

Invariance Properties in the Dynamic Gaussian Copula Model

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Abstract We prove that the default times (or any of their minima) in the dynamic Gaussian copula model are invariance times in the sense of Crépey and Song (2015b), with related invariance probability measures different from the pricing measure. This reflects a departure from the immersion property, whereby the default intensities of the surviving names and therefore the value of credit protection spike at default times. These properties are in line with the wrong-way risk feature of counterparty risk embedded in credit derivatives, i.e. the adverse dependence between the default risk of a counterparty and the underlying credit derivative exposure.

Keywords: counterparty credit risk, wrong-way risk, Gaussian copula, dynamic copula, immersion property, invariance time, CDS.

Mathematics Subject Classification: 91G40, 60G07.

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1 Introduction

This paper deals with the mathematics of the dynamic Gaussian copula (DGC) model of Crépey, Jeanblanc, and Wu (2013) (see also Crépey, Bielecki, and Brigo (2014, Part IV) and Crépey and Nguyen (2015)). As developed in Crépey et al. (2014, Section 7.3.3), this model yields a dynamic meaning to the ad hoc bump sensitivities that were used by traders for hedging CDO tranches by CDS contracts before the crisis. From a more topical perspective, it can be used for counterparty risk computations on CDS portfolios. Related models include the one-period Merton model of Fermanian and Vigneron (2013, Section 6) or other variants commonly used in credit and counterparty risk softwares.

The dynamic Gaussian copula model has been assessed from an engineering perspective in previous work but a detailed mathematical study has been deferred to the present paper.

1.1 Invariance Times and Probability Measures

We work on a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$. The following conditions are studied in Crépey and Song (2015a, 2015b).

Given a \mathbb{G} stopping time τ and a subfiltration \mathbb{F} of \mathbb{G} , \mathbb{F} and \mathbb{G} satisfying the usual conditions, let S denote the Azéma supermartingale of τ , i.e. $S_t = \mathbb{Q}(\tau > t | \mathcal{F}_t)$, $t \geq 0$.

Condition (B). Any \mathbb{G} predictable process U admits an \mathbb{F} predictable reduction, i.e. an \mathbb{F} predictable process, denoted by U' , that coincides with U on $\llbracket 0, \tau \rrbracket$.

Remark 1.1 If $S_T > 0$ holds almost surely for some constant T , then any inequality between two \mathbb{G} predictable processes on $(0, \tau]$ implies the same inequality between their \mathbb{F} predictable reductions on $(0, T]$ (see Song (2014a, Lemma 6.1)); in particular, the process U' is uniquely determined on $(0, T]$.

For any left-limited process Y , we denote by $Y^{\tau-} = JY + (1 - J)Y_{\tau-}$ the process Y stopped before τ .

Condition (A). There exist a probability measure \mathbb{P} equivalent to \mathbb{Q} on \mathcal{F}_T such that (\mathbb{F}, \mathbb{P}) local martingales stopped before τ are (\mathbb{G}, \mathbb{Q}) local martingales on $[0, T]$.

If the conditions (B) and (A) are satisfied, then we say that τ is an invariance time and \mathbb{P} is an invariance probability measure. With respect to a standard progressive enlargement setup, the conditions (B) and (A) provide more flexibility by allowing for a full model filtration \mathbb{G} larger than \mathbb{F} progressively enlarged by τ and by allowing playing with the measure \mathbb{P} .

2 Dynamic Gaussian Copula Model

In the paper we prove that the default times τ_i (or any of their minima) in the DGC model are invariance times, with related invariance probability measures different from \mathbb{Q} . This reflects a departure from the immersion property, whereby the default intensities of the surviving names and therefore the value of credit protection spike at default times, consistent with empirical facts. This feature makes the DGC model appropriate for dealing with counterparty risk on credit derivatives (notably, portfolios of CDS contracts) traded between a bank and its counterparty, respectively labeled as -1 and 0 , and referencing credit names 1 to n , for some positive integer n . Accordingly, we introduce

$$N = \{-1, 0, 1, \dots, n\} \text{ and } N^* = \{1, \dots, n\}$$

and we focus on $\tau = \tau_{-1} \wedge \tau_0$ in the paper. However, analog properties hold for any minimum of the τ_i and, in particular, for the τ_i themselves.

2.1 The model

We consider a family of independent standard linear Brownian motions Z and Z^i , $i \in N$. For $\varrho \in [0, 1)$, we define

$$B_t^i := \sqrt{\varrho}Z_t + \sqrt{1 - \varrho}Z_t^i. \quad (2.1)$$

Let ς be a continuous function on \mathbb{R}_+ with $\int_{\mathbb{R}_+} \varsigma^2(s)ds = 1$. We assume that $\alpha^2(t) := \int_t^{+\infty} \varsigma^2(s)ds > 0$ for all $t \geq 0$. For any $i \in N$, let h_i be a continuously differentiable strictly increasing function from \mathbb{R}_+^* to \mathbb{R} , with derivative denoted by \dot{h}_i , such that $\lim_{s \downarrow 0} h_i(s) = -\infty$ and $\lim_{s \uparrow +\infty} h_i(s) = +\infty$. We define

$$\tau_i := h_i^{-1}\left(\int_0^{+\infty} \varsigma(u)dB_u^i\right) = h_i^{-1}\left(\sqrt{\varrho}\int_0^{+\infty} \varsigma(u)dZ_u + \sqrt{1 - \varrho}\int_0^{+\infty} \varsigma(u)dZ_u^i\right), \quad (2.2)$$

for $i \in N$. The random times $(\tau_i)_{i \in N}$ follow the standard one-factor Gaussian copula model of Li (2000) (a DGC model in abbreviation), with correlation parameter ϱ and with marginal survival function $\Phi \circ h_i$ of τ_i , where Φ is the standard normal survival function

$$\Phi(t) := \int_t^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} dx, \quad t \in \mathbb{R}.$$

Notice that, if $\varrho < 1$, the τ_i avoid each other:

$$\mathbb{Q}[\tau_i = \tau_j] = 0, \text{ for } i, j \in N, i \neq j.$$

2.2 Density Property

By multivariate density model, we mean a model with a conditional density of the default times given some reference market filtration, in the sense of the condition (DH) in page 1800 of Pham (2010). This is the multivariate extension of the notion of a density time, first introduced in an initial enlargement setup in Jacod (1987) and revisited in a progressive enlargement setup, under the name of initial time, in Jeanblanc and Le Cam (2009) and El Karoui, Jeanblanc, and Jiao (2010).

We now prove that every DGC model is a multivariate density model with respect to the natural filtration $\mathbb{B} = (\mathcal{B}_t)_{t \geq 0}$ of the Brownian motions $Z, Z^i, i \in N$. We introduce the following processes.

$$m_t^i := \int_0^t \varsigma(u) dB_u^i \quad \text{and} \quad \bar{m}_t^i := \int_t^\infty \varsigma(u) dB_u^i = h(\tau_i) - m_t^i, \quad i \in N.$$

Recall that $\alpha^2(t) = \int_t^{+\infty} \varsigma^2(v) dv$ is assumed positive for all $t \geq 0$. The standard normal survival function is denoted by Φ , while the standard normal density function is denoted by

$$\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$

Theorem 2.1 *The dynamic Gaussian copula model is a multivariate density model of default times (with respect to the filtration \mathbb{B}), with conditional Lebesgue density*

$$p_t(t_i, i \in N) := \partial_{t_{-1}} \dots \partial_{t_n} \mathbb{Q}(\tau_i < t_i, i \in N \mid \mathcal{B}_t)$$

of the $\tau_i, i \in N$, given, for any nonnegative $t_i, i \in N$, and $t \in \mathbb{R}_+$, by

$$p_t(t_i, i \in N) = \int_{\mathbb{R}} \phi(y) \prod_{i \in N} \phi\left(\frac{h_i(t_i) - m_t^i + \alpha(t)\sqrt{\varrho}y}{\alpha(t)\sqrt{1-\varrho}}\right) \frac{\dot{h}_i(t_i)}{\alpha(t)\sqrt{1-\varrho}} dy. \quad (2.3)$$

Proof. The conditional density function p given \mathcal{B}_t can be computed thanks to the independence of increments of the processes $Z, Z^i, i \in N$. Actually, for any $t \geq 0$,

$$\tau_i = h_i^{-1}(m_t^i + \sqrt{\varrho}\xi + \sqrt{1-\varrho}\xi_i), \quad i \in N,$$

where ξ is a real normal random variable with variance α_t^2 , and $(\xi_j)_{j \in N}$ is a $|N|$ -dimensional centered Gaussian vector with homogeneous marginal variances α_t^2 and null correlations, independent of ξ , and the family $\xi, \xi_i, i \in N$, is independent of \mathcal{B}_t . See Crépey et al. (2014, page 172)¹. ■

¹ Or Crépey et al. (2013, page 3) in the journal version.

2.3 Computation of the intensity processes

Notice that the Brownian filtration \mathbb{B} is not a model filtration, because the random times τ_i are not stopping times of \mathbb{B} . For a DGC model, the model filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is taken to be the progressive enlargement of the Brownian filtration \mathbb{B} by the $\tau_i, i \in N$, augmented so as to satisfy the usual conditions:

$$\mathcal{G}_t = \cap_{s > t} (\mathcal{B}_s \vee \bigvee_{i \in N} \sigma(\tau_i \wedge s)), \quad t \geq 0. \quad (2.4)$$

Note that the τ_i are \mathcal{B}_∞ measurable. In this section we prove that they are totally inaccessible \mathbb{G} stopping times with intensities that we compute explicitly.

For $I \subseteq N$ and $j \in N$, we define:

$$\rho^I = \frac{\varrho}{|I|\varrho + 1}, \quad (\sigma^I)^2 = (1 - \varrho) \frac{|I|\varrho + 1}{|I|\varrho + 1 - \varrho}, \quad \lambda^I = \frac{\varrho}{(|I| - 1)\varrho + 1},$$

$$Z_t^{j,I}(u) = \frac{h_j(u) - m_t^j}{\alpha(t)} - \lambda^I \sum_{i \in I} \frac{\bar{m}_t^i}{\alpha(t)}.$$

For $t \geq 0$, let

$$\mathcal{I}_t = \{i \in N : \tau_i \leq t\},$$

(representing the set of obligors in N that are in default at time t) and

$$\rho_t = \rho^{\mathcal{I}_t}, \quad \sigma_t = \sigma^{\mathcal{I}_t}, \quad \mathcal{J}_t = N \setminus \mathcal{I}_t.$$

For $\sigma > 0$ and $\rho \in [0, 1]$, for $J \subseteq N$, we define functions

$$\Phi_{J,\rho,\sigma}(\mathbf{z}_J) := \mathbb{Q}(\xi_j > z_j, j \in J), \quad \psi_{J,\rho,\sigma}^j(\mathbf{z}_J) := -\frac{\partial_{z_j} \Phi_{J,\rho,\sigma}}{\Phi_{J,\rho,\sigma}}(\mathbf{z}_J), \quad j \in J, \quad (2.5)$$

where $\mathbf{z}_J = (z_j)_{j \in J}$ is a real function on J , and $(\xi_j)_{j \in N}$ is a $(n + 2)$ -dimensional centered Gaussian vector with homogeneous marginal variance σ^2 and pairwise correlation ρ . Note the following:

Lemma 2.2 *For $I = N \setminus J$, the family of random variables*

$$\left(\xi_j - \frac{\rho}{(|I| - 1)\rho + 1} \sum_{i \in I} \xi_i \right)_{j \in J}$$

defines a centered Gaussian vector independent of $\sigma(\xi_i, i \in I)$, of marginal variance and respectively of the correlation

$$\sigma^2(1 - \rho) \frac{|I|\rho + 1}{|I|\rho + 1 - \rho} \quad \text{and respectively} \quad \frac{\rho}{|I|\rho + 1}. \quad \blacksquare \quad (2.6)$$

Lemma 2.3 *For $u > 0$,*

$$\mathbb{E}[\mathbf{1}_{\{u_j < \tau_j, j \in J\}} | \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)] = \Phi_{J,\rho^I,\sigma^I}(Z_t^{j,I}(u_j), j \in J). \quad (2.7)$$

Proof. For $j \in J$ and $u_j \in \mathbb{R}$, the condition $u_j < \tau_j$ is equivalent to

$$Z_t^{j,I}(u_j) = \frac{h_j(u_j) - m_t^j}{\alpha(t)} - \lambda^I \sum_{i \in I} \frac{\bar{m}_t^i}{\alpha(t)} < \frac{\bar{m}_t^j}{\alpha(t)} - \lambda^I \sum_{i \in I} \frac{\bar{m}_t^i}{\alpha(t)} \quad (2.8)$$

Noting that $m_t^j \in \mathcal{B}_t$, $\bar{m}_t^i \in \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)$, $i \in I$, the desired result follows by an application of Lemma 2.2. ■

Lemma 2.4 For $t > 0$, for $I \subseteq N$, $J = N \setminus I$ (the sets I, J being considered with their natural order),

$$\begin{aligned} & \{\tau_i \leq t < \tau_j : i \in I, j \in J\} \cap \mathcal{G}_t \\ &= \{\tau_i \leq t < \tau_j : i \in I, j \in J\} \cap (\mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)), \end{aligned} \quad (2.9)$$

where $\boldsymbol{\tau}_I$ denotes the vector $(\tau_i)_{i \in I}$. ■

Proof. Let the $\tau_{(i)}$ be the increasing ordering of the τ_i , with also $\tau_{(0)} = 0$ and $\tau_{(n+1)} = \infty$. According to the optional splitting formula which holds in any multivariate density model of default times (see Song (2014b)), for any \mathbb{G} optional process Y , there exists a $\mathcal{O}(\mathbb{B}) \otimes \mathcal{B}([0, \infty]^n)$ -measurable functions $Y^{(i)}$, $i \in N$, such that

$$Y = \sum_{i=0}^n Y^{(i)}(\tau_{-1} \uparrow \tau_{(i)}, \dots, \tau_n \uparrow \tau_{(i)}) \mathbb{1}_{[\tau_{(i)}, \tau_{(i+1)})}, \quad (2.10)$$

where $a \uparrow b$ denotes a if $a \leq b$ and ∞ if $a > b$, for $a, b \in [0, \infty]$. Since $\mathcal{G}_t = \sigma(Y_t)$ and $Y^{(i)}(\tau_{-1} \uparrow \tau_{(i)}, \dots, \tau_n \uparrow \tau_{(i)}) \mathbb{1}_{[\tau_{(i)}, \tau_{(i+1)})}$ is a function of \mathcal{B}_t and $\boldsymbol{\tau}_I$ on $\{\tau_i \leq t < \tau_j : i \in I, j \in J\}$, this shows (2.9). ■

Theorem 2.5 For any $j \in N$, τ_j admits a (\mathbb{G}, \mathbb{Q}) intensity given, for $t \in \mathbb{R}_+$, by:

$$\gamma_t^j := \mathbb{1}_{\{t < \tau_j\}} \frac{\dot{h}_j(t)}{\alpha(t)} \psi_{\mathcal{J}_t, \rho_t, \sigma_t}^j(Z_t^{j, \mathcal{I}_t}(t), j \in \mathcal{J}_t). \quad (2.11)$$

Proof. Let $l \in N$. For bounded \mathcal{B}_t measurable functions F , for measurable bounded function f , for $0 \leq t \leq s < \infty$, we look at

$$\mathbb{E}[Ff(\boldsymbol{\tau}_I) \mathbb{1}_{\{\tau_i \leq t < \tau_j : i \in I, j \in J\}} \mathbb{1}_{\{t < \tau_l \leq s\}}].$$

We need only to consider $l \in J$. Then, using (2.7) to pass to the third line and conditioning in conjunction with the tower rule to pass to the fourth line:

$$\begin{aligned} & \mathbb{E}[Ff(\boldsymbol{\tau}_I) \mathbb{1}_{\{\tau_i \leq t < \tau_j, i \in I, j \in J\}} \mathbb{1}_{\{s < \tau_l\}}] \\ &= \mathbb{E}[Ff(\boldsymbol{\tau}_I) \mathbb{1}_{\{\tau_i \leq t, i \in I\}} \mathbb{E}[\mathbb{1}_{\{t < \tau_j, j \in J\}} \mathbb{1}_{\{s < \tau_l\}} | \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)]] \\ &= \mathbb{E}[Ff(\boldsymbol{\tau}_I) \mathbb{1}_{\{\tau_i \leq t, i \in I\}} \Phi_{J, \rho^I, \sigma^I}(Z_t^{j, I}(u_j), j \in J)] \\ & \quad \text{where } u_j = t \text{ except } u_l = s, \\ &= \mathbb{E}[Ff(\boldsymbol{\tau}_I) \mathbb{1}_{\{\tau_i \leq t < \tau_j, i \in I, j \in J\}} \frac{\Phi_{J, \rho^I, \sigma^I}(Z_t^{j, I}(u_j), j \in J)}{\Phi_{J, \rho^I, \sigma^I}(Z_t^{j, I}(t), j \in J)}] \\ &= \mathbb{E}[Ff(\boldsymbol{\tau}_I) \mathbb{1}_{\{\tau_i \leq t < \tau_j, i \in I, j \in J\}} \frac{\Phi_{\mathcal{J}_t, \rho_t, \sigma_t}(Z_t^{j, \mathcal{I}_t}(u_j), j \in \mathcal{J}_t)}{\Phi_{\mathcal{J}_t, \rho_t, \sigma_t}(Z_t^{j, \mathcal{I}_t}(t), j \in \mathcal{J}_t)}]. \end{aligned} \quad (2.12)$$

With the formula (2.9), we conclude

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{t < \tau_l \leq s\}} | \mathcal{G}_t] &= \mathbb{E}[\mathbb{1}_{\{t < \tau_l\}} | \mathcal{G}_t] - \mathbb{E}[\mathbb{1}_{\{s < \tau_l\}} | \mathcal{G}_t] \\ &= \mathbb{1}_{\{t < \tau_l\}} \left(1 - \frac{\Phi_{\mathcal{J}_t, \rho_t, \sigma_t}(Z_t^{j, \mathcal{I}_t}(u_j), j \in \mathcal{J}_t)}{\Phi_{\mathcal{J}_t, \rho_t, \sigma_t}(Z_t^{j, \mathcal{I}_t}(t), j \in \mathcal{J}_t)}\right).\end{aligned}$$

The stated result follows by an application of the Laplace formula of Dellacherie (1972, Theorem T54). ■

2.4 Computation of the drift of the Brownian motion

We study now the processes $B^i, i \in N$, in the filtration \mathbb{G} . Thanks to Theorem 2.1, the DGC model is a multivariate density model. We have therefore the following consequence, according to Jacod (1987).

Lemma 2.6 *The processes $B^i, i \in N$, are \mathbb{G} semimartingales.*

Another consequence of the multivariate density property is the martingale representation property according to Jeanblanc and Song (2013, Theorem 6.4).

Theorem 2.7 *Let W^i , for $i \in N$, denote the martingale part in \mathbb{G} of B^i . Let*

$$dM_t^i = d\mathbb{1}_{\tau_i \leq t} - \gamma_t^i dt, \quad t > 0,$$

where the process γ^i is defined in (2.11). Then, the martingale representation property holds in \mathbb{G} with respect to $(W^i, M^i, i \in N)$.

This section is devoted to the computation of the martingales W^i . We begin with the following remark on the Gaussian processes B^i (cf. Lemma 2.2).

Lemma 2.8 *For $J \subseteq N$ and $I = N \setminus J$, the family of processes*

$$\left(B^j - \frac{\varrho}{(|I| - 1)\varrho + 1} \sum_{i \in I} B^i \right)_{j \in J} \quad (2.13)$$

is a continuous Levy process independent of $\sigma(B^i, i \in I)$, of marginal variance and correlation

$$(\sigma^I)^2 = (1 - \varrho) \frac{|I|\varrho + 1}{|I|\varrho + 1 - \varrho} \quad \text{and} \quad \rho^I = \frac{\varrho}{|I|\varrho + 1}. \quad \blacksquare \quad (2.14)$$

Proof. This follows by computing the brackets of the continuous local martingales and applying the Levy processes characterization. ■

Lemma 2.9 *For $k \in I$,*

$$\mathbb{E}[B_{s'}^k - B_s^k | \mathcal{B}_t \vee \sigma(\tau_I)] = \left(\frac{1}{\alpha_t^2} \int_s^{s'} \zeta_u du \right) \bar{m}_t^k.$$

Proof. For $k \in I$, for $0 \leq t \leq s \leq s' < \infty$,

$$B_{s'}^k - B_s^k - \left(\frac{1}{\alpha_t^2} \int_s^{s'} \varsigma_u du \right) \bar{m}_t^k \quad (2.15)$$

is a centered Gaussian random variable, independent of \bar{m}_t^k , with variance

$$(s' - s) - 2 \frac{1}{\alpha_t^2} \left(\int_s^{s'} \varsigma_u du \right)^2 + \frac{1}{\alpha_t^2} \left(\int_s^{s'} \varsigma_u du \right)^2 = (s' - s) - \frac{1}{\alpha_t^2} \left(\int_s^{s'} \varsigma_u du \right)^2.$$

Hence, for $k \in I$,

$$\begin{aligned} & \mathbb{E}[B_{s'}^k - B_s^k | \mathcal{B}_t \vee \sigma(\tau_I)] = \mathbb{E}[B_{s'}^i - B_s^i | \mathcal{B}_t \vee \sigma(\bar{m}_t^i, i \in I)] \\ &= \mathbb{E}[B_{s'}^i - B_s^i | \mathcal{B}_t \vee \sigma(\bar{m}_t^k) \vee \sigma(\xi_i, i \in I \setminus \{k\})] \\ & \quad \text{where } \xi_i \text{ form a Gaussian family independent of } \mathcal{B}_t \vee \sigma(B^k), \text{ constructed with (2.13),} \\ &= \mathbb{E}[B_{s'}^i - B_s^i | \sigma(\bar{m}_t^k)] = \left(\frac{1}{\alpha_t^2} \int_s^{s'} \varsigma_u du \right) \bar{m}_t^k \text{ because of (2.15). } \blacksquare \end{aligned}$$

Theorem 2.10 Define, for $J \subseteq N$ and $k \in J$, the function

$$\mathfrak{b}_J(\mathbf{z}, x) := \mathbb{E}[\mathbb{1}_{\{z_j < \xi_j, j \in J\}} (\xi_k + x)]$$

where $x \in \mathbb{R}$, $\mathbf{z} = (z_j, j \in J)$ and $(\xi_j, j \in J)$ is a Gaussian family of marginal variance $(\sigma^I)^2$ and of correlation ρ^I . For any $k \in N$, define the process, $t \geq 0$,

$$\beta_t^k := \frac{\varsigma(t)}{\alpha(t)} \left(\mathbb{1}_{\{k \in \mathcal{I}_t\}} \frac{\bar{m}_t^k}{\alpha(t)} + \mathbb{1}_{\{k \notin \mathcal{I}_t\}} \frac{\mathfrak{b}_{\mathcal{J}_t}((Z_t^{j, \mathcal{I}_t}(t), j \in \mathcal{J}_t), \lambda^{\mathcal{I}_t} \sum_{i \in \mathcal{I}_t} \frac{\bar{m}_t^i}{\alpha(t)})}{\Phi_{\mathcal{J}_t, \rho_t, \sigma_t}(Z_t^{j, \mathcal{I}_t}(t), j \in \mathcal{J}_t)} \right).$$

Then, $W^k = B^k - \beta^k \cdot \lambda$.

Proof. For $0 \leq t \leq t' < \infty$, for bounded \mathcal{B}_t measurable functions F , for measurable bounded function f , we compute

$$\begin{aligned} & \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t < \tau_j, i \in I, j \in J\}} (B_{s'}^k - B_s^k)] \\ &= \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t, i \in I\}} \mathbb{1}_{\{t < \tau_j, j \in J\}} \mathbb{E}[(B_{s'}^k - B_s^k) | \mathcal{B}_t \vee \sigma(\tau_N)]] \\ &= \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t, i \in I\}} \mathbb{1}_{\{t < \tau_j, j \in J\}} \left(\frac{1}{\alpha_t^2} \int_s^{s'} \varsigma_u du \right) \bar{m}_t^k]. \end{aligned}$$

If $k \in I$, $\bar{m}_t^k \in \mathcal{B}_t \vee \sigma(\tau_I)$. If $k \notin I$,

$$\begin{aligned} & \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t, i \in I\}} \mathbb{1}_{\{t < \tau_j, j \in J\}} \left(\frac{1}{\alpha_t^2} \int_s^{s'} \varsigma_u du \right) \bar{m}_t^k] \\ &= \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t, i \in I\}} \left(\frac{1}{\alpha(t)} \int_s^{s'} \varsigma_u du \right) \mathbb{E}[\mathbb{1}_{\{t < \tau_j, j \in J\}} \frac{\bar{m}_t^k}{\alpha(t)} | \mathcal{B}_t \vee \sigma(\tau_I)]] \\ &= \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t, i \in I\}} \left(\frac{1}{\alpha(t)} \int_s^{s'} \varsigma_u du \right) \\ & \quad \mathbb{E}[\mathbb{1}_{\{Z_t^{j, I}(t) < \frac{\bar{m}_t^j}{\alpha(t)} - \lambda^I \sum_{i \in I} \frac{\bar{m}_t^i}{\alpha(t)}, j \in J\}} \left(\frac{\bar{m}_t^k}{\alpha(t)} - \lambda^I \sum_{i \in I} \frac{\bar{m}_t^i}{\alpha(t)} + \lambda^I \sum_{i \in I} \frac{\bar{m}_t^i}{\alpha(t)} \right) | \mathcal{B}_t \vee \sigma(\tau_I)]] \\ &= \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t, i \in I\}} \left(\frac{1}{\alpha(t)} \int_s^{s'} \varsigma_u du \right) \mathbb{E}[\mathbb{1}_{\{Z_t^{j, I}(t) < \xi_j, j \in J\}} (\xi_k + \lambda^I \sum_{i \in I} \frac{\bar{m}_t^i}{\alpha(t)})]] \\ &= \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t, i \in I\}} \left(\frac{1}{\alpha(t)} \int_s^{s'} \varsigma_u du \right) \mathfrak{b}_J((Z_t^{j, I}(t), j \in J), \lambda^I \sum_{i \in I} \frac{\bar{m}_t^i}{\alpha(t)})] \\ &= \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t, i \in I\}} \mathbb{1}_{\{t < \tau_j, j \in J\}} \left(\frac{1}{\alpha(t)} \int_s^{s'} \varsigma_u du \right) \frac{\mathfrak{b}_J((Z_t^{j, I}(t), j \in J), \lambda^I \sum_{i \in I} \frac{\bar{m}_t^i}{\alpha(t)})}{\Phi_{J, \rho^I, \sigma^I}(Z_t^{j, I}(t), j \in J)}] \end{aligned}$$

This, combined with the formula (2.9), implies

$$\mathbb{E}[(B_{s'}^k - B_s^k) | \mathcal{G}_t] = \left(\frac{1}{\alpha(t)} \int_s^{s'} \varsigma_u du \right) \times \left(\mathbb{1}_{\{k \in \mathcal{I}_t\}} \frac{\bar{m}_t^k}{\alpha(t)} + \mathbb{1}_{\{k \notin \mathcal{I}_t\}} \frac{\mathbf{b}_{\mathcal{J}_t}((Z_t^{j, \mathcal{I}_t}(t), j \in \mathcal{J}_t), \lambda^{\mathcal{I}_t} \sum_{i \in \mathcal{I}_t} \frac{\bar{m}_t^i}{\alpha(t)})}{\Phi_{\mathcal{J}_t, \rho_t, \sigma_t}(Z_t^{j, \mathcal{I}_t}(t), j \in \mathcal{J}_t)} \right).$$

We obtain the drift of B^k in \mathbb{G} given by the differential of the above with respect to Lebesgue measure:

$$\frac{\varsigma(t)}{\alpha(t)} \left(\mathbb{1}_{\{k \in \mathcal{I}_t\}} \frac{\bar{m}_t^k}{\alpha(t)} + \mathbb{1}_{\{k \notin \mathcal{I}_t\}} \frac{\mathbf{b}_{\mathcal{J}_t}((Z_t^{j, \mathcal{I}_t}(t), j \in \mathcal{J}_t), \lambda^{\mathcal{I}_t} \sum_{i \in \mathcal{I}_t} \frac{\bar{m}_t^i}{\alpha(t)})}{\Phi_{\mathcal{J}_t, \rho_t, \sigma_t}(Z_t^{j, \mathcal{I}_t}(t), j \in \mathcal{J}_t)} \right) dt. \blacksquare$$

3 Reduced Model

We now study the invariance property based on the DGC model. In this regard, the market condition before the default event of the bank or of its counterparty is modeled by the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, where \mathcal{F}_t is

$$\bigcap_{s > t} (\mathcal{B}_s \vee \bigvee_{i \in N^*} (\sigma(\tau_i \wedge s))), \quad (3.1)$$

augmented to satisfy the usual condition, while the default time is marked by the random time $\tau_{-1} \wedge \tau_0$.

Notice that because of the multivariate density property of the family of $(\tau^j, j \in N^*)$ with respect to the filtration \mathbb{B} , the computations we have made in \mathbb{G} in the previous section can be made similarly in \mathbb{F} . In particular, the splitting formula such as (2.9) holds in the filtration \mathbb{F} in a similar way: for $t > 0$, $I \subseteq N^*$ and $J = N^* \setminus I$,

$$\begin{aligned} & \{\tau_i \leq t < \tau_j : i \in I, j \in J\} \cap \mathcal{F}_t \\ &= \{\tau_i \leq t < \tau_j : i \in I, j \in J\} \cap \mathcal{B}_t \vee \sigma(\tau_I). \end{aligned} \quad (3.2)$$

Also the processes $B^k, k \in N$, are \mathbb{F} semimartingales and the random times $\tau_j, j \in N^*$, are \mathbb{F} totally inaccessible stopping times, as stated in the following lemma. For $t > 0$, set

$$\mathcal{I}_t^* = \{i \in N^* : \tau_i \leq t\}, \quad \mathcal{J}_t^* = N^* \setminus \mathcal{I}_t^*, \quad \rho_t^* = \rho^{\mathcal{I}_t^*}, \quad \sigma_t^* = \sigma^{\mathcal{I}_t^*}.$$

Lemma 3.1 *For any $k \in N$, the process $\bar{W}_t^k = B_t^k - \int_0^t \bar{\beta}_s^k ds, t \geq 0$ is an \mathbb{F} local martingale, where $\bar{\beta}^k$ is defined as*

$$\bar{\beta}_t^k := \frac{\varsigma(t)}{\alpha(t)} \left(\mathbb{1}_{\{k \in \mathcal{I}_t^*\}} \frac{\bar{m}_t^k}{\alpha(t)} + \mathbb{1}_{\{k \notin \mathcal{I}_t^*\}} \frac{\mathbf{b}_{\mathcal{J}_t^*}((Z_t^{j, \mathcal{I}_t^*}(t), j \in \mathcal{J}_t^*), \lambda^{\mathcal{I}_t^*} \sum_{i \in \mathcal{I}_t^*} \frac{\bar{m}_t^i}{\alpha(t)})}{\Phi_{\mathcal{J}_t^*, \rho_t^*, \sigma_t^*}(Z_t^{j, \mathcal{I}_t^*}(t), j \in \mathcal{J}_t^*)} \right).$$

Also, for $j \in N^*$, τ_j is an \mathbb{F} totally inaccessible stopping time and the process $d\bar{M}_t^j = d\mathbb{1}_{\tau_j \leq t} - \bar{\gamma}_t^j dt$, $t > 0$, is an \mathbb{F} local martingale, where the process $\bar{\gamma}^i$ is defined as

$$\bar{\gamma}_t^j := \mathbb{1}_{\{t < \tau_j\}} \frac{\dot{h}_j(t)}{\alpha(t)} \psi_{\mathcal{J}_t^*, \rho_t^*, \sigma_t^*}^j(Z_t^{j, \mathcal{I}_t^*}(t), j \in \mathcal{J}_t^*). \quad (3.3)$$

Moreover, the family of processes \bar{W}^k , $k \in N$ and \bar{M}^j , $j \in N^*$, has the martingale representation property in the filtration \mathbb{F} .

3.1 The Azéma supermartingale

We need to compute the Azéma supermartingale of the random time $\tau_{-1} \wedge \tau_0$ in the filtration \mathbb{F} , i.e.,

$$\mathbb{E}[\mathbb{1}_{\{t < \tau_{-1} \wedge \tau_0\}} | \mathcal{F}_t], \quad t \geq 0.$$

The following computation is carried out in the same way as (2.12).

Lemma 3.2 *The Azéma supermartingale of the random time $\tau_{-1} \wedge \tau_0$ in the filtration \mathbb{F} is given by*

$$S_t := \frac{\Phi_{\mathcal{J}_t^* \cup \{-1, 0\}, \rho_t^*, \sigma_t^*}(Z_t^{j, \mathcal{I}_t^*}(t), j \in \mathcal{J}_t^* \cup \{-1, 0\})}{\Phi_{\mathcal{J}_t^*, \rho_t^*, \sigma_t^*}(Z_t^{j, \mathcal{I}_t^*}(t), j \in \mathcal{J}_t^*)}, \quad t \geq 0. \quad (3.4)$$

In particular, the Azéma supermartingale S is strictly positive.

Proof. For bounded \mathcal{B}_t measurable functions F , for measurable bounded function f ,

$$\begin{aligned} & \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t < \tau_j, i \in I, j \in J\}} \mathbb{1}_{\{t < \tau_{-1} \wedge \tau_0\}}] \\ &= \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t: i \in I\}} \mathbb{E}[\mathbb{1}_{\{t < \tau_j: j \in J\}} \mathbb{1}_{\{t < \tau_{-1} \wedge \tau_0\}} | \mathcal{B}_t \vee \sigma(\tau_I)]] \\ &= \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t: i \in I\}} \Phi_{J \cup \{-1, 0\}, \rho^I, \sigma^I}(Z_t^{j, I}(t) : j \in J \cup \{-1, 0\})] \\ &= \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t < \tau_j: i \in I, j \in J\}} \frac{\Phi_{J \cup \{-1, 0\}, \rho^I, \sigma^I}(Z_t^{j, I}(t) : j \in J \cup \{-1, 0\})}{\Phi_{J, \rho^I, \sigma^I}(Z_t^{j, I}(t) : j \in J)}] \\ &= \mathbb{E}[Ff(\tau_I) \mathbb{1}_{\{\tau_i \leq t < \tau_j: i \in I, j \in J\}} \frac{\Phi_{\mathcal{J}_t^* \cup \{-1, 0\}, \rho_t^*, \sigma_t^*}(Z_t^{j, \mathcal{I}_t^*}(t) : j \in \mathcal{J}_t^* \cup \{-1, 0\})}{\Phi_{\mathcal{J}_t^*, \rho_t^*, \sigma_t^*}(Z_t^{j, \mathcal{I}_t^*}(t) : j \in \mathcal{J}_t^*)}], \end{aligned}$$

where conditioning and the tower rule was used in the next-to-last identity. With the formula (3.2), we conclude

$$\mathbb{E}[\mathbb{1}_{\{t < \tau_{-1} \wedge \tau_0\}} | \mathcal{F}_t] = \frac{\Phi_{\mathcal{J}_t^* \cup \{-1, 0\}, \rho_t^*, \sigma_t^*}(Z_t^{j, \mathcal{I}_t^*}(t) : j \in \mathcal{J}_t^* \cup \{-1, 0\})}{\Phi_{\mathcal{J}_t^*, \rho_t^*, \sigma_t^*}(Z_t^{j, \mathcal{I}_t^*}(t) : j \in \mathcal{J}_t^*)}. \blacksquare$$

Let Q^c denote the continuous martingale component of the (\mathbb{F}, \mathbb{Q}) Azéma supermartingale S .

Lemma 3.3 *The process $\nu := \frac{1}{S} \cdot Q^c$ satisfies*

$$d\nu_t = \sum_{j \in \mathcal{J}_{t-}^* \cup \{-1, 0\}} \psi_{\mathcal{J}_{t-}^* \cup \{-1, 0\}, \rho_t^*, \sigma_t^*}^j(Z_t^{j, \mathcal{I}_{t-}^*}(t) : j \in \mathcal{J}_t^* \cup \{-1, 0\}) d\zeta_t^{j, \mathcal{I}_{t-}^*} \\ - \sum_{j \in \mathcal{J}_{t-}^*} \psi_{\mathcal{J}_{t-}^*, \rho_t^*, \sigma_t^*}^j(Z_t^{j, \mathcal{I}_{t-}^*}(t) : j \in \mathcal{J}_t^*) d\zeta_t^{j, \mathcal{I}_{t-}^*},$$

where, for $I \subseteq N^*$, $\zeta_t^{j, I}$ denotes the martingale part of $\left(-\frac{1}{\alpha(t)} dm_t^j + \frac{\rho}{(|I|-1)\varrho+1} \sum_{i \in I} \frac{1}{\alpha(t)} dm_t^i\right)$ in \mathbb{F} .

Proof. To obtain dQ_t^c , we apply Itô calculus to the expression (3.4) of S on every random intervals where \mathcal{I}_{t-}^* is constant. Note that, knowing t is in such an interval, $\tau_{\mathcal{I}_{t-}^*}$ is in \mathcal{F}_t . Note also that the jumps on S_t , produced when \mathcal{I}_{t-}^* changes value, have no impact here, because Q^c is a continuous local martingale. Now, dividing the result by S , we prove the lemma. ■

3.2 \mathbb{F} reductions of β^k, γ^j

Lemma 3.4 *The condition (B) holds between the couple $\mathbb{F} \subseteq \mathbb{G}$.*

Proof. To check the condition (B), by the monotone class theorem, we only need consider the elementary \mathbb{G} predictable processes of the form $U = \nu f(\tau_c \wedge s, \tau_b \wedge s) \mathbb{1}_{(s, t]}$, for an \mathcal{F}_s measurable random variable F and a Borel function f . Since $U \mathbb{1}_{(0, \tau]} = F f(s, s) \mathbb{1}_{(s, t]} \mathbb{1}_{(0, \tau]}$, we may take $U' = F f(s, s) \mathbb{1}_{(s, t]}$ in the condition (B). ■

Look at now the reduction of the processes β^k, γ^j in the filtration \mathbb{F} . Notice that, for $t < \tau_{-1} \wedge \tau_0$,

$$\mathcal{I}_t = \mathcal{I}_t^*, \quad \mathcal{J}_t = \mathcal{J}_t^* \cup \{-1, 0\}.$$

We have, therefore, the following lemma.

Lemma 3.5 *An \mathbb{F} reduction of $\gamma^j, j \in N^*$, is:*

$$\tilde{\gamma}_t^j := \mathbb{1}_{\{t < \tau_j\}} \frac{\dot{h}_j(t)}{\alpha(t)} \psi_{\mathcal{J}_t^* \cup \{-1, 0\}, \rho_t^*, \sigma_t^*}^j(Z_t^{j, \mathcal{I}_t^*}(t), j \in \mathcal{J}_t^* \cup \{-1, 0\}). \quad (3.5)$$

Similarly, an \mathbb{F} reduction of $\beta^k, k \in N$, is:

$$\tilde{\beta}_t^k := \frac{\varsigma(t)}{\alpha(t)} \times \left(\mathbb{1}_{\{k \in \mathcal{I}_t^*\}} \frac{\tilde{m}_t^k}{\alpha(t)} + \mathbb{1}_{\{k \notin \mathcal{I}_t^*\}} \frac{\mathfrak{b}_{\mathcal{J}_t^* \cup \{-1, 0\}}((Z_t^{j, \mathcal{I}_t^*}(t), j \in \mathcal{J}_t^* \cup \{-1, 0\}), \lambda^{\mathcal{I}_t^*} \sum_{i \in \mathcal{I}_t^*} \frac{\tilde{m}_t^i}{\alpha(t)})}{\Phi_{\mathcal{J}_t^* \cup \{-1, 0\}, \rho_t^*, \sigma_t^*}(Z_t^{j, \mathcal{I}_t^*}(t), j \in \mathcal{J}_t^* \cup \{-1, 0\})} \right).$$

We note that the processes $\gamma^j, \tilde{\gamma}^j$ and $\beta^k, \tilde{\beta}^k$ are càdlàg. The next result shows that the process β^k (and consequently $\tilde{\beta}$) is linked with $\tilde{\beta}^k$ through the process ν .

Lemma 3.6 For $k \in N$,

$$\int_0^t \tilde{\beta}_s^k ds = \int_0^t \bar{\beta}_s^k ds + \langle B^k, \nu \rangle_t, \quad t \in [0, \tau_{-1} \wedge \tau_0].$$

Proof. Notice that B^k is a continuous process. By the Jeulin–Yor formula (see e.g. Del-lacherie, Maisonneuve, and Meyer (1992, no 77 Remarques b))),

$$B_t^k - \int_0^t \bar{\beta}_s^k ds - \langle B^k, \nu \rangle_t, \quad t \in [0, \tau_{-1} \wedge \tau_0],$$

defines a \mathbb{G} local martingale. But, it is known that the drift of B^k in \mathbb{G} is $\int_0^t \beta_s^k ds, t \geq 0$, according to Theorem 2.10. We conclude that

$$\int_0^t \tilde{\beta}_s^k ds = \int_0^t \beta_s^k ds = \int_0^t \bar{\beta}_s^k ds + \langle B^k, \nu \rangle_t$$

for $t \in [0, \tau_{-1} \wedge \tau_0]$. ■

This discussion on the \mathbb{F} reductions is intended to give an idea of the way that the invariance probability measure between \mathbb{F} and \mathbb{G} can be constructed. Actually, knowing the \mathbb{F} reductions $\tilde{\beta}^k, \tilde{\gamma}^j$ of respectively β^k and γ^j , our strategy for constructing an invariance probability measure \mathbb{P} becomes clear. It is enough to find a probability measure \mathbb{Q} equivalent to \mathbb{P} on \mathcal{F}_T (given a constant $T > 0$) such that the (\mathbb{F}, \mathbb{Q}) drift of $B^k, k \in N$, is $\tilde{\beta}^k$ and the (\mathbb{F}, \mathbb{Q}) compensator of $\tau^j, j \in N^*$, has the density process $\tilde{\gamma}^j$. (In fact, we take account of the avoidance of $\tau_{-1} \wedge \tau_0$ to $\tau_j, j \in N^*$, and the martingale representation property in \mathbb{F} .)

To make the above strategy work, the estimates of the following lemma will be essential.

Lemma 3.7 There exists a constant $C > 0$ such that, for $0 \leq r \leq t$ and $j \in N^*$,

$$\langle \nu \rangle_t \leq C \left(\sum_{i \in N} \sup_{0 < s \leq t} |m_s^i|^2 + 1 \right) t, \quad (3.6)$$

$$\tilde{\gamma}_r^j \vee \tilde{\gamma}_r^j \leq C \left(\sum_{i \in N} \sup_{0 < s \leq t} |m_s^i| + 1 \right), \quad (3.7)$$

$$\tilde{\gamma}_r^j \ln(\tilde{\gamma}_r^j \vee \tilde{\gamma}_r^j) \leq C \sum_{i \in N} \sup_{0 < s \leq t} (|m_s^i| + 1) \ln(|m_s^i| + 1).$$

Proof. Applying Lemma A.2 to the formula (3.3) and noting that the function α , continuous and positive, is bounded away from 0 on $[0, T]$, we obtain, for positive constants C (that may change from place to place),

$$\langle \nu \rangle_t \leq C \int_0^t \left(\sum_{I \subseteq N} \sum_{j \in N \setminus I} |Z_s^{j,I}(s)| + 1 \right)^2 ds \leq C \left(\sum_{I \subseteq N} \sum_{j \in N \setminus I} \sup_{0 < s \leq t} |Z_s^{j,I}(s)| + 1 \right)^2 t,$$

which yields (3.6). Applying Lemma A.2 to the formulas of $\tilde{\gamma}^j$ in (3.3) and of $\tilde{\gamma}^j$ in (3.5), we obtain the first line in (3.7), whence the second line follows from

$$\begin{aligned} \tilde{\gamma}_r^j \ln(\tilde{\gamma}_r^j \vee \tilde{\gamma}_r^j) &\leq C(\max_{i \in N} \sup_{0 < s \leq t} |m_s^i| + 1) \ln(C(\max_{i \in N} \sup_{0 < s \leq t} |m_s^i| + 1)) \\ &= \max_{i \in N} \sup_{0 < s \leq t} C(|m_s^i| + 1) \ln(C|m_s^i| + 1). \blacksquare \end{aligned}$$

Notice that the processes $\tilde{\gamma}^j$ are strictly positive. Consider the \mathbb{F} local martingale $\mu := \nu + \sum_{j \in N^*} \left(\frac{\tilde{\gamma}^j}{\tilde{\gamma}^j} - 1\right) \cdot \bar{M}^j$. As a consequence of Lemma 3.7, we have:

Lemma 3.8 *The Doléans-Dade exponential $\mathcal{E}(\mu)$ is a true (\mathbb{F}, \mathbb{Q}) martingale.*

Proof. Following Lepingle and Mémin (1978, Theorem III.1), we consider

$$\sum_{j \in N^*} \left(\left(1 + \left(\frac{\tilde{\gamma}_{\tau_j^-}^j}{\tilde{\gamma}_{\tau_j^-}^j} - 1\right)\right) \ln\left(1 + \left(\frac{\tilde{\gamma}_{\tau_j^-}^j}{\tilde{\gamma}_{\tau_j^-}^j} - 1\right)\right) - \left(\frac{\tilde{\gamma}_{\tau_j^-}^j}{\tilde{\gamma}_{\tau_j^-}^j} - 1\right) \right) \mathbf{1}_{\{\tau_j \leq t\}},$$

and its \mathbb{F} predictable dual projection

$$\begin{aligned} M_t &:= \sum_{j \in N^*} \int_0^t \left(\left(1 + \left(\frac{\tilde{\gamma}_{s^-}^j}{\tilde{\gamma}_{s^-}^j} - 1\right)\right) \ln\left(1 + \left(\frac{\tilde{\gamma}_{s^-}^j}{\tilde{\gamma}_{s^-}^j} - 1\right)\right) - \left(\frac{\tilde{\gamma}_{s^-}^j}{\tilde{\gamma}_{s^-}^j} - 1\right) \right) \tilde{\gamma}_{s^-}^j ds \\ &= \sum_{j \in N^*} \int_0^t \left(\tilde{\gamma}_{s^-}^j (\ln(\tilde{\gamma}_{s^-}^j) - \ln(\tilde{\gamma}_{s^-}^j)) - (\tilde{\gamma}_{\tau_j^-}^j - \tilde{\gamma}_{s^-}^j) \right) ds. \end{aligned}$$

Combining Lemma 3.7 and Lemma A.3, we prove that $e^{\frac{1}{2}\langle \nu \rangle_t + M_t}$ is \mathbb{Q} integrable for a sufficiently small $t_0 > 0$. According to Lepingle and Mémin (1978, Theorem III.1), we conclude $\mathbb{E}[\mathcal{E}(\mu)_{t_0}] = 1$. Repeat this argument with the conditional probability $\mathbb{Q}[\cdot | \mathcal{F}_{t_0}]$ replacing \mathbb{Q} , we prove that $\mathbb{E}[\mathcal{E}(\mu)_{2t_0} | \mathcal{F}_{t_0}] = 1$. Iterating the argument, we arrive at $\mathbb{E}[\mathcal{E}(\mu)_{kt_0}] = 1$ for any integer $k > 0$. \blacksquare

3.3 The invariance probability measure

We have proved that $\mathcal{E}(\mu)$ is a (\mathbb{F}, \mathbb{Q}) true martingale. We can then define a new probability measure $\mathbb{P} := \mathcal{E}(\mu) \cdot \mathbb{Q}$ on \mathcal{F}_T .

Notice that, as demonstrated in Émery (1980), a general stochastic integral with respect to a local martingale need not be a local martingale. This fact justifies the distinction, in the proof of the next result, between the stochastic integral in the sense of semimartingales and the stochastic integral in the sense of local martingales. See also Delbaen and Schachermayer (1994).

Theorem 3.9 *The probability measure \mathbb{P} is an invariance probability measure for the DGC model $(\tau_{-1} \wedge \tau_0, \mathbb{F}, \mathbb{G}, \mathbb{Q})$ on the horizon $[0, T]$ for any constant $T > 0$.*

Proof. By the Girsanov theorem, the intensity of τ_i , $i \in N^*$, in \mathbb{F} under \mathbb{P} becomes $\tilde{\gamma}^i$, while the drift of B^k , $k \in N$, in \mathbb{F} under \mathbb{P} becomes $(\bar{\beta}^k \cdot \lambda + \langle B^k, \nu \rangle)$.

Let us prove that the probability measure $\mathbb{P} = \mathcal{E}(\mu) \cdot \mathbb{Q}$ is an invariance probability measure for the quadruplet $(\tau_{-1} \wedge \tau_0, \mathbb{F}, \mathbb{G}, \mathbb{Q})$ on the horizon $[0, T]$ for any constant $T > 0$. According to Crépey and Song (2015b, Corollary C.1), we only need to consider the locally bounded (\mathbb{F}, \mathbb{P}) local martingales. We begin with the stopped process

$$(B^k - \bar{\beta}^k \cdot \lambda - \langle B^k, \nu \rangle)^{\tau_{-1} \wedge \tau_0^-} = (B^k - \beta^k \cdot \lambda)^{\tau_{-1} \wedge \tau_0} = (W^k)^{\tau_{-1} \wedge \tau_0},$$

thanks to Lemma 3.6, which is a (\mathbb{G}, \mathbb{Q}) local martingales. The (\mathbb{G}, \mathbb{Q}) local martingale property of

$$(\mathbb{1}_{[\tau^j, \infty)} - \tilde{\gamma}^j \cdot \lambda)^{\tau_{-1} \wedge \tau_0^-} = (\mathbb{1}_{[\tau^j, \infty)} - \gamma^j \cdot \lambda)^{\tau_{-1} \wedge \tau_0} = (M^j)^{\tau_{-1} \wedge \tau_0}$$

(cf. Lemma 3.5) also is an evidence.

Notice that, as we did in Lemma 3.1 under the probability \mathbb{Q} , we can prove that the family of processes

$$B^k - \bar{\beta}^k \cdot \lambda - \langle B^k, \nu \rangle \text{ and } \mathbb{1}_{[\tau^j, \infty)} - \tilde{\gamma}^j \cdot \lambda, \quad k \in N, j \in N^*,$$

has the martingale representation property in the filtration \mathbb{F} under \mathbb{P} . Because of this property, any locally bounded (\mathbb{F}, \mathbb{P}) local martingale X is a stochastic integral in \mathbb{F} of the processes $B^k - \bar{\beta}^k \cdot \lambda - \langle B^k, \nu \rangle$ and of $\mathbb{1}_{[\tau^j, \infty)} - \tilde{\gamma}^j \cdot \lambda$ under the probability measure \mathbb{P} . The natural idea is to say, then, $X^{\tau_{-1} \wedge \tau_0^-}$ is the stochastic integral, in the sense of local martingale in \mathbb{G} , of the processes $(W^k)^{\tau_{-1} \wedge \tau_0}$ and of $(M^j)^{\tau_{-1} \wedge \tau_0}$ under the probability \mathbb{Q} , so that $X^{\tau_{-1} \wedge \tau_0^-}$ itself is a (\mathbb{G}, \mathbb{Q}) local martingale. However, knowing the discussion in Jeulin and Yor (1979) about “*faux amis*” regarding enlargement of filtration and stochastic integrals, we have to be careful.

We can argue as follows. Let us denote by \widehat{W}^k , $k \in N$, the process $B^k - \bar{\beta}^k \cdot \lambda - \langle B^k, \nu \rangle$, and respectively by \widehat{M}^j , $j \in N^*$, the process $\mathbb{1}_{[\tau^j, \infty)} - \tilde{\gamma}^j \cdot \lambda$. We consider separately the case of X continuous, or the case of X purely discontinuous. When X is a continuous (\mathbb{F}, \mathbb{P}) local martingale, X is the (\mathbb{F}, \mathbb{P}) stochastic integral, in the sense of local martingale, of an \mathbb{F} predictable $n + 2$ -dimensional process $H = (H^k, k \in N)$ with respect to the $n + 2$ -dimensional process $(\widehat{W}^k, k \in N)$. Since the matrix $(\frac{d\langle \widehat{W}^k, \widehat{W}^{k'} \rangle_t}{dt}, k, k' \in N)$ is uniformly positive-definite, for every $k \in N$, H^k is individually (\mathbb{F}, \mathbb{P}) integrable with respect to the one-dimensional Brownian motion \widehat{W}^k in the sense of local martingale. By Girsanov theorem, H^k is integrable with respect to $\langle \widehat{W}^k, \nu \rangle$ under \mathbb{Q} , so that H^k is (\mathbb{F}, \mathbb{Q}) integrable with respect to \widehat{W}^k in the sense of semimartingale. As the hypothesis (H²) holds between $\mathbb{F} \subseteq \mathbb{G}$ under \mathbb{Q} , Jeulin (1980, Proposition 2.1) implies that H^k is (\mathbb{G}, \mathbb{Q}) integrable with respect to \widehat{W}^k in the sense of semimartingale. Applying He, Wang, and Yan (1992, Theorem 9.16), we see that, in fact, H^k is (\mathbb{G}, \mathbb{Q}) integrable with respect to \widehat{W}^k in the sense of local martingale. By Jeanblanc and Song (2013, Lemma 2.1), the stochastic integrals in the sense of the (\mathbb{F}, \mathbb{Q}) local martingales and in the sense of the (\mathbb{G}, \mathbb{Q}) local martingales are the same, hence

$$X^{\tau_{-1} \wedge \tau_0^-} = \sum_{k \in N} (H^k \cdot \widehat{W}^k)^{\tau_{-1} \wedge \tau_0^-} = \sum_{k \in N} (H^k \cdot W^k)^{\tau_{-1} \wedge \tau_0},$$

which is a (\mathbb{G}, \mathbb{Q}) local martingale.

Consider now the case of X purely discontinuous. Without loss of generality we suppose that X is bounded. Then, X is the (\mathbb{F}, \mathbb{P}) stochastic integral (in the sense of local martingale) of an \mathbb{F} predictable n -dimensional process $K = (K^j, j \in N^*)$ with respect to the n -dimensional process $(\widehat{M}^j, j \in N^*)$. The processes $\widehat{M}^j, j \in N^*$, have disjoint jump times with jump amplitude 1. This implies, on the one hand, that K^j is integrable with respect to \widehat{M}^j individually. On the other hand, as X is bounded, the random variables $K_{\tau_j}, j \in N^*$, are bounded. That means that we can assume that K itself is bounded (cf. He, Wang, and Yan (1992, Theorem 7.23)). As a consequence, K is automatically (\mathbb{G}, \mathbb{Q}) integrable with respect to $(\widehat{M}^j, j \in N^*)$ in the sense of local martingale. By Jeanblanc and Song (2013, Lemma 2.1) again,

$$X^{\tau_{-1} \wedge \tau_0^-} = \sum_{j \in N^*} (K^j \cdot \widehat{M}^j)^{\tau_{-1} \wedge \tau_0^-} = \sum_{j \in N^*} (K^j \cdot M^j)^{\tau_{-1} \wedge \tau_0},$$

which is a (\mathbb{G}, \mathbb{Q}) local martingale.

The invariance property of the probability measure \mathbb{P} in regard to the quadruplet $(\tau_{-1} \wedge \tau_0, \mathbb{F}, \mathbb{G}, \mathbb{Q})$ on the horizon $[0, T]$ for any constant $T > 0$ is proved. ■

3.4 Alternative Proof of the Condition (A)

Notice that, if we only want to establish the condition (A) in the DGC model, we can have a very short proof based on the sufficiency condition of Crépey and Song (2015b, Theorem 5.1), as it will be shown in this section. However, Theorem 3.9 has a better merit, because it gives an explicit construction of the invariance probability measure \mathbb{P} .

Lemma 3.10 *The (\mathbb{G}, \mathbb{Q}) intensity γ of the random time $\tau_{-1} \wedge \tau_0$ is given by*

$$\gamma = \mathbb{1}_{[0, \tau_{-1} \wedge \tau_0]}(\gamma^{-1} + \gamma^0).$$

Proof. This has been proved, for example, in Crépey and Song (2016, Lemma 6.2). ■

Theorem 3.11 *The condition (A) holds in the DGC model $(\tau_{-1} \wedge \tau_0, \mathbb{F}, \mathbb{G}, \mathbb{Q})$ on the horizon $[0, T]$ for any constant $T > 0$.*

Proof. According to Crépey and Song (2015b, Theorem 5.1), we only need to prove the exponential integrability of $\int_0^{\tau \wedge T} \gamma_s ds$, which can be done in the same way as we prove Lemma 3.8. ■

4 Wrong Way Risk

As visible in (2.11), the default intensities of the surviving names spike at defaults in the DGC model. This is very much related to the departure from the immersion property in this model, i.e. the fact that the invariance probability measures are different from the pricing measure \mathbb{Q} . As a consequence, the value of CDS protection spikes in

the model, consistent with what is observed in the market. This makes the DGC model appropriate for dealing with counterparty risk on credit derivatives, notably portfolios of CDS contracts traded between a bank and its counterparty, respectively labeled as -1 and 0 , and bearing on reference firms $i = 1, \dots, n$.

To illustrate this numerically, in this concluding section of the paper, we study the valuation adjustment accounting for counterparty and funding risks (total valuation adjustment TVA) on one CDS between the bank and the counterparty on a third reference firm.

In Figure 4.1, the left graph shows the TVA computed as a function of the correlation parameter ρ in a DGC model with three names (i.e. $n = 1$): the bank, its counterparty and the reference credit name of the CDS. The different curves correspond to different levels of credit spread λ of bank: the higher λ , the higher the funding costs for the bank, resulting in higher TVAs. All numbers are computed by a Monte Carlo scheme dubbed “FT scheme of order 3” in Crépey and Nguyen (2015, Section 6.1). FT refers to Fujii and Takahashi (2012a,2012b). The numerical parameters are set as in Crépey and Nguyen (2015, Section 6.1), to which we also refer the reader for a complete description of the CDS contract, of the numerical scheme that is used and of other numerical experiments involving CDS portfolios (as opposed to a single contract here).

The right graph of Figure 4.1 shows the analog of the left graph, but in a fake DGC model, where we deliberately ignore the impact of the default of the counterparty in the value of the CDS at time τ (technically, in the notation of Crépey and Song (2016, (6.7)), we replace $(\tilde{P}_t^e + \tilde{\Delta}_t^e)$ by P_{t-} in the coefficient \hat{f}), in order to remove the wrong-way risk feature from the DGC model. We can see from the figure that, for large ρ , the corresponding fake TVA numbers are five to ten times smaller than the “true” TVA levels that can be seen in the left panel. In addition of being much smaller for large ρ , the fake DGC TVA numbers in the right panel are now mostly decreasing with ρ , showing by comparison with the previous DGC results that it is actually the wrong-way risk that explains the “systemic” increasing feature observed in the “true” DGC model.

A Gaussian Estimates

In this appendix we derive Gaussian estimates used in the study of the DGC model of Sect. 2.

Lemma A.1 *Given a positive decreasing continuously differentiable function Γ on \mathbb{R}_+ such that*

$$\int_{\mathbb{R}_+} t^d \Gamma(t) dt < \infty \text{ and } \lim_{t \uparrow \infty} t^{d-1} \Gamma(t) \rightarrow 0,$$

for some integer $d \geq 0$, we write $g(y) = -\frac{\Gamma'(y)}{\Gamma(y)}$, $G(y) = \int_y^\infty t^d \Gamma(t) dt$. Let $\bar{y} \geq 0$ and $\alpha, \epsilon > 0$.

(i) *If $g(y) \geq \alpha y$ for $y > \bar{y}$, then*

$$G(y) \leq \left(\frac{1}{\alpha} + \epsilon\right) y^{d-1} \Gamma(y) \text{ for } y > \bar{y} \vee \sqrt{|d-1| \left(\frac{1}{\epsilon \alpha^2} + \frac{1}{\alpha}\right)}.$$

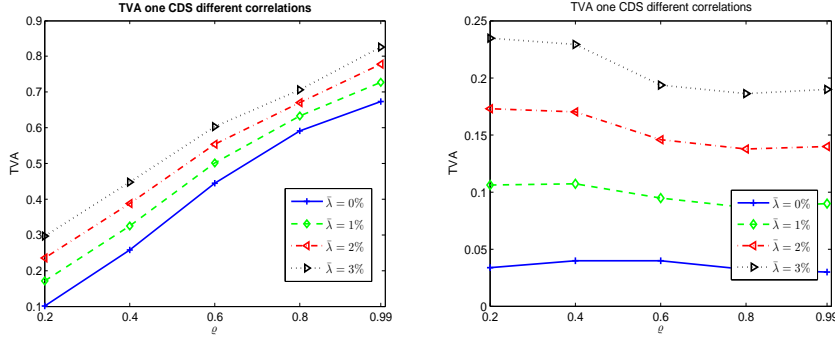


Fig. 4.1 *Left:* TVA on one CDS as a function of the correlation parameter ρ and for different bank credit spreads $\bar{\lambda}$, in a DGC model with three names: the bank, its counterparty and the reference credit name of the CDS. *Right:* Analog results in a fake DGC model without wrong-way-risk.

(ii) If $g(y) \leq \alpha y$ for $y > \bar{y}$, then

$$G(y) \geq \left(\frac{1}{\alpha} - \epsilon\right)y^{d-1}\Gamma(y) \quad \text{for } y > \bar{y} \vee \sqrt{|d-1|\left(\frac{1}{\epsilon\alpha^2} - \frac{1}{\alpha}\right)}.$$

Proof. We only prove (i), for (ii) is similar. For every positive continuously differentiable function φ on $(0, +\infty)$,

$$\begin{aligned} (G(y) - \varphi(y)\Gamma(y))' &= -y^d\Gamma(y) - \varphi'(y)\Gamma(y) + \varphi(y)g(y)\Gamma(y) \\ &= (\varphi(y)g(y) - y^d - \varphi'(y))\Gamma(y) \geq (\alpha y\varphi(y) - y^d - \varphi'(y))\Gamma(y) \end{aligned}$$

for $y \geq \bar{y}$. For $\varphi(y) = \left(\frac{1}{\alpha} + \epsilon\right)y^{d-1}$,

$$\begin{aligned} \alpha y\varphi(y) - y^d - \varphi'(y) &= (1 + \epsilon\alpha)y^d - y^d - \left(\frac{1}{\alpha} + \epsilon\right)(d-1)y^{d-2} = \\ &= \epsilon\alpha y^d - \left(\frac{1}{\alpha} + \epsilon\right)(d-1)y^{d-2} = \left(\epsilon\alpha y^2 - \left(\frac{1}{\alpha} + \epsilon\right)(d-1)\right)y^{d-2}. \end{aligned}$$

Therefore, if $y > \bar{y} \vee \sqrt{|d-1|\left(\frac{1}{\epsilon\alpha^2} + \frac{1}{\alpha}\right)}$, then $(G(y) - \varphi(y)\Gamma(y))' \geq \alpha y\varphi(y) - y^d - \varphi'(y) \geq 0$. But $\lim_{y \uparrow \infty} (G(y) - \varphi(y)\Gamma(y)) = 0$, hence $G(y) - \varphi(y)\Gamma(y) \leq 0$. ■

We use the notation (2.5) as well as Φ and ϕ for the standard normal survival and density functions. By a first application of Lemma A.1, to the standard normal density $\Gamma = \phi$, we recover the following classical inequalities on $\psi = \frac{\phi}{\Phi}$: for any constant $c > 1$,

$$c^{-1}y \leq \psi(y) \leq cy, \quad y > y_0, \quad (\text{A.1})$$

for some $y_0 > 0$ depending on c . The following estimate, where c and y_0 are as here, can be seen as a multivariate extension of the right hand side inequality in (A.1).

Lemma A.2 *There exist constants a and b such that, for every $j \in J$,*

$$0 \leq \psi_{\rho,\sigma}^j(\mathbf{z}) \leq a + b\|\mathbf{z}\|_\infty. \quad (\text{A.2})$$

Proof. By conditional independence of the components of a multivariate Gaussian vector with homogeneous pairwise correlation ϱ , we have $\Phi_{\rho,\sigma}(\mathbf{z}) = \int_{\mathbb{R}} \Gamma(y) dy$, where $\Gamma(y) = \prod_{l \in J} \Phi\left(\frac{z_l + \sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}\right) \phi(y)$. Hence

$$\psi_{\rho,\sigma}^j(\mathbf{z}) = \frac{1}{\sigma\sqrt{1-\rho}} \int_{\mathbb{R}} w_{\rho,\sigma}(\mathbf{z}, y) \psi\left(\frac{z_j + \sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}\right) dy, \quad (\text{A.3})$$

where $w_{\rho,\sigma}(\mathbf{z}, y) = \frac{\Gamma(y)}{\Phi_{\rho,\sigma}(\mathbf{z})}$. Straightforward computations yield

$$g(t) = -\frac{\Gamma'(t)}{\Gamma(t)} = \sum_{l \in J} \psi\left(\frac{z_l + \sigma\sqrt{\rho}t}{\sigma\sqrt{1-\rho}}\right) \frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} + t \geq t,$$

whereas for $t > \max_{l \in J} \frac{1}{\sigma\sqrt{\rho}}(\sigma\sqrt{1-\rho}y_0 - z_l)$ and $t > \frac{1}{\sigma\sqrt{\rho}} \max_{l \in J} z_l$, we have

$$g(t) \leq \sum_{l \in J} c \frac{z_l + \sigma\sqrt{\rho}t}{\sigma\sqrt{1-\rho}} \frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} + t \leq \bar{\alpha}t,$$

with $\bar{\alpha} := \sum_{l \in J} 2c \frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} \frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} + 1 \geq 1$. Applying Lemma A.1(i) with $d = 1, \alpha = 1$ and $\epsilon = 1$, respectively (ii) with $d = 0, \alpha = \bar{\alpha}$ and $\epsilon = \frac{1}{2\bar{\alpha}}$, yields

$$\int_y^\infty t\Gamma(t)dt \leq 2\Gamma(y), \quad y > 0, \quad \text{respectively} \quad \int_y^\infty \Gamma(t)dt \geq \frac{1}{2\bar{\alpha}y} \Gamma(y), \quad y > \bar{y} \vee \frac{1}{\sqrt{\bar{\alpha}}},$$

where $\bar{y} = \frac{1}{\sigma\sqrt{\rho}} \max_{l \in J} |z_l| + \frac{1}{\sigma\sqrt{\rho}} \sigma\sqrt{1-\rho}y_0$. Thus, setting $y_1 = \bar{y} + 1 = \frac{1}{\sigma\sqrt{\rho}} \max_{l \in J} |z_l| + \frac{1}{\sigma\sqrt{\rho}} \sigma\sqrt{1-\rho}y_0 + 1$,

$$\begin{aligned} \int_0^\infty t\Gamma(t)dt &= \int_0^{y_1} t\Gamma(t)dt + \int_{y_1}^\infty t\Gamma(t)dt \leq y_1 \int_0^{y_1} \Gamma(t)dt + 2\Gamma(y_1) \\ &\leq y_1 \int_0^{y_1} \Gamma(t)dt + 4\bar{\alpha}y_1 \int_{y_1}^\infty \Gamma(t)dt \leq (1 + 4\bar{\alpha}) \int_0^\infty \Gamma(t)dt, \end{aligned}$$

i.e.

$$\int_0^\infty tw_{\rho,\sigma}(\mathbf{z}, t)dt \leq (1 + 4\bar{\alpha})y_1. \quad (\text{A.4})$$

Now, by (A.3) and the right hand side inequality in (A.1),

$$\begin{aligned} 0 &\leq \sigma\sqrt{1-\rho} \psi_{\rho,\sigma}^j(\mathbf{z}) \\ &\leq \int_{\mathbb{R}} \left(\frac{1}{\Phi(y_0)} \mathbb{1}_{\left\{\frac{z_j + \sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}} \leq y_0\right\}} + c \frac{z_j + \sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}} \mathbb{1}_{\left\{\frac{z_j + \sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}} > y_0\right\}} \right) w_{\rho,\sigma}(\mathbf{z}, y) dy \\ &= \left(\frac{1}{\Phi(y_0)} + \frac{cz_j}{\sigma\sqrt{1-\rho}} \right) + \frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} \int_{\mathbb{R}} \mathbb{1}_{\{\sigma\sqrt{\rho}y > \sigma\sqrt{1-\rho}y_0 - z_j\}} y w_{\rho,\sigma}(\mathbf{z}, y) dy \\ &\leq \left(\frac{1}{\Phi(y_0)} + \frac{cz_j}{\sigma\sqrt{1-\rho}} \right) + \frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} \int_0^\infty y w_{\rho,\sigma}(\mathbf{z}, y) dy, \end{aligned} \quad (\text{A.5})$$

so that by substitution of (A.4) into (A.5)

$$0 \leq \sigma \sqrt{1-\rho} \psi_{\rho,\sigma}^j(\mathbf{z}) \leq \left(\frac{1}{\Phi(y_0)} + \frac{cz_j}{\sigma \sqrt{1-\rho}} \right) + \frac{c\sigma \sqrt{\rho}}{\sigma \sqrt{1-\rho}} (1+4\bar{\alpha})y_1. \blacksquare$$

Lemma A.3 Let $m_t = \int_0^t \varsigma(s) dB_s$, where B is a univariate standard Brownian motion and ς is a square integrable function with unit L^2 norm. For any constant $q > 0$, $e^{q \sup_{0 \leq s \leq t} m_s^2}$ is integrable for sufficiently small t .

Proof. The process $(m_t)_{t \geq 0}$ is equal in law to a time changed Brownian motion $(W_{\bar{t}})_{t \geq 0}$, where W is a univariate standard Brownian motion and $\bar{t} = \int_0^t \varsigma^2(s) ds$ goes to 0 with t . Thus, it suffices to show the result with m replaced by W . Let r_t be the density function of the law of $\sup_{0 \leq s \leq t} |W_s|$ and let $R_t(y) = \int_y^\infty r_t(x) dx$, $y > 0$, so that

$$\mathbb{E}[e^{q \sup_{0 \leq s \leq t} W_s^2}] = \int_0^\infty e^{qy^2} r_t(y) dy = -[R_t(y) e^{qy^2}]_0^\infty + 2q \int_0^\infty y R_t(y) e^{qy^2} dy \quad (\text{A.6})$$

and (using the reflection principle of the Brownian motion)

$$\begin{aligned} R_t(y) &= \mathbb{Q}[\sup_{0 \leq s \leq t} (W_s^+ + W_s^-) > y] \leq \mathbb{Q}[\sup_{0 \leq s \leq t} W_s^+ > \frac{y}{2}] + \mathbb{Q}[\sup_{0 \leq s \leq t} W_s^- > \frac{y}{2}] \\ &= 2\mathbb{Q}[\sup_{0 \leq s \leq t} W_s > \frac{y}{2}] = 2\mathbb{Q}[|W_t| > \frac{y}{2}] = 2\mathbb{Q}[|W_1| > \frac{y}{2\sqrt{t}}] = 4\Phi\left(\frac{y}{2\sqrt{t}}\right), \end{aligned}$$

where by the left hand side in (A.1)

$$\Phi\left(\frac{y}{2\sqrt{t}}\right) \frac{y}{2\sqrt{t}} \leq c\phi\left(\frac{y}{2\sqrt{t}}\right) = \frac{c}{\sqrt{2\pi}} e^{-\frac{y^2}{8t}}, \quad \frac{y}{2\sqrt{t}} > y_0.$$

Therefore, for $\frac{1}{8t} > q$, both terms are finite in the right hand side of (A.6), which shows the result. \blacksquare

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References

- Crépey, S., T. R. Bielecki, and D. Brigo (2014). *Counterparty Risk and Funding: A Tale of Two Puzzles*. Chapman & Hall/CRC Financial Mathematics Series.
- Crépey, S., M. Jeanblanc, and D. L. Wu (2013). Informationally dynamized Gaussian copula. *International Journal of Theoretical and Applied Finance* 16(2), 1350008 (29 pages).
- Crépey, S. and T. M. Nguyen (2015). Nonlinear Monte Carlo schemes for counterparty risk on credit derivatives. In *Challenges in Derivatives Markets*, Springer Proceedings in Mathematics. Springer. Forthcoming.
- Crépey, S. and S. Song (2015a). BSDEs of counterparty risk. *Stochastic Processes and their Applications* 125(8), 3023–3052.

- Crépey, S. and S. Song (2015b). Invariance times. Submitted (available on <https://math.maths.univ-evry.fr/crepey>).
- Crépey, S. and S. Song (2016). Counterparty risk and funding: Immersion and beyond. *Finance and Stochastics* 20(4), 901–930.
- Delbaen, F. and W. Schachermayer (1994). A General Version of the Fundamental Theorem of Asset Pricing. *Mathematische Annalen*, 463–520.
- Dellacherie, C. (1972). *Capacités et Processus stochastiques*. Springer.
- Dellacherie, C., B. Maisonneuve, and P.-A. Meyer (1992). *Probabilités et Potentiel, Chapitres XVII-XXIV*. Hermann.
- El Karoui, N., M. Jeanblanc, and Y. Jiao (2010). What happens after a default: The conditional density approach. *Stochastic Processes and their Applications* 120(7), 1011–1032.
- Émery, M. (1980). Compensation de processus non localement intégrable. In *Séminaire de Probabilités XIV*, Lecture Notes Math. 784, pp. 152–160. Springer.
- Fermanian, J.-D. and O. Vigneron (2013). On break-even correlation: the way to price structured credit derivatives by replication. *Quantitative Finance*. Forthcoming, DOI:10.1080/14697688.2013.812233.
- Fujii, M. and A. Takahashi (2012a). Analytical approximation for non-linear FBSDEs with perturbation scheme. *International Journal of Theoretical and Applied Finance* 15(5), 1250034 (24 pages).
- Fujii, M. and A. Takahashi (2012b). Perturbative expansion of FBSDE in an incomplete market with stochastic volatility. *Quarterly Journal of Finance* 2(3), 1250015 (24 pages).
- He, S.-W., J.-G. Wang, and J.-A. Yan (1992). *Semimartingale Theory and Stochastic Calculus*. CRC Press.
- Jacod, J. (1987). *Grossissement initial, Hypothèse (H) et théorème de Girsanov*. Lecture Notes in Mathematics 1118. Springer.
- Jeanblanc, M. and Y. Le Cam (2009). Progressive enlargement of filtrations with initial times. *Stochastic Processes and their Applications* 119, 2523–2543.
- Jeanblanc, M. and S. Song (2013). Martingale representation theorem in progressively enlarged filtrations. *Stochastic Processes and their Applications* 125(11), 4242–4271.
- Jeulin, T. (1980). *Semi-Martingales et Grossissement d'une Filtration*. Lecture Notes in Mathematics 833. Springer.
- Jeulin, T. and M. Yor (1979). Inégalité de Hardy, semimartingales, et faux-amis. In *Séminaire de Probabilités XIII, 1977/78*, Lecture Notes Math. 721, pp. 332–359. Springer.
- Lepingle, D. and J. Mémin (1978). Sur l'intégrabilité uniforme des martingales exponentielles. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 42, 175–203.
- Li, D. (2000). On default correlation: A copula function approach. *Journal of Fixed Income* 9(4), 43–54.
- Pham, H. (2010). Stochastic control under progressive enlargement of filtrations and applications to multiple defaults risk management. *Stochastic Processes and their Applications* 120, 1795–1820.

-
- Song, S. (2014a). Local martingale deflators for asset processes stopped at a default time s^t or just before s^{t-} . arXiv:1405.4474.
- Song, S. (2014b). Optional splitting formula in a progressively enlarged filtration. *ESAIM Probability and Statistics* 18, 829–853.