2 A Priori Estimate

Proposition 4.1 *If (Y, Z, U) is a square integrable solution to the ABSDE (21), then*

$$\left\|(Y, Z, U)\right\|_2^2 \le C\,\mathbb{E}\left(|\xi|^2 + \int_0^T |f(t,0)|^2 dt\right). \tag{22}$$

Proof. For $t \in [0, T]$, set $\Gamma_t = e^{\kappa t}$, for some to-be-determined nonnegative constant κ. An application of the Itô formula to $\Gamma|Y|^2$ yields

$$\begin{aligned}\Gamma_t|Y_t|^2 + \int_t^T \Gamma_s|Z_s|^2 ds + \int_t^T \Gamma_s|U_s|_s^2 ds &= \Gamma_T|\xi|^2 - \kappa\int_t^T \Gamma_s|Y_s|^2 ds \\ &\quad + 2\int_t^T \Gamma_s Y_s f\big(s, Y_s, Z_s, U_s, \rho_s(Z, U)\big) ds \\ &\quad - 2\int_t^T \Gamma_s Y_s Z_s dW_s - \int_t^T \int_E \Gamma_s\big(|Y_{s-} + U_s(e)|^2 - |Y_{s-}|^2\big) M(ds, de).\end{aligned} \tag{23}$$

By Assumption 4.2 on f and the Young inequality, we have for any $\epsilon > 0$

$$\begin{aligned}&2Y_s f\big(s, Y_s, Z_s, U_s, \rho_s(Z, U)\big) \\ &\le 2|Y_s||f(s,0)| + 2c_f|Y_s||Z_s| + 2c_f|Y_s||U_s|_s + 2|c_f||Y_s||\rho_s(Z, U) - \rho_s(0)| \\ &\le \left(1 + 8c_f^2 + c_f^2\epsilon^{-1}\right)|Y_s|^2 + |f(s,0)|^2 + \frac{1}{4}|Z_s|^2 + \frac{1}{4}|U_s|_s^2 + \epsilon|\rho_s(Z, U) - \rho_s(0)|^2.\end{aligned} \tag{24}$$

Besides, Assumption 4.1 implies

$$\begin{aligned}\mathbb{E}_t\Big[\int_t^T \epsilon\Gamma_s|\rho_s(Z, U) - \rho_s(0)|^2 ds\Big] &\le \epsilon c_\rho^2\,\mathbb{E}_t\Big[\int_t^T \Gamma_s\mathbb{E}_s\Big(\int_s^{(s+1)\wedge T} (|Z_u|^2 + |U_u|_u^2) du\Big) ds\Big] \\ &\le \epsilon c_\rho^2\mathbb{E}_t\left[\int_t^T \Gamma_u\big(|Z_u|^2 + |U_u|_u^2\big) du\right].\end{aligned} \tag{25}$$

Plugging (25) into (24), which in turn goes into (23), we compute for $\epsilon = \frac{1}{4c_\rho^2}$

$$\Gamma_t|Y_t|^2 + \frac{1}{2}\mathbb{E}_t\big[\int_t^T (\Gamma_s|Z_s|^2 + |U_s|_s^2)ds\big] + \mathbb{E}_t\big[\int_t^T \left(\kappa - (1 + 8c_f^2 + 4c_f^2 c_\rho^2)\right)\Gamma_s|Y_s|^2 ds\big]$$
$$\leq \mathbb{E}_t\big[\Gamma_T|\xi|^2\big] + \mathbb{E}_t\big[\int_t^T \Gamma_s|f(s,0)|^2 ds\big]. \tag{26}$$

Setting $\kappa = 1 + 8c_f^2 + 4c_f^2 c_\rho^2 > 0$, we deduce

$$\sup_{t\in[0,T]} \mathbb{E}\big[|Y_t|^2\big] + \mathbb{E}\big[\int_0^T (|Z_s|^2 + |U_s|_s^2)ds\big] \leq C\mathbb{E}\left(|\xi|^2 + \int_0^T |f(s,0)|^2 ds\right). \tag{27}$$

On the other hand, as $Y \in \mathcal{S}_2$ and $Z \in \mathcal{H}_2$, the stochastic integral $\int_0^t Y_s Z_s dW_s$ is a uniformly integrable martingale. Therefore we deduce from the Burkholder-Davis-Gundy and Young inequalities that

$$\mathbb{E}\big[\sup_{t\in[0,T]}\big|\int_t^T Y_s Z_s dW_s\big|\big] \leq \frac{1}{8}\mathbb{E}\big(\sup_{t\in[0,T]}|Y_t|^2\big) + 2C\mathbb{E}\big[\int_0^T |Z_s|^2 ds\big].$$

By the same line of argument, the stochastic integral with respect to M in (23) is a uniformly integrable martingale and we have

$$\mathbb{E}\big[\sup_{t\in[0,T]}\big|\int_t^T \int_E Y_{s-}U_s(e)M(ds,de)\big|\big] \quad \leq \frac{1}{8}\mathbb{E}\big(\sup_{t\in[0,T]}|Y_t|^2\big) + 2C\mathbb{E}\big[\int_0^T |U_s|_s^2 ds\big].$$

Plugging these estimates together with (27) in (23) yields

$$\mathbb{E}\left(\sup_{t\in[0,T]}|Y_t|^2\right) \leq C\mathbb{E}\left(|\xi|^2 + \int_0^T |f(s,0)|^2 ds\right),$$

which, in conjunction with (27), concludes the demonstration. ∎

4.3 Existence, Uniqueness, and Picard Approximation

We now consider the existence and uniqueness of a solution to the ABSDE (21). We use a contraction argument, which yields a Picard approximation scheme of the solution as a by product.

Theorem 4.1 *The ABSDE (21) admits a unique square integrable solution.*

***Proof*.** For (V, N) given in $\mathcal{H}_2 \times \widehat{\mathcal{H}}_2$, we consider the following monotone BSDE:

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s, U_s, \rho_s(V,N)\right)ds$$
$$- \int_t^T \int_E U_s(e)M(ds,de) - \int_t^T Z_s dW_s,\ 0 \leq t \leq T.$$

By Theorem 1 in Kruse and Popier (2016), this equation has a unique square integrable solution (Y, Z, U). Let Φ be the map from the Banach space $\mathcal{H}_2 \times \widehat{\mathcal{H}}_2$ into itself defined by $\Phi(V, N) = (Z, U)$. Let $(Z, U) = \Phi(V, N)$ and $(Z', U') = \Phi(V', N')$, for (V, N) and (V', N') in $\mathcal{H}_2 \times \widehat{\mathcal{H}}_2$. For $X = Y, Z, U, V$, or N, we write $\delta X = X - X'$. As in the proof of Proposition 4.1, letting $\Gamma_t = e^{\kappa t}$ and applying Ito's formula to $\Gamma|\delta Y|^2$ yields

$$\mathbb{E}[\Gamma_t|\delta Y_t|^2] + \mathbb{E}[\int_t^T \Gamma_s(|\delta Z_s|^2 + |\delta U_s|_s^2 + \kappa|\delta Y_s|^2)ds] = 2\int_t^T \mathbb{E}[\Gamma_s \delta Y_s \delta f_s]ds, \qquad (28)$$

where

$$\delta f_s = f(s, Y_s, Z_s, U_s, \rho_s(V, N)) - f(s, Y_s', Z_s', U_s', \rho_s(V', N')).$$

Given Assumptions 4.1 on ρ and 4.2 on f, using (24), we have

$$\begin{aligned} 2\int_t^T \mathbb{E}[\Gamma_s \delta Y_s \delta f_s]ds &\le \mathbb{E}[\int_t^T \Gamma_s \left(8c_f^2 + 4c_f^2 c_\rho^2\right)|\delta Y_s|^2 ds] \\ &+ \frac{1}{4}\mathbb{E}[\int_t^T \Gamma_s(|\delta Z_s|^2 + |\delta U_s|_s^2)ds] + \frac{1}{4c_\rho^2}\mathbb{E}[\int_t^T |\rho_s(V, N) - \rho_s(V', N')|^2 ds]. \end{aligned} \qquad (29)$$

Similarly as in (25), we obtain

$$\mathbb{E}[\int_t^T |\rho_s(V, N) - \rho_s(V', N')|^2 ds] \le c_\rho^2 \mathbb{E}\Big[\int_t^T \Gamma_u(|\delta V_u|^2 + |\delta N_u|_u^2)du\Big]. \qquad (30)$$

Plugging the resulting estimate into (29), in turn cast into (28), yields, for $\kappa = 1 + 8c_f^2 + 4c_f^2 c_\rho^2$ as in the proof of Proposition 4.1,

$$\mathbb{E}[\int_0^T \Gamma_s(|\delta Z_s|^2 + |\delta U_s|_s^2)ds] \le \frac{1}{3}\mathbb{E}[\int_0^T \Gamma_s(|\delta V_s|^2 + |\delta N_s|_s^2)ds]. \qquad (31)$$

Hence (noting that Γ is bounded away from 0) Φ is a contraction on the Banach space $\mathcal{H}_2 \times \widehat{\mathcal{H}}_2$. This entails the existence of a unique fix-point of Φ, which immediately implies the existence of a unique square integrable solution to (21). ∎

Corollary 4.1 *The Picard iterative scheme defined, for $n \ge 1$, by*

$$\begin{aligned} Y_t^n = \xi &+ \int_t^T f\left(s, Y_s^n, Z_s^n, U_s^n, \rho_s(Z^{n-1}, U^{n-1})\right) ds \\ &- \int_t^T Z_s^n dW_s - \int_t^T \int_E U_s^n(e) M(ds, de),\ 0 \le t \le T, \end{aligned}$$

starting from $(Y^0, Z^0, U^0) = (0, 0, 0)$, converges in $\mathcal{S}_2 \times \mathcal{H}_2 \times \widehat{\mathcal{H}}_2$ to the square integrable solution of the ABSDE (21).

***Proof*.** Setting $\Gamma_t = e^{\kappa t}$ with $\kappa = 1 + 8c_f^2 + 4c_f^2 c_\rho^2$ and reasoning as in the proof of Theorem 4.1 yields

$$\mathbb{E}\left[\int_0^T \Gamma_s(|Z_s^n - Z_s^{n-1}|^2 + |U_s^n - U_s^{n-1}|_s^2)ds\right] \leq \left(\frac{1}{3}\right)^{n-1} \mathbb{E}\left[\int_0^T \Gamma_s(|Z_s^1|^2 + |U_s^1|_s^2)ds\right].$$

Combining this estimate with standard Burkholder-Davis-Gundy arguments as in the proof of Proposition 4.1 leads to

$$\left\|(Y^n, Z^n, U^n) - (Y^{n-1}, Z^{n-1}, U^{n-1})\right\|_2^2 \leq C\left(\frac{1}{3}\right)^{n-1},$$

which implies the result. ∎

Remark 4.2 A similar argumentation ensures the convergence in $\mathcal{S}_2 \times \mathcal{H}_2 \times \widehat{\mathcal{H}}_2$ of the following more explicit Picard iteration:

$$\begin{aligned} Y_t^n = \xi &+ \int_t^T f\left(s, Y_s^n, Z_s^{n-1}, U_s^{n-1}, \rho_s(Z^{n-1}, U^{n-1})\right) ds \\ &- \int_t^T Z_s^n dW_s - \int_t^T \int_E U_s^n(e) M(ds, de),\ 0 \leq t \leq T. \end{aligned}$$

The convergence of a fully explicit scheme is a priori not available, unless the driver f is not only monotone, but in fact Lipschitz in y. ∎

4.4 The FVA ABSDE Assuming $\mathrm{EC} = \mathrm{ES}$ is Well Posed

Back to our financial XVA setup:

Lemma 4.3 *The mapping*

$$\rho : (t, Z, U) \mapsto \mathrm{ES}_t\left[\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s\left(d\mu_s + Z_s dW_s + \int_E U_s(e) M(ds, de)\right)\right] \quad (32)$$

(cf. (10)–(11)*) satisfies Assumption 4.1.*

***Proof*.** that By Lemma 4.1, we have

$$\begin{aligned} |\rho_t(Z', U') - \rho_t(Z, U)|^2 \leq \frac{1}{(1-\alpha)^2}\bigg(\mathbb{E}_t\bigg| &\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s(Z'_s - Z_s)dW_s \\ &+ \int_t^{(t+1)\wedge T}\int_E \beta_t^{-1}\beta_s(U'_s(e) - U_s(e))M(ds, de)\bigg|\bigg)^2. \end{aligned}$$

Hence, using Jensen's inequality,

$$|\rho_t(Z', U') - \rho_t(Z, U)|^2 \leq \frac{C}{(1-\alpha)^2}\mathbb{E}_t\Big[\int_t^{(t+1)\wedge T} |Z'_s - Z_s|^2 ds + \int_t^{(t+1)\wedge T} |U'_s - U_s|_s^2 ds\Big],$$

as r is bounded from below. ∎

Note that, as CVA and ES are nonnegative, (33) below is satisfied as soon as $(P^{+})^2 \in \mathcal{H}^1$.

Proposition 4.2 *If* $r \in \mathcal{H}_1$ *and*

$$\Big(\Big(P - \mathrm{CVA} - \mathrm{ES}_{\cdot}\big[\int_{\cdot}^{(\cdot+1)\wedge T} \beta_{\cdot}^{-1}\beta_s d\mu_s\big]\Big)^{+}\Big)^2 \in \mathcal{H}_1, \tag{33}$$

then the FVA *ABSDE* (16) *admits a unique square integrable solution.*

Proof. Given the square integrable martingale representation property postulated in Sect. 2.1, the FVA ABSDE (16) for $\mathrm{FVA} \in \mathcal{S}_2$ with $m(\mathrm{FVA}) \in \mathcal{S}_2$ is equivalent to the following ABSDE for $(\mathrm{FVA}, Z, U) \in \mathcal{S}_2 \times \mathcal{H}_2 \times \widehat{\mathcal{H}}_2$:

$$\begin{aligned}\mathrm{FVA}_t = \mathbb{E}_t \int_t^T \beta_t^{-1}\beta_s\lambda_s\Big(&P_s - \mathrm{CVA}_s - \mathrm{FVA}_s \\ &- \mathrm{ES}_s\Big[\int_s^{s+1} \beta_s^{-1}\beta_u\big(d\mu_u + Z_s dW_s + \int_E U_s(e)M(ds,de)\big)\Big]\Big)^{+} ds,\ 0 \le t \le T.\end{aligned} \tag{34}$$

We want to apply Theorem 4.1 to (34) with ρ as in (32), which by Lemma 4.3 satisfies Assumption 4.1, $\xi = 0$, and

$$f : (t, y, z, u, \varrho) \mapsto \lambda_t\Big(P_t - \mathrm{CVA}_t - y - \varrho\Big)^{+} - r_t y,$$

which we simply denote by $f(t, y, \varrho)$.

By the same computations as in the proof of Proposition 3.1, the coefficient f satisfies Assumptions 4.2(i). Moreover, for any arguments t, y, ϱ, ϱ', we have

$$\begin{aligned}|f(t,y,\varrho) - f(t,y,\varrho')| &= |\lambda_t|\Big|\big(P_t - \mathrm{CVA}_t - y - \varrho\big)^{+} - \big(P_t - \mathrm{CVA}_t - y - \varrho'\big)^{+}\Big| \\ &\le C\big|\varrho - \varrho'\big|.\end{aligned}$$

Hence Assumption 4.2(ii) holds. Next, under (33), Assumption 4.2(iii) is satisfied because

$$f(t, y, 0, 0, \rho_t(0)) = \lambda_t\Big(P_t - \mathrm{CVA}_t - y - \mathrm{ES}_t\big[\int_t^{(t+1)\wedge T} \beta_t^{-1}\beta_s d\mu_s\big]\Big)^{-r_t y},\ 0 \le t \le T.$$

Finally, the terminal condition $\xi = 0$ obviously satisfies Assumption 4.2(iv). Therefore, by application of Theorem 4.1, the ABSDE (34) admits a unique solution (FVA, Z, U) in $\mathcal{S}_2 \times \mathcal{H}_2 \times \widehat{\mathcal{H}}_2$, i.e. the FVA ABSDE (16) admits a unique square integrable solution. ∎

Remark 4.3 As opposed to the base case of Sect. 3.3 where EC was ignored as a funding source (cf. Remark 3.2), the dependency with respect to the $\mathrm{ES}_s[\ldots]$ term in the FVA ABSDE (16) goes in the wrong direction for comparison to hold in general (cf. Peng and Yang (2009, Example 5.2)). ∎

Next, we consider the following Picard iterative scheme:

$$L^{(0)} = z + \mu - \mu_0,\ \mathrm{FVA}^{(0)} = 0 \tag{35}$$

and, for $n \geq 1$,

$$\begin{aligned}
&\mathrm{CA}^{(n)} = \mathrm{CVA} + \mathrm{FVA}^{(n)}\ \text{, where}\\
&\mathrm{FVA}^{(n)}_t = \mathbb{E}_t \int_t^T \beta_t^{-1}\beta_s\lambda_s\Big(P_s - \mathrm{CA}^{(n)}_s - \mathrm{ES}_s\big(\int_s^{s+1} \beta_s^{-1}\beta_u dL^{(n-1)}_u\big)\Big)^+ ds,\ t \in (0,T]\\
&L^{(n)}_0 = z \text{ and, for } t \in (0,T],\\
&dL^{(n)}_t = d\mathrm{CA}^{(n)}_t dt - r_t \mathrm{CA}^{(n)}_t dt + Q_t\delta(dt)\\
&\qquad\quad + \lambda_t\Big(P_t - \mathrm{CA}^{(n)}_t - \mathrm{ES}_t\big(\int_t^{t+1} \beta_t^{-1}\beta_s dL^{(n-1)}_s\big)\Big)^+ dt.
\end{aligned} \tag{36}$$

Proposition 4.3 *Under the assumptions of Proposition 4.2, the sequence* $(\mathrm{FVA}^{(n)})$ *converges in* $\mathcal{S}_2$ *to the square integrable solution* FVA *of* (16).

Proof. By the same computations as in the proof of Lemma 4.2 based on the definition of $\mathrm{FVA}^{(n)}$ in the second line of (36), we have, for $n \geq 1$,

$$dL^{(n)}_t = d\mu_t + dm(\mathrm{FVA}^{(n)})_t,\ 0 \leq t \leq T,$$

which also holds for $n = 0$, by (35). Consequently, $\mathrm{ES}_s\big(\int_s^{s+1} \beta_s^{-1}\beta_u dL^{(n-1)}_u\big)$ in (36) is nothing but $\mathrm{ES}_s\big(\int_s^{s+1} \beta_s^{-1}\beta_u(d\mu_u + dm(\mathrm{FVA}^{(n-1)})_u)\big)$. Hence, we have the equivalence between (36) and the following sequence of ABSDEs: $\mathrm{FVA}^{(0)} = 0$ and, for $n \geq 1$,

$$\begin{aligned}
\mathrm{FVA}^{(n)}_t = \mathbb{E}_t \int_t^T &\beta_t^{-1}\beta_s\lambda_s\Big(P_s - \mathrm{CVA}_s - \mathrm{FVA}^{(n)}_s\\
&- \mathrm{ES}_s\Big[\int_s^{s+1} \beta_s^{-1}\beta_u d\big(\mu_u + m(\mathrm{FVA}^{(n-1)})_u\big)\Big]\Big)^+ ds,\ 0 \leq t \leq T.
\end{aligned} \tag{37}$$

The convergence follows by Corollary 4.1 applied for the same data as Proposition 4.2. ∎

Remark 4.4 If r is bounded, then the FVA ABSDE (16) has a Lipschitz coefficient, so that a fully explicit variant of the scheme (36), with $\mathrm{CA}^{(n-1)}$ instead of $\mathrm{CA}^{(n)}$ in the right hand sides, converges as well (cf. Remark 4.2).

When r is only bounded from below, the ABSDE satisfied by $(\beta\mathrm{FVA})$ is still Lipschitz, so that a fully explicit scheme for $(\beta\mathrm{FVA})$ converges in $\mathcal{S}^2$ to it. ∎

5 Accounting for the KVA in Economic Capital

Sect. 4 accounts for the FVA reduction provided by the possibility to use economic capital as a source of funding, where economic capital is simply meant as ES, as if there was no KVA.

But, in fact, there is a KVA that we assume given by (2), or, equivalently,

$$\mathrm{KVA}_t = h\mathbb{E}_t \int_t^T \bar{\beta}_t^{-1}\bar{\beta}_s \mathrm{EC}_s ds,\ 0 \le t \le T, \tag{38}$$

where we set $\bar{\beta} = e^{-\int_0^\cdot (r_t+h)dt}$.

Accounting for the KVA component in economic capital and the corresponding consistency condition (3) that EC $\geq$ KVA, we redefine EC as, instead of ES in (13)–(14),

$$\mathrm{EC} = \max(\mathrm{ES}, \mathrm{KVA}), \tag{39}$$

so that (38) appears at first sight as a KVA BSDE with monotone[9] coefficient. However, as a consequence of (39), the bank trading loss process L depends itself on the KVA. Namely, the overall XVA problem is now written as (cf. (13)–(14))

$$\begin{aligned}
&L_0 = z \text{ and, for } t \in (0,T],\\
&dL_t = d\mathrm{CA}_t + Q_t\delta(dt) + \Big(\lambda_t\big(P_t - \mathrm{CA}_t - \max(\mathrm{ES}_t(\int_t^{t+1} \beta_t^{-1}\beta_s dL_s), \mathrm{KVA}_t)\big)^+\\
&\qquad\quad - r_t\mathrm{CA}_t\Big)dt\\
&\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}, \text{ where, for } t \in [0,T],\\
&\mathrm{FVA}_t = \mathbb{E}_t \int_t^T \beta_t^{-1}\beta_s\lambda_s\Big(P_s - \mathrm{CA}_s - \max\big(\mathrm{ES}_s(\int_s^{s+1} \beta_s^{-1}\beta_u dL_u), \mathrm{KVA}_s\big)\Big)^+ ds\\
&\mathrm{KVA}_t = h\mathbb{E}_t \int_t^T \bar{\beta}_t^{-1}\bar{\beta}_s \max\big(\mathrm{ES}_s(\int_s^{s+1} \beta_s^{-1}\beta_u dL_u), \mathrm{KVA}_s\big)ds,\ 0 \le t \le T.
\end{aligned} \tag{40}$$

By a straightforward adaptation of the proof of Lemma 4.2, (40) is equivalent to the following (FVA, KVA) system of equations: For $t \in [0,T]$,

$$\begin{aligned}
\mathrm{FVA}_t &= \mathbb{E}_t \int_t^T \beta_t^{-1}\beta_s\lambda_s\Big(P_s - \mathrm{CVA}_s - \mathrm{FVA}_s\\
&\qquad - \max\big(\mathrm{ES}_s(\int_s^{s+1} \beta_s^{-1}\beta_u dL_u), \mathrm{KVA}_s\big)\Big)^+ ds,\\
\mathrm{KVA}_t &= h\mathbb{E}_t \int_t^T \bar{\beta}_t^{-1}\bar{\beta}_s \max\Big(\mathrm{ES}_s\big(\int_s^{s+1} \beta_s^{-1}\beta_u dL_u\big), \mathrm{KVA}_s\Big)ds,
\end{aligned} \tag{41}$$

where

$$dL_t = d\mu_t + dm(\mathrm{FVA})_t. \tag{42}$$

[9] Because of unbounded r.

5.1 ABSDE System Setup

Given ρ as in Assumption 4.1, we now need to establish the well-posedness of a system of ABSDEs of the form

$$\begin{cases} Y_T = \xi,\ \bar{Y}_T = \chi \text{ and, for } t \le T, \\ -dY_t = f\Big(t, \bar{Y}_t, Y_t, Z_t, U_t, \rho_t(Z,U)\Big)dt - Z_t dW_t - \displaystyle\int_E U_t(e)M(dt,de), \\ -d\bar{Y}_t = g\left(t, \bar{Y}_t, \bar{Z}_t, \bar{U}_t, \rho_t(Z,U)\right)dt - \bar{Z}_t dW_t - \displaystyle\int_E \bar{U}_t(e)M(dt,de), \end{cases} \tag{43}$$

for some $\mathcal{G}_T$ measurable terminal conditions ξ, χ and some $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathcal{L}_0)\otimes\mathcal{B}(\mathbb{R})$ and $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}(\mathcal{L}_0)\otimes\mathcal{B}(\mathbb{R})$ measurable coefficients f and g such that, abbreviating $\rho_t(0,0)$ into $\rho_t(0)$, $f(t,0,0,0,0,\rho_t(0))$ into $f(t,0)$, and $g(t,0,0,0,\rho_t(0))$ into $g(t,0)$:

Assumption 5.1 *(i) For every arguments $t, z, u, y, \bar{y}$, the functions (for every fixed ω) $f(t,\bar{y},\cdot,z,u,\varrho)$ and $g(t,\cdot,\bar{z},\bar{u},\varrho)$ are continuous. Moreover, there exists positive constants c_m and $\bar{c}_m$ such that*

$$(f(t,\bar{y},y,z,u,\varrho) - f(t,\bar{y},y',z,u,\varrho)(y-y') \le c_m(y-y')^2$$
$$(g(t,\bar{y},\bar{z},\bar{u},\varrho) - f(t,\bar{y}',\bar{z},\bar{u},\varrho)(\bar{y}-\bar{y}') \le \bar{c}_m(\bar{y}-\bar{y}')^2$$

holds uniformly;

(ii) There exists a positive constant c_f such that, for any arguments $t, \bar{y}, \bar{y}', y, \varrho, \varrho', z, z', u, u'$:

$$|f(t,\bar{y},y,z,u,\varrho) - f(t,\bar{y}',y,z',u',\varrho')| \le c_f(|z-z'| + |u-u'|_t + |\varrho-\varrho'| + |\bar{y}-\bar{y}'|);$$

(iii) There exists a positive constant c_g such that, for any t, y, y', z, z', u, u',

$$|g(t,\bar{y},\bar{z},\bar{u},\varrho) - g(t,\bar{y}',\bar{z}',\bar{u}',\varrho'))| \le c_g(|\bar{z}-\bar{z}'| + |\bar{u}-\bar{u}'|_t + |\varrho-\varrho'|);$$

(iv) $\sup_{|y|\le c}|f(\cdot,0,y,0,0,\rho_\cdot(0)) - f(\cdot,0)|$ and $\sup_{|\bar{y}|\le c}|g(\cdot,\bar{y},0,0,\rho_\cdot(0)) - g(\cdot,0)|$ (for every $c>0$), as well as $(f(\cdot,0))^2$ and $(g(\cdot,0))^2$, are in $\mathcal{H}_1$;

(v) $\mathbb{E}\big[|\xi|^2 + |\chi|^2\big] < +\infty$. ∎

A solution $(Y,Z,U,\bar{Y},\bar{Z},\bar{U})$ to (43) is said **square integrable** if $(Y,Z,U,\bar{Y},\bar{Z},\bar{U})$ is in $(\mathcal{S}_2\times\mathcal{H}_2\times\widehat{\mathcal{H}}_2)^2$.

As in Remark 4.1, we suppose without loss of generality that $c_m = \bar{c}_m = 0$ in Assumption 5.1(i).

5.2 A priori Estimate

Proposition 5.1 *If* $(Y, Z, U, \bar{Y}, \bar{Z}, \bar{U})$ *is a square integrable solution to the ABSDE system* (43), *then*

$$\begin{aligned}\|(Y,Z,U)\|_2^2 + \|(\bar{Y},\bar{Z},\bar{U})\|_2^2 \leq \\ C\,\mathbb{E}\left[|\xi|^2 + |\chi|^2 + \int_0^T (|f(s,0)|^2 + |g(s,0)|^2)ds\right].\end{aligned} \tag{44}$$

Proof. Let $\Gamma_t = e^{\kappa t}$, for some to-be-determined nonnegative constant κ. Viewing $(\bar{Y}, \bar{Z}, \bar{U})$ as a solution to a classical monotone BSDE parameterized by (Z, U), usual estimates obtained by applications of Ito's formula to $\Gamma \bar{Y}^2$, for any $\epsilon_1 > 0$, imply

$$\begin{aligned}&\Gamma_t|\bar{Y}_t|^2 + \frac{1}{2}\mathbb{E}_t\int_t^T \Gamma_s(|\bar{Z}_s|^2 + |\bar{U}_s|_s^2)ds \\ &\qquad + \mathbb{E}_t\int_t^T \Big(\kappa - 1 - 4c_g^2 - \epsilon_1^{-1}c_g^2\Big)\Gamma_s|\bar{Y}_s|^2 ds \\ &\leq \mathbb{E}_t[\Gamma_T|\chi|^2] + \mathbb{E}_t\int_t^T \Gamma_s|g(s,0)|^2 ds + \epsilon_1\mathbb{E}_t\int_t^T \Gamma_s|\rho_s(Z,U) - \rho_s(0)|^2 ds \\ &\leq \mathbb{E}_t[\Gamma_T|\chi|^2] + \mathbb{E}_t\int_t^T \Gamma_s|g(s,0)|^2 ds + \epsilon_1 c_\rho^2\mathbb{E}_t\int_t^T \Gamma_s(|Z_s|^2 + |U_s|_s^2)ds,\end{aligned} \tag{45}$$

where the last inequality follows from Assumption 4.1 together with similar computations as in (25). In particular, for $\epsilon_1 = \frac{1}{4c_\rho^2}$ and $\kappa \geq 1 + 4c_g^2 + 4c_g^2c_\rho^2$, (45) implies

$$\mathbb{E}_t\int_t^T \Gamma_s|\bar{Y}_s|^2 ds \leq \mathbb{E}_t[\Gamma_T\chi] + \mathbb{E}_t\int_t^T \Gamma_s|g(s,0)|^2 ds + \frac{1}{4}\mathbb{E}_t\int_t^T \Gamma_s(|Z_s|^2 + |U_s|_s^2)ds. \tag{46}$$

Besides, viewing (Y, Z, U) as a solution to an ABSDE of the form (21) parameterized by $\bar{Y}$, similar computations as in (26), for $\kappa = \max(1 + 4c_g^2 + 4c_g^2c_\rho^2; 1 + 5c_f^2 + 8c_f^2c_\rho^2)$, yields

$$\begin{aligned}&\Gamma_t|Y_t|^2 + \frac{3}{8}\mathbb{E}_t\big[\int_t^T \Gamma_s(|Z_s|^2 + |U_s|_s^2)ds\big] \leq \\ &\qquad \mathbb{E}_t\big[\Gamma_T|\xi|^2\big] + \mathbb{E}_t\big[\int_t^T \Gamma_s|f(s,0)|^2 ds\big] + \mathbb{E}_t\Big[\int_t^T \Gamma_s|\bar{Y}_s|^2 ds\Big].\end{aligned}$$

Replacing the last term by its majorant from (46) yields

$$\begin{aligned}&\Gamma_t|Y_t|^2 + \frac{1}{8}\mathbb{E}_t\big[\int_t^T \Gamma_s(|Z_s|^2 + |U_s|_s^2)ds\big] \leq \\ &\qquad + \mathbb{E}_t\big[\Gamma_T(|\xi|^2 + |\chi|^2)\big]\mathbb{E}_t\big[\int_t^T \Gamma_s(|f(s,0)|^2 + |g(s,0)|^2)ds\big].\end{aligned}$$

The result follows by plugging this estimate into (45) and using the usual Burkholder Davis Gundy type estimations as in the proof of Proposition 4.1. ∎

5.3 Existence, Uniqueness, and Picard Approximation

Theorem 5.1 *The ABSDE system* (43) *has a unique square integrable solution* $(Y, Z, U, \bar{Y}, \bar{Z}, \bar{U})$.

Proof. For any given $(V, N) \in \mathcal{H}_2 \times \widehat{\mathcal{H}}_2$, we consider the BSDE system

$$\begin{cases} Y_T = \xi,\ \bar{Y}_T = \chi \text{ and, for } t \le T \\ -\,dY_t = f\Big(t, \bar{Y}_t, Y_t, Z_t, U_t, \rho_t(V, N)\Big)dt - Z_t dW_t - \displaystyle\int_E U_t(e) M(dt, de), \\ -\,d\bar{Y}_t = g\left(t, \bar{Y}_t, \bar{Z}_t, \bar{U}_t, \rho_t(V, N)\right)dt - \bar{Z}_t dW_t - \displaystyle\int_E \bar{U}_t(e) M(dt, de). \end{cases} \tag{47}$$

As in the proof of Theorem 4.1, the existence of a unique square integrable solution $(Y, Z, U, \bar{Y}, \bar{Z}, \bar{U})$ to (47) is ensured by classical results for monotone BSDEs, which allows to define a map Ψ from $\mathcal{H}_2 \times \widehat{\mathcal{H}}_2$ into itself by $\Psi(V, N) = (Z, U)$.

Let $\Psi(V, N) = (Z, U)$ and $\Psi(V', N') = (Z', U')$, for (Z, U) and (Z', U') in $\mathcal{H}_2 \times \widehat{\mathcal{H}}_2$. For $X = Y, Z, U, \bar{Y}, \bar{Z}$, or $\bar{U}$, we denote $\delta X = X' - X$. Introducing $\Gamma_t = e^{\kappa t}$ with $\kappa = \max(1 + 4c_g^2 + 4c_g^2 c_\rho^2; 1 + 5c_f^2 + 8c_f^2 c_\rho^2)$ as in the proof of Proposition 5.1, estimates similar to (46) imply

$$\mathbb{E}_t \int_t^T \Gamma_s |\delta\bar{Y}_s|^2 ds \le \frac{1}{4}\mathbb{E}_t \int_t^T \Gamma_s (|\delta V_s|^2 + |\delta N_s|_s^2) ds. \tag{48}$$

As in (47), we compute

$$\begin{aligned} &\Gamma_t |\delta Y_t|^2 + \frac{1}{2}\mathbb{E}_t\Big[\int_t^T \Gamma_s (|\delta Z_s|^2 + |\delta U_s|_s^2) ds\Big] \\ &\quad \le \mathbb{E}_t\Big[\int_t^T \Gamma_s |\delta\bar{Y}_s|^2 ds\Big] + \frac{1}{8}\mathbb{E}_t\Big[\int_t^T \Gamma_s (|\delta V_s|^2 + |\delta N_s|_s^2) ds\Big], \end{aligned} \tag{49}$$

Combining (48) and (49), we deduce

$$\mathbb{E}\Big[\int_0^T \Gamma_s (|\delta Z_s|^2 + |\delta U_s|_s^2) ds\Big] \le \frac{1}{2}\mathbb{E}\Big[\int_0^T \Gamma_s (|\delta V_s|^2 + |\delta N_s|_s^2) ds\Big]. \tag{50}$$

Therefore, Ψ is a contraction on the Banach space $\mathcal{H}_2 \times \widehat{\mathcal{H}}_2$, so that it has a unique a fix-point (Z, U) in $\mathcal{H}_2 \times \widehat{\mathcal{H}}_2$. Hence, in view of Proposition 5.1, the ABSDE system (43) has a unique solution $(Y, Z, U, \bar{Y}, \bar{Z}, \bar{U})$ in $(\mathcal{S}_2 \times \mathcal{H}_2 \times \widehat{\mathcal{H}}_2)^2$. ∎

Corollary 5.1 *The sequence* $(Y^n, Z^n, U^n, \bar{Y}^n, \bar{Z}^n, \bar{U}^n)$ *defined, for* $n \ge 1$, *by*

$$\begin{cases} Y_T^n = \xi,\ \bar{Y}_T^n = \chi \text{ and, for } t \le T \\ -\,dY_t^n = f\Big(t, \bar{Y}_t^n, Y_t^n, Z_t^n, U_t^n, \rho_t(Z^{n-1}, U^{n-1})\Big)dt - Z_t^n dW_t - \displaystyle\int_E U_t^n(e) M(dt, de), \\ -\,d\bar{Y}_t^n = g\left(t, \bar{Y}_t^n, \bar{Z}_t^n, \bar{U}_t^n, \rho_t(Z^{n-1}, U^{n-1})\right)dt - \bar{Z}_t^n dW_t - \displaystyle\int_E \bar{U}_t^n(e) M(dt, de), \end{cases}$$

starting from $(Y^0, Z^0, U^0) = (\bar{Y}^0, \bar{Z}^0, \bar{U}^0) = (0, 0, 0)$*, converges in* $(\mathcal{S}_2 \times \mathcal{H}_2 \times \widehat{\mathcal{H}}_2)^2$ *to the square integrable solution* $(Y, Z, U, \bar{Y}, \bar{Z}, \bar{U})$ *of the ABSDE system* (43).

Proof. From (50) in the proof of Theorem 5.1, we obtain

$$\begin{aligned}&\mathbb{E}\big[\int_0^T \Gamma_s(|Z_s^{n+1} - Z_s^n|^2 + |U_s^{n+1} - U_s^n|_s^2)ds\big]\\ &\qquad \leq \Big(\frac{1}{2}\Big)^n \mathbb{E}\big[\int_0^T \Gamma_s(|Z_s^1|^2 + |U_s^1|_s^2)ds\big].\end{aligned}$$

In view of the a priori estimate of Proposition 5.1, we deduce

$$\begin{aligned}&\|(Y^{n+1}, Z^{n+1}, U^{n+1}) - (Y^n, Z^n, U^n)\|_2^2 + \|(\bar{Y}^{n+1}, \bar{Z}^{n+1}, \bar{U}^{n+1}) - (\bar{Y}^n, \bar{Z}^n, \bar{U}^n)\|_2^2\\ &\qquad \leq C\Big(\frac{1}{2}\Big)^n,\end{aligned}$$

which implies the $(\mathcal{S}_2 \times \mathcal{H}_2 \times \widehat{\mathcal{H}}_2)^2$ convergence of the Picard iteration to $(Y, Z, U, \bar{Y}, \bar{Z}, \bar{U})$. ∎

Remark 5.1 By the same line of argument, the following more explicit Picard iteration:

$$\begin{cases} Y_T^n = \xi,\ \bar{Y}_T^n = \chi \text{ and, for } t \leq T \\ -\,dY_t^n = f\Big(t, \bar{Y}_t^n, Y_t^n, Z_t^{n-1}, U_t^{n-1}, \rho_t(Z^{n-1}, U^{n-1})\Big)dt - \int_E U_t^n(e)M(dt, de) - Z_t^n dW_t, \\ -\,d\bar{Y}_t^n = g\left(t, \bar{Y}_t^{n-1}, \bar{Z}_t^{n-1}, \bar{U}_t^{n-1}, \rho_t(Z^{n-1}, U^{n-1})\right)dt - \int_E \bar{U}_t^n(e)M(dt, de) - \bar{Z}_t^n dW_t, \end{cases}$$

converges likewise. The convergence of a fully explicit scheme is not available unless f is not only monotone, but in fact Lipschitz in y (cf. Remark 4.2). ∎

5.4 The (FVA,KVA) ABSDE System is Well-Posed

Back to the (FVA, KVA) system (41):

Proposition 5.2 *If* $r \in \mathcal{H}_1$ *and*

$$\Big(\mathrm{ES}_\cdot\big[\int_\cdot^{(\cdot+1)\wedge T} \beta_\cdot^{-1}\beta_s d\mu_s\big]\Big)^2,\ \Big(\Big(P - \mathrm{CVA} - \mathrm{ES}_\cdot\big[\int_\cdot^{(\cdot+1)\wedge T} \beta_\cdot^{-1}\beta_s d\mu_s\big]\Big)^+\Big)^2 \in \mathcal{H}_1, \quad (51)$$

then the (FVA, KVA) *system* (41) *admits a unique square integrable solution (componentwise).*

Proof. Given the square integrable martingale representation postulated in Sect. 2.1, this results from Theorem 5.1 applied with ρ defined by (32), $\xi = \chi = 0$ and

$$\begin{aligned}f(t, \bar{y}, y, z, u, \varrho) &= \lambda_t\Big(P_t - \mathrm{CVA}_t - y - \max\big(\varrho, \bar{y}\big)\Big)^+ - r_t y\\ g(t, \bar{y}, \bar{z}, \bar{u}, \varrho) &= h \max\big(\varrho, \bar{y}\big) - (r_t + h)\bar{y}.\end{aligned}$$

Indeed, Assumptions 4.1 and 5.1(i)-(ii)-(iii) are satisfied as in the proof of Proposition 4.2 and Assumption 5.1(iv) follows from (51), whereas Assumption 5.1(v) is straightforward. ∎

Remark 5.2 In the base case ignoring the use of economic capital as a funding source, Albanese and Crépey (2017, Proposition 7.1) (or Theorem 7.1 there in the case of a defaultable bank) establishes that the KVA in the sense of the monotonous coefficient BSDE (38)-(39), where L in $\mathrm{ES} = \mathrm{ES}_t(\int_t^{t+1} \beta_t^{-1}\beta_s dL_s)$ is exogenous in this base case, and the ensuing economic capital process as per (39), are optimal: EC thus defined is minimal among the set of economic capital processes C satisfying the admissibility conditions $C \geq \mathrm{ES}$ and $C \geq$ the ensuing cost of remunerating shareholder capital at risk at the hurdle rate h.

But this result, as well as its consequence that the corresponding KVA is the cheapest admissible remuneration of shareholder capital at risk, relies on the comparison theorem for the corresponding KVA BSDE. In the present setup where KVA and EC are intertwined and comparison does not hold (cf. Remark 5.2), we only have admissibility of our XVA equations, whereas minimality is an open issue.

However, it was also noted after equation (32) in Albanese, Caenazzo, and Crépey (2017) that in most cases we have numerically that EC = ES, i.e. the constraint EC $\geq$ KVA is not binding in (39): See Figure 2 for illustration (referring the reader to Albanese, Caenazzo, and Crépey (2017, Section 5.2) for every detail). The inequality would only stop holding if the hurdle rate h was very large and the term structure of EC started out very small and had a sharp peak in a few years, which would be quite unusual for a portfolio held on a run-off basis, as considered in XVA computations, which tends to amortize in time. As the coupling between our KVA and FVA equations in (41) is only via this constraint, we conjecture that our XVA equations are practically close to optimality, including in the more realistic setup where capital is deemed usable for funding purposes. ∎

The Picard iteration corresponding to the (FVA, KVA) system (41) is written as

$$
\begin{aligned}
&\mathrm{FVA}^{(0)} = \mathrm{KVA}^{(0)} = 0,\ dL^{(0)} = d\mu \text{ and, for every } n \geq 1,\\
&\mathrm{CA}^{(n)} = \mathrm{CVA} + \mathrm{FVA}^{(n)} \text{ , where, for } t \in (0, T],\\
&\mathrm{KVA}_t^{(n)} = h\mathbb{E}_t \int_t^T \bar{\beta}_t^{-1}\bar{\beta}_s \max\big(\mathrm{ES}_s(\int_s^{s+1} \beta_s^{-1}\beta_u dL_u^{(n-1)}), \mathrm{KVA}_s^{(n)}\big)ds\\
&\mathrm{FVA}_t^{(n)} = \mathbb{E}_t \int_t^T \beta_t^{-1}\beta_s\lambda_s\Big(P_s - \mathrm{CA}_s^{(n)} - \max\big(\mathrm{ES}_t(\int_t^{t+1} \beta_t^{-1}\beta_u dL_u^{(n-1)}), \mathrm{KVA}_t^{(n)}\big)\Big)^+ ds\\
&L_0^{(n)} = z \text{ and, for } t \in (0, T],\\
&dL_t^{(n)} = d\mathrm{CA}_t^{(n)}dt - r_t\mathrm{CA}_t^{(n)}dt + Q_t\delta(dt)\\
&\qquad + \lambda_t\Big(P_t - \mathrm{CA}_t^{(n)} - \max\big(\mathrm{ES}_t(\int_t^{t+1} \beta_t^{-1}\beta_u dL_u^{(n-1)}), \mathrm{KVA}_s^{(n)}\big)\Big)^+ dt.
\end{aligned}
\tag{52}
$$

Proposition 5.3 *Under the assumptions of Proposition 5.2, the sequence* $(\mathrm{FVA}^{(n)}, \mathrm{KVA}^{(n)})$ *converges in* $\mathcal{S}_2 \times \mathcal{S}_2$ *to the square integrable solution (componentwise)* $(\mathrm{FVA}, \mathrm{KVA})$ *of the ABSDE system* (41).

Proof. As in the proof of Proposition 4.3, we have the equivalence between (52) and the sequence $(\mathrm{FVA}^{(n)}, \mathrm{KVA}^{(n)})$ defined by the following system of equations: $\mathrm{FVA}^{(0)} = \mathrm{KVA}^{(0)} = 0$, $dL^{(0)} = d\mu$, and, for any $n \geq 1$ and $0 \leq t \leq T$,

$$\begin{aligned}
\mathrm{KVA}_t^{(n)} &= h\mathbb{E}_t \int_t^T \bar{\beta}_t^{-1}\bar{\beta}_s \max\big(\mathrm{ES}_s(\int_s^{s+1} \beta_s^{-1}\beta_u dL_u^{(n-1)}), \mathrm{KVA}_s^{(n)}\big)ds \\
\mathrm{FVA}_t^{(n)} &= \mathbb{E}_t \int_t^T \beta_t^{-1}\beta_s\lambda_s\Big(P_s - \mathrm{CVA}_s - \mathrm{FVA}_s^{(n)} \\
&\qquad - \max\big(\mathrm{ES}_s\Big[\int_s^{s+1} \beta_s^{-1}\beta_u\big(d\mu_u + dm(\mathrm{FVA}^{(n-1)})_u\big)\Big], \mathrm{KVA}_s^{(n)}\big)\Big)^+ ds \\
dL_t^{(n)} &= d\mu_t + dm(\beta\mathrm{FVA}^{(n)})_t.
\end{aligned} \tag{53}$$

Hence the result follows from Corollary 5.1 applied for the same data as Proposition 5.2. ∎

The immediate analog of Remark 4.4 (cf. also Remark 5.1) applies equally here.

6 The Case of a Defaultable Bank

In reality banks are defaultable and counterparty risk is related to cash flows or valuations linked to either counterparty default or the default of the bank itself. This section extends our previous results to the case of a defaultable bank.

Accounting for the default time of the bank τ and assuming instantaneous liquidations, the time horizon of the problem becomes $\bar{\tau} = \tau \wedge T$. A key distinction appears between the cash flows received by the bank prior τ and the cash flows received by the bank during the default resolution period starting at τ. Indeed the first stream of cash flows affects the bank shareholders, whereas the second stream of cash flows only affects creditors. For accepting a new deal, bank shareholders need to be at least indifferent given the first stream of cash flows only. Hence, cash flows need to be stopped before the default time τ in all equations (see Albanese and Crépey (2017, Section 3.2))).

For any left-limited process Y, we denote by $Y^{\tau-} = \mathbb{1}_{[0,\tau)}Y + \mathbb{1}_{[\tau,+\infty)}Y_{\tau-}$ the process Y stopped before time τ.

6.1 Reduction of Filtration Setup

In order to deal with equations stopped before τ, we suppose that the full model filtration $\mathbb{G}$ is the enlargement of a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ satisfying the usual assumptions as well as

$$\forall t \geq 0 \text{ and } B \in \mathcal{G}_t, \ \exists A \in \mathcal{F}_t \text{ such that } A \cap \{t < \tau\} = B \cap \{t < \tau\}. \tag{54}$$

Then (cf. Song (2016)):

- For any $\mathbb{G}$ stopping time θ, there exists an $\mathbb{F}$ stopping time θ', which we call the $\mathbb{F}$ reduction of θ, such that $\{\theta < \tau\} = \{\theta' < \tau\} \subseteq \{\theta = \theta'\}$;
- Any $\mathbb{G}$ optional (resp. predictable) process Y admits an $\mathbb{F}$ optional (resp. predictable) reduction Y' such that $\mathbb{1}_{[0,\tau)}Y = \mathbb{1}_{[0,\tau)}Y'$ (resp. $\mathbb{1}_{(0,\tau]}Y = \mathbb{1}_{(0,\tau]}Y'$).

We introduce the Azéma supermartingale $\mathsf{S}_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t), t > 0$ of τ and we assume $\mathsf{S}_T > 0$ a.s., so that $\mathbb{F}$ optional reductions are uniquely defined on $[0, T]$ (see e.g. Crépey and Song (2017a, Lemma 2.3)). Hence, in particular:

$$\text{Two } \mathbb{F} \text{ optional processes that coincide before } \tau \text{ coincide on } [0, T]. \tag{55}$$

We assume further that τ has a $(\mathbb{G}, \mathbb{Q})$ intensity $\gamma\mathbb{1}_{(0,\tau]}$ and we identify λ, the funding spread of the bank, to the instantaneous credit spread process $(1 - \bar{R})\gamma$, where $\bar{R}$ is a constant recovery rate of the bank. Assuming that $e^{\int_0^{\bar\tau} \gamma_s ds}$ is $\mathbb{Q}$ integrable, the results of Crépey and Song (2017b) (see in particular Theorem 4.1 there) show the existence of a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that we have the following bijections inverse to each other:

$$\mathcal{M}^\circ \underset{.^{\tau-}}{\overset{.'}{\rightleftarrows}} \mathcal{M}' \tag{56}$$

$$\mathcal{S}^\circ \underset{.^{\tau-}}{\overset{.'}{\rightleftarrows}} \mathcal{S}', \tag{57}$$

where $\mathcal{S}'$ is the space of the $(\mathbb{F}, \mathbb{P})$ semimartingales on $[0, T]$, $\mathcal{S}^\circ$ is the space of the $(\mathbb{G}, \mathbb{Q})$ semimartingales on $[0, \bar{\tau}]$ without jump at τ, $\mathcal{M}'$ and $\mathcal{M}^\circ$ are their respective subspaces of local martingales, and $\cdot'$ stands for the $\mathbb{F}$ optional reduction.

Example 6.1 The situation where $\mathbb{P} = \mathbb{Q}$ corresponds to the case of a default time τ with a nonincreasing and predictable Azéma supermartingale S. The subcase where S is also continuous yields the class of pseudo-stopping times with the avoidance property (see Nikeghbali and Yor (2005) and Crépey and Song (2017a, Sect. 4.1)). The latter includes Cox times (see Bielecki et al. (2009, Chapter 3)), the class of default times most commonly used in applications. ∎

Still writing as before $\mathbb{E}$ and $\mathbb{E}_t$ for the $\mathbb{Q}$ expectation and $(\mathcal{G}_t, \mathbb{Q})$ conditional expectation, and $m(S)$ for the $(\mathbb{G}, \mathbb{Q})$ canonical local martingale component of a $(\mathbb{G}, \mathbb{Q})$ special semimartingale S, we further denote by $\mathbb{E}'$ and $\mathbb{E}'_t$ the $\mathbb{P}$ expectation and $(\mathcal{F}_t, \mathbb{P})$ conditional expectation, and by $m'(\widetilde{S})$ the $(\mathbb{F}, \mathbb{P})$ canonical local martingale component of an $(\mathbb{F}, \mathbb{P})$ special semimartingale $\widetilde{S}$. We also denote by

- $\mathcal{S}_2^\gamma$, the space of real valued càdlàg $\mathbb{G}$ adapted processes Y such that $Y = Y^{\tau-}$ and
$$\|Y\|^2_{\mathcal{S}_2^\gamma} = \mathbb{E}\Big[Y_0^2 + \int_0^T e^{\int_0^s \gamma_u du}\mathbb{1}_{\{s<\tau\}} d(Y_s^*)^2\Big] < \infty, \tag{58}$$
where $Y_t^* = \sup_{s\in[0,t]} |Y_s|$;

- $\mathcal{S}_2'$, the space of real valued càdlàg $\mathbb{F}$ adapted processes $\widetilde{Y}$ such that

$$\|\widetilde{Y}\|^2_{\mathcal{S}_2'} = \mathbb{E}'\big[\sup_{t\in[0,T]} \widetilde{Y}_t^2\big] < \infty;$$

- $\mathcal{H}_1'$, the space of real valued $\mathbb{F}$ progressive processes $\widetilde{X}$ such that $\mathbb{E}'\big[\int_0^T |\widetilde{X}_t|dt\big] < \infty$.

Additional results in Crépey and Song (2017b) show that we have the isometry (cf. (57))

$$\mathcal{S}_2^{\gamma} \;\underset{\cdot^{\tau-}}{\overset{\cdot'}{\rightleftarrows}}\; \mathcal{S}_2', \tag{59}$$

which, in view of (56), induces the analogous isometry on their respective subspaces of local martingales.

In this context, the notion of a **square integrable solution** to an XVA BSDE with respect to $(\mathbb{F},\mathbb{P})$ (in which case, to avoid ambiguity, we write $\widetilde{\mathrm{XVA}}$ rather than XVA) is defined as in previous sections (see the end of Sect. 2), but with respect to $(\mathbb{F},\mathbb{P})$, i.e. as an $(\mathbb{F},\mathbb{P})$ special semimartingale solution $\widetilde{\mathrm{XVA}}$ in $\mathcal{S}_2'$ with $m'(\widetilde{\mathrm{XVA}})$ in $\mathcal{S}_2'$; a **square integrable solution** to an XVA BSDE with respect to $(\mathbb{G},\mathbb{Q})$ is redefined as a $(\mathbb{G},\mathbb{Q})$ special semimartingale solution XVA in $\mathcal{S}_2^{\gamma}$ with $m(\mathrm{XVA})$ in $\mathcal{S}_2^{\gamma}$.

In view of the above, we assume that the default time of the client is an $\mathbb{F}$ stopping time and that the processes r, λ, and P (respectively Q) are $\mathbb{F}$ predictable (respectively $\mathbb{F}$ optional), without loss of generality, as it is always possible to consider their $\mathbb{F}$ reductions instead of them in all the $(\mathbb{F},\mathbb{P})$ equations below.

Finally, we now assume that $\int_0^T \beta_t Q_t \delta(dt)$ **is $\mathbb{P}$ integrable** (instead of $\mathbb{Q}$ integrable before).

6.2 Ignoring the Fungibility of Economic Capital and Variation Margin

In the base case of Sect. 3.3 where the fungibility of economic capital and variation margin is ignored, stopping all cash flows before τ in the right-hand side of (5) results in the following amended form of the CA equation (6) (written in differential form):

$$\begin{aligned}&\mathrm{CA}_T = 0 \text{ on } \{T<\tau\} \text{ and, for } t\in[0,\bar{\tau}],\\ &d\mathrm{CA}_t^{\tau-} = -\mathbb{1}_{\{t<\tau\}} Q_t\delta(dt) - \Big(\lambda_t\big(P_t - \mathrm{CA}_t\big)^+ - r_t\mathrm{CA}_t\Big)dt + dL_t,\end{aligned} \tag{60}$$

for some $(\mathbb{G},\mathbb{Q})$ local martingale L. Equivalently, we have the following equations for the CVA and FVA terms in the decomposition CA = CVA + FVA that is implicit in (60) (cf. (6)):

$$\begin{aligned}&\mathrm{CVA}_T = 0 \text{ on } \{T<\tau\} \text{ and, for } t\in[0,\bar{\tau}],\\ &d\mathrm{CVA}_t^{\tau-} = -\mathbb{1}_{\{t<\tau\}} Q_t\delta(dt) - r_t\mathrm{CVA}_t dt + d\mu_t,\end{aligned} \tag{61}$$

for some $(\mathbb{G},\mathbb{Q})$ local martingale μ, and

$$\begin{aligned}&\mathrm{FVA}_T = 0 \text{ on } \{T<\tau\} \text{ and, for } t\in[0,\bar{\tau}],\\ &d\mathrm{FVA}_t^{\tau-} = -\Big(\lambda_t\big(P_t - \mathrm{CVA}_t - \mathrm{FVA}_t\big)^+ - r_t\mathrm{FVA}_t\Big)dt + dm(\mathrm{FVA}^{\tau-})_t.\end{aligned} \tag{62}$$

These $(\mathbb{G},\mathbb{Q})$ CVA and FVA BSDEs stopped before τ can be tackled by reduction of filtration as follows. We consider the following $(\mathbb{F},\mathbb{P})$ BSDEs for some $\mathbb{F}$ semimartingales $\widetilde{\mathrm{CVA}}$ and $\widetilde{\mathrm{FVA}}$ on $[0,T]$:

$$\begin{aligned}&\widetilde{\mathrm{CVA}}_T = 0 \text{ and, for } t\in(0,T],\\ &d\widetilde{\mathrm{CVA}}_t = -Q_t\delta(dt) - r_t\widetilde{\mathrm{CVA}}_t dt + d\widetilde{\mu}_t,\end{aligned} \tag{63}$$

for some $(\mathbb{F},\mathbb{P})$ local martingale $\widetilde{\mu}$, and

$$\begin{aligned}&\widetilde{\mathrm{FVA}}_T = 0 \text{ and, for } t\in(0,T],\\ &d\widetilde{\mathrm{FVA}}_t = -\Big(\lambda_t\big(P_t - \widetilde{\mathrm{CVA}}_t - \widetilde{\mathrm{FVA}}_t\big)^+ - r_t\widetilde{\mathrm{FVA}}_t\Big)dt + dm'(\widetilde{\mathrm{FVA}})_t.\end{aligned} \tag{64}$$

Proposition 6.1 **(i)** *The* $(\mathbb{G},\mathbb{Q})$ CVA *BSDE* (61) *and the* $(\mathbb{F},\mathbb{P})$ $\widetilde{\mathrm{CVA}}$ *BSDE* (63) *have unique solutions in their respective spaces of semimartingale solutions, related by the bijection* (57) *whereas* μ *and* $\widetilde{\mu}$ *are related through* (56)*, with*

$$\widetilde{\mathrm{CVA}}_t = \mathbb{E}'_t\int_t^T \beta_t^{-1}\beta_s Q_s\delta(ds),\ t\in[0,T]; \tag{65}$$

(ii) *The* $(\mathbb{G},\mathbb{Q})$ FVA *BSDE* (62) *and the* $(\mathbb{F},\mathbb{P})$ $\widetilde{\mathrm{FVA}}$ *BSDE* (64) *are equivalent in their respective spaces of semimartingales through the bijection* (57) *and in their respective subspaces of square integrable solutions through the isometry* (59)*.*

Assuming that r *and* $\big((P-\widetilde{\mathrm{CVA}})^+\big)^2$ *are in* $\mathcal{H}'_1$*, then the* $(\mathbb{F},\mathbb{P})$ $\widetilde{\mathrm{FVA}}$ *BSDE* (64) *and the* $(\mathbb{G},\mathbb{Q})$ FVA *BSDE* (62) *have unique solutions in their respective spaces of square integrable solutions.*

Proof. **(i)** Assuming that (CVA,μ) satisfies (61), then $(\widetilde{\mathrm{CVA}},\widetilde{\mu}) = (\mathrm{CVA}',\mu')$ satisfies the second line in (63) on $[0,\bar{\tau}]$, hence on $[0,T]$, by (55). In addition, in view of the bijection (56), $\widetilde{\mu}=\mu'$ is an $(\mathbb{F},\mathbb{P})$ local martingale. Last, taking the $\mathcal{F}_T$ conditional expectation in the first line of (61) yields

$$0 = \mathbb{E}[\mathrm{CVA}_T\mathbb{1}_{\{T<\tau\}}|\mathcal{F}_T] = \mathbb{E}[\mathrm{CVA}'_T\mathbb{1}_{\{T<\tau\}}|\mathcal{F}_T] = \mathrm{CVA}'_T\mathsf{S}_T,$$

where $\mathsf{S}_T>0$, hence $\widetilde{\mathrm{CVA}}_T = \mathrm{CVA}'_T = 0$.

Conversely, assuming that $(\widetilde{\mathrm{CVA}},\widetilde{\mu})$ satisfies (63), then $(\mathrm{CVA},\mu) = (\widetilde{\mathrm{CVA}}^{\tau-},\widetilde{\mu}^{\tau-})$ satisfies (61) and $\mu=\widetilde{\mu}^{\tau-}$ is a $(\mathbb{G},\mathbb{Q})$ local martingale, by virtue of the bijection (56).

Hence, the CVA and $\widetilde{\mathrm{CVA}}$ equations are equivalent in their respective spaces of semimartingale solutions. Moreover, having assumed that $\int_0^T\beta_tQ_t\delta(dt)$ is $\mathbb{P}$ integrable,

the $(\mathbb{F},\mathbb{P})$ $\widetilde{\mathrm{CVA}}$ BSDE (63) has obviously (65) for unique solution.
(ii) The equivalence between the FVA and $\widetilde{\mathrm{FVA}}$ equations in their respective spaces of semimartingale solutions can be established much like the equivalence regarding the CVA in the above. In view of the isometry (59), this implies their further equivalence in their respective spaces of square integrable solutions. The second part of the result then follows by an application of Proposition 3.1, applied under $(\mathbb{F},\mathbb{P})$, to the $(\mathbb{F},\mathbb{P})$ $\widetilde{\mathrm{FVA}}$ BSDE (63). ∎

In the sequel, the CVA, $\widetilde{\mathrm{CVA}}$, μ, and $\widetilde{\mu}$ processes are as in Proposition 6.1(i).

6.3 Using Economic Capital as a Funding Source

Next we extend to a defaultable bank the situation of Sect. 4.4, accounting for the FVA reduction provided by the possibility to use economic capital as a source of funding, where economic capital is simply meant as the expected shortfall of the trading loss of the bank over one year, as if there was no KVA.

We denote by $\mathrm{ES}'_t(\widetilde{\ell})$ the $(\mathcal{F}_t,\mathbb{P})$ conditional expected shortfall of level α of an $\mathcal{F}_T$ measurable, $\mathbb{P}$ integrable random variable (loss) $\widetilde{\ell}$. Capital calculations are typically performed "on a going-concern basis," i.e. disregarding the default of the bank itself. Accordingly, our reference definition for economic capital is now $\mathrm{ES}'_t(\int_t^{t+1}\beta_t^{-1}\beta_s dL'_s)$, where L is the trading loss of the bank, with $\mathbb{F}$ optional reduction L'.

Accounting for the possible use of economic capital as a funding source (but ignoring the KVA at this stage), the CA process satisfies the following $(\mathbb{G},\mathbb{Q})$ BSDE on $[0,\bar{\tau}]$, which extends (14) to the case of a defaultable bank (written in differential form):

$$
\begin{aligned}
&\mathrm{CA}_T = 0 \text{ on } \{T<\tau\} \text{ and, for } t\in[0,\bar{\tau}],\\
&d\mathrm{CA}^{\tau-}_t = -\mathbb{1}_{\{t<\tau\}}Q_t\delta(dt) - \Big(\lambda_t\big(P_t - \mathrm{ES}'_t(\int_t^{t+1}\beta_t^{-1}\beta_s dL'_s) - \mathrm{CA}_t\big)^+ - r_t\mathrm{CA}_t\Big)dt + dL_t,
\end{aligned}
\tag{66}
$$

for some $(\mathbb{G},\mathbb{Q})$ local martingale L on $[0,\bar{\tau}]$. The CVA being given as in Proposition 6.1(i), we are looking for the CA process or, equivalently, for the FVA term in the decomposition CA = CVA + FVA.

First we consider the following $(\mathbb{F},\mathbb{P})$ $\widetilde{\mathrm{CA}}$ BSDE on $[0,T]$:

$$
\begin{aligned}
&\widetilde{\mathrm{CA}}_T = 0 \text{ and, for } t\in(0,T],\\
&d\widetilde{\mathrm{CA}}_t = -Q_t\delta(dt) - \Big(\lambda_t\big(P_t - \mathrm{ES}'_t(\int_t^{t+1}\beta_t^{-1}\beta_s d\widetilde{L}_s) - \widetilde{\mathrm{CA}}_t\big)^+ - r_t\widetilde{\mathrm{CA}}_t\Big)dt + d\widetilde{L}_t,
\end{aligned}
\tag{67}
$$

for some $(\mathbb{F},\mathbb{P})$ local martingale $\widetilde{L}$ on $[0,T]$.

Proposition 6.2 *The* $(\mathbb{G},\mathbb{Q})$ CA *BSDE* (66) *and the* $(\mathbb{F},\mathbb{P})$ $\widetilde{\mathrm{CA}}$ *BSDE* (67) *are equivalent in their respective spaces of semimartingale solutions. Specifically, if* (CA,L) *solves* (66)*, then* $(\widetilde{\mathrm{CA}},\widetilde{L}) = (\mathrm{CA}',L')$ *solves* (67)*. Conversely, if* $(\widetilde{\mathrm{CA}},\widetilde{L})$ *solves* (67)*, then* $(\mathrm{CA},L) = (\widetilde{\mathrm{CA}},\widetilde{L})^{\tau-}$ *solves* (66)*.*

***Proof*.** Similar to the proof of Proposition 6.1(i). ∎

Next we consider the following $(\mathbb{F},\mathbb{P})$ $\widetilde{\mathrm{FVA}}$ BSDE on $[0,T]$ (with $\widetilde{\mathrm{CVA}}$ and $\widetilde{\mu}$ as in (65)):

$$\begin{aligned}&\widetilde{\mathrm{FVA}}_T = 0 \text{ and, for } t \in (0,T],\\ &d\widetilde{\mathrm{FVA}}_t = -\Big(\lambda_t\big(P_t - \mathrm{ES}'_t(\int_t^{t+1}\beta_t^{-1}\beta_s(d\widetilde{\mu}_s + dm'(\widetilde{\mathrm{FVA}})_s)) - \widetilde{\mathrm{CVA}}_t - \widetilde{\mathrm{FVA}}_t\big)^+ \\ &\qquad\qquad - r_t\widetilde{\mathrm{FVA}}_t\Big)dt + dm'(\widetilde{\mathrm{FVA}})_t.\end{aligned} \tag{68}$$

Proposition 6.3 *The* $\widetilde{\mathrm{CA}}$ *BSDE* (67) *and the* $\widetilde{\mathrm{FVA}}$ *ABSDE* (68) *are equivalent in the sense of* $(\mathbb{F},\mathbb{P})$ *semimartingale solutions through the correspondence*

$$\widetilde{\mathrm{CA}} = \widetilde{\mathrm{CVA}} + \widetilde{\mathrm{FVA}},\ d\widetilde{L} = d\widetilde{\mu} + dm'(\widetilde{\mathrm{FVA}}). \tag{69}$$

If r *and* $\big((P - \widetilde{\mathrm{CVA}})^+\big)^2$ *are in* $\mathcal{H}'_1$*, then the* $(\mathbb{F},\mathbb{P})$ $\widetilde{\mathrm{FVA}}$ *BSDE* (68) *has a unique square integrable solution.*

***Proof*.** The first and second part follow by respective applications of Lemma 4.2 and Proposition 4.2 under $(\mathbb{F},\mathbb{P})$. ∎

The $\mathcal{S}'_2$ convergence of suitable Picard iterative schemes (e.g. the $(\mathbb{F},\mathbb{P})$ analog of (36)) to the square integrable solution $\widetilde{\mathrm{FVA}}$ of (68) follows by application of Proposition 4.3 under $(\mathbb{F},\mathbb{P})$.

6.4 Accounting for the KVA in Economic Capital

Taking the KVA into consideration, we redefine economic capital at time t as (cf. (39))

$$\mathrm{EC} = \max(\mathrm{ES}', \mathrm{KVA}),$$

instead of ES' simply in (66), which becomes:

$$\begin{aligned}&\mathrm{CA}_T = 0 \text{ on } \{T < \tau\} \text{ and, for } t \in (0,\bar{\tau}],\\ &d\mathrm{CA}_t^{\tau-} = -\mathbb{1}_{\{t<\tau\}}Q_t\delta(dt)\\ &\quad - \Big(\lambda_t\Big(P_t - \mathrm{CA}_t - \max\big(\mathrm{ES}'_t(\int_t^{t+1}\beta_t^{-1}\beta_s dL'_s), \mathrm{KVA}_t\big)\Big)^+ - r_t\mathrm{CA}_t\Big)dt + dL_t,\end{aligned} \tag{70}$$

for some $(\mathbb{G},\mathbb{Q})$ local martingale L, to be considered jointly with the following KVA BSDE:

$$\begin{aligned}&\mathrm{KVA}_T = 0 \text{ on } \{T < \tau\} \text{ and, for } t \in (0,\bar{\tau}],\\ &d\mathrm{KVA}_t^{\tau-} = -\Big(h\max\big(\mathrm{ES}'_t(\int_t^{t+1}\beta_t^{-1}\beta_s dL'_s), \mathrm{KVA}_t\big) - (r_t + h)\mathrm{KVA}_t\Big)dt + dm(\mathrm{KVA}^{\tau-})_t.\end{aligned} \tag{71}$$

Next, we consider the following auxiliary $(\mathbb{F},\mathbb{P})$ ABSDE system:

$$
\begin{aligned}
&\widetilde{\mathrm{CA}}_T = 0 \text{ and, for } t\in(0,T],\\
&d\widetilde{\mathrm{CA}}_t = -Q_t\delta(dt) - \Big(\lambda_t\Big(P_t - \max\big(\mathrm{ES}'_t(\int_t^{t+1}\beta_t^{-1}\beta_s d\widetilde{L}_s), \widetilde{\mathrm{KVA}}_t\big) - \widetilde{\mathrm{CA}}_t\Big)^+ - r_t\widetilde{\mathrm{CA}}_t\Big)dt\\
&\qquad\quad + d\widetilde{L}_t,
\end{aligned}
\tag{72}
$$

for some $(\mathbb{F},\mathbb{P})$ local martingale $\widetilde{L}$, along with

$$
\begin{aligned}
&\widetilde{\mathrm{KVA}}_T = 0 \text{ and, for } t\in(0,T],\\
&d\widetilde{\mathrm{KVA}}_t = -\Big(h\max\big(\mathrm{ES}'_t(\int_t^{t+1}\beta_t^{-1}\beta_s d\widetilde{L}_s), \widetilde{\mathrm{KVA}}_t\big) - (r_t+h)\widetilde{\mathrm{KVA}}_t\Big)dt + dm'(\widetilde{\mathrm{KVA}})_t.
\end{aligned}
\tag{73}
$$

Proposition 6.4 *The* $(\mathbb{G},\mathbb{Q})$ $(\mathrm{CA},\mathrm{KVA})$ *ABSDE system* (70)–(71) *and the* $(\mathbb{F},\mathbb{P})$ $(\widetilde{\mathrm{CA}},\widetilde{\mathrm{KVA}})$ *ABSDE system* (72)–(73) *are equivalent, in their respective (componentwise) spaces of componentwise semimartingale solutions, through the bijection* (57) *(componentwise).*

Proof. Similar to the proof of Proposition 6.1. ∎

Next, we consider the system (for $\widetilde{\mathrm{CVA}}$and $\widetilde{\mu}$ as in (65)):

$$
\begin{aligned}
&\widetilde{\mathrm{FVA}}_T = 0 \text{ and, for } t\in(0,T],\\
&d\widetilde{\mathrm{FVA}}_t = -\Big(\lambda_t\Big(P_t - \max\big(\mathrm{ES}'_t(\int_t^{t+1}\beta_t^{-1}\beta_s(d\widetilde{\mu}_s + dm'(\widetilde{\mathrm{FVA}})_s)), \widetilde{\mathrm{KVA}}_t\big)\\
&\qquad\qquad - \widetilde{\mathrm{CVA}}_t - \widetilde{\mathrm{FVA}}_t\Big)^+ - r_t\widetilde{\mathrm{FVA}}_t\Big)dt + dm'(\widetilde{\mathrm{FVA}})_t,
\end{aligned}
\tag{74}
$$

along with

$$
\begin{aligned}
&\widetilde{\mathrm{KVA}}_T = 0 \text{ and, for } t\in(0,T],\\
&d\widetilde{\mathrm{KVA}}_t = -\Big(h\max\big(\mathrm{ES}'_t(\int_t^{t+1}\beta_t^{-1}\beta_s(d\widetilde{\mu}_s + dm'(\widetilde{\mathrm{FVA}})_s)), \widetilde{\mathrm{KVA}}_t\big)\\
&\qquad\quad - (r_t+h)\widetilde{\mathrm{KVA}}_t\Big)dt + dm'(\widetilde{\mathrm{KVA}})_t.
\end{aligned}
\tag{75}
$$

Proposition 6.5 *The* $(\mathbb{F},\mathbb{P})$ $(\widetilde{\mathrm{CA}},\widetilde{\mathrm{KVA}})$ *ABSDE system* (72)–(73) *and the* $(\mathbb{F},\mathbb{P})$ $(\widetilde{\mathrm{FVA}},\widetilde{\mathrm{KVA}})$ *ABSDE system* (74)–(75) *are equivalent, in the sense of* $(\mathbb{F},\mathbb{P})$ *compenentwise semimartingale solutions, through the formal correspondence* (69).

Assuming that r *and* $((P-\widetilde{\mathrm{CVA}})^+)^2$ *are in* $\mathcal{H}'_1$*, then the* $(\mathbb{F},\mathbb{P})$ $(\widetilde{\mathrm{FVA}},\widetilde{\mathrm{KVA}})$ *ABSDE system* (74)–(75) *has a unique square integrable solution (componentwise).*

Proof. Similar to the proof of the CVA Proposition 6.1(i) for the first part and by an application of Proposition 5.2 under $(\mathbb{F},\mathbb{P})$ for the second part. ∎

The $\mathcal{S}_2'$ (componentwise) convergence of a suitable Picard iterative scheme (e.g. the $(\mathbb{F},\mathbb{P})$ analog of (52)) to the square integrable (compenentwise) solution $(\widetilde{\mathrm{FVA}},\widetilde{\mathrm{KVA}})$ of (74)–(75) follows by application of Proposition 5.3 under $(\mathbb{F},\mathbb{P})$. Used in conjunction with the approximation

$$\mathrm{ES}_0'(\cdot) \approx \mathrm{ES}_t'(\cdot) \tag{76}$$

in all equations, this results in a heuristic scheme, biased due to the approximation (76), for the numerical solution by nested Monte Carlo simulation of the $(\mathrm{FVA},\mathrm{KVA})$ system (41). This is the numerical scheme that was used for producing Figures 1 and 2.

If, in a Markov simulation setup, one was able to "learn" the conditional risk measures in (41) similarly as conditional expectations can be regressed in classical BSDE numerical schemes, then one could consider unbiased numerical schemes for (41). Note that the convergence of the scheme would have to be established without BSDE comparison results, as comparison does not hold for ABSDEs in general (cf. Remark 4.3). This is left for future research.

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