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INVARIANCE PROPERTIES IN THE DYNAMIC GAUSSIAN COPULA MODEL *

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Abstract. We prove that the default times (or any of their minima) in the dynamic Gaussian copula model of Crépey, Jeanblanc, and Wu (2013) are invariance times in the sense of Crépey and Song (2017), with related invariance probability measures different from the pricing measure. This reflects a departure from the immersion property, whereby the default intensities of the surviving names and therefore the value of credit protection spike at default times. These properties are in line with the wrong-way risk feature of counterparty risk embedded in credit derivatives, i.e. the adverse dependence between the default risk of a counterparty and an underlying credit derivative exposure.

Keywords: counterparty credit risk, wrong-way risk, Gaussian copula, dynamic copula, immersion property, invariance time, CDS.

Mathematics Subject Classification: 91G40, 60G07.

1. Introduction

This paper deals with the mathematics of the dynamic Gaussian copula (DGC) model of Crépey, Jeanblanc, and Wu (2013) (see also Crépey, Bielecki, and Brigo (2014, Chapter 7) and Crépey and Nguyen (2016)). As developed in Crépey et al. (2014, Section 7.3.3), this model yields a dynamic meaning to the ad hoc bump sensitivities that were used by traders for hedging CDO tranches by CDS contracts before the subprime crisis. From a more topical perspective, it can be used for counterparty risk computations on CDS portfolios. Related models include the one-period Merton model of Fermanian and Vigneron (2015, Section 6) or other variants commonly used in credit and counterparty risk softwares.

The dynamic Gaussian copula model has been assessed from an engineering perspective in previous work, but a detailed mathematical study, including explicit computation of the main model primitives, has been deferred to the present paper.

1.1. Invariance Times and Probability Measures

We work on a filtered probability space $(\Omega, \mathbb{G}, \mathcal{A}, \mathbb{Q})$. Given a $\mathbb{G}$ stopping time τ and a subfiltration $\mathbb{F}$ of $\mathbb{G}$, $\mathbb{F}$ and $\mathbb{G}$ satisfying the usual conditions, let J and S denote the survival indicator process of τ and its optional projection known as the Azéma supermartingale of τ, i.e.

$$J_t = \mathbb{1}_{\{\tau > t\}},\ \ S_t = {}^oJ_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t),\ \ t \geq 0.$$

The following conditions are studied in Crépey and Song (2015,2017).

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Condition (B). Any $\mathbb{G}$ predictable process U admits an $\mathbb{F}$ predictable reduction, i.e. an $\mathbb{F}$ predictable process, denoted by U', that coincides with U on $\rrbracket 0, \tau \rrbracket$.

For any left-limited process Y, we denote by $Y^{\tau-} = JY + (1-J)Y_{\tau-}$ the process Y stopped before τ.

Condition (A). Given a constant time horizon $T > 0$, there exists a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ such that $(\mathbb{F}, \mathbb{P})$ local martingales stopped before τ are $(\mathbb{G}, \mathbb{Q})$ local martingales on $[0, T]$.

If the conditions (B) and (A) are satisfied, then we say that τ is an invariance time and $\mathbb{P}$ is an invariance probability measure. If, in addition, $S_T > 0$ almost surely, then $\mathbb{F}$ predictable reductions are uniquely defined on $(0, T]$ and any inequality between two $\mathbb{G}$ predictable processes on $(0, \tau]$ implies the same inequality between their $\mathbb{F}$ predictable reductions on $(0, T]$ (see Song (2014a, Lemma 6.1)); invariance probability measures are uniquely defined on $\mathcal{F}_T$, so that one can talk of the invariance probability measure $\mathbb{P}$ (as the specification of an invariance probability measure outside $\mathcal{F}_T$ is immaterial anyway).

2. Dynamic Gaussian Copula Model

In the paper we prove that, given a constant time horizon $T > 0$, the default times τ_i (or any of their minima) in the DGC model are invariance times, with related invariance probability measures $\mathbb{P}$ uniquely defined and not equal to $\mathbb{Q}$ on $\mathcal{F}_T$. This reflects a departure from the immersion property, whereby the default intensities of the surviving names and therefore the value of credit protection spike at default times, as observed in practice. This feature makes the DGC model appropriate for dealing with counterparty risk on credit derivatives (notably, portfolios of CDS contracts) traded between a bank and its counterparty, respectively labeled as -1 and 0, and referencing credit names 1 to n, for some positive integer n. Accordingly, we introduce

$$N = \{-1, 0, 1, \dots, n\} \text{ and } N^\star = \{1, \dots, n\}$$

and we focus on $\tau = \tau_{-1} \wedge \tau_0$ in the paper. However, analog properties hold for any minimum of the τ_i and, in particular, for the τ_i themselves.

2.1. The model

We consider a family of independent standard linear Brownian motions Z and $Z^i, i \in N$. For $\varrho \in [0, 1)$, we define

$$B_t^i = \sqrt{\varrho} Z_t + \sqrt{1-\varrho} Z_t^i. \tag{2.1}$$

Let ς be a continuous function on $\mathbb{R}_+$ with $\int_{\mathbb{R}_+} \varsigma^2(s)ds = 1$ and $\alpha^2(t) = \int_t^{+\infty} \varsigma^2(s)ds > 0$ for all $t \in \mathbb{R}_+$. For any $i \in N$, let h_i be a continuously differentiable strictly increasing function from $\mathbb{R}_+^*$ to $\mathbb{R}$, with derivative denoted by $\dot{h}_i$, such that $\lim_{s\downarrow 0} h_i(s) = -\infty$ and $\lim_{s\uparrow +\infty} h_i(s) = +\infty$. We define

$$\tau_i = h_i^{-1}\big(\int_0^{+\infty} \varsigma(u) dB_u^i \big) = h_i^{-1}\big(\sqrt{\varrho} \int_0^{+\infty} \varsigma(u) dZ_u + \sqrt{1-\varrho} \int_0^{+\infty} \varsigma(u) dZ_u^i \big), \tag{2.2}$$

for $i \in N$. The random times $(\tau_i)_{i\in N}$ follow the standard one-factor Gaussian copula model of Li (2000) (a DGC model in abbreviation), with correlation parameter ϱ and with marginal survival function $\Phi \circ h_i$ of τ_i, where

$$\Phi(t) = \int_t^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx, \ t \in \mathbb{R}$$

is the standard normal survival function. Note that, if $\varrho < 1$, the τ_i avoid each other:

$$\mathbb{Q}(\tau_i = \tau_j) = 0, \text{ for any } i \neq j \text{ in } N.$$

2.2. Density Property

By multivariate density default model, we mean a model with an $\mathbb{F}$ conditional density of the default times (see e.g. the condition (DH) in Pham (2010, page 1800)), given some reference subfiltration $\mathbb{F}$ of $\mathbb{G}$. This is the multivariate extension of the notion of a density time, first introduced in an initial enlargement setup in Jacod (1987) and revisited in a progressive enlargement setup in Jeanblanc and Le Cam (2009) (under the name of initial time) and El Karoui, Jeanblanc, and Jiao (2010,2015b,2015a).

First we prove that the DGC model is a multivariate density model with respect to the natural filtration $\mathbb{B} = (\mathcal{B}_t)_{t\geq 0}$ of the Brownian motions Z and $Z^i, i \in N$. We introduce the following processes.

$$m_t^i = \int_0^t \varsigma(u) dB_u^i \text{ and } \bar{m}_t^i = \int_t^\infty \varsigma(u) dB_u^i = h(\tau_i) - m_t^i,\ i \in N.$$

The standard normal density function is denoted by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},\ x \in \mathbb{R}.$$

Theorem 2.1. *The dynamic Gaussian copula model is a multivariate density model of default times (with respect to the filtration $\mathbb{B}$), with conditional Lebesgue density*

$$p_t(t_i, i \in N) = \partial_{t_{-1}} \dots \partial_{t_n} \mathbb{Q}(\tau_i < t_i,\ i \in N \,|\, \mathcal{B}_t)$$

of the τ_i, $i \in N$, given, for any nonnegative t_i, $i \in N$, and $t \in \mathbb{R}_+$, by

$$p_t(t_i,\ i \in N) = \int_{\mathbb{R}} \phi(y) \prod_{i \in N} \phi\left(\frac{h_i(t_i) - m_t^i + \alpha(t)\sqrt{\varrho} y}{\alpha(t)\sqrt{1-\varrho}}\right) \frac{\dot{h}_i(t_i)}{\alpha(t)\sqrt{1-\varrho}} dy. \tag{2.3}$$

Proof. The conditional density function p given $\mathcal{B}_t$ can be computed thanks to the independence of increments of the processes $Z, Z^i, i \in N$. Actually, for any $t \geq 0$, we can write

$$\tau_i = h_i^{-1}(m_t^i + \sqrt{\varrho}\xi + \sqrt{1-\varrho}\xi_i),\ i \in N,$$

where ξ is a real normal random variable with variance α_t^2, where $(\xi_j)_{j \in N}$ is a centered Gaussian vector independent of ξ with homogeneous marginal variances α_t^2 and zero pairwise correlations, and where the family $\xi, \xi_i, i \in N$, is independent of $\mathcal{B}_t$. See Crépey et al. (2014, page 172)[1]. ∎

2.3. Computation of the intensity processes

Note that the τ_i are $\mathcal{B}_\infty$ measurable, but they are not $\mathbb{B}$ stopping times. In the DGC model, the full model filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is taken as the progressive enlargement of the Brownian filtration $\mathbb{B}$ by the $\tau_i, i \in N$, augmented so as to satisfy the usual conditions, i.e.

$$\mathcal{G}_t = \cap_{s>t}(\mathcal{B}_s \vee \bigvee_{i \in N} \sigma(\tau_i \wedge s)),\ \ t \geq 0. \tag{2.4}$$

In this section we prove that the τ_i are totally inaccessible $\mathbb{G}$ stopping times with intensities that we compute explicitly.

[1] Or Crépey et al. (2013, page 3) in the journal version.

For $I \subseteq N$ and $j \in N$, we define:

$$\rho^I = \frac{\varrho}{|I|\varrho+1}, \quad (\sigma^I)^2 = (1-\varrho)\frac{|I|\varrho+1}{|I|\varrho+1-\varrho}, \quad \lambda^I = \frac{\varrho}{(|I|-1)\varrho+1},$$
$$Z_t^{j,I}(u) = \frac{h_j(u)-m_t^j}{\alpha(t)} - \lambda^I \sum_{i\in I} \frac{\bar{m}_t^i}{\alpha(t)}.$$

For $t \geq 0$, let

$$\mathcal{I}_t = \{i \in N : \tau_i \leq t\}$$

(representing the set of obligors in N that are in default at time t) and let

$$\rho_t = \rho^{\mathcal{I}_t},\ \sigma_t = \sigma^{\mathcal{I}_t},\ \mathcal{J}_t = N \setminus \mathcal{I}_t.$$

For $\sigma > 0$, $\rho \in [0,1]$ and $J \subseteq N$, we define the functions

$$\Phi_{J,\rho,\sigma}(\mathbf{z}_J) = \mathbb{Q}(\xi_j > z_j, j \in J),\ \ \psi^j_{J,\rho,\sigma}(\mathbf{z}_J) = -\frac{\partial_{z_j}\Phi_{J,\rho,\sigma}}{\Phi_{J,\rho,\sigma}}(\mathbf{z}_J),\ j \in J, \tag{2.5}$$

where $\mathbf{z}_J = (z_j)_{j\in J}$ is a real vector and $(\xi_j)_{j\in N}$ is a centered Gaussian vector with homogeneous marginal variances σ^2 and pairwise correlations ρ. Note the following:

Lemma 2.1. *For $I = N \setminus J$, the family of random variables*

$$\left(\xi_j - \frac{\rho}{(|I|-1)\rho+1}\sum_{i\in I}\xi_i\right)_{j\in J}$$

defines a centered Gaussian vector independent of $\sigma(\xi_i, i \in I)$, with homogeneous marginal variances and pairwise correlations, respectively given as

$$\sigma^2(1-\rho)\frac{|I|\rho+1}{|I|\rho+1-\rho} \quad \textit{and} \quad \frac{\rho}{|I|\rho+1}. \blacksquare \tag{2.6}$$

Lemma 2.2. *For $u > 0$,*

$$\mathbb{E}[\mathbb{1}_{\{u_j<\tau_j, j\in J\}}|\mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)] = \Phi_{J,\rho^I,\sigma^I}(Z_t^{j,I}(u_j), j \in J). \tag{2.7}$$

Proof. For $j \in J$ and $u_j \in \mathbb{R}$, the condition $u_j < \tau_j$ is equivalent to

$$Z_t^{j,I}(u_j) = \frac{h_j(u_j)-m_t^j}{\alpha(t)} - \lambda^I\sum_{i\in I}\frac{\bar{m}_t^i}{\alpha(t)} \ < \ \frac{\bar{m}_t^j}{\alpha(t)} - \lambda^I\sum_{i\in I}\frac{\bar{m}_t^i}{\alpha(t)} \tag{2.8}$$

Noting that $m_t^j \in \mathcal{B}_t, \bar{m}_t^i \in \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I), i \in I$, the desired result follows by an application of Lemma 2.1. $\blacksquare$

Lemma 2.3. *For every $t > 0$ and $I \subseteq N$ we have, writing $J = N \setminus I$ and $\boldsymbol{\tau}_I = (\tau_i)_{i\in I}$:*

$$\{\tau_i \leq t < \tau_j : i \in I, j \in J\} \cap \mathcal{G}_t = \{\tau_i \leq t < \tau_j : i \in I, j \in J\} \cap (\mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)). \tag{2.9}$$

Proof. Let the $\tau_{(i)}$ be the increasing ordering of the τ_i, with also $\tau_{(0)} = 0$ and $\tau_{(n+1)} = \infty$. According to the optional splitting formula which holds in any multivariate density model of default times (see Song (2014b)), for any $\mathbb{G}$ optional process Y, there exists a $\mathcal{O}(\mathbb{B}) \otimes \mathcal{B}([0,\infty]^n)$-measurable functions $Y^{(i)},\ i \in N$, such that

$$Y = \sum_{i=0}^{n} Y^{(i)}(\tau_{-1} \nmid \tau_{(i)}, \ldots, \tau_n \nmid \tau_{(i)})\mathbb{1}_{[\tau_{(i)},\tau_{(i+1)})}, \tag{2.10}$$

where $a \nmid b$ denotes a if $a \leq b$ and ∞ if $a > b$, for $a, b \in [0, \infty]$. Since $\mathcal{G}_t = \sigma(Y_t)$ and $Y^{(i)}(\tau_1 \nmid \tau_{(i)}, \ldots, \tau_n \nmid \tau_{(i)})\mathbb{1}_{[\tau_{(i)}, \tau_{(i+1)})}$ is a function of $\mathcal{B}_t$ and $\boldsymbol{\tau}_I$ on $\{\tau_i \leq t < \tau_j : i \in I, j \in J\}$, this implies (2.9). ∎

Theorem 2.2. *For any $j \in N$, τ_j admits a $(\mathbb{G}, \mathbb{Q})$ intensity given by*

$$\gamma_t^j = \mathbb{1}_{\{t < \tau_j\}} \frac{\dot{h}_j(t)}{\alpha(t)} \psi^j_{\mathcal{J}_t, \rho_t, \sigma_t}\big(Z_t^{j, \mathcal{I}_t}(t), j \in \mathcal{J}_t\big), \ t \in \mathbb{R}_+. \tag{2.11}$$

Proof. Let $l \in N$. For bounded $\mathcal{B}_t$ measurable functions F, for measurable bounded function f, for $0 \leq t \leq s < \infty$, we look at

$$\mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \leq t < \tau_j : i \in I, j \in J\}}\mathbb{1}_{\{t < \tau_l \leq s\}}].$$

We need only to consider $l \in J$. Then, using (2.7) to pass to the third line and conditioning in conjunction with the tower rule to pass to the fourth line:

$$\begin{array}{ll} & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \leq t < \tau_j, i \in I, j \in J\}}\mathbb{1}_{\{s < \tau_l\}}] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \leq t, i \in I\}}\mathbb{E}[\mathbb{1}_{\{t < \tau_j, j \in J\}}\mathbb{1}_{\{s < \tau_l\}} | \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)]] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \leq t, i \in I\}}\Phi_{J, \rho^I, \sigma^I}(Z_t^{j, I}(u_j), j \in J)] \\ & \text{where } u_j = t \text{ except } u_l = s, \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \leq t < \tau_j, i \in I, j \in J\}}\frac{\Phi_{J, \rho^I, \sigma^I}(Z_t^{j, I}(u_j), j \in J)}{\Phi_{J, \rho^I, \sigma^I}(Z_t^{j, I}(t), j \in J)}] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \leq t < \tau_j, i \in I, j \in J\}}\frac{\Phi_{\mathcal{J}_t, \rho_t, \sigma_t}(Z_t^{j, \mathcal{I}_t}(u_j), j \in \mathcal{J}_t)}{\Phi_{\mathcal{J}_t, \rho_t, \sigma_t}(Z_t^{j, \mathcal{I}_t}(t), j \in \mathcal{J}_t)}]. \end{array} \tag{2.12}$$

With the formula (2.9), we conclude

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{\{t < \tau_l \leq s\}} | \mathcal{G}_t] &= \mathbb{E}[\mathbb{1}_{\{t < \tau_l\}} | \mathcal{G}_t] - \mathbb{E}[\mathbb{1}_{\{s < \tau_l\}} | \mathcal{G}_t] \\ &= \mathbb{1}_{\{t < \tau_l\}}(1 - \frac{\Phi_{\mathcal{J}_t, \rho_t, \sigma_t}(Z_t^{j, \mathcal{I}_t}(u_j), j \in \mathcal{J}_t)}{\Phi_{\mathcal{J}_t, \rho_t, \sigma_t}(Z_t^{j, \mathcal{I}_t}(t), j \in \mathcal{J}_t)}). \end{aligned}$$

The stated result follows by an application of the Laplace formula of Dellacherie (1972, Chapter V, Theorem T54) (see also Dellacherie and Doléans-Dade (1971) or Knight (1991)). ∎

2.4. Computation of the drift of the Brownian motion

Next we study the processes $B^i, i \in N$, in the filtration $\mathbb{G}$. Thanks to Theorem 2.1, the DGC model is a multivariate density model. According to Jacod (1987), this implies the following:

Lemma 2.4. *The processes $B^i, i \in N$, are $\mathbb{G}$ semimartingales.* ∎

By virtue of Jeanblanc and Song (2013, Theorem 6.4), another consequence of the multivariate density property is the martingale representation property.

Theorem 2.3. *Let W^i, for $i \in N$, denote the martingale part in $\mathbb{G}$ of B^i. Let*

$$dM_t^i = d\mathbb{1}_{\tau_i \leq t} - \gamma_t^i dt, \ \ t > 0,$$

where the process γ^i is defined in (2.11). Then, the martingale representation property holds in $\mathbb{G}$ with respect to $(W^i, M^i, i \in N)$.

This section is devoted to the computation of the martingales W^i. We begin with the following remark on the Gaussian processes B^i (cf. Lemma 2.1).

Lemma 2.5. *For $J \subseteq N$ and $I = N \setminus J$, the family of processes*

$$\left(B^j - \frac{\varrho}{(|I|-1)\varrho + 1}\sum_{i\in I} B^i\right)_{j\in J} \tag{2.13}$$

is a continuous Lévy process (multivariate Brownian motion with drift) independent of $\sigma(B^i, i \in I)$, of homogeneous marginal variances and pairwise correlations, equal to, respectively,

$$(\sigma^I)^2 = (1-\varrho)\frac{|I|\varrho + 1}{|I|\varrho + 1 - \varrho} \text{ and } \rho^I = \frac{\varrho}{|I|\varrho + 1}. \blacksquare \tag{2.14}$$

Proof. This follows by computing the brackets of the continuous local martingales and applying the Lévy processes characterization. ∎

Lemma 2.6. *For $k \in I$ and $0 \le t \le s \le s'$,*

$$\mathbb{E}[B^k_{s'} - B^k_s | \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)] = (\tfrac{1}{\alpha_t^2}\int_s^{s'}\varsigma_u du)\, \bar{m}^k_t.$$

Proof. For $k \in I$, for $0 \le t \le s \le s' < \infty$,

$$B^k_{s'} - B^k_s - (\frac{1}{\alpha_t^2}\int_s^{s'}\varsigma_u du)\, \bar{m}^k_t \tag{2.15}$$

is a centered Gaussian random variable, independent of $\bar{m}^k_t$, with variance

$$(s'-s) - 2\frac{1}{\alpha_t^2}(\int_s^{s'}\varsigma_u du)^2 + \frac{1}{\alpha_t^2}(\int_s^{s'}\varsigma_u du)^2 = (s'-s) - \frac{1}{\alpha_t^2}(\int_s^{s'}\varsigma_u du)^2.$$

Hence, for $k \in I$,

$$\begin{aligned} & \mathbb{E}[B^k_{s'} - B^k_s | \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)] = \mathbb{E}[B^k_{s'} - B^k_s | \mathcal{B}_t \vee \sigma(\bar{m}^i_t, i \in I)] \\ = \; & \mathbb{E}[B^k_{s'} - B^k_s | \mathcal{B}_t \vee \sigma(\bar{m}^k_t) \vee \sigma(\xi_i, i \in I \setminus \{k\})] \\ & \text{where } \xi_i \text{ form a Gaussian family independent of } \mathcal{B}_t \vee \sigma(B^k), \text{ constructed with (2.13)}, \\ = \; & \mathbb{E}[B^k_{s'} - B^k_s | \sigma(\bar{m}^k_t)] = (\tfrac{1}{\alpha_t^2}\int_s^{s'}\varsigma_u du)\, \bar{m}^k_t \text{ because of (2.15).} \blacksquare \end{aligned}$$

In the sequel we find it sometimes convenient to denote stochastic integration (or integration against measures) by $\centerdot$ and the Lebesgue measure on the half-line by $\boldsymbol{\lambda}$.

Theorem 2.4. *For $J \subseteq N$ and $k \in J$, define the function*

$$\mathfrak{b}^k_J(\mathbf{z}, x) = \mathbb{E}[\mathbb{1}_{\{z_j < \xi_j, j \in J\}}(\xi_k + x)], \;\; x \in \mathbb{R}, \, \mathbf{z} = (z_j, j \in J),$$

where $(\xi_j, j \in J)$ is a Gaussian family of homogeneous marginal variances $(\sigma^I)^2$ and pairwise correlations ρ^I. For any $k \in N$, define the process

$$\beta^k_t = \frac{\varsigma(t)}{\alpha(t)}\left(\mathbb{1}_{\{k\in\mathcal{I}_t\}}\frac{\bar{m}^k_t}{\alpha(t)} + \mathbb{1}_{\{k\notin\mathcal{I}_t\}}\frac{\mathfrak{b}^k_{\mathcal{J}_t}((Z^{j,\mathcal{I}_t}_t(t), j \in \mathcal{J}_t),\; \lambda^{\mathcal{I}_t}\sum_{i\in\mathcal{I}_t}\frac{\bar{m}^i_t}{\alpha(t)})}{\Phi_{\mathcal{J}_t,\rho_t,\sigma_t}(Z^{j,\mathcal{I}_t}_t(t), j \in \mathcal{J}_t)}\right), \;\; t \in \mathbb{R}_+.$$

Then, $W^k = B^k - \beta^k \centerdot \boldsymbol{\lambda}$.

Proof. For $0 \le t \le s \le s' < \infty$, for any bounded $\mathcal{B}_t$ measurable function F and measurable bounded function f, we compute

$$\begin{array}{rl} & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t<\tau_j, i\in I, j\in J\}}(B^k_{s'}-B^k_s)] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t, i\in I\}}\mathbb{1}_{\{t<\tau_j, j\in J\}}\mathbb{E}[(B^k_{s'}-B^k_s)|\mathcal{B}_t\vee\sigma(\boldsymbol{\tau}_N)]] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t, i\in I\}}\mathbb{1}_{\{t<\tau_j, j\in J\}}(\frac{1}{\alpha^2_t}\int_s^{s'}\varsigma_u du)\,\bar{m}^k_t]. \end{array}$$

If $k \in I$, $\bar{m}^k_t \in \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)$. If $k \notin I$,

$$\begin{array}{rl} & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t, i\in I\}}\mathbb{1}_{\{t<\tau_j, j\in J\}}(\frac{1}{\alpha^2_t}\int_s^{s'}\varsigma_u du)\,\bar{m}^k_t] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t, i\in I\}}(\frac{1}{\alpha(t)}\int_s^{s'}\varsigma_u du)\,\mathbb{E}[\mathbb{1}_{\{t<\tau_j, j\in J\}}\frac{\bar{m}^k_t}{\alpha(t)}|\mathcal{B}_t\vee\sigma(\boldsymbol{\tau}_I)]] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t, i\in I\}}(\frac{1}{\alpha(t)}\int_s^{s'}\varsigma_u du) \\ & \mathbb{E}[\mathbb{1}_{\{Z^{j,I}_t(t)<\frac{\bar{m}^j_t}{\alpha(t)}-\lambda^I\sum_{i\in I}\frac{\bar{m}^i_t}{\alpha(t)}, j\in J\}}(\frac{\bar{m}^k_t}{\alpha(t)}-\lambda^I\sum_{i\in I}\frac{\bar{m}^i_t}{\alpha(t)}+\lambda^I\sum_{i\in I}\frac{\bar{m}^i_t}{\alpha(t)})|\mathcal{B}_t\vee\sigma(\boldsymbol{\tau}_I)]] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t, i\in I\}}(\frac{1}{\alpha(t)}\int_s^{s'}\varsigma_u du)\,\mathbb{E}[\mathbb{1}_{\{Z^{j,I}_t(t)<\xi_j, j\in J\}}(\xi_k+\lambda^I\sum_{i\in I}\frac{\bar{m}^i_t}{\alpha(t)})]] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t, i\in I\}}(\frac{1}{\alpha(t)}\int_s^{s'}\varsigma_u du)\,\mathfrak{b}^k_J((Z^{j,I}_t(t), j\in J),\ \lambda^I\sum_{i\in I}\frac{\bar{m}^i_t}{\alpha(t)})] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i\le t, i\in I\}}\mathbb{1}_{\{t<\tau_j, j\in J\}}(\frac{1}{\alpha(t)}\int_s^{s'}\varsigma_u du)\,\frac{\mathfrak{b}^k_J((Z^{j,I}_t(t), j\in J),\ \lambda^I\sum_{i\in I}\frac{\bar{m}^i_t}{\alpha(t)})}{\Phi_{J,\rho^I,\sigma^I}(Z^{j,I}_t(t), j\in J)}] \end{array}$$

This, combined with the formula (2.9), implies

$$\mathbb{E}[(B^k_{s'}-B^k_s)|\mathcal{G}_t] = (\frac{1}{\alpha(t)}\int_s^{s'}\varsigma_u du)\times \left(\mathbb{1}_{\{k\in\mathcal{I}_t\}}\frac{\bar{m}^k_t}{\alpha(t)} + \mathbb{1}_{\{k\notin\mathcal{I}_t\}}\frac{\mathfrak{b}^k_{\mathcal{J}_t}((Z^{j,\mathcal{I}_t}_t(t), j\in\mathcal{J}_t),\ \lambda^{\mathcal{I}_t}\sum_{i\in\mathcal{I}_t}\frac{\bar{m}^i_t}{\alpha(t)})}{\Phi_{\mathcal{J}_t,\rho_t,\sigma_t}(Z^{j,\mathcal{I}_t}_t(t), j\in\mathcal{J}_t)}\right).$$

The $\mathbb{G}$ drift of B^k is obtained as the differential of the above with respect to Lebesgue measure, i.e.

$$\frac{\varsigma(t)}{\alpha(t)}\left(\mathbb{1}_{\{k\in\mathcal{I}_t\}}\frac{\bar{m}^k_t}{\alpha(t)} + \mathbb{1}_{\{k\notin\mathcal{I}_t\}}\frac{\mathfrak{b}^k_{\mathcal{J}_t}((Z^{j,\mathcal{I}_t}_t(t), j\in\mathcal{J}_t),\ \lambda^{\mathcal{I}_t}\sum_{i\in\mathcal{I}_t}\frac{\bar{m}^i_t}{\alpha(t)})}{\Phi_{\mathcal{J}_t,\rho_t,\sigma_t}(Z^{j,\mathcal{I}_t}_t(t), j\in\mathcal{J}_t)}\right)dt. \blacksquare$$

3. Reduced DGC Model

We now study the invariance properties of the DGC model. In this perspective, the market information before the default event of the bank or of its counterparty is modeled by the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$, where

$$\mathcal{F}_t = \cap_{s>t}\big(\mathcal{B}_s \vee \bigvee_{i\in N^\star}\big(\sigma(\tau_i\wedge s)\big), \tag{3.1}$$

augmented so as to satisfy the usual conditions.

Because of the multivariate density property of the family of $(\tau^j, j \in N^\star)$ with respect to the filtration $\mathbb{B}$ (same proof as Theorem 2.1), the computations we have made in $\mathbb{G}$ in the previous section can be made similarly in $\mathbb{F}$. In particular, the following splitting formula holds (cf. (2.9)): for any $t > 0$ and $I \subseteq N^\star$, writing $J = N^\star \setminus I$,

$$\{\tau_i \le t < \tau_j : i \in I, j \in J\} \cap \mathcal{F}_t = \{\tau_i \le t < \tau_j : i \in I, j \in J\} \cap \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I). \tag{3.2}$$

Moreover, the so-called condition (H') holds, i.e. the processes $B^k, k \in N$, are $\mathbb{F}$ semimartingales, and the random times $\tau_j, j \in N^\star$, are $\mathbb{F}$ totally inaccessible stopping times, as stated in the following lemma. For $t > 0$, let

$$\mathcal{I}_t^\star = \{i \in N^\star : \tau_i \le t\},\ \mathcal{J}_t^\star = N^\star \setminus \mathcal{I}_t^\star,\ \rho_t^\star = \rho^{\mathcal{I}_t^\star},\ \sigma_t^\star = \sigma^{\mathcal{I}_t^\star}.$$

Lemma 3.1. *For any $k \in N$, the process $\bar{W}_t^k = B_t^k - \int_0^t \bar{\beta}_s^k ds,\ t \ge 0$ is an $\mathbb{F}$ local martingale, where*

$$\bar{\beta}_t^k = \frac{\varsigma(t)}{\alpha(t)} \left(\mathbb{1}_{\{k \in \mathcal{I}_t^\star\}} \frac{\bar{m}_t^k}{\alpha(t)} + \mathbb{1}_{\{k \notin \mathcal{I}_t^\star\}} \frac{\mathfrak{b}^k_{\mathcal{J}_t^\star}((Z_t^{j,\mathcal{I}_t^\star}(t), j \in \mathcal{J}_t^\star),\ \lambda^{\mathcal{I}_t^\star} \sum_{i \in \mathcal{I}_t^\star} \frac{\bar{m}_t^i}{\alpha(t)})}{\Phi_{\mathcal{J}_t^\star, \rho_t^\star, \sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t), j \in \mathcal{J}_t^\star)} \right),\ t \in \mathbb{R}_+.$$

For $j \in N^\star$, τ_j is an $\mathbb{F}$ totally inaccessible stopping time and the process $d\bar{M}_t^j = d\mathbb{1}_{\tau_j \le t} - \bar{\gamma}_t^j dt,\ \ t > 0$, is an $\mathbb{F}$ local martingale, where

$$\bar{\gamma}_t^j = \mathbb{1}_{\{t < \tau_j\}} \frac{\dot{h}_j(t)}{\alpha(t)} \psi^j_{\mathcal{J}_t^\star, \rho_t^\star, \sigma_t^\star}\big(Z_t^{j,\mathcal{I}_t^\star}(t), j \in \mathcal{J}_t^\star\big),\ t \in \mathbb{R}_+. \tag{3.3}$$

The family of processes $\bar{W}^k, k \in N$ and $\bar{M}^j, j \in N^\star$, has the martingale representation property in the filtration $\mathbb{F}$. ∎

3.1. The Azéma supermartingale

Our next aim is to compute the Azéma supermartingale of the random time $\tau_{-1} \wedge \tau_0$ in the filtration $\mathbb{F}$, i.e.,

$$\mathbb{E}[\mathbb{1}_{\{t < \tau_{-1} \wedge \tau_0\}} | \mathcal{F}_t],\ t \ge 0.$$

Lemma 3.2. *The Azéma supermartingale of the random time $\tau_{-1} \wedge \tau_0$ in the filtration $\mathbb{F}$ is given by*

$$S_t = \frac{\Phi_{\mathcal{J}_t^\star \cup \{-1,0\}, \rho_t^\star, \sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t), j \in \mathcal{J}_t^\star \cup \{-1,0\})}{\Phi_{\mathcal{J}_t^\star, \rho_t^\star, \sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t), j \in \mathcal{J}_t^\star)},\ t \ge 0. \tag{3.4}$$

In particular, the Azéma supermartingale S is positive.

Proof. For any bounded $\mathcal{B}_t$ measurable functions F and measurable bounded function f, we compute (cf. (2.12))

$$\begin{array}{rl} & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \le t < \tau_j, i \in I, j \in J\}} \mathbb{1}_{\{t < \tau_{-1} \wedge \tau_0\}}] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \le t : i \in I\}} \mathbb{E}[\mathbb{1}_{\{t < \tau_j : j \in J\}} \mathbb{1}_{\{t < \tau_{-1} \wedge \tau_0\}} | \mathcal{B}_t \vee \sigma(\boldsymbol{\tau}_I)]] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \le t : i \in I\}} \Phi_{J \cup \{-1,0\}, \rho^I, \sigma^I}(Z_t^{j,I}(t) : j \in J \cup \{-1,0\})] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \le t < \tau_j : i \in I, j \in J\}} \frac{\Phi_{J \cup \{-1,0\}, \rho^I, \sigma^I}(Z_t^{j,I}(t) : j \in J \cup \{-1,0\})}{\Phi_{J, \rho^I, \sigma^I}(Z_t^{j,I}(t) : j \in J)}] \\ = & \mathbb{E}[Ff(\boldsymbol{\tau}_I)\mathbb{1}_{\{\tau_i \le t < \tau_j : i \in I, j \in J\}} \frac{\Phi_{\mathcal{J}_t^\star \cup \{-1,0\}, \rho_t^\star, \sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t) : j \in \mathcal{J}_t^\star \cup \{-1,0\})}{\Phi_{\mathcal{J}_t^\star, \rho_t^\star, \sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t) : j \in \mathcal{J}_t^\star)}], \end{array}$$

where conditioning and the tower rule are used in the next-to-last identity. With the formula (3.2), we conclude

$$\mathbb{E}[\mathbb{1}_{\{t < \tau_{-1} \wedge \tau_0\}} | \mathcal{F}_t] = \frac{\Phi_{\mathcal{J}_t^\star \cup \{-1,0\}, \rho_t^\star, \sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t) : j \in \mathcal{J}_t^\star \cup \{-1,0\})}{\Phi_{\mathcal{J}_t^\star, \rho_t^\star, \sigma_t^\star}(Z_t^{j,\mathcal{I}_t^\star}(t) : j \in \mathcal{J}_t^\star)}. \ ∎$$

Let $\nu = \frac{1}{S} \centerdot S^c$, where S^c denotes the continuous martingale component of the $(\mathbb{F}, \mathbb{Q})$ Azéma supermartingale S.

Lemma 3.3. *We have*

$$\begin{aligned} d\nu_t &= \sum_{j\in\mathcal{J}^\star_{t-}\cup\{-1,0\}} \psi^j_{\mathcal{J}^\star_{t-}\cup\{-1,0\},\rho^\star_t,\sigma^\star_t}\big(Z^{j,\mathcal{I}^\star_{t-}}_t(t) : j\in\mathcal{J}^\star_t\cup\{-1,0\}\big)d\zeta^{j,\mathcal{I}^\star_t}_t \\ &\quad -\sum_{j\in\mathcal{J}^\star_{t-}} \psi^j_{\mathcal{J}^\star_{t-},\rho^\star_t,\sigma^\star_t}\big(Z^{j,\mathcal{I}^\star_{t-}}_t(t) : j\in\mathcal{J}^\star_t\big)d\zeta^{j,\mathcal{I}^\star_t}_t, \end{aligned}$$

where, for $I\subseteq N^\star$, $\zeta^{j,I}_t$ denotes the martingale part of $\Big(-\frac{1}{\alpha(t)}dm^j_t + \frac{\varrho}{(|I|-1)\varrho+1}\sum_{i\in I}\frac{1}{\alpha(t)}dm^i_t\Big)$ in $\mathbb{F}$.

Proof. To obtain dS^c_t (which is then divided by S_t), it suffices to apply Itô calculus to the expression (3.4) of S on every random interval where $\mathcal{I}^\star_{t-}$ is constant. Note that, knowing t is in such an interval, $\boldsymbol{\tau}_{\mathcal{I}^\star_{t-}}$ is in $\mathcal{F}_t$. Also note that the jumps of S_t triggered by the jumps of $\mathcal{I}^\star_{t-}$ have no impact here, because S^c is a continuous local martingale. ∎

3.2. $\mathbb{F}$ **reductions of** β^k,γ^j

Lemma 3.4. *The triplet $(\tau_{-1}\wedge\tau_0,\mathbb{F},\mathbb{G})$ satisfies the condition (B).*

Proof. To check the condition (B), by the monotone class theorem, we only need consider the elementary $\mathbb{G}$ predictable processes of the form $U=\nu f(\tau_{-1}\wedge s,\tau_0\wedge s)\mathbb{1}_{(s,t]}$, for an $\mathcal{F}_s$ measurable random variable F and a Borel function f. Since $U\mathbb{1}_{(0,\tau]} = Ff(s,s)\mathbb{1}_{(s,t]}\mathbb{1}_{(0,\tau]}$, we may take $U' = Ff(s,s)\mathbb{1}_{(s,t]}$ in the condition (B). ∎

Next we consider the reduction of the processes β^k,γ^j in the filtration $\mathbb{F}$. Notice that, for $t<\tau_{-1}\wedge\tau_0$,

$$\mathcal{I}_t=\mathcal{I}^\star_t,\ \mathcal{J}_t=\mathcal{J}^\star_t\cup\{-1,0\}.$$

Therefore, the following lemma holds.

Lemma 3.5. *The $\mathbb{F}$ reduction of $\gamma^j, j\in N^\star$, is*

$$\widetilde{\gamma}^j_t = \mathbb{1}_{\{t<\tau_j\}}\frac{\dot{h}_j(t)}{\alpha(t)}\psi^j_{\mathcal{J}^\star_t\cup\{-1,0\},\rho^\star_t,\sigma^\star_t}\big(Z^{j,\mathcal{I}^\star_t}_t(t), j\in\mathcal{J}^\star_t\cup\{-1,0\}\big),\ t\in\mathbb{R}_+. \tag{3.5}$$

Similarly, the $\mathbb{F}$ reduction of $\beta^k, k\in N$, is

$$\widetilde{\beta}^k_t = \frac{\varsigma(t)}{\alpha(t)}\times\left(\mathbb{1}_{\{k\in\mathcal{I}^\star_t\}}\frac{\bar{m}^k_t}{\alpha(t)} + \mathbb{1}_{\{k\notin\mathcal{I}^\star_t\}}\frac{\mathfrak{b}^k_{\mathcal{J}^\star_t\cup\{-1,0\}}\big((Z^{j,\mathcal{I}^\star_t}_t(t), j\in\mathcal{J}^\star_t\cup\{-1,0\}),\ \lambda^{\mathcal{I}^\star_t}\sum_{i\in\mathcal{I}^\star_t}\frac{\bar{m}^i_t}{\alpha(t)}\big)}{\Phi_{\mathcal{J}^\star_t\cup\{-1,0\},\rho^\star_t,\sigma^\star_t}\big(Z^{j,\mathcal{I}^\star_t}_t(t), j\in\mathcal{J}^\star_t\cup\{-1,0\}\big)}\right),\ t\in\mathbb{R}_+. \ ∎$$

Note that the processes $\gamma^j,\widetilde{\gamma}^j$ and $\beta^k,\widetilde{\beta}^k$ are càdlàg. The next result shows that the process β^k (and consequently $\widetilde{\beta}$) is linked with $\bar{\beta}^k$ through the process ν.

Lemma 3.6. *For $k\in N$,*

$$\int_0^t\widetilde{\beta}^k_s ds = \int_0^t\bar{\beta}^k_s ds + \langle B^k,\nu\rangle_t,\ t\in[0,\tau_{-1}\wedge\tau_0].$$

Proof. Notice that B^k is a continuous process. By the Jeulin–Yor formula (see e.g. Dellacherie, Maisonneuve, and Meyer (1992, no 77 Remarques b))),

$$B^k_t - \int_0^t\bar{\beta}^k_s ds - \langle B^k,\nu\rangle_t,\ t\in[0,\tau_{-1}\wedge\tau_0],$$

defines a $\mathbb{G}$ local martingale. But, acccording to Theorem 2.4, the drift of B^k in $\mathbb{G}$ is $\int_0^t\beta^k_s ds, t\geq 0$. We conclude that

$$\int_0^t\widetilde{\beta}^k_s ds = \int_0^t\beta^k_s ds = \int_0^t\bar{\beta}^k_s ds + \langle B^k,\nu\rangle_t$$

for $t \in [0, \tau_{-1} \wedge \tau_0]$. ∎

Knowing the $\mathbb{F}$ reductions $\widetilde{\beta}^k$ and $\widetilde{\gamma}^j$ of β^k and γ^j, in view of the martingale representation property in $\mathbb{F}$, accounting also for the avoidance of $\tau_{-1} \wedge \tau_0$ and $\tau_j, j \in N^\star$, the strategy for constructing an invariance probability measure $\mathbb{P}$ becomes clear. It is enough to find a probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_T$ (given a constant $T > 0$) such that the $(\mathbb{F}, \mathbb{P})$ drift of $B^k, k \in N$, is $\widetilde{\beta}^k$ and the $(\mathbb{F}, \mathbb{P})$ compensator of $\tau^j, j \in N^\star$, has the density process $\widetilde{\gamma}^j$.

To implement this idea, the following estimates will be useful.

Lemma 3.7. *There exists a constant $C > 0$ such that*

$$\langle \nu \rangle_t \leq C(\sum_{i \in N} \sup_{0<s\leq t} |m^i_s|^2 + 1)t \tag{3.6}$$

and for $0 \leq r \leq t$ and $j \in N^\star$

$$\begin{aligned} \widetilde{\gamma}^j_r \vee \bar{\gamma}^j_r &\leq C(\sum_{i \in N} \sup_{0<s\leq t} |m^i_s| + 1), \\ \widetilde{\gamma}^j_r \ln(\widetilde{\gamma}^j_r \vee \bar{\gamma}^j_r) &\leq C \sum_{i \in N} \sup_{0<s\leq t} (|m^i_s| + 1) \ln(|m^i_s| + 1). \end{aligned} \tag{3.7}$$

Proof. Applying Lemma A.2 to the formula (3.3) and noting that the function α, continuous and positive, is bounded away from 0 on $[0, T]$, we obtain, for positive constants C that may change from place to place,

$$\langle \nu \rangle_t \leq C \int_0^t (\sum_{I \subseteq N} \sum_{j \in N \setminus I} |Z^{j,I}_s(s)| + 1)^2 ds \leq C(\sum_{I \subseteq N} \sum_{j \in N \setminus I} \sup_{0<s\leq t} |Z^{j,I}_s(s)| + 1)^2 t,$$

which yields (3.6). Applying Lemma A.2 to the formulas (3.3) for $\bar{\gamma}^j$ and (3.5) for $\widetilde{\gamma}^j$, we obtain the first line in (3.7)), whence the second line follows from

$$\begin{aligned} \widetilde{\gamma}^j_r \ln(\widetilde{\gamma}^j_r \vee \bar{\gamma}^j_r) &\leq C(\max_{i \in N} \sup_{0<s\leq t} |m^i_s| + 1) \ln\left(C(\max_{i \in N} \sup_{0<s\leq t} |m^i_s| + 1)\right) \\ &= \max_{i \in N} \sup_{0<s\leq t} C(|m^i_s| + 1) \ln(C|m^i_s| + 1). \end{aligned}$$ ∎

Notice that the processes $\bar{\gamma}^j$ are positive. Consider the $\mathbb{F}$ local martingale $\mu = \nu + \sum_{j \in N^\star} (\frac{\widetilde{\gamma}^j_-}{\bar{\gamma}^j_-} - 1) \centerdot \bar{M}^j$.

Lemma 3.8. *The Doléans-Dade exponential $\mathcal{E}(\mu)$ is a true $(\mathbb{F}, \mathbb{Q})$ martingale.*

Proof. Following Lepingle and Mémin (1978, Theorem III.1), we consider

$$\sum_{j \in N^\star} \left((1 + (\frac{\widetilde{\gamma}^j_{\tau_j-}}{\bar{\gamma}^j_{\tau_j-}} - 1)) \ln(1 + (\frac{\widetilde{\gamma}^j_{\tau_j-}}{\bar{\gamma}^j_{\tau_j-}} - 1)) - (\frac{\widetilde{\gamma}^j_{\tau_j-}}{\bar{\gamma}^j_{\tau_j-}} - 1)\right) \mathbb{1}_{\{\tau_j \leq t\}},$$

and its $\mathbb{F}$ predictable dual projection

$$\begin{aligned} M_t &= \sum_{j \in N^\star} \int_0^t \left((1 + (\frac{\widetilde{\gamma}^j_{s-}}{\bar{\gamma}^j_{s-}} - 1)) \ln(1 + (\frac{\widetilde{\gamma}^j_{s-}}{\bar{\gamma}^j_{s-}} - 1)) - (\frac{\widetilde{\gamma}^j_{\tau_j-}}{\bar{\gamma}^j_{s-}} - 1)\right) \bar{\gamma}^j_{s-} ds \\ &= \sum_{j \in N^\star} \int_0^t \left(\widetilde{\gamma}^j_{s-} (\ln(\widetilde{\gamma}^j_{s-}) - \ln(\bar{\gamma}^j_{s-})) - (\widetilde{\gamma}^j_{\tau_j-} - \bar{\gamma}^j_{s-})\right) ds. \end{aligned}$$

Combining Lemma 3.7 and Lemma A.3, we prove that $e^{\frac{1}{2}\langle \nu \rangle_t + M_t}$ is $\mathbb{Q}$ integrable for a sufficiently small $t = t_0 > 0$. According to Lepingle and Mémin (1978, Theorem III.1), we conclude that $\mathbb{E}[\mathcal{E}(\mu)_{t_0}] = 1$. The same argument applied with the conditional probability $\mathbb{Q}[\cdot|\mathcal{F}_{t_0}]$ instead of $\mathbb{Q}$ proves that $\mathbb{E}[\mathcal{E}(\mu)_{2t_0}|\mathcal{F}_{t_0}] = 1$. Iterating, we arrive at $\mathbb{E}[\mathcal{E}(\mu)_{kt_0}] = 1$ for any integer $k > 0$. ∎

3.3. **The invariance probability measure**

We have proved that $\mathcal{E}(\mu)$ is an $(\mathbb{F},\mathbb{Q})$ true martingale. We can then define a new probability measure $\mathbb{P}=\mathcal{E}(\mu).\mathbb{Q}$ on $\mathcal{F}_T$.

Theorem 3.1. *The probability measure $\mathbb{P}$ is an invariance probability measure for the DGC model $(\tau_{-1}\wedge\tau_0,\mathbb{F},\mathbb{G},\mathbb{Q})$ on the horizon $[0,T]$, for any constant $T>0$.*

Proof. By the Girsanov theorem, the intensity of τ_i, $i\in N^\star$, in $\mathbb{F}$ under $\mathbb{P}$ is $\widetilde{\gamma}^i$, while the drift of $B^k, k\in N$, in $\mathbb{F}$ under $\mathbb{P}$ is $(\bar{\beta}^k\centerdot\boldsymbol{\lambda}+\langle B^k,\nu\rangle)$.

Given a constant $T>0$, let us prove that the probability measure $\mathbb{P}$ such that $\frac{d\mathbb{P}}{d\mathbb{Q}}=\mathcal{E}(\mu)$ is an invariance probability measure for the quadruplet $(\tau_{-1}\wedge\tau_0,\mathbb{F},\mathbb{G},\mathbb{Q})$. According to Crépey and Song (2017, Corollary C.1), we only need to consider the locally bounded $(\mathbb{F},\mathbb{P})$ local martingales P in the condition (A). We write

$$\widehat{W}^k=B^k-\bar{\beta}^k\centerdot\boldsymbol{\lambda}-\langle B^k,\nu\rangle\ k\in N,\ \text{and}\ \widehat{M}^j=\mathbb{1}_{[\tau^j,\infty)}-\widetilde{\gamma}^j\centerdot\boldsymbol{\lambda},\ k\in N,\ j\in N^\star.$$

Thanks to Lemma 3.6, the stopped process

$$(\widehat{W}^k)^{\tau_{-1}\wedge\tau_0-}=(B^k-\beta^k\centerdot\boldsymbol{\lambda})^{\tau_{-1}\wedge\tau_0}=(W^k)^{\tau_{-1}\wedge\tau_0}$$

is a $(\mathbb{G},\mathbb{Q})$ local martingale. The $(\mathbb{G},\mathbb{Q})$ local martingale property of

$$(\widehat{M}^j)^{\tau_{-1}\wedge\tau_0-}=(\mathbb{1}_{[\tau^j,\infty)}-\gamma^j\centerdot\boldsymbol{\lambda})^{\tau_{-1}\wedge\tau_0}=(M^j)^{\tau_{-1}\wedge\tau_0}$$

(cf. Lemma 3.5) is clear.

As we did in Lemma 3.1 under the probability $\mathbb{Q}$, it can be proven that the family of processes $\widehat{W}^k$, $k\in N$, and $\widehat{M}^j$, $j\in N^\star$, has the martingale representation property in the filtration $\mathbb{F}$ under $\mathbb{P}$. Hence any $(\mathbb{F},\mathbb{P})$ local martingale P is an stochastic integral in $\mathbb{F}$ of the processes $\widehat{W}^k$ and of $\widehat{M}^j$ under the probability measure $\mathbb{P}$. The natural idea is to say, then, $P^{\tau_{-1}\wedge\tau_0-}$ is the stochastic integral in $\mathbb{G}$ of the processes $(W^k)^{\tau_{-1}\wedge\tau_0}$ and of $(M^j)^{\tau_{-1}\wedge\tau_0}$ under the probability $\mathbb{Q}$, so that $P^{\tau_{-1}\wedge\tau_0-}$ itself is a $(\mathbb{G},\mathbb{Q})$ local martingale. However, knowing the discussion in Jeulin and Yor (1979) about "*faux amis*" regarding enlargement of filtration and stochastic integrals, we have to be careful. More precisely, we need to distinguish between the stochastic integral in the sense of semimartingales and the stochastic integral in the sense of local martingales, recalling from Émery (1980) (cf. also Delbaen and Schachermayer (1994, Theorem 2.9)) that a stochastic integral in the sense of semimartingales with respect to a local martingale need not be a local martingale.

We can argue as follows. We consider separately the cases of continuous and purely discontinuous P. When P is a continuous $(\mathbb{F},\mathbb{P})$ local martingale, P is the $(\mathbb{F},\mathbb{P})$ stochastic integral, in the sense of local martingales, of an $\mathbb{F}$ predictable $(n+2)$ dimensional process $H=(H^k,k\in N)$ with respect to the $(n+2)$ dimensional process $(\widehat{W}^k,k\in N)$. Since the matrix $(\frac{d\langle\widehat{W}^k,\widehat{W}^{k'}\rangle_t}{dt},k,k'\in N)$ is uniformly positive-definite, for every $k\in N$, H^k is individually $(\mathbb{F},\mathbb{P})$ integrable with respect to the one dimensional Brownian motion $\widehat{W}^k$ in the sense of local martingales (see Jacod and Shiryaev (2003, Chapter III, Section 4)). Moreover, as $\langle H^k\centerdot\widehat{W}^k,\nu\rangle$ exists under $\mathbb{P}$ (noting $H^k\centerdot\widehat{W}^k$ is continuous), by the Girsanov theorem (see He, Wang, and Yan (1992, Theorem 12.13)), $\langle H^k\centerdot\widehat{W}^k,\nu\rangle$ is of locally integrable total variation under $\mathbb{Q}$. Hence H^k is integrable with respect to $\langle\widehat{W}^k,\nu\rangle$ under $\mathbb{Q}$. Moreover the $(\mathbb{F},\mathbb{Q})$ martingale part of $\widehat{W}^k$ is an $(\mathbb{F},\mathbb{Q})$ Brownian motion, hence the $(\mathbb{F},\mathbb{Q})$ integrability of H^k against this martingale part reduces to the a.s. finiteness of $(H^k)^2\centerdot\boldsymbol{\lambda}$, which holds under $\mathbb{P}$ and therefore under $\mathbb{Q}$. In sum, H^k is $(\mathbb{F},\mathbb{Q})$ integrable with respect to $\widehat{W}^k$ in the sense of semimartingales. As the hypothesis (H') holds between $\mathbb{F}\subseteq\mathbb{G}$ under $\mathbb{Q}$ (see after (3.2)), Jeulin (1980, Proposition 2.1) implies that H^k is $(\mathbb{G},\mathbb{Q})$ integrable with respect to $\widehat{W}^k$ in the sense of semimartingales. By Jeanblanc and Song (2013, Lemma 2.1), the stochastic integrals in the sense of the $(\mathbb{F},\mathbb{Q})$ semimartingales and in the sense of the $(\mathbb{G},\mathbb{Q})$ semimartingales are the same, hence $P=\sum_{k\in N}H^k\centerdot\widehat{W}^k$ also holds in the sense of $(\mathbb{G},\mathbb{Q})$ semimartingales. By Lemma 3.6, $(\widehat{W}^k)^{\tau_{-1}\wedge\tau_0}=(W^k)^{\tau_{-1}\wedge\tau_0}$ is a $(\mathbb{G},\mathbb{Q})$ local martingale. Applying He, Wang, and Yan (1992, Theorem 9.16), we conclude that, in fact, H^k is $(\mathbb{G},\mathbb{Q})$

integrable with respect to $(\widehat{W}^k)^{\tau_{-1}\wedge\tau_0}$ in the sense of local martingales. Hence

$$P^{\tau_{-1}\wedge\tau_0-}=\sum_{k\in N}(H^k\centerdot\widehat{W}^k)^{\tau_{-1}\wedge\tau_0-}=\sum_{k\in N}(H^k\centerdot W^k)^{\tau_{-1}\wedge\tau_0}$$

is a $(\mathbb{G},\mathbb{Q})$ local martingale.

Consider now the case of P purely discontinuous. Without loss of generality we suppose that the locally bounded process P is in fact bounded. Then, P is the $(\mathbb{F},\mathbb{P})$ stochastic integral (in the sense of local martingale) of an $\mathbb{F}$ predictable n dimensional process $K=(K^j,j\in N^\star)$ with respect to the n dimensional process $(\widehat{M}^j,j\in N^\star)$. The processes $\widehat{M}^j,j\in N^\star$, have disjoint jump times with jump amplitude 1. This implies that K^j is integrable with respect to $\widehat{M}^j$ individually. Moreover, as P is bounded, the random variables $K_{\tau_j},j\in N^\star$, are bounded, hence K itself is bounded (cf. He et al. (1992, Theorem 7.23)). As a consequence, K is automatically $(\mathbb{G},\mathbb{Q})$ integrable with respect to $(\widehat{M}^j,j\in N^\star)$ in the sense of local martingale. By Jeanblanc and Song (2013, Lemma 2.1) again,

$$P^{\tau_{-1}\wedge\tau_0-}=\sum_{j\in N^\star}(K^j\centerdot\widehat{M}^j)^{\tau_{-1}\wedge\tau_0-}=\sum_{j\in N^\star}(K^j\centerdot M^j)^{\tau_{-1}\wedge\tau_0},$$

which is a $(\mathbb{G},\mathbb{Q})$ local martingale. ∎

3.4. **Alternative Proof of the Condition (A)**

Theorem 3.1 yields an explicit construction of the invariance probability measure $\mathbb{P}$ in the DGC model. If we only want to establish the condition (A), i.e. the existence of $\mathbb{P}$, a shorter proof is available based on the sufficiency condition of Crépey and Song (2017, Theorem 5.1).

Lemma 3.9. *The $(\mathbb{G},\mathbb{Q})$ intensity γ of the random time $\tau_{-1}\wedge\tau_0$ is given by*

$$\gamma=\mathbb{1}_{[0,\tau_{-1}\wedge\tau_0]}(\gamma^{-1}+\gamma^0).$$

***Proof*.** This follows from, for example, Crépey and Song (2016a, Lemma 6.2). ∎

Theorem 3.2. *The condition (A) holds in the DGC model $(\tau_{-1}\wedge\tau_0,\mathbb{F},\mathbb{G},\mathbb{Q})$.*

***Proof*.** Given a constant horizon $T>0$, according to Crépey and Song (2017, Theorem 5.1), we only need to prove the exponential integrability of $\int_0^{\tau\wedge T}\gamma_s ds$, which can be done similarly to the proof of Lemma 3.8. ∎

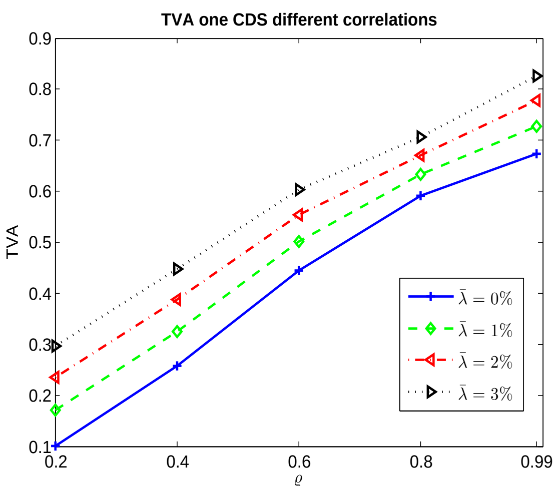
4. Wrong Way Risk

As visible in (2.11), the default intensities of the surviving names spike at defaults in the DGC model. This is very much related to the departure from the immersion property in this model, i.e. the fact that the invariance probability measure $\mathbb{P}$ is not equal to the pricing measure $\mathbb{Q}$ on $\mathcal{F}_T$. This 'wrong way risk' feature (cf. Crépey and Song (2016b)) makes the DGC model appropriate for dealing with counterparty risk on credit derivatives, notably portfolios of CDS contracts traded between a bank and its counterparty, respectively labeled as -1 and 0, and bearing on reference firms $i=1,\dots,n$.

To illustrate this numerically, in this concluding section of the paper, we study the valuation adjustment accounting for counterparty and funding risks (total valuation adjustment TVA) embedded in one CDS between a bank and its counterparty on a third reference firm.

In Figure 1, the left graph shows the TVA computed as a function of the correlation parameter ϱ in a DGC model of the three credit names (hence $n=1$): the bank, its counterparty and the reference credit name of the CDS. The different curves correspond to different levels of credit spread $\bar{\lambda}$ of bank: the higher $\bar{\lambda}$, the higher the funding costs for the bank, resulting in higher TVAs. All the TVA numbers are computed by a Monte Carlo scheme dubbed "FT scheme of order 3" in Crépey and Nguyen (2016, Section 6.1). FT refers to Fujii and Takahashi (2012a,2012b). The numerical parameters are set as in Crépey and Nguyen (2016, Section 6.1), to which we refer the reader for a complete description of the CDS contract, of the FT numerical scheme and of other numerical experiments involving CDS portfolios (as opposed to a single contract here).

The right panel of Figure 1 shows the analog of the left graph, but in a fake DGC model, where we deliberately ignore the impact of the default of the counterparty in the valuation of the CDS at time $\tau_{-1} \wedge \tau_0$ (technically, in the notation of Crépey and Song (2016a, Equation (6.7)), we replace $(\widetilde{P}^e_t + \widetilde{\Delta}^e_t)$ by P_{t-} in the coefficient $\widehat{f}$), in order to kill the wrong-way risk feature of the DGC model. We can see from the figure that, for large ϱ, the corresponding fake TVA numbers are five to ten times smaller than the "true" TVA levels that can be seen in the left panel. In addition of being much smaller for large ϱ, the fake DGC TVA numbers in the right panel are mostly decreasing with ϱ. This shows that the wrong-way risk feature of the DGC model is indeed responsible for the "systemic" increasing pattern observed in the left panel.



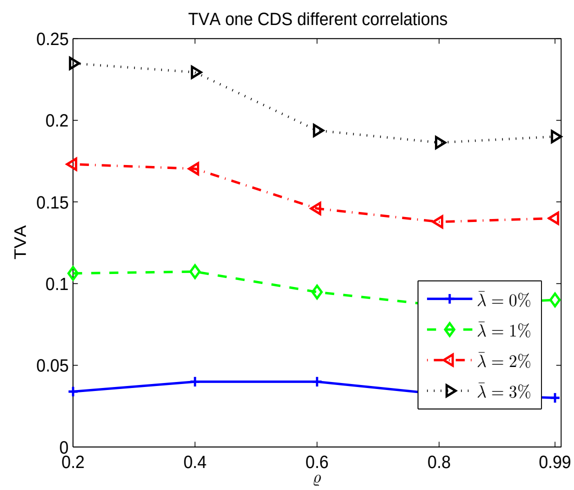


Figure 1. *Left*: TVA on one CDS as a function of the correlation parameter ϱ and for different bank credit spreads $\bar\lambda$, in a DGC model with three names: the bank, its counterparty and the reference credit name of the CDS. *Right*: Analog results in a fake DGC model without wrong-way-risk.

A. Gaussian Estimates

In this appendix we derive the Gaussian estimates that are used in the proofs of Lemmas 3.7 and 3.8.

Lemma A.1. *Given a positive decreasing continuously differentiable function Γ on $\mathbb{R}_+$ such that*

$$\int_{\mathbb{R}_+} t^d \Gamma(t) dt < \infty \;\; \text{and} \;\; \lim_{t\uparrow\infty} t^{d-1}\Gamma(t) \to 0,$$

for some integer $d \geq 0$, we write $g(y) = -\frac{\Gamma'(y)}{\Gamma(y)}$, $G(y) = \int_y^\infty t^d \Gamma(t)dt$. Let $\bar y \geq 0$ and $\alpha, \epsilon > 0$.
(i) *If $g(y) \geq \alpha y$ for $y > \bar y$, then*

$$G(y) \leq (\frac{1}{\alpha} + \epsilon) y^{d-1}\Gamma(y) \quad \text{for} \quad y > \bar y \vee \sqrt{|d-1|(\frac{1}{\epsilon\alpha^2} + \frac{1}{\alpha})}.$$

(ii) *If $g(y) \leq \alpha y$ for $y > \bar y$, then*

$$G(y) \geq (\frac{1}{\alpha} - \epsilon) y^{d-1}\Gamma(y) \quad \text{for} \quad y > \bar y \vee \sqrt{|d-1|(\frac{1}{\epsilon\alpha^2} - \frac{1}{\alpha})}.$$

Proof. (i) For every positive continuously differentiable function φ on $(0,+\infty)$,

$$\begin{aligned}(G(y)-\varphi(y)\Gamma(y))' &= -y^d\Gamma(y)-\varphi'(y)\Gamma(y)+\varphi(y)g(y)\Gamma(y)\\ &= (\varphi(y)g(y)-y^d-\varphi'(y))\Gamma(y)\geq(\alpha y\varphi(y)-y^d-\varphi'(y))\Gamma(y)\end{aligned}$$

for $y\geq\bar{y}$. For $\varphi(y)=(\frac{1}{\alpha}+\epsilon)y^{d-1}$,

$$\begin{aligned}&\alpha y\varphi(y)-y^d-\varphi'(y)=(1+\epsilon\alpha)y^d-y^d-(\frac{1}{\alpha}+\epsilon)(d-1)y^{d-2}=\\ &\epsilon\alpha y^d-(\frac{1}{\alpha}+\epsilon)(d-1)y^{d-2}=(\epsilon\alpha y^2-(\frac{1}{\alpha}+\epsilon)(d-1))y^{d-2}.\end{aligned}$$

Therefore, if $y>\bar{y}\vee\sqrt{|d-1|(\frac{1}{\epsilon\alpha^2}+\frac{1}{\alpha})}$, then $(G(y)-\varphi(y)\Gamma(y))'\geq\alpha y\varphi(y)-y^d-\varphi'(y)\geq 0$. But $\lim_{y\uparrow\infty}(G(y)-\varphi(y)\Gamma(y))=0$, hence $G(y)-\varphi(y)\Gamma(y)\leq 0$.

(ii) We again begin with

$$\begin{aligned}(G(y)-\varphi(y)\Gamma(y))' &= (\varphi(y)g(y)-y^k-\varphi'(y))\Gamma(y)\\ &\leq (\varphi(y)(\alpha y+\alpha')-y^k-\varphi'(y))\Gamma(y)\end{aligned}$$

for $y\geq\bar{y}$. We conclude as in (i) based on $\varphi(y)=(\frac{1}{\alpha}-\epsilon)y^{k-1}$, assuming $\frac{1}{\alpha}>\epsilon$ (otherwise (ii) obviously holds). ∎

We use the notation (2.5) as well as Φ and ϕ for the standard normal survival and density functions. By a first application of Lemma A.1, to the standard normal density $\Gamma=\phi$, we recover the following classical inequalities on $\psi=\frac{\phi}{\Phi}$: for any constant $c>1$,

$$c^{-1}y\leq\psi(y)\leq cy,\ y>y_0, \tag{A.1}$$

for some $y_0>0$ depending on c. The following estimate, where c and y_0 are as here, can be seen as a multivariate extension of the right hand side inequality in (A.1).

Lemma A.2. *There exist constants a and b such that, for every $j\in J$,*

$$0\leq\psi^j_{\rho,\sigma}(\mathbf{z})\leq a+b||\mathbf{z}||_\infty. \tag{A.2}$$

Proof. By conditional independence of the components of a multivariate Gaussian vector with homogeneous pairwise correlation ϱ, we have $\Phi_{\rho,\sigma}(\mathbf{z})=\int_{\mathbb{R}}\Gamma(y)dy$, where $\Gamma(y)=\prod_{l\in J}\Phi\left(\frac{z_l+\sigma\sqrt{\rho}\,y}{\sigma\sqrt{1-\rho}}\right)\phi(y)$. Hence

$$\psi^j_{\rho,\sigma}(\mathbf{z})=\frac{1}{\sigma\sqrt{1-\rho}}\int_{\mathbb{R}}w_{\rho,\sigma}(\mathbf{z},y)\psi\left(\frac{z_j+\sigma\sqrt{\rho}\,y}{\sigma\sqrt{1-\rho}}\right)dy, \tag{A.3}$$

where $w_{\rho,\sigma}(\mathbf{z},y)=\frac{\Gamma(y)}{\Phi_{\rho,\sigma}(\mathbf{z})}$. Straightforward computations yield

$$g(t)=-\frac{\Gamma'(t)}{\Gamma(t)}=\sum_{l\in J}\psi(\frac{z_l+\sigma\sqrt{\rho}t}{\sigma\sqrt{1-\rho}})\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}+t\geq t,$$

whereas for $t>\max_{l\in J}\frac{1}{\sigma\sqrt{\rho}}(\sigma\sqrt{1-\rho}y_0-z_l)$ and $t>\frac{1}{\sigma\sqrt{\rho}}\max_{l\in J}z_l$, we have

$$g(t)\leq\sum_{l\in J}c\frac{z_l+\sigma\sqrt{\rho}t}{\sigma\sqrt{1-\rho}}\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}+t\leq\bar{\alpha}t,$$

with $\bar\alpha = \sum_{l\in J} 2c\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}\frac{\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}} + 1 \ge 1$. Applying Lemma A.1(i) with $d = 1, \alpha = 1$ and $\epsilon = 1$, respectively (ii) with $d = 0$, $\alpha = \bar\alpha$ and $\epsilon = \frac{1}{2\bar\alpha}$, yields

$$\int_y^\infty t\Gamma(t)dt \le 2\Gamma(y),\ \ y > 0, \quad \text{respectively} \int_y^\infty \Gamma(t)dt \ge \frac{1}{2\bar\alpha y}\Gamma(y),\ \ y > \bar y \vee \frac{1}{\sqrt{\bar\alpha}},$$

where $\bar y = \frac{1}{\sigma\sqrt{\rho}}\max_{l\in J}|z_l| + \frac{1}{\sigma\sqrt{\rho}}\sigma\sqrt{1-\rho}y_0$. Thus, setting $y_1 = \bar y + 1 = \frac{1}{\sigma\sqrt{\rho}}\max_{l\in J}|z_l| + \frac{1}{\sigma\sqrt{\rho}}\sigma\sqrt{1-\rho}y_0 + 1$,

$$\begin{aligned}\int_0^\infty t\Gamma(t)dt &= \int_0^{y_1} t\Gamma(t)dt + \int_{y_1}^\infty t\Gamma(t)dt \le y_1\int_0^{y_1}\Gamma(t)dt + 2\Gamma(y_1)\\ &\le y_1\int_0^{y_1}\Gamma(t)dt + 4\bar\alpha y_1\int_{y_1}^\infty \Gamma(t)dt \le (1+4\bar\alpha)\int_0^\infty \Gamma(t)dt,\end{aligned}$$

i.e.

$$\int_0^\infty t w_{\rho,\sigma}(\mathbf{z},t)dt \le (1+4\bar\alpha)y_1. \tag{A.4}$$

Now, by (A.3) and the right hand side inequality in (A.1),

$$\begin{aligned}0 &\le \sigma\sqrt{1-\rho}\,\psi^j_{\rho,\sigma}(\mathbf{z})\\ &\le \int_{\mathbb{R}}\left(\frac{1}{\Phi(y_0)}\mathbb{1}_{\{\frac{z_j+\sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}\le y_0\}} + c\frac{z_j+\sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}\mathbb{1}_{\{\frac{z_j+\sigma\sqrt{\rho}y}{\sigma\sqrt{1-\rho}}> y_0\}}\right)w_{\rho,\sigma}(\mathbf{z},y)dy\\ &= \left(\frac{1}{\Phi(y_0)} + \frac{cz_j}{\sigma\sqrt{1-\rho}}\right) + \frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}\int_{\mathbb{R}}\mathbb{1}_{\{\sigma\sqrt{\rho}y>\sigma\sqrt{1-\rho}y_0 - z_j\}}y w_{\rho,\sigma}(\mathbf{z},y)dy\\ &\le \left(\frac{1}{\Phi(y_0)} + \frac{cz_j}{\sigma\sqrt{1-\rho}}\right) + \frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}\int_0^\infty y w_{\rho,\sigma}(\mathbf{z},y)dy,\end{aligned} \tag{A.5}$$

so that by substitution of (A.4) into (A.5)

$$0 \le \sigma\sqrt{1-\rho}\,\psi^j_{\rho,\sigma}(\mathbf{z}) \le \left(\frac{1}{\Phi(y_0)} + \frac{cz_j}{\sigma\sqrt{1-\rho}}\right) + \frac{c\sigma\sqrt{\rho}}{\sigma\sqrt{1-\rho}}(1+4\bar\alpha)y_1. \blacksquare$$

Lemma A.3. *Let $m_t = \int_0^t \varsigma(s)dB_s$, where B is a a univariate standard Brownian motion and ς is a square integrable function with unit L^2 norm. For any constant $q > 0$, $e^{q\sup_{0\le s\le t} m_s^2}$ is integrable for sufficiently small t.*

Proof. The process $(m_t)_{t\ge0}$ is equal in law to a time changed Brownian motion $(W_{\bar t})_{t\ge0}$, where W is a a univariate standard Brownian motion and $\bar t = \int_0^t \varsigma^2(s)ds$ goes to 0 with t. Thus, it suffices to show the result with m replaced by W. Let r_t be the density function of the law of $\sup_{0\le s\le t}|W_s|$ and let $R_t(y) = \int_y^\infty r_t(x)dx, y > 0$, so that

$$\mathbb{E}[e^{q\sup_{0\le s\le t}W_s^2}] = \int_0^\infty e^{qy^2}r_t(y)dy = -[R_t(y)e^{qy^2}]_0^\infty + 2q\int_0^\infty yR_t(y)e^{qy^2}dy \tag{A.6}$$

and (using the reflection principle of the Brownian motion)

$$\begin{aligned}R_t(y) &= \mathbb{Q}[\sup_{0\le s\le t}(W_s^+ + W_s^-) > y] \le \mathbb{Q}[\sup_{0\le s\le t}W_s^+ > \tfrac{y}{2}] + \mathbb{Q}[\sup_{0\le s\le t}W_s^- > \tfrac{y}{2}]\\ &= 2\mathbb{Q}[\sup_{0\le s\le t}W_s > \tfrac{y}{2}] = 2\mathbb{Q}[|W_t| > \tfrac{y}{2}] = 2\mathbb{Q}[|W_1| > \tfrac{y}{2\sqrt t}] = 4\Phi(\tfrac{-y}{2\sqrt t}),\end{aligned}$$

where by the left hand side in (A.1)

$$\Phi(\frac{y}{2\sqrt{t}})\frac{y}{2\sqrt{t}} \leq c\phi(\frac{y}{2\sqrt{t}}) = \frac{c}{\sqrt{2\pi}}e^{-\frac{y^2}{8t}}, \ \ \frac{y}{2\sqrt{t}} > y_0.$$

Therefore, for $\frac{1}{8t} > q$, both terms are finite in the right hand side of (A.6), which shows the result. ∎

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