CALIBRATION OF THE LOCAL VOLATILITY IN A GENERALIZED BLACK–SCHOLES MODEL USING TIKHONOV REGULARIZATION

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Abstract. Following an approach introduced by Lagnado and Osher (1997), we study Tikhonov regularization applied to an inverse problem important in mathematical finance, that of calibrating, in a generalized Black–Scholes model, a local volatility function from observed vanilla option prices.

We first establish $W_p^{1,2}$ estimates for the Black–Scholes and Dupire equations with measurable ingredients. Applying general results available in the theory of Tikhonov regularization for ill-posed nonlinear inverse problems, we then prove the stability of this approach, its convergence towards a minimum norm solution of the calibration problem (which we assume to exist), and discuss convergence rates issues.

Key words. Options, calibration, ill-posed nonlinear inverse problem, Tikhonov regularization, parameter estimation, $W_p^{1,2}$ estimates

AMS subject classifications. 35K15, 35Q80, 35R05, 35R30

1. Introduction. A quantity of fundamental importance to the trading of options on a stock S, is the stochastic component in the evolution of the stock price, the so-called *volatility.* Obtaining estimates for the volatility is a major challenge for market finance. Unlike historical estimates of the volatility, based upon observations of the time-series of the stock price, calibration estimates rely upon the anticipations of the trading agents reflected in the prices of the traded option products derived from S. We consider in this article Tikhonov regularization applied to a widely studied inverse problem in mathematical finance, that of calibrating a local volatility function from a given set of option prices, in a generalized Black–Scholes model.

This calibration problem has received intensive study in the last ten years, see for instance [19, 17, 18, 36, 1, 11, 8, 35, 2, 29, 24, 7, 14, 15, 4] and references therein. Notable approaches include entropy regularization (Avellaneda *et al* [2]) or parametrix expansion (Bouchouev and Isakov [8]). In this paper, we shall focus upon the Tikhonov regularization method, following an approach introduced by Lagnado and Osher [29]. Jackson, Süli and Howison [24] have devised an implementation of this method with splines. Bodurtha and Jermakyan use linearization [7]. However, while most previous approaches adopt a numerical and empirical point of view, our aim is to establish a rigorous theoretical ground for this inverse problem, in a Partial Differential Equation framework.

Work corresponding to a first stage of this research has been published in my PhD Thesis [14, Part IV] (in French), while a preliminary version of this article has been published as a CMAP Internal Research Report [15]. A further article will address an implementation of the method in a trinomial tree (explicit finite differences) setting, and report numerical experiments illustrating the stability of the local volatility function thus calibrated [16].

2. Preliminaries. In this section, we will give an informal presentation of the calibration problem and of the Tikhonov rwe shall ation method, provide an overview of the paper and define the main notations and general assumptions.

2.1. Generalized Black–Scholes model. In market finance, a European *call* (respectively *put*) option with maturity date T and exercise price K, on an underlying

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asset S, denotes a right to buy (respectively sell), at price K, a unit of S at time T. Let us then consider a theoretical financial market, with two traded assets: cash, with constant interest-rate r, and a risky stock, with diffusion price process

$$dS_t = S_t(\rho(t, S_t)dt + \sigma(t, S_t)dW_t) \ , \quad t > t_0 \ ; \quad S_{t_0} = S_0 \ .$$

Here W means a standard Brownian motion. Moreover, the stock is assumed to yield a continuously compounded dividend at constant rate q. Suppose finally, that the market is liquid, non arbitrable, and perfect. These assumptions mean, respectively, that first, there are always buyers and sellers, second, there can be no opportunity that a riskless investment can earn more than the interest-rate of the economy r, and third, there are no restrictions of any kind on the sales, neither transaction costs. Under these assumptions the market is complete. This means that any option can be duplicated by a portfolio of cash and stock. Moreover, a European call/put on S has a theoretical fair price within the model, that we shall denote by $\Pi^{+/-}_{T,K}(t_0, S_0; r, q, a)$, where $a \equiv \sigma^2/2$, and

$$\Pi^{+/-}_{T,K}(t_0, S_0; r, q, a) = e^{-r(T-t_0)} \mathrm{E}^{t_0,S_0}_P (S_T - K)^{+/-} \ . \tag{2.1}$$

Here P denotes the so-called *risk-neutral* probability, under which

$$dS_t = S_t((r-q)dt + \sigma(t, S_t)dW_t) \ , \quad t > t_0 \ ; \quad S_{t_0} = S_0 \ . \tag{2.2}$$

Alternatively to the probabilistic representation (2.1), the prices $\Pi^{+/-}$ can be given as the solution to a differential equation. One can use either the *Black–Scholes* backward parabolic equation, in the variables (t_0, S_0), which is

$$\begin{cases} -\partial_t \Pi - (r-q)S\partial_S \Pi - a(t,S)S^2\partial^2_{S^2}\Pi + r\Pi \ = \ 0, \quad t < T \\ \Pi|_T \ \equiv \ (S-K)^{+/-} \ , \end{cases} \tag{2.3}$$

or the *Dupire* forward parabolic equation, in the variables (T, K), given by

$$\begin{cases} \partial_T \Pi - (q-r)K\partial_K \Pi - a(T,K)K^2\partial^2_{K^2}\Pi + q\Pi \ = \ 0, \quad T > t_0 \\ \Pi|_{t_0} \ \equiv \ (S_0-K)^{+/-} \ . \end{cases} \tag{2.4}$$

We will show in lemma 4.1 and theorem 4.3 that equations (2.1) or (2.3)–(2.4) hold for an arbitrary measurable, positively bounded local volatility function a. However, let us give a less formal insight by recalling the Black–Scholes seminal analysis [6], valid in the special case where the volatility depends on time alone. We consider a self-financing portfolio, short one option and long $\partial_S\Pi$ shares of the underlying stock. The value V of the risky component of the portfolio then evolves as

$$\begin{aligned} dV_t &= -d\Pi(t, S_t) + \partial_S\Pi(dS_t + qS_t dt) \\ &= -(\partial_t\Pi - qS\partial_S\Pi + aS^2\partial^2_{S^2}\Pi)dt \ , \end{aligned}$$

from Itô's lemma. Since V has a deterministic rate of return, absence of opportunity of arbitrage implies that this rate equals the riskless interest-rate r. Otherwise said,

$$-\partial_t\Pi + qS\partial_S\Pi - aS^2\partial^2_{S^2}\Pi = r(-\Pi + S\partial_S\Pi) \ ,$$

whence (2.3). As for (2.1), it can be viewed as the Feynman–Kac representation for the solution of (2.3). Notice that this analysis does not rely on the specific character of the payoff of the call or put option. However, the oposite is true for (2.4). It is indeed, as noticed by Dupire [19], a Fokker–Planck equation integrated twice with respect to the space variable K, using moreover the formal identity

$$\partial^2_{K^2}(S_0 - K)^{+/-} \equiv \delta_{S_0}(K) \ ,$$

where δ_{S_0} denotes the Dirac mass at S_0.

2.2. Direct and inverse problems. In the special case where the volatility, $a \equiv \sigma^2/2$, is a constant, or a function of time alone, explicit formulas for the prices $\Pi^{+/-}$ are known (see Black and Scholes [6] or Merton [32]). But in the case of a general local volatility function $a(t, S)$, one must turn to finite differences or a Monte-Carlo procedure based upon equations (2.3)–(2.4) or (2.1). Moreover, observation teaches that no constant or merely time dependent local volatility function is consistent with most sets of market quotes. This phenomenon is known by market practitioners as the *smile of implied volatility*.

However, in practice it is not the local volatility that is known, but the prices themselves. In fact the local volatility is the only quantity in (2.1) or (2.3)–(2.4) which cannot be obtained from the market. Indeed r and q, as well as, to some extent, Π, can all be retrieved from market-quoted quantities. Consequently, one usually wishes to solve the inverse problem, that of finding $a(t, S)$ such that the theoretical prices given by (2.1) or (2.3)–(2.4) match the observed option prices. We thus use liquid quotations of actively traded options, which are usually referred to as *vanilla* options, as a way to extract information about the future behavior of the underlying asset. The calibrated local volatility function is then used by risk managers or traders to value risk exposure, or price *exotic* (non vanilla) options and calculate hedge ratios consistently with the market.

This is the problem we shall be concerned with here. In particular, there are two cases which are commonly considered in the literature, and we shall treat both in parallel. In the first one, this matching is required to occur on the actual, hence finite, set of pairs (T, K) with observed prices. In the second case, the matching is required to occur over all (T, K) such that $T \geq t_0,\ K > 0$. This makes sense, for example, if the actual set of observed prices has been interpolated. To distinguish between these two cases, we shall refer to the the first as the *discrete*, and the second as the *continuous*, calibration problem.

2.3. The Tikhonov regularization method. Both the discrete and continuous calibration problems are ill-posed. This is the case in the continuous calibration problem because the solution depends in an unstable way upon the data, and in the discrete calibration problem because the full surface $a(t, S)$ is simply underdetermined by the discrete data. It is then necessary to introduce stabilizing procedures in the reconstruction method for the local volatility function. One of these is known as the Tikhonov regularization method [39, 21]. The idea is to tackle the calibration problem as a minimization problem, where the cost criterion to be minimized is

$$J_\alpha\left(a\right) \equiv d\left(\Pi\left(a\right), \pi\right)^2 + \alpha\rho\left(a, a_0\right)^2 \ .$$

Here $d\left(\Pi\left(a\right), \pi\right)$ denotes a distance between the model prices $\Pi\left(a\right)$ and the observed prices π, α is the regularization parameter, and ρ is a penalty designed to keep a close to the *prior* a_0, which reflects *a priori* information about a. Following Lagnado and Osher [29], we shall choose $\rho\left(a, a_0\right)^2 \equiv \|a - a_0\|^2_{H^1}$, where

$$\|u\|^2_{H^1} \ \equiv \ \int\int u^2 + \|\nabla u\|^2,$$

which is the H^1-(squared) norm of u with logarithmic variables $t, y = \ln(S)$.

2.4. Overview. We first study, in an appropriate functional analysis setting, Black–Scholes and Dupire linear parabolic equations with measurable ingredients (§3 and 4). These are linear uni-dimensional equations in nondivergence form, with positively bounded dominant coefficients. We thus extend well known results when the dominant coefficient a is a constant, or a regular function. Mixing the probabilistic pointwise and L_p estimates of Krylov [26] with the analytic $W_p^{1,2}$ estimates of Fabes and Stroock and Varadhan [38], we obtain $W_p^{1,2}$ estimates for the equations with source terms. Using the theory of L_p-viscosity solutions [10, 13], we then show that our equations admit unique solutions, for which we provide a probabilistic representation (theorems 4.2 and 4.3).

Proposition 5.1 sums up the main properties of the pricing functional Π useful for the study of the calibration problems, namely: compactness, twice Gateaux differentiability and stability with respect to perturbations of parameters. We can then apply the general theory of Tikhonov regularization for ill-posed nonlinear inverse problems [21, 22, 27, 33, 34], both to the continuous and discrete calibration problems. We thus prove the stability of the method for arbitrary values of the regularization parameter (§5). Assuming the existence of a solution of the calibration problem, we prove the convergence of the method towards an a_0-minimum norm solution when the regularization parameter tends to 0, and we exhibit conditions sufficient to ensure convergence rates in $\mathrm{O}(\sqrt{\delta})$, where δ is the data noise (§6).

2.5. Main notations and general assumptions. To avoid too many repetitions, we define now a set of notations and related general assumptions, that will be assumed to hold throughout the paper. When stronger assumptions are required, they will be stated explicitely in the body of the paper.

General notation.

$x \wedge y$, $x \vee y$: $\min(x, y)$, $\max(x, y)$.
x^+, x^-: $\max(x, 0)$, $\max(-x, 0)$.
C, C', ... $C \equiv C_\rho\ (\rho_1, \ldots, \rho_n)$: Constants C, C', ... depending upon nothing but the parameters $\rho, \rho_1, \ldots, \rho_n$.

One should be aware that these constants may vary with the context. We will also use the notation "$\equiv$" for "denotes", or "equals identically" (that is, equality between functions), according to the context.

Mathematical finance.

S, $y = \ln(S)$: Lognormal underlying diffusion, in financial and logarithmic variables.
$q, r \in [0, R]$: Dividend yield attached to S, short rate of the economy.
$a \equiv \sigma^2/2$, a_0: local volatility function, prior a_0 on a.
$\underline{a}$, $\overline{a}$, $\hat{a}$: Bounds on a_0 and a such that $0 < \underline{a} < \overline{a}$, $\hat{a} \equiv (\underline{a} + \overline{a})/2$.
$\overline{p} \equiv \overline{p}(\underline{a}, \overline{a})$: A real in $]2, 3[$ depending upon $\underline{a}$ and $\overline{a}$; see theorem 4.2.
W: Standard Brownian motion.
$Q =]\underline{t}, \overline{T}[\times \mathbb{R}$: A plane strip on which a is defined, in logarithmic variables.
$(t_0, y_0), (T, k)$: Points in $\overline{Q}$, with $t_0 \le T$.
$\overline{y}_0, \overline{k}$: Bounds on $|y_0|$, $|k|$.
Q_{t_0}, Q^T: $Q \cap \{t > t_0\}$, $Q \cap \{t < T\}$.
$\overline{Q}_{t_0}$, $\overline{Q}^T$: closures of Q_{t_0}, Q^T.
$\Pi^{+/-}_{T,K}(t_0, S_0; r, q, a)$, $\Pi^{+/-}_{T,k}(t_0, y_0; r, q, a)$: The price, in a generalized Black–Scholes model, for a European call/put option with maturity T and exercise price

$K = e^k$, at the current phase $t_0, S_0 = e^{y_0}$, in financial and logarithmic variables.

$\gamma_{t_0,y_0}(t, y; r, q, a)$**:** Transition probability density discounted at rate r (that is, $e^{-r(t-t_0)}\times$ the density), for the underlying diffusion in logarithmic variable y.

$BS^{+/-}_{Q^T}(k; r, q, a)$, $BS'_{Q^T}(r, q, a; \Gamma)$, $DUP^{+/-}_{Q_{t_0}}(y_0; r, q, a)$**:** Black–Scholes call/put equation on Q^T, Black–Scholes derived equation with source term Γ, Dupire call/put equation on Q_{t_0}; see §3.2.

To alleviate notations, r, q, a will sometimes be abbreviated to a; $\Pi^{+/-}_{T,K}(t_0, S_0; a)$ or $\Pi^{+/-}_{T,k}(t_0, y_0; a)$, to $\Pi^{+/-}$; $BS^{+/-}_{Q^T}(k; a)$, $BS'_{Q^T}(a; \Gamma)$, $DUP^{+/-}_{Q_{t_0}}(y_0; a)$ and $\gamma_{t_0,y_0}(t, y; a)$, to $BS^{+/-}$, BS', $DUP^{+/-}$ and γ, respectively.

In the case of the call option, we shall sometimes drop the $^+$ superscript. For instance, by default, Π will refer to Π^+.

Functional analysis.

Ω**:** Regular by parts, open plane area.

p, θ**:** Real $p \in]2, +\infty[$, $\theta \equiv 1 - 2/p > 0$.

$L_p(\Omega)$,$L_{p,loc}(\Omega)$, $H^1(\Omega)$, $H^2(\Omega)$, $W^1_p(\Omega)$, $W^{1,2}_p(\Omega)$, $W^{1,2}_{p,loc}(\Omega)$, $\mathcal{C}^0_\theta(\overline{\Omega})$, $\mathcal{D}(\overline{\Omega})$**:** Sobolev spaces on Ω; see §3.1

Γ**:** Element of $L_p(Q)$.

$\mathcal{M}_Q(\underline{a}, \overline{a})$**:** Set of real measurable functions on Q with bounds $\underline{a}$ and $\overline{a}$.

$a_0 + H^1_Q(\underline{a}, \overline{a})$**:** Set of functions in $a_0 + H^1(Q)$ with bounds $\underline{a}$ and $\overline{a}$.

h, h'**:** Elements of $H^1(Q)$.

$\mathcal{E} \rightarrow$**:** Convergence in the topology of the space $\mathcal{E}$.

$\|X\|$, $\|X\|_{\mathcal{E}}$**:** Euclidean norm of X, norm of X in the surrounding normed space $\mathcal{E}$.

$\langle X, Y\rangle$, $\langle X, Y\rangle_{\mathcal{E}}$**:** Inner product of X and Y in the surrounding Euclidean space, Hilbert space $\mathcal{E}$.

$d\Pi(a).h$**:** Derivative in the direction h of the functional Π, at the local volatility function a.

$d\Pi(a)^\star$**:** Adjoint operator of $h \mapsto d\Pi(a).h$; see §6.2.

$\nabla J(a)$**:** Gateaux derivative of the cost criterion J at the local volatility function a.

For instance, if J denotes a cost criterion on a Hilbert space $\mathcal{E}$, then in our notations:

$$\langle \nabla J(a)\ , h\rangle_{\mathcal{E}} \ = \ dJ(a).h\ , \quad h \in \mathcal{E}\ .$$

In the same way, the general assumptions we have made above on a and a_0 can be stated as

$$a_0, a \in \mathcal{M}_Q(\underline{a}, \overline{a})\ .$$

Finally, we shall refer to the statements in remark 3.5 and lemma 4.1.3, as *symmetry* and *parity*, respectively.

3. Strong solutions of parabolic problems.

3.1. Functional spaces and Sobolev embeddings. Let us first introduce some Hilbert and Banach spaces, which we shall use as spaces of local volatility functions and solutions of Black–Scholes and Dupire equations.

Given the open plane area Ω, we shall denote by $\mathcal{D}(\overline{\Omega})$ the space of traces on Ω of regular functions with compact support in the plane. We will use the usual Hilbert spaces $H^2(\Omega) \subset H^1(\Omega) \subset L_2(\Omega)$, and the Banach spaces $\mathcal{C}^0_\theta(\overline{\Omega})$, $L_p(\Omega)$, $W^1_p(\Omega)$, $W^{1,2}_p(\Omega)$, where

$$\|u\|_{\mathcal{C}^0_\theta(\overline{\Omega})} = \sup_{(t,y)\in\overline{\Omega}} |u| \; + \sup_{(t,y)\neq(t',y')\in\overline{\Omega}} \frac{|u(t,y)-u(t',y')|}{|t-t'|^\theta + |y-y'|^\theta};$$
$$\|u\|_{W^1_p(\Omega)} = \|u\|_{L_p(\Omega)} \; + \; \|\partial_t u\|_{L_p(\Omega)} \; + \; \|\partial_y u\|_{L_p(\Omega)};$$
$$\|u\|_{W^{1,2}_p(\Omega)} = \|u\|_{L_p(\Omega)} \; + \; \|\partial_t u\|_{L_p(\Omega)} \; + \; \|\partial_y u\|_{L_p(\Omega)} \; + \; \|\partial^2_{y^2} u\|_{L_p(\Omega)} \; .$$

Finally, we shall denote by $W^{1,2}_{p,loc}(\Omega)$ the localized Fréchet space of functions which belong to $W^{1,2}_p(\Omega')$ for every regular open bounded subset Ω' with $\overline{\Omega}' \subset \Omega$.

Now we have the following Sobolev embeddings, for which the reader is referred for instance to Larrouturou and Lions [30]:

1. For Ω bounded or half-plane,

$$W^1_p(\Omega) \hookrightarrow C^0_\theta(\overline{\Omega}) \; . \tag{3.1}$$

This embedding notably implies the existence of a unique continuous extension up to the boundary for the strong solutions introduced at definition 3.1.1 below.

2. For Ω bounded,

$$H^1(\Omega) \hookrightarrow L_p(\Omega) \; . \tag{3.2}$$

This embedding, called the Rellich–Kondrakov embedding, is *compact*, which means that it maps weakly convergent sequences into strongly convergent ones.

Let us now present the definitions of a solution of a Partial Differential Equation that we shall need. For more about these definitions, the reader is referred to Ladyzhenskaya *et al* [28], Crandall *et al* [13], Wang [40], Caffarelli *et al* [10] and Crandall *et al* [12].

Definition 3.1. *Let there be a linear parabolic equation on $\overline{\Omega}$, with measurable ingredients and a continuous boundary condition on $\partial_p\Omega$, the parabolic boundary of Ω.*

1. *We call a function a* strong solution in $L_p(\Omega)$*, or a* $L_p(\Omega)$-solution*, if it is a function in $W^{1,2}_p(\Omega)$, that satisfies the boundary condition, and solves the equation almost everywhere. We also use this definition with $W^{1,2}_{p,loc}(\Omega)$ to define an $L_{p,loc}(\Omega)$-solution.*
2. *We call a function a* $L_{p,loc}(\Omega)$-viscosity solution*, if it is a continuous function on $\overline{\Omega}$, that satisfies the boundary condition, and solves the equation in the viscosity meaning for test functions in $W^{1,2}_{p,loc}(\Omega)$.*

The relations between these definitions of a solution are as follows (see Crandall *et al* [13]):

1. A $L_{p,loc}(\Omega)$-solution is a $L_{p,loc}(\Omega)$-viscosity solution.
2. Conversely, a $L_{p,loc}(\Omega)$-viscosity solution that belongs to $W^{1,2}_{p,loc}(\Omega)$ is a $L_{p,loc}(\Omega)$-solution.

The following theorem gathers the main properties of the Sobolev spaces on plane strips that we shall need.

Theorem 3.2.

1. *$H^1(Q)$ is continuously embedded in $L_p(Q)$;*
2. *$\mathcal{D}(\overline{Q})$ is dense in $L_p(Q)$, $H^1(Q)$, $H^2(Q)$;*

3. *The application*

$$\mathcal{D}(\overline{Q}) \times \mathcal{D}(\overline{Q}) \ni (u, v) \mapsto (u|_{\partial Q}, \partial_n v) \in L_2(\partial Q)^2 \ ,$$

where $\partial_n v$ *denotes the normal derivative, admits a unique linear continuous extension, called* trace, *from* $H^1(Q) \times H^2(Q)$ *to* $L_2(\partial Q)^2$.

4. *The set of traces on* ∂Q *of functions of* $H^1(Q) \times H^2(Q)$ *forms a dense subset of* $L^2(\partial Q)^2$, *and we have the so-called* generalized Green formula, *for every* $(u, v) \in H^1(Q) \times H^2(Q)$:

$$- \int\int_Q u \ (\Delta v) \ = \ \int\int_Q \langle \nabla u, \nabla v \rangle \ - \ \int_{\partial Q} u \ \partial_n v \ .$$

Proof. These properties result from the analogous properties well known on open half-planes (see for instance Larrouturou and P.L. Lions [30], Bensoussan and J.L. Lions [3]). For the details, the reader is referred to Crépey [14, theorem F.1] and the proof given therein. □

In the upcoming proofs, we shall often be able to proceed by density thanks to the following lemma.

Lemma 3.3. *There exists Lipschitzian functions* $a_n \in \mathcal{M}_Q(\underline{a}, \overline{a})$ $(n \in \mathbb{N}^\star)$, *such that* a_n *converges to* a *in* $L_{p,loc}(Q)$ *when* $n \to +\infty$.

Proof. This follows from standard mollification with compact support, applied to a prolongated by zero outside Q (see for instance Brézis [9]). □

3.2. Black–Scholes, Dupire and derived equations. Let us now introduce the main equations in this work.

Definition 3.4.

1. *We define the Black–Scholes call/put equation,* $BS^{+/-}_{Q^T}(k; r, q, a)$, *with backward logarithmic variables* $(t, y) \in \overline{Q}^T$, *parameterized by* (T, k), *as*

$$\begin{cases} -\partial_t \Pi - (r - q - a(t,y))\, \partial_y \Pi - a(t,y) \partial^2_{y^2} \Pi + r\Pi \ = \ 0 \quad on\ Q_T \ , \\ \Pi|_T = (e^y - e^k)^{+/-} \ . \end{cases}$$

We also define the Black–Scholes derived equation with source term Γ, $BS'_{Q^T}(r, q, a; \Gamma)$, *as*

$$\begin{cases} -\partial_t (\delta\Pi) - (r - q - a(t,y))\, \partial_y (\delta\Pi) - a(t,y) \partial^2_{y^2} (\delta\Pi) + r\Pi \ = \ \Gamma \quad on\ Q_T \ , \\ \delta\Pi|_T \ \equiv \ 0 \ . \end{cases}$$

2. *We define the Dupire call/put equation,* $DUP^{+/-}_{Q_{t_0}}(y_0; r, q, a)$, *with forward logarithmic variables* (T, k), *at the current phase* (t_0, y_0), *as*

$$\begin{cases} \partial_T \Pi_{T,k} - (q - r - a(T,k)) \partial_k \Pi_{T,k} - a(T,k) \partial^2_{k^2} \Pi_{T,k} + q\Pi_{T,k} \ = \ 0 \quad on\ Q_{t_0} \ , \\ \Pi|_{t_0} \ \equiv \ (e^{y_0} - e^k)^{+/-} \ . \end{cases}$$

3. *Finally, we define the diffusion underlying the previous problems, with logarithmic variables, as*

$$dy_t \ = \ (r - q - \frac{\sigma(t, y_t)^2}{2}) \ dt + \sigma(t, y_t) \ dW_t \ , \qquad y_{t_0} \ = \ y_0 \ . \tag{3.3}$$

REMARK 3.5 (Symmetry). Changing moreover the direction of time T, via $\tau \equiv \overline{T} + t_0 - T$, $\check{\phi}(\tau, k) \equiv \phi(T, k)$ for any function ϕ, then $DUP^{+/-}_{Q_{t_0}}(y_0; r, q, a)$ becomes $BS^{-/+}_{Q_{t_0}}(y_0; q, r, \check{a})$.

LEMMA 3.6.

1. *(Black–Scholes and Dupire equations) Equations $BS^{+/-}$ have at most one $L_{p,loc}(Q^T)$-solution Π such that $|\Pi| \le K \vee S$.*

2. *(Derived equations) For any $L_p(Q^T)$-solution $\delta\Pi$ of BS', we have:*

$$\|\delta\Pi\|_{\mathcal{C}^0_\theta(\overline{Q}^T)} \ \le \ C' \ \|\delta\Pi\|_{W^{1,2}_p(Q^T)} \ , \tag{3.4}$$

where $C' \equiv C'_p$. Moreover, $\delta\Pi$ is also the unique $L_{p,loc}(Q^T)$-solution of BS' which converges to 0 when $|y| \to +\infty$, uniformly with t.

Proof.

1. Given two such solutions Π and Π', let us define $\delta\Pi \equiv e^{-2y+\rho t}(\Pi - \Pi')$, where $\rho = r + 2\overline{a}$. By linearity, $\delta\Pi$ is a $L_{p,loc}(Q^T)$-solution of

$$\begin{cases} -\partial_t \delta\Pi - (r - q + 3a)\, \partial_y \delta\Pi - a\partial^2_{y^2}\delta\Pi + (2q + 2\overline{a} - 2a)\delta\Pi = 0 \\ \delta\Pi|_T \ \equiv \ 0 \ . \end{cases} \tag{3.5}$$

Moreover, let us fix $\varepsilon > 0$. One can choose $Y_\varepsilon \ge 1/\varepsilon$ such that for $|y| \ge Y_\varepsilon$, we have $|\delta\Pi(t, y)| \le 2e^{-2y+\rho t}(K \vee e^y) \le \varepsilon$, uniformly with $t \in [\underline{t}, T]$. Then, $|\delta\Pi| \le \varepsilon$ on $Q^T \cap \{|y| \le Y_\varepsilon\}$, by the maximum principle in Crandall *et al* [13, proposition 2.6]. So $\delta\Pi \equiv 0$ on Q^T, by passage to the limit when $\varepsilon \to 0$.

2. By the same maximum principle as above, we have uniqueness in the class of $L_{p,loc}(Q^T)$-solutions of BS' which converge to 0 when $|y| \to +\infty$, uniformly with t. Now, let us be given a $L_p(Q^T)$-solution $\delta\Pi$ of BS'. Since the solution $\delta\Pi$ is continuous on $\overline{Q}^T$ and vanishes at T, it may be identified with an element of $W^1_p(\Omega)$, where $\Omega \equiv]\underline{t}, +\infty[\times\mathbb{R}$, by extension with 0 on the right of T. Estimate (3.4) then follows from the Sobolev embedding (3.1), on the half-plane Ω. Finally, $\delta\Pi \in \mathcal{C}^0_\theta(\overline{Q}^T) \cap L_p(Q^T)$ converges to 0 when $|y| \to +\infty$, uniformly with t.

□

4. Existence, uniqueness, and probabilistic representation of solutions.

4.1. Diffusion. The following lemma links the price of a European call/put with the discounted expectation of the corresponding payoff, in a generalized Black–Scholes model.

LEMMA 4.1.

1. *The diffusion equation (2.2) has a unique weak solution on $]t_0, \overline{T}[$:*

$$S_t \ = \ S_0 \ e^{(r-q)(t-t_0)} \ \exp\left(\int_{t_0}^t \sigma(s, S_s) dW_s - \frac{1}{2}\int_{t_0}^t \sigma^2(s, S_s) ds\right) \ , \quad t \in]t_0, \overline{T}[$$

where the last exponential is a martingale, under the risk-neutral probability P. In particular,

$$E^{t_0, S_0}_P S_t = S_0 e^{(r-q)(t-t_0)} \ , \ t \in]t_0, \overline{T}[\ . \tag{4.1}$$

2. *The price $\Pi^{+/-}$ equals the payoff expectation of the call/put at T, discounted at rate r:*

$$\Pi^{+/-} = e^{-r(T-t_0)} E^{t_0, S_0}_P \left(S_T - K\right)^{+/-}$$

under the risk-neutral probability P. In particular, $0 \leq \Pi \leq S_0$.

3. *Denoting $\Pi^+ - \Pi^-$ by $\delta\Pi$, we have*

$$\delta\Pi \equiv S_0 e^{-q(T-t_0)} - K e^{-r(T-t_0)} \ .$$

This relation, known as call/put parity*, notably implies that $\partial^2_{S^2}\delta\Pi$, $\partial^2_{K^2}\delta\Pi$, $(\partial^2_{y^2} - \partial_y)\delta\Pi$ and $(\partial^2_{k^2} - \partial_k)\delta\Pi$, all vanish identically.*

Proof.

1. See for instance Strook and Varadhan [38, exercise 7.3.3] and Karatzas and Shreve [25, problem 5.6.15 and corollary 3.5.13].

2. and 3. The expression for $\Pi^{+/-}$ then follows from Karatzas and Shreve [25, §5.8.A]. Using this expression, the bounds on Π and the call/put parity proceed from (4.1).

□

4.2. Derived hedge equations with source terms. The following theorem and the estimate (4.3) therein, are the cornerstones of this article. The difficulty comes from the lack of regularity of the local volatility function a, that is merely required to be measurable and positively bounded. But this turns out to be sufficient, in the present uni-dimensional linear framework. Recall that Γ denotes an element of $L_p(Q)$.

THEOREM 4.2. *There exists $\overline{p} \equiv \overline{p}(\underline{a}, \overline{a}) \in]2, 3[$, such that if $p \in]2, \overline{p}[$, then, when (t, y) varies within $\overline{Q}^T$,*

$$\delta\Pi(t, y) \ = \ E_P^{t,y} \int_{s=t}^{T} e^{-r(s-t)} \Gamma(s, y_s) \ ds \tag{4.2}$$

is the only $L_p(Q^T)$-solution, or $L_{p,loc}(Q^T)$-solution converging to 0 when $|y| \to +\infty$, uniformly with t, of $BS'_{Q^T}(a; \Gamma)$.

Moreover,

$$\|\delta\Pi\|_{\mathcal{C}^0_\theta(\overline{Q}^T)} \ \leq \ C' \ \|\delta\Pi\|_{W_p^{1,2}(Q^T)} \ \leq \ C'C \ \|\Gamma\|_{L_p(Q^T)} \ , \tag{4.3}$$

where $C' \equiv C'_p$ is as in (3.4), and $C \equiv C_p(\underline{t}, \overline{T}; R, \underline{a}, \overline{a})$.

Proof. For the moment, $p \in]2, +\infty[$. We first show that for $\varphi \in W_p^{1,2}(Q^T)$,

$$\|\varphi\|_{W_p^{0,1}(Q^T)} \ \leq \ C_p \ \|\varphi\|^{1/2}_{W_p^{0,2}(Q^T)} \ \|\varphi\|^{1/2}_{L_p(Q^T)} \ . \tag{4.4}$$

Inequality (4.4) can be more readily seen on the following equivalent norms:

$$\|\varphi\|^p_{\widetilde{W}_p^{0,j}(Q^T)} \ \equiv \ \sum_{k \leq j} \|\partial^k_{y^k}\varphi\|^p_{L_p(Q^T)} \ , \ 0 \leq j \leq 2 \ .$$

Indeed, by integration over time of a classic Sobolev inequality (see for instance Bensoussan and J.L. Lions [3, Chapter 2, equation (5.8)]):

$$\begin{aligned} \|\varphi\|^p_{\widetilde{W}_p^{0,1}(Q^T)} &= \int_{t=\underline{t}}^{T} \|\varphi(t, \cdot)\|^p_{\widetilde{W}_p^1(\mathbb{R})} \ dt \\ &\leq C_p^p \int_{t=\underline{t}}^{T} \|\varphi(t, \cdot)\|^{p/2}_{\widetilde{W}_p^2(\mathbb{R})} \ \|\varphi(t, \cdot)\|^{p/2}_{L_p(\mathbb{R})} \ dt \end{aligned}$$

$$\leq C_p^p \left(\int_{t=\underline{t}}^{T} \|\varphi(t,\cdot)\|^p_{\widetilde{W}^2_p(\mathbb{R})} \, dt \right)^{1/2} \left(\int_{t=\underline{t}}^{T} \|\varphi(t,\cdot)\|^p_{L_p(\mathbb{R})} \, dt \right)^{1/2}$$
$$= C_p^p \, \|\varphi\|^{p/2}_{\widetilde{W}^{0,2}_p(Q^T)} \, \|\varphi\|^{p/2}_{L_p(Q^T)} \ ,$$

by the Cauchy–Schwarz inequality. This shows (4.4), which in turn implies

$$\|\varphi\|_{W^{0,1}_p(Q^T)} \ \leq \ rC_p \|\varphi\|_{W^{0,2}_p(Q^T)} + C_p(r)\|\varphi\|_{L_p(Q^T)} \tag{4.5}$$

for every fixed $r > 0$, provided $C_p(r) \leq C_p/4r$.

On the other hand, since (3.3) admits a unique weak solution (see lemma 4.1.1), then from Krylov [26, proof of theorem 2.4.5.a and theorem 2.4.1]

$$E_P^{t,y} \int_t^T e^{-r(s-t)} |\Gamma(s, y_s)| \, ds \leq C \, \|\Gamma\|_{L_p(Q^T)} \ , \tag{4.6}$$

where $C \equiv C_p(\underline{t}, T, R, \underline{a}, \overline{a})$.

We now assume that φ is a $L_p(Q^T)$-solution of $BS'_{Q^T}(a;\, \Gamma)$. For $\varepsilon > 0$, let τ_ε denote the exit time of $Q^T \cap \{|y| \leq 1/\varepsilon\}$ for the y-process (3.3). It can be shown that (4.6) implies the following probabilistic representation:

$$E_P^{t,y} e^{-r(\tau_\varepsilon - t)} \varphi(\tau_\varepsilon, y_{\tau_\varepsilon}) - \varphi(t,y) \ = \ - E_P^{t,y} \int_{s=t}^{\tau_\varepsilon} e^{-r(s-t)} \Gamma(s, y_s) \, ds \ . \tag{4.7}$$

This has been shown by Bensoussan and J.L. Lions [3, Chapter 2, §8.3] in a variational context. We do not reproduce the proof here, though it proceeds in a similar fashion, using regularization and classic Itô's formula.

When $\varepsilon \to 0$, τ_ε almost surely converges to T. Moreover, φ is bounded and continuous. Estimate (4.6) then implies, through dominated convergence on the left hand side and right hand side of (4.7),

$$\varphi(t,y) = E_P^{t,y} \int_{s=t}^{T} e^{-r(s-t)} \Gamma(s, y_s) \, ds. \tag{4.8}$$

Then, from Krylov [26, theorem 2.4.5.a]:

$$\|\varphi\|_{L_p(Q^T)} \ \leq \ C \, \|\Gamma\|_{L_p(Q^T)} \ , \tag{4.9}$$

where $C \equiv C_p(\underline{t}, T, R, \underline{a}, \overline{a})$. The probabilistic representation (4.8), for any *a priori* $L_p(Q^T)$-solution φ of $BS'_{Q^T}(a;\Gamma)$, also shows the consistency of such *a priori* solutions across various values of $p > 2$.

Moreover, by linearity, such an *a priori* solution φ is the $L_p(Q^T)$-solution of the equation $-\partial_t \varphi - \hat{a} \partial^2_{y^2} \varphi \ = \ \hat{\Gamma}$, where

$$\hat{\Gamma} \ = \ \Gamma - r\varphi + (r - q - a(t,y))\partial_y \varphi + (a - \hat{a})\partial^2_{y^2} \varphi \ ,$$

with homogeneous terminal condition. Therefore, following Stroock–Varadhan [38, exercise 7.3.3 and p.211], we have the following estimate:

$$\|\partial^2_{y^2}\varphi\|_{L_p(Q^T)} \ \leq \ C_p(\hat{a}) \times \left(\|\Gamma\|_{L_p(Q^T)} + R\|\varphi\|_{L_p(Q^T)} + (R + \overline{a})\|\partial_y \varphi\|_{L_p(Q^T)} + \frac{1}{2}(\overline{a} - \underline{a})\|\partial^2_{y^2}\varphi\|_{L_p(Q^T)} \right) \tag{4.10}$$

where $C_p(\hat{a})$ is a log-convex, hence continuous, function of $1/p$, also defined at $p = 2$, such that

$$C_{p=2}(\hat{a}) = \frac{1}{\hat{a}} < \frac{2}{\overline{a}}.$$

Therefore one can choose $\overline{p} \equiv \overline{p}(\underline{a}, \overline{a}) \in]2, 3[$ such that $C_p(\hat{a}) \leq \frac{2}{\hat{a}}$ if $p \in]2, \overline{p}[$. Estimate (4.3), at least with T instead of $\overline{T}$ in C, then results from (4.10), (4.5), (4.9) and (3.4). We will refer to the estimate (4.3) with T instead of $\overline{T}$ in C, as *the temporary version* of estimate (4.3).

The existence of a $L_p(Q^T)$-solution φ of $BS'_{Q^T}(a;\,\Gamma)$ in the special case where $\Gamma \in \mathcal{D}(\overline{Q}^T)$ follows by density using lemma 3.3 as follows. Define $p' \equiv (2+p)/2$. Following Fabes [23], $BS'_{Q^T}(a_n;\,\Gamma)$ admits a $L_p(Q^T) \cap L_{p'}(Q^T)$-solution φ_n. By the temporary version of estimate (4.3), and successive extractions, one can find a subsequence $\varphi_{n'}$ that converges to a limit φ, weakly in $W_p^{1,2}(Q^T)$ or $W_{p'}^{1,2}(Q^T)$ and locally uniformly on $\overline{Q}^T$. By $W_p^{1,2}(Q^T)$-weak passage to the limit, φ inherits the temporary version of estimate (4.3). Then φ is a $L_p(Q^T)$-solution of $BS'_{Q^T}(a;\,\Gamma)$, by lemma A.1. The general case where $\Gamma \in L_p(Q^T)$ straightaway follows by density using theorem 3.2.2.

Let us now consider the solution $\tilde{\varphi}$ of $BS'_Q(a;\,\widetilde{\Gamma})$, where $\widetilde{\Gamma} \equiv \Gamma/0$ on the left/right of T. By linearity and uniqueness of solutions of BS', $\tilde{\varphi}$ vanishes on Q_T, and $\tilde{\varphi}$ is equal to φ on Q^T. Therefore, the estimate (4.3) for φ on Q^T results from the temporary version of estimate (4.3) for $\tilde{\varphi}$ on Q. □

4.3. Homogeneous valuation equations. The following theorem is formally well known. When the local volatility function a is Hölderian (with logarithmic variables), it has indeed been justified by many authors. For instance, Dupire [19] or Bouchouev and Isakov [8] use Partial Differential Equation arguments involving fundamental solutions. Alternatively, El Karoui [20] or Crépey [14, §4.1, Part IV] use probabilistic arguments involving local time. We also refer the reader to Crépey [14, §4.1, Part IV] or Berestycki, Busca and Florent [4] for results in the case where a is uniformly continuous. Here, we prove the more general case where $a \in \mathcal{M}_Q(\underline{a}, \overline{a})$. This is indeed the case that will be relevant for the study of the calibration problems.

THEOREM 4.3. *Assume* $p \in]2, \overline{p}[$. *Then:*

1. *The call price*

$$\overline{Q}^T \ni (t, y) \mapsto \Pi_{T,k}(t, y; a),$$

is the unique $L_{p,loc}(Q^T)$*-solution between 0 and* S*, of* $BS_{Q^T}(k; a)$*. Moreover, it is convex and nondecreasing with respect to* S*, nondecreasing with the local volatility, and it converges to* 0 *when* $S \to 0$*, uniformly with* t*.*

2. *The call price*

$$\overline{Q}_{t_0} \ni (T, k) \mapsto \Pi_{T,k}(t_0, y_0; a),$$

is the unique $L_{p,loc}(Q_{t_0})$*-solution between 0 and* S_0*, of* $DUP_{Q_{t_0}}(y_0; a)$*. Moreover, it is convex and nonincreasing with respect to* K*, nondecreasing with the local volatility, and it converges to* 0 *when* $K \to +\infty$*, uniformly with* T*. Finally, for almost every* $t > t_0$*, the* y*-process (3.3) admits a transition probability density between* t_0 *and* t*. Discounting this density at rate* r*, it becomes*

$$\gamma_{t_0,y_0}(t, y; a) \equiv e^{-y}(\partial^2_{y^2} - \partial_y)\Pi_{t,y}(t_0, y_0; a)\ . \tag{4.11}$$

Proof. We proceed by density from the known case of a Lipschitzian function a_n approximating a as in lemma 3.3. Denoting $(p+\overline{p})/2$ by p', let $\hat{\Pi}$, respectively Π_n, be the strong solution in $L_{p,loc}(Q^T) \cap L_{p',loc}(Q^T)$ between 0 and S, of $BS_{Q^T}(k; \hat{a})$, respectively $BS_{Q^T}(k; a_n)$.

Since $2 < p < p' < \overline{p} < 3$, it is well known that

$$(\partial^2_{y^2} - \partial_y)\hat{\Pi} \in L_p(Q^T) \cap L_{p'}(Q^T)$$

(see for instance Crépey [14, remark 4.1, Part IV]). Therefore, using theorem 4.2, there exists a $L_p(Q^T) \cap L_{p'}(Q^T)$-solution $\delta\Pi$ of $BS'_{Q^T}(a; \Gamma)$, where $\Gamma \equiv (a - \hat{a})(\partial^2_{y^2} - \partial_y)\hat{\Pi}$. By linearity, $\Pi \equiv \hat{\Pi} + \delta\Pi$ is then a strong solution in $L_{p,loc}(Q^T) \cap L_{p',loc}(Q^T)$ of $BS_{Q^T}(k; a)$. Moreover,

$$(\partial^2_{y^2} - \partial_y)\Pi \equiv (\partial^2_{y^2} - \partial_y)\hat{\Pi} + (\partial^2_{y^2} - \partial_y)\delta\Pi \in L_p(Q^T) \cap L_{p'}(Q^T)\ .$$

Denote $\Pi_n - \hat{\Pi}$ by $\delta_n\Pi$. By linearity, symmetry, parity, and the results of the theorem in the Lipschitzian case, $\delta_n\Pi$ converges to 0 when $|y| \to +\infty$, uniformly with t, and $\delta_n\Pi$ is a strong solution in $L_{p,loc}(Q^T) \cap L_{p',loc}(Q^T)$ of $BS'_{Q^T}(a_n; \Gamma_n)$, where $\Gamma_n \equiv (a_n - \hat{a})(\partial^2_{y^2} - \partial_y)\hat{\Pi}$. Therefore, by theorem 4.2, $\delta_n\Pi$ is the strong solution in $L_p(Q^T) \cap L_{p'}(Q^T)$ of $BS'_{Q^T}(a_n; \Gamma_n)$. So $\Pi_n - \Pi = \delta_n\Pi - \delta\Pi$ is the strong solution in $L_p(Q^T) \cap L_{p'}(Q^T)$ of $BS'_{Q^T}(a_n; \Gamma'_n)$, where

$$\Gamma'_n \equiv \Gamma_n - \Gamma + (a_n - a)(\partial^2_{y^2} - \partial_y)\delta\Pi = (a_n - a)(\partial^2_{y^2} - \partial_y)\Pi.$$

Furthermore, Γ'_n converges to 0 in $L_p(Q^T)$ when $n \to +\infty$. Indeed, having fixed $\varepsilon > 0$, let us choose a subset $Q_\varepsilon \equiv Q^T \cap \{|y| \le Y_\varepsilon\}$ such that $\|(\partial^2_{y^2} - \partial_y)\Pi\|_{L_p(Q^c_\varepsilon)} \le \varepsilon$, where $Q^c_\varepsilon \equiv Q^T \setminus Q_\varepsilon$. By Hölder's inequality, it follows thanks to lemma 3.3 that

$$\|\Gamma'_n\|^p_{L_p(Q^T)} \le (\|(\partial^2_{y^2} - \partial_y)\Pi\|^p_{L_{p'}(Q^T)} + (\overline{a} - \underline{a})^p)\varepsilon^p,$$

for n large enough.

Using estimate (4.3) applied to $\Pi_n - \Pi$, Π then inherits the bounds on Π_n. So $BS^+_{Q^T}(k; a)$ admits a $L_{p,loc}(Q^T)$-solution $\Pi^+ \equiv \Pi$ between 0 and S. Similarly, $BS^-_{Q^T}(k; a)$ admits a $L_{p,loc}(Q^T)$-solution Π^- between 0 and K. We also have symmetric solutions $\Pi^{+/-}_{T,k}$ for $DUP^{+/-}_{Q_{t_0}}(y_0; a)$. Moreover, $\Pi^{+/-} \equiv \Pi^{+/-}_{T,k}$, by passage to the limits in the analogous identities at fixed n. Furthermore, by lemma 3.6.1, these solutions $\Pi^{+/-}$ and $\Pi^{+/-}_{T,k}$ are the only ones between the required bounds.

The probabilistic representation for Π^- then results from a generalized integrated Itô's formula, as in the proof of theorem 4.2. Since $\Pi^{+/-}$ is the limit of the $\Pi^{+/-}_n$, the probabilistic representation for Π^+ then follows from those for Π^- and $\Pi^{+/-}_n$, using also the call/put parity at a and a_n.

$\Pi^{+/-}$ and $\Pi^{+/-}_{T,k}$ then inherit the monotonicity and convexity properties valid at fixed n, by passage to the limit locally uniform over (t, y) and (T, k), respectively. The asymptotic results follow from those, already known, at constant volatility $\underline{a}$ or $\overline{a}$, and from the monotonicity with respect to a.

Finally, by standard arguments developed for instance in Stroock and Varadhan [38, proof of theorem 9.1.9, p. 224], estimate (4.3), or merely (4.6), valid for all $\Gamma \in L_p(Q^T)$, enforces the existence of a transition probability density between t_0 and t for the process y, for almost every $t > t_0$.

Then, by general arguments set out for instance in Crépey [14, §4.1, Part IV], independent of the Lipschitzian assumption on a therein, the discounted density for the process S is $\partial^2_{S^2}\Pi_{t,S}(t_0, S_0; a)$, whence, after change of variables, the expression for γ. □

The following proposition gathers a few consequences of the previous results that will be useful in the following study of the calibration problems. The proposition is stated for $\Pi \equiv \Pi^+$. The analogous statements for $\Pi \equiv \Pi^-$ follow by parity. We then also have the symmetric statements in the variables (T, k). Recall that h and h' denote elements of $H^1(Q)$.

PROPOSITION 4.4. *Assume* $p \in]2, \overline{p}[$.

1. *Then*

$$\|(\partial^2_{y^2} - \partial_y)\Pi\|_{L_p(Q^T)} \;\leq\; C_p \;, \tag{4.12}$$

where $C_p \equiv C_p(\underline{t}, \overline{T}, \overline{k}; R, \underline{a}, \overline{a})$.

2. *The price* Π *is locally* θ*-Hölderian, jointly with respect to* $(t_0, y_0), (T, k)$*, uniformly with* $q, r \in [0, R]$, $a \in \mathcal{M}_Q(\underline{a}, \overline{a})$.

3. *Further define* $p' = (2 + p)/2$, $p'' = (2 + p')/2$, *and* $\Gamma \equiv h(\partial^2_{y^2} - \partial_y)\Pi$. *Then*

$$\|\Gamma\|_{L_{p'}(Q^T)} \;\leq\; C'_{p'} \; \|h\|_{H^1(Q)} \;,$$

where $C'_{p'} \equiv C'_{p'}(\underline{t}, \overline{T}, \overline{k}; R, \underline{a}, \overline{a})$. *Let then* $d\Pi$, *or* $d\Pi_{T,k}(\cdot; a).h$, *be the* $L_{p'}(Q^T)$*-solution of* $BS'_{Q^T}(a; \Gamma)$. *Furthermore, let* Γ' *and* $d\Pi'$ *be defined as* Γ *and* $d\Pi$ *with* h' *instead of* h, *and*

$$d\Gamma \;\equiv\; h'(\partial^2_{y^2} - \partial_y)d\Pi \;+\; h(\partial^2_{y^2} - \partial_y)d\Pi' \;.$$

Then

$$\|d\Gamma\|_{L_{p''}(Q^T)} \;\leq\; C''_{p''} \; \|h\|_{H^1(Q)} \; \|h'\|_{H^1(Q)} \;,$$

where $C''_{p''} \equiv C''_{p''}(\underline{t}, \overline{T}, \overline{k}; R, \underline{a}, \overline{a})$. *We shall then denote by* $d^2\Pi$, *or* $d^2\Pi_{T,k}(\cdot; a).(h, h')$, *the* $L_{p''}(Q^T)$*-solution of* $BS'_{Q^T}(a; d\Gamma)$.

4. *We have:*

$$\begin{aligned}
\|d\Pi\|_{\mathcal{C}^0_\theta(\overline{Q}^T)} &\leq C' \; \|d\Pi\|_{W^{1,2}_p(Q^T)} \leq \; C'C \; \|h\|_{H^1(Q)} \\
\|d^2\Pi\|_{\mathcal{C}^0_\theta(\overline{Q}^T)} &\leq C' \; \|d^2\Pi\|_{W^{1,2}_p(Q^T)} \leq C'C \; \|h\|_{H^1(Q)}\|h'\|_{H^1(Q)} \;,
\end{aligned}$$

where $C' \equiv C'_p$ *is as in (3.4), and* $C \equiv C_p(\underline{t}, \overline{T}, \overline{k}; R, \underline{a}, \overline{a})$. *Moreover, if* $a + h \in \mathcal{M}_Q(\underline{a}, \overline{a})$, *let us define, for* $\varepsilon \in]0, 1[$:

$$\begin{aligned}
\varepsilon^{-1}\delta_\varepsilon\Pi &\equiv \varepsilon^{-1}[\Pi_{T,k}(\cdot; a + \varepsilon h) - \Pi_{T,k}(\cdot; a)] \\
\varepsilon^{-1}\delta_\varepsilon d\Pi &\equiv \varepsilon^{-1}[d\Pi_{T,k}(\cdot; a + \varepsilon h).h' - d\Pi_{T,k}(\cdot; a).h'] \;.
\end{aligned}$$

When $\varepsilon \to 0$, $\varepsilon^{-1}\delta_\varepsilon\Pi$ *and* $\varepsilon^{-1}\delta_\varepsilon d\Pi$ *converge in* $\mathcal{C}^0_\theta(\overline{Q}^T) \cap W^{1,2}_p(Q^T)$ *respectively to* $d\Pi$ *and* $d^2\Pi$.

5. *Assume furthermore that* a, *and for* $n \in \mathbb{N}^\star$, a_n, *belong to* $a_0 + H^1_Q(\underline{a}, \overline{a})$, *where* $a_n - a$ *converges to* 0 *weakly in* $H^1(Q)$ *when* $n \to +\infty$. *Then* $\Pi_n \equiv \Pi_{T,k}(\cdot; a_n)$ *converges to* $\Pi \equiv \Pi_{T,k}(\cdot; a)$ *in* $\mathcal{C}^0_\theta(\overline{Q}^T) \cap W^{1,2}_p(Q^T)$.

Notice that $d\Pi$ and $d^2\Pi$ in this proposition are well defined, by theorem 4.2.

Proof. The proof is deferred to Appendix B. □

5. Stability.

5.1. The ill-posed calibration problems. Let us now give a rigorous statement of the calibration problems. From now on, we assume $p \in]2, \overline{p}[$, and we shall denote by $\overset{\circ}{W}{}_p^{1,2}(Q_{t_0})$ the set of functions in $W_p^{1,2}(Q_{t_0})$ that vanish at time t_0. We also fix a finite subset $\mathcal{F} \subset Q_{t_0}$ with $|\mathcal{F}| = M \in \mathbb{N}^\star$. Then we define the following nonlinear *pricing functional*:

$$a_0 + H_Q^1(\underline{a}, \overline{a}) \ni a \overset{\Pi}{\longmapsto} \Pi\,(a) \in \Pi_0 + \overset{\circ}{W}{}_p^{1,2}(Q_{t_0}) ,$$

where Π_0, respectively $\Pi(a)$, denotes the $L_{p,loc}(Q_{t_0})$-solution between 0 and S_0 of $DUP_{Q_{t_0}}(y_0; a_0)$, respectively $DUP_{Q_{t_0}}(y_0; a)$. Recall that $a_0 \in \mathcal{M}_Q(\underline{a}, \overline{a})$ denotes the *prior* on a.

PROPOSITION 5.1. *The pricing functional* Π *and the restriction* $\Pi_{|\mathcal{F}}$ *are well defined, on the closed convex subset* $a_0 + H_Q^1(\underline{a}, \overline{a})$ *of* $a_0 + H^1(Q)$*. Moreover:*

1. *(Compactness)* Π *and* $\Pi_{|\mathcal{F}}$ *map weakly convergent sequences into strongly convergent ones.*
2. *(Differentiability)* Π *and* $\Pi_{|\mathcal{F}}$ *are twice Gateaux differentiable.*
3. *(Perturbations of the operator)* $\Pi_{|\mathcal{F}}$ *has* θ*-Hölderian dependence with respect to* (t_0, y_0) *and* $\mathcal{F}$*.*

Proof. By theorems 4.2 and 4.3, Π and $\Pi_{|\mathcal{F}}$ are well defined. Now, points 1, 2 and 3 respectively follow from the results symmetric to proposition 4.4.5, 4.4.4 and 4.4.2 in the variables (T, k) . □

DEFINITION 5.2. *By* the continuous calibration problem *with data*

$$\widetilde{\Pi} \in \Pi_0 + \overset{\circ}{W}{}_p^{1,2}(Q_{t_0}) ,$$

respectively the discrete calibration problem *with data* $\pi \in \mathbb{R}^M$*, we shall mean, finding an* $a \in a_0 + H_Q^1(\underline{a}, \overline{a})$*, such that:*

$$\widetilde{\Pi}_{T,k} = \Pi_{T,k}(t_0, y_0; a) \ , \ (T, k) \in Q_{t_0}$$

respectively

$$\pi_{T,k} = \Pi_{T,k}(t_0, y_0; a) \ , \ (T, k) \in \mathcal{F} \ .$$

Data for which this is possible will be said to be calibrateable.

REMARK 5.3. To fix notations, we thus consider the calibration problems with European call option prices. However, by symmetry and parity, all the results below extend straightaway to the following situations:

1. (Continuous problem) Calibration from European put option prices.
2. (Discrete problem) Calibration from European call and put option prices.

A nonlinear inverse problem is said to be *ill-posed* at any data set around which the direct operator (here, the pricing functional Π or $\Pi_{|\mathcal{F}}$) is not continuously invertible.

THEOREM 5.4. *For every continuous function* $a \in a_0 + H_Q^1(\underline{a}, \overline{a})$*, the continuous calibration problem is ill-posed at* $\widetilde{\Pi} \equiv \Pi(a)$*, and the discrete calibration problem is ill-posed at* $\pi \equiv \Pi_{|\mathcal{F}}(a)$*.*

Proof. See Appendix C. □

5.2. Stabilization by Tikhonov regularization. The best known stabilization method, for ill-posed nonlinear inverse problems, is Tikhonov regularization [39, 21], which we now consider. The properties of the nonlinear pricing functional Π, summed up at proposition 5.1, will allow us to apply the general theory surveyed, for instance, in Engl *et al* [21, Chapter 10].

In practice, market prices π are defined as bid-ask spreads. Moreover, $\widetilde{\Pi}$ depends on an interpolation procedure. Therefore, the actual set of observed prices, or input data for the calibration, π^δ or $\widetilde{\Pi}^\delta$, is only known up to some noise δ. Moreover, any numerical procedure used to tackle the discrete calibration problem, entails some computational burden η. Furthermore, the local volatility function is calibrated at the current phase (t_0, y_0) and set $\mathcal{F}$, for later use at the perturbed phase (t_0^μ, y_0^μ) and set $\mathcal{F}_\mu$. The Tikhonov regularization method allows one to overcome such data noise, computational burden and perturbations of the operator.

Definition 5.5. *(Continuous problem) By an* α-solution *of the continuous calibration problem with prior* a_0 *and noisy data*

$$\widetilde{\Pi}^\delta \in \Pi_0 + \overset{\circ}{W}{}_p^{1,2}(Q_{t_0}) ,$$

we shall mean, in $a_0 + H_Q^1(\underline{a}, \overline{a})$*, any* a_α^δ *such that for every* a*:*

$$J_\alpha^\delta\left(a_\alpha^\delta\right) \le J_\alpha^\delta(a)$$

where

$$2J_\alpha^\delta(a) \equiv \left\| \Pi(t_0, y_0, a) - \widetilde{\Pi}^\delta \right\|^2_{W_p^{1,2}(Q_{t_0})} + \alpha \left\| a - a_0 \right\|^2_{H^1(Q)} .$$

(Discrete problem) By an α-solution *of the discrete calibration problem with prior* a_0*, noisy data* $\pi^\delta \in \mathbb{R}^M$*, perturbed parameters* $(t_0^\mu, y_0^\mu) \in Q$*,* $\mathcal{F}_\mu \subset Q_{t_0^\mu}$ *with* $|\mathcal{F}_\mu| = M$*, and computational burden* $\eta \ge 0$*, we shall mean, in* $a_0 + H_Q^1(\underline{a}, \overline{a})$*, any* $a_\alpha^{\delta,\mu,\eta}$ *such that for every* a*:*

$$J_\alpha^{\delta,\mu}\left(a_\alpha^{\delta,\mu,\eta}\right) \le J_\alpha^{\delta,\mu}(a) + \eta$$

where

$$2J_\alpha^{\delta,\mu}(a) \equiv \left\| \Pi_{|\mathcal{F}_\mu}(t_0^\mu, y_0^\mu, a) - \pi^\delta \right\|^2_{\mathbb{R}^M} + \alpha \left\| a - a_0 \right\|^2_{H^1(Q)} .$$

Such α-solutions do exist, because of proposition 5.1.1. We shall not address in this paper the problem of the uniqueness of the unregularized calibration problems, or of the regularized problems for arbitrary values of the regularization parameter α. However, at least for the discrete problem, one has the following result when α tends to $+\infty$. The intuition behind this result is that when α tends to $+\infty$, the regularization term becomes dominant and enforces the convexity of the cost criterion as a whole.

Theorem 5.6. *There exists* $C \equiv (1 + \pi^\delta) M C_p(\underline{t}, \overline{y}_0, \overline{T}, \overline{k}; R, \underline{a}, \overline{a})$*, such that the cost criterion* $J \equiv J_\alpha^{\delta,\mu}$ *is C-strongly convex on* $a_0 + H_Q^1(\underline{a}, \overline{a})$*, for every* $\alpha \ge 2C$*. Here,* $\overline{y}_0$ *and* $\overline{k}$ *denote bounds on* $|y_0^\mu|$ *and* $|k|$ *for* $(T, k) \in \mathcal{F}_\mu$*.*

$J_\alpha^{\delta,\mu}$ then admits a unique minimum, that depends continuously upon (t_0^μ, y_0^μ), $\mathcal{F}_\mu$ and π^δ. Otherwise said, the minimization problem of $J_\alpha^{\delta,\mu}$ is well posed in the sense of Hadamard.

Proof. By the chain rule, we have:

$$\begin{aligned} d^2J(a).(h,h') &\equiv \alpha\langle h,h'\rangle_{H^1(Q)} \\ &+ \sum_{(T,k)\in\mathcal{F}_\mu} d\Pi_{T,k}(t_0^\mu,y_0^\mu;a).h\ d\Pi_{T,k}(t_0^\mu,y_0^\mu;a).h' \\ &+ \sum_{(T,k)\in\mathcal{F}_\mu} \left(\Pi_{T,k}(t_0^\mu,y_0^\mu;a)-\pi^\delta\right)\ d^2\Pi_{T,k}(t_0^\mu,y_0^\mu;a).(h,h')\ . \end{aligned}$$

For $a,b\in a_0+H^1_Q(\underline{a},\overline{a})$, and $\varepsilon\in]0,1[$, let us define $a_\varepsilon\equiv(1-\varepsilon)a+\varepsilon b$, $J_\varepsilon\equiv J(a_\varepsilon)$. Using proposition 4.4.4 and the bound $e^{y_0^\mu}$ on $|\Pi|$, it follows, denoting by $'$ the derivative with respect to ε:

$$\langle\nabla J(b)-\nabla J(a),b-a\rangle_{H^1(Q)} = J_1'-J_0' = \int_0^1 J_\varepsilon'' d\varepsilon$$

$$= \int_0^1 d^2J(a_\varepsilon).(b-a,b-a) \geq (\alpha-(1+e^{y_0^\mu}+\pi^\delta)MC)\ \|b-a\|^2_{H^1(Q)}\ ,$$

where $C\equiv C_p(\underline{t},\overline{T},\overline{k};R,\underline{a},\overline{a})$. □

Moreover, Tikhonov regularized solutions of the calibration problems at arbitrary level $\alpha>0$ are *stable*, in the following meaning.

THEOREM 5.7. *(Stability, continuous problem) Assume* $\widetilde{\Pi}^{\delta_n}\to\widetilde{\Pi}^\delta$ *when* $n\to+\infty$*. Then any sequence of* α*-solutions* $a_\alpha^{\delta_n}$ *admits a subsequence which converges towards an* α*-solution* a_α^δ*.*

(Stability, discrete problem) Assume

$$\pi^{\delta_n}\ ,\ (t_0^{\mu_n},y_0^{\mu_n})\ ,\ \mathcal{F}_{\mu_n}\ ,\ \eta_n\longrightarrow\pi^\delta\ ,\ (t_0^\mu,y_0^\mu)\ ,\ \mathcal{F}_\mu\ ,\ \eta\equiv 0$$

when $n\to+\infty$*. Then any sequence of* α*-solutions* $a_\alpha^{\delta_n,\mu_n,\eta_n}$ *admits a subsequence which converges towards an* α*-solution* $a_\alpha^{\delta,\mu,\eta\equiv 0}$*.*

Notice that this convergence is strong in $H^1(Q)$.

Proof. Using proposition 5.1.1, this results directly from theorem 2.1 in Engl *et al* [22], supplemented by remark 3.4 in Binder *et al* [5], for the continuous problem. For the discrete problem, the proof is an immediate adaptation of the one in [22, theorem 2.1], using propositions 5.1.1 and 5.1.3. □

6. Convergence and convergence rates.

6.1. Convergence. We are going to see that the Tikhonov regularization method behaves as an approximating scheme for the pseudo-inverse of Π or $\Pi_{|\mathcal{F}}$. By *pseudo-inverse*, we mean the operator that maps calibrateable data $\widetilde{\Pi}$ or π, to an element a which minimizes $\|a-a_0\|$ over the set of all pre-images of $\widetilde{\Pi}$ or π through Π or $\Pi_{|\mathcal{F}}$.

DEFINITION 6.1 (a_0-MNS). *Given calibrateable data, we shall call an* a_0-minimal norm solution *(a_0-MNS) of the calibration problem, any solution* a *that minimizes* $\|a-a_0\|$ *over the set of all solutions.*

Such an a_0–MNS a exists, for all calibrateable data. But it may be nonunique, since the pricing functional Π is nonlinear.

THEOREM 6.2. *(Convergence, continuous problem) Given calibrateable data* $\widetilde{\Pi}$*, suppose that*

$$\left\|\widetilde{\Pi}-\widetilde{\Pi}^{\delta_n}\right\|_{W_p^{1,2}(Q_{t_0})}\leq\delta_n \quad \textit{for} \quad n\in\mathbb{N}$$

$$\alpha_n\quad,\quad \delta_n^2/\alpha_n \quad\longrightarrow 0\quad \textit{when}\quad \to+\infty.$$

Then any sequence $a_{\alpha_n}^{\delta_n}$ *admits a subsequence which converges towards an* a_0*-MNS* a*. Moreover,* $a_{\alpha_n}^{\delta_n} \to a$*, if* a *is the unique* a_0*-MNS of the calibration problem at* $\widetilde{\Pi}$*.*

(Convergence, discrete problem) Given calibrateable data π*, suppose that*

$$\left\|\pi - \pi^{\delta_n}\right\|_{\mathbb{R}^M} \leq \delta_n, \quad |t_0 - t_0^{\mu_n}| \vee |y_0 - y_0^{\mu_n}| \vee \|\mathcal{F} - \mathcal{F}_{\mu_n}\| \leq \mu_n \ \ \textit{for} \ \ n \in \mathbb{N}$$
$$\alpha_n, \quad \delta_n^2/\alpha_n, \quad \mu_n^{2\theta}/\alpha_n, \quad \eta_n/\alpha_n \longrightarrow 0 \ \textit{when} \ \ n \to +\infty.$$

Then any sequence $a_{\alpha_n}^{\delta_n,\mu_n,\eta_n}$ *admits a subsequence which converges towards an* a_0*-MNS* a*. Moreover,* $a_{\alpha_n}^{\delta_n,\mu_n,\eta_n} \to a$*, if* a *is the unique* a_0*-MNS of the calibration problem at* π*.*

Proof. Using proposition 5.1.1, this follows directly from theorem 2.3 in Engl *et al* [22], supplemented by remark 3.4 in Binder *et al* [5], for the continuous problem. For the discrete problem, it results for instance from Kunisch and Geymayer [27, proposition 1], using propositions 5.1.1 and 5.1.3. □

Following Engl *et al* [21, proposition 3.11 and remark 3.12], there can be, for the convergence of such regularized schemes towards solutions of ill-posed inverse problems, no uniform rate over all calibrateable data. In fact, this presents a generic character for any method of resolution, Tikhonov or otherwise. It is therefore important to be able to specialize subsets of $a_0 + H_Q^1(\underline{a}, \overline{a})$, on which such uniform rates may be exhibited.

6.2. Convergence rates. We first have the following abstract statement. Let $d\Pi_{|\mathcal{F}}(a)^\star$ and $d\Pi(a)^\star$ denote the adjoints of the operators $d\Pi_{|\mathcal{F}}(a)$ and $d\Pi(a)$, respectively. That is to say, by definition:

$$\langle h \,, d\Pi_{|\mathcal{F}}(a)^\star \lambda \rangle_{H^1(Q)} \;=\; \sum_{(T,k)\in\mathcal{F}} \lambda_{T,k} d\Pi_{T,k}(a).h \;; \quad (h,\lambda) \in H^1(Q) \times \mathbb{R}^M$$

respectively

$$\langle h \,, d\Pi(a)^\star \lambda \rangle_{H^1(Q)} \;=\; \langle d\Pi(a).h \,, \lambda \rangle_{W_p^{1,2}(Q_{t_0}), W_\rho^{1,2}(Q_{t_0})} \;; \quad (h,\lambda) \in H^1(Q) \times W_\rho^{1,2}(Q_{t_0})$$

where $p^{-1} + \rho^{-1} = 1$, and where the last bracket denotes the *duality bracket* between λ and $d\Pi(a).h$.

Theorem 6.3. *(Convergence rates, continuous problem) There exists* $C_p \equiv C_p(\underline{t}, \overline{y}_0, \overline{T}; R, \underline{a}, \overline{a})$*, such that for every* a_0*-MNS* a *of the calibration problem at* $\widetilde{\Pi}$ *with*

$$a - a_0 = d\Pi(a)^\star \lambda \tag{6.1}$$

for some $\|\lambda\|_{W_\rho^{1,2}(Q_{t_0})} \leq C_p$*, then*

$$\|a_\alpha^\delta - a\|_{H^1(Q)} = \mathrm{O}(\delta^{\frac{1}{2}}),$$

whenever

$$\left\|\widetilde{\Pi} - \widetilde{\Pi}^\delta\right\|_{W_p^{1,2}(Q_{t_0})} \leq \delta \quad , \quad \alpha \sim \delta \,.$$

(Convergence rates, discrete problem) There exists $C_p \equiv C_p(\underline{t}, \overline{y}_0, \overline{T}; R, \underline{a}, \overline{a})$*, such that for every* a_0*-MNS* a *of the calibration problem at* π *with*

$$a - a_0 = d\Pi_{|\mathcal{F}}(t_0, y_0; a)^\star \lambda \tag{6.2}$$

for some $\|\lambda\|_{\mathbb{R}^M} \le C_p/\sqrt{M}$, *then*

$$\|a_\alpha^{\delta,\mu,\eta} - a\|_{H^1(Q)} = \mathrm{O}(\delta^{\frac{1}{2}} + \mu^{\frac{\theta}{2}}),$$

whenever

$$\left\|\pi - \pi^\delta\right\|_{\mathbb{R}^M} \le \delta \ , \ |t_0 - t_0^\mu| \vee |y_0 - y_0^\mu| \vee \|\mathcal{F} - \mathcal{F}_\mu\| \le \mu \ , \ \alpha \quad \sim \quad \delta \ \vee \ \mu^\theta \ , \ \eta = \mathrm{O}\left(\delta^2\right) \ .$$

Therefore a is the only a_0-MNS satisfying condition (6.1) or (6.2).

Proof. (Continuous problem) Using propositions 5.1.1 and 5.1.2, this follows from Engl *et al* [21, theorem 10.4 and remark 10.5], by noticing that the proof therein readily extends from their Hilbert → Hilbert to our Hilbert → reflexive Banach setting, by reading duality brackets instead of inner products.

(Discrete problem) Using proposition 5.1, this follows from Kunisch and Geymayer [27, theorem 2 and remark iv p. 86]. □

REMARK 6.4. Kunisch and Geymayer [27, theorem 2] assume that a belongs to the interior of $a_0+H^1_Q(\underline{a},\overline{a})$. However, this cannot be realized in our case. Indeed, $a_0+H^1_Q(\underline{a},\overline{a})$ has empty interior. But this assumption is not used as long as discretization of the source space is not dealt with.

Except in the trivial case where $a \equiv a_0$, conditions (6.1)–(6.2) may seem rather abstract. Whether there is a neighborhood around a_0 such that they are satisfied, is an open question. However, in the case where a is uniformly continuous with respect to its space variable y, one can derive a more explicit formulation of (6.2). In the following, let $\widetilde{\nabla}\Pi_{T,k}$, not to be mistaken with the Gateaux derivative of Π in $H^1(Q)$, denote the following function on Q, parameterized by (t_0, y_0, T, k) and a :

$$\widetilde{\nabla}\Pi_{T,k}(t,y) \ \equiv \ \mathbf{1}_{\{t_0<t<T\}} e^{-y}(\partial^2_{y^2} - \partial_y)\Pi_{t,y}(t_0,y_0;a)(\partial^2_{y^2} - \partial_y)\Pi_{T,k}(t,y;a) \ .$$

LEMMA 6.5. *For* $(T,k) \in \mathcal{F}$,

$$d\Pi_{T,k}(t_0,y_0;a) \ . \ h \ = \ \int\!\!\int_Q \widetilde{\nabla}\Pi_{T,k} \ h \ .$$

Proof. Indeed, this is just the probabilistic representation (4.2) for $d\Pi$, given the expression for γ in theorem 4.3.2 and the L_p-estimate on Γ in proposition 4.4.3. □

THEOREM 6.6. *Let* $a \in a_0 + H^1_Q(\underline{a},\overline{a})$ *be uniformly continuous with respect to its space variable* y*. Then:*

1. $\widetilde{\nabla}\Pi_{T,k} \in L_2(Q)$, *for* $(T,k) \in \mathcal{F}$.
2. $\Lambda \equiv d\Pi_{|\mathcal{F}}(a)^\star\lambda$ *is the unique solution in* $H^2(Q)$ *of the following problem:*

$$\left\{ \begin{array}{l} \Lambda - \Delta\Lambda = \sum_{(T,k)\in\mathcal{F}} \lambda_{T,k} \widetilde{\nabla}\Pi_{T,k} \ , \quad Q\text{-}a.e. \\ \partial_n\Lambda \ = \ 0 \ , \ \partial Q\text{-}a.e. \end{array} \right. \tag{6.3}$$

3. *Condition (6.2) means that (6.3) holds with* $\Lambda \equiv a - a_0$, *for some*

$$\|\lambda\|_{\mathbb{R}^M} \ \le \ C_p(\underline{t}, \overline{y}_0, \overline{T}; R, \underline{a}, \overline{a})/\sqrt{M}.$$

Notice that by theorem 3.2.3, the normal derivative $\partial_n\Lambda \in L_2(\partial Q)$ is well defined, for $\Lambda \in H^2(Q)$.

Proof.

1. According to proposition 4.4.3,

$$(\partial^2_{y^2} - \partial_y)\Pi_{T,k}(t, y; a) \in L_p(]\frac{t_0 + T}{2}, T[\times\mathbb{R}).$$

On the other hand, we have by Stroock–Varadhan [38, theorem 9.1.9, equation (1.35)]:

$$e^{-y}(\partial^2_{y^2} - \partial_y)\Pi_{t,y}(t_0, y_0; a) = \gamma_{t_0,y_0}(t, y; a) \in L_q(]\frac{t_0 + T}{2}, T[\times\mathbb{R}),$$

for every $1 \le q < +\infty$. More precisely,

$$\|\gamma_{t_0,y_0}(\cdot; a)\|_{L_q(]\frac{t_0+T}{2},T[\times\mathbb{R})} \le C^\omega_q(\underline{t}, \overline{T}, R, \underline{a}, \overline{a}) \ ,$$

where ω denotes a modulus of continuity of a with respect to y. Hence $\widetilde{\nabla}\Pi_{T,k} \in L_2(]\frac{t_0+T}{2}, T[\times\mathbb{R})$, by Hölder's inequality. By symmetry and parity, we can conclude that $\widetilde{\nabla}\Pi_{T,k} \in L_2(Q)$.

2. Therefore, using lemma 6.5, the adjunction relations for Λ can be written as

$$\langle\Lambda \ , h\rangle_{H^1(Q)} \ = \ \sum_{(T,k)\in\mathcal{F}} \lambda_{T,k} \ \langle\widetilde{\nabla}\Pi_{T,k}, h\rangle_{L_2(Q)} \ , \quad h \in H^1(Q) \ . \tag{6.4}$$

It is then known that the adjoint $\Lambda \in H^1(Q)$ belongs in fact to $H^2(Q)$ — see for instance Bensoussan–J.L. Lions [3, theorem 5.10, Chapter 2 and the footnote on page 96]. We can then apply the generalized Green formula to identity (6.4) and conclude in a classic way, using theorem 3.2.4; see for example Larrouturou–P.L. Lions [30, p. 150, step 6, Interpretation of the variational formulation].

3. Immediate from 2.

□

Remark 6.7.

1. The condition in theorem 6.6.3, that ensures a convergence rate in $O(\delta^{\frac{1}{2}} + \mu^{\frac{\theta}{2}})$, is very severe, since its implies that $(\mathrm{Id} - \Delta).(a - a_0)$ belongs to the $\le M$-dimensional subspace of $L_2(Q)$ spanned by the $\widetilde{\nabla}\Pi_{T,k}$, $(T, k) \in \mathcal{F}$, for sufficiently small coefficients $\lambda_{T,k}$. Notice also that the severity of this condition tends to relax when M increases.

2. This condition is both a closedness and smoothness condition of a with respect to a_0, which says that, as already noted elsewhere, "Tikhonov regularization can only resolve smooth details fast" [37, p. 611]. Indeed, one then has the following $H^2(Q)$-estimate from regularity theory for elliptic equations (method of tangential translations, see for instance Brézis [9, p.181 and 184])

$$\|a - a_0\|_{H^2(Q)} \ \le \ \sqrt{M} \ C^\omega_p(\underline{t}, \overline{y}_0, \overline{T}; R, \underline{a}, \overline{a}) \ \|\lambda\|_{\mathbb{R}^M}$$

where ω denotes a modulus of continuity of a with respect to y.

3. At least in the Hilbert → Hilbert setting of the discrete problem, there exist conditions stronger than (6.2) ensuring better convergence rates, typically in $O(\delta^{\frac{2}{3}})$, see for instance [33, 34, 21]. But these conditions require that a be interior to the domain of definition of the direct operator — see for instance Neubauer [33, equation (2.5)]. As already observed above, this cannot be realized in our case. Indeed, $H^1_Q(\underline{a}, \overline{a})$ has empty interior. Nonetheless, the reader is referred to Neubauer and Scherzer [37, §3] for a special case in which an $O(\delta^{\frac{2}{3}})$ convergence rate is proved, although the domain of definition of the direct operator has empty interior.

7. Conclusion. Having established $W_p^{1,2}$ estimates for Black–Scholes and Dupire equations with measurable ingredients, we have shown that the problem of inverting observed vanilla option prices into a local volatility function, in a generalized Black–Scholes model, fits into the frame of the Tikhonov regularization method. Moreover, this holds true both when the option prices form a continuum, and when they consist of a finite set. We were then able to derive results for stability, convergence and convergence rates for this method. Discretization and effective implementation, as well as numerical results, are left for later publication [16]. In addition, further work will bear on an extension of the numerical implementation to the problem of calibration from American option prices. With respect to this, an open question is whether the theoretical results obtained in the present article relating to calibration from European option prices, in a generalized Black–Scholes model, may be extended to calibration from American option prices. Another more incidental open question, is whether the continuity assumption is necessary in theorem 6.6.

Appendix A. A technical lemma. The following lemma justifies the passage to the limit at the end of the proof of theorem 4.2. Although it is an adaptation of theorem 3.8 in Caffarelli *et al* [10], using also theorem 2.8 in Crandall *et al* [13], we give the proof in detail for completeness. The notations are the same as above.

LEMMA A.1. *Let us be given* $\Gamma \in \mathcal{D}(\overline{Q}^T)$, *and* $2 < p' < p$. *For* $n \in \mathbb{N}$, *let* φ_n *be a* $L_{p',loc}(Q^T)$*-solution of* $BS'_{Q^T}(a_n;\Gamma)$, *where* a_n *is a Lipschitzian approximation of* a *as in lemma 3.3. Assume the existence of a function* $\varphi \in W^{1,2}_{p,loc}(Q^T)$ *such that* $\varphi_n \to \varphi$ *when* $n \to +\infty$, *locally uniformly on* $\overline{Q}^T$. *Then* φ *is a* $L_{p,loc}(Q^T)$*-solution of* $BS'_{Q^T}(a;\Gamma)$.

Proof. The proof proceeds by contradiction. Assume that φ is, say, no $L_{p,loc}(Q^T)$-viscosity subsolution of $BS'_{Q^T}(a;\Gamma)$. Therefore, there exist open nonempty bounded intervals I and J, a rectangle $Q' = I \times J \subseteq Q^T$ centered at a point $(t_0, y_0) \in Q^T$, and a test function $\psi \in W_p^{1,2}(Q')$, such that:

$$-\partial_t\psi - (r-q-a(t,y))\,\partial_y\psi - a(t,y)\partial^2_{y^2}\psi + r\varphi \;>\; \Gamma + \varepsilon \quad \text{on} \quad Q' \tag{A.1}$$

$$(\varphi-\psi)(t_0,y_0) = 0,\ \varphi - \psi < -\delta \quad \text{on} \quad \partial_p Q' \;. \tag{A.2}$$

Moreover, due to the Hölderian character of φ and ψ through the Sobolev embedding (3.1) on Q', one can assume:

$$\varphi - \psi < -\frac{\delta}{2} \text{ on } \partial_p Q'' \tag{A.3}$$

for some subrectangle Q'' with the same properties as Q', and $\overline{Q}'' \subset Q'$.

We are going to construct a sequence of functions ψ_n (hence, $\psi+\psi_n) \in W^{1,2}_{p',loc}(Q')$, such that

$$\psi_n \;\to\; 0 \;\text{ in } L_\infty(Q'') \text{ as } n \to \infty \tag{A.4}$$

and for n large enough

$$\begin{aligned}-\partial_t(\psi+\psi_n) \quad &- (r-q-a_n(t,y))\,\partial_y(\psi+\psi_n) - a_n(t,y)\partial^2_{y^2}(\psi+\psi_n) + r\varphi_n \\ &\geq \Gamma + \varepsilon \text{ on } Q''.\end{aligned} \tag{A.5}$$

Then by (A.2), (A.3), (A.4), and the assumed local uniform convergence of φ_n to φ, $\varphi_n - (\psi+\psi_n)$ will be larger at (t_0,y_0) than anywhere else on $\partial_p Q''$, for n large enough.

In view of (A.5), this contradicts the assumption that φ_n is a $L_{p',loc}(Q^T)$-viscosity solution of $BS'_{Q^T}(a_n;\Gamma)$.

To construct ψ_n, notice that by (A.1), we have on Q', for ψ_n arbitrary in $W^{1,2}_{p',loc}(Q')$:

$$\begin{aligned}&-\partial_t(\psi+\psi_n)-(r-q-a_n(t,y))\,\partial_y(\psi+\psi_n)-a_n(t,y)\partial^2_{y^2}(\psi+\psi_n)+r\varphi_n-\Gamma\\&\geq\varepsilon+(a-a_n)(\partial^2_{y^2}-\partial_y)\psi-r(\varphi-\varphi_n)\\&\quad-\partial_t\psi_n-(r-q-a_n(t,y))\,\partial_y\psi_n-a_n(t,y)\partial^2_{y^2}\psi_n\\&\geq\varepsilon+\Gamma_n-\partial_t\psi_n-(R+\overline{a})\,|\partial_y\psi_n|-\overline{a}\left(\partial^2_{y^2}\psi_n\right)^+ +\underline{a}\left(\partial^2_{y^2}\psi_n\right)^-\end{aligned}$$

where

$$\Gamma_n\equiv(a-a_n)(\partial^2_{y^2}-\partial_y)\psi-r(\varphi-\varphi_n)\ \rightarrow\ 0\text{ in }L_{p'}(Q')\text{ as }n\rightarrow\infty.$$

Now, choose ψ_n to be, by theorem 2.8 in Crandall *et al* [13], the $L_{p',loc}(Q')$-solution of the following problem:

$$\begin{cases}\partial_t\psi_n+(R+\overline{a})\,|\partial_y\psi_n|+\overline{a}\left(\partial^2_{y^2}\psi_n\right)^+-\underline{a}\left(\partial^2_{y^2}\psi_n\right)^-\ =\ \Gamma_n\quad\text{on }Q'\\ \psi_n\ =\ 0\text{ on }\partial_pQ'\ ,\end{cases}$$

with estimate:

$$\|\psi_n\|_{W^{1,2}_{p'}(Q'')}\ \leq\ C\ \|\Gamma_n\|_{L_{p'}(Q')}\ ,$$

$C\equiv C_{p'}(R,\underline{a},\overline{a},Q',Q'')$ independent of n. Considering the Sobolev embedding (3.1) on Q'', this furnishes the desired sequence ψ_n. □

Appendix B. Proof of proposition 4.4.

1. By theorem 4.3.1, Let us consider Π, respectively $\hat{\Pi}$, the $L_{p,loc}(Q^T)$-solution between 0 and S of $BS_{Q^T}(k;a)$, respectively $BS_{Q^T}(k;\hat{a})$. Then by linearity, symmetry, parity and the asymptotic results in theorem 4.3.1, $\delta\Pi\equiv\Pi-\hat{\Pi}$ converges to 0 when $|y|\rightarrow+\infty$, uniformly with t, and $\delta\Pi$ is a $L_{p,loc}(Q^T)$-solution of $BS'_{Q^T}(a;\Gamma)$, where $\Gamma\equiv(a-\hat{a})(\partial^2_{y^2}-\partial_y)\hat{\Pi}$. Now, it is well known that

$$\|(\partial^2_{y^2}-\partial_y)\hat{\Pi}\|_{L_p(Q^T)}\ \leq\ C_p(\underline{t},\overline{T},\overline{k},R,\underline{a},\overline{a})$$

(see for instance Crépey [14, remark 4.1, Part IV]). Hence (4.12), via (4.3).

2. Let us be given $(t_0,y_0),(t'_0,y'_0),(T,k),(T',k')\in\overline{Q}$, where $t_0\leq t'_0$; $|y_0|,|y'_0|\leq\overline{y}_0$; $|k|,|k'|\leq\overline{k}$; $0<\varepsilon\leq T-t_0,T'-t'_0$. Define Π, $\hat{\Pi}$, $\delta\Pi$ as above. Then using the estimates (4.3), (4.12) and the results symmetric in the variables (T,k), and using also well known results related to $\hat{\Pi}$ which is explicitely given by the Black–Scholes formula, it follows:

$$\begin{aligned}&|\Pi_{T,k}(t_0,y_0)-\Pi_{T',k'}(t'_0,y'_0)|\\&\leq|\Pi_{T,k}(t_0,y_0)-\Pi_{T',k'}(t_0,y_0)|+|\Pi_{T',k'}(t_0,y_0)-\Pi_{T',k'}(t'_0,y'_0)|\\&\leq|\delta\Pi_{T,k}(t_0,y_0)-\delta\Pi_{T',k'}(t_0,y_0)|+|\hat{\Pi}_{T,k}(t_0,y_0)-\hat{\Pi}_{T',k'}(t_0,y_0)|\\&+|\delta\Pi_{T',k'}(t_0,y_0)-\delta\Pi_{T',k'}(t'_0,y'_0)|+|\hat{\Pi}_{T',k'}(t_0,y_0)-\hat{\Pi}_{T',k'}(t'_0,y'_0)|\\&\leq\|\delta\Pi_{\cdot}(t_0,y_0)\|_{\mathcal{C}^0_\theta(\overline{Q}_{t_0})}\left(|T-T'|^\theta+|k-k'|^\theta\right)+C^\varepsilon_p(\underline{t},\overline{y}_0,\overline{T},\overline{k};R,\underline{a},\overline{a})\,(|T-T'|+|k-k'|)\\&+\|\delta\Pi_{T',k'}(\cdot)\|_{\mathcal{C}^0_\theta(\overline{Q}^{T'})}\left(|t_0-t'_0|^\theta+|y_0-y'_0|^\theta\right)+C^\varepsilon_p(\underline{t},\overline{y}_0,\overline{T},\overline{k};R,\underline{a},\overline{a})\,(|t_0-t'_0|+|y_0-y'_0|)\\&\leq C'_p(C_p(\underline{t},\overline{y}_0,\overline{T};R,\underline{a},\overline{a})\vee C_p(\underline{t},\overline{T},\overline{k};R,\underline{a},\overline{a}))\left(|T-T'|^\theta+|k-k'|^\theta+|t_0-t'_0|^\theta+|y_0-y'_0|^\theta\right)\\&+C^\varepsilon_p(\underline{t},\overline{y}_0,\overline{T},\overline{k};R,\underline{a},\overline{a})\,(|T-T'|+|k-k'|+|t_0-t'_0|+|y_0-y'_0|)\ .\end{aligned}$$

3. Using (4.12), we obtain if ${p'}^{-1} = p^{-1} + \rho^{-1}$, by Hölder's inequality

$$\begin{aligned} \|h(\partial^2_{y^2} - \partial_y)\Pi\|_{L_{p'}(Q^T)} &\le \|h\|_{L_\rho(Q^T)} \ \|(\partial^2_{y^2} - \partial_y)\Pi\|_{L_p(Q^T)} \\ &\le C'_{p'} \ \|h\|_{H^1(Q)} \ , \end{aligned}$$

through the Sobolev embedding in theorem 3.2.1. Using estimates (4.12) for Π and (4.3) for $d\Pi$ and $d\Pi'$, we obtain similarly:

$$\begin{aligned} \|d\Gamma\|_{L_{p''}(Q^T)} &\le \|h'(\partial^2_{y^2} - \partial_y)d\Pi\|_{L_{p''}(Q^T)} \ + \ \|h(\partial^2_{y^2} - \partial_y)d\Pi'\|_{L_{p''}(Q^T)} \\ &\le \|h'\|_{L_\nu(Q^T)} \ \|(\partial^2_{y^2} - \partial_y)d\Pi\|_{L_{p'}(Q^T)} \\ &\quad + \ \|h\|_{L_\nu(Q^T)} \ \|(\partial^2_{y^2} - \partial_y)d\Pi'\|_{L_{p'}(Q^T)} \\ &\le C''_{p''} \ \|h\|_{H^1(Q)} \ \|h'\|_{H^1(Q)} \ . \end{aligned}$$

4. The estimates for $d\Pi$ and $d^2\Pi$ result from point 3 and theorem 4.2. Let us additionally suppose that $a + h \in \mathcal{M}_Q(\underline{a}, \overline{a})$. By linearity as above, $\varepsilon^{-1}\delta_\varepsilon\Pi$ is the $L_p(Q^T)$-solution of $BS'_{Q^T}(a + \varepsilon h; \Gamma)$, and $\varepsilon^{-1}\delta_\varepsilon\Pi$ converges in $\mathcal{C}^0_\theta(\overline{Q}^T) \cap W^{1,2}_p(Q^T)$, when $\varepsilon \to 0$, towards the solution $d\Pi$ of $BS'_{Q^T}(a; \Gamma)$. Similarly, $\varepsilon^{-1}\delta_\varepsilon d\Pi$ is the $L_p(Q^T)$-solution of $BS'_{Q^T}(a + \varepsilon h; d\Gamma_\varepsilon)$, where

$$\begin{aligned} d\Gamma_\varepsilon \equiv\ & h(\partial^2_{y^2} - \partial_y)d\Pi_{T,k}(\cdot; a).h' \\ & + \ h'(\partial^2_{y^2} - \partial_y)\left[\varepsilon^{-1}(\Pi_{T,k}(\cdot; a + \varepsilon h) - \Pi_{T,k}(\cdot; a))\right] \ . \end{aligned}$$

Now, $d\Gamma_\varepsilon$ converges in $L_p(Q^T)$ to $d\Gamma$ when $\varepsilon \to 0$. Therefore, $\varepsilon^{-1}\delta_\varepsilon d\Pi$ converges in $\mathcal{C}^0_\theta(\overline{Q}^T) \cap W^{1,2}_p(Q^T)$ to $d^2\Pi$ when $\varepsilon \to 0$.

5. Having fixed $\varepsilon > 0$, and $2 < p < p' < \overline{p}$, define ρ such that $p^{-1} = {p'}^{-1} + \rho^{-1}$. By (4.12), we can choose a subset $Q_\varepsilon \equiv Q^T \cap \{|y| \le Y_\varepsilon\}$ such that $\|(\partial^2_{y^2} - \partial_y)\Pi\|_{L_p(Q^c_\varepsilon)} \le \varepsilon$, where $Q^c_\varepsilon \equiv Q^T \setminus Q_\varepsilon$. By the assumed weak convergence of $a_n - a$ to 0, and the Sobolev compact embedding (3.2), $a_n - a$ converges to 0 in $L_\rho(Q_\varepsilon)$. Denoting ${\Gamma'}_n \equiv (a_n - a)(\partial^2_{y^2} - \partial_y)\Pi$, it follows, in the same manner as in the proof of theorem 4.3, that ${\Gamma'}_n$ converges to 0 in $L_p(Q^T)$. The $L_p(Q^T)$-solution $\Pi_n - \Pi$ of $BS'_{Q^T}(a_n; {\Gamma'}_n)$ then converges to 0 in $\mathcal{C}^0_\theta(\overline{Q}^T) \cap W^{1,2}_p(Q^T)$ when $n \to +\infty$, by theorem 4.2.

Appendix C. Proof of theorem 5.4.

We are going to construct, for $n \in \mathbb{N}^\star$, $a_n \in a_0 + H^1_Q(\underline{a}, \overline{a})$ which takes values in the vicinity of a, such that, when $n \to +\infty$, $a_n - a$ converges to 0 weakly in $H^1(Q)$. Hence, by proposition 5.1.1, $\Pi_\cdot(t_0, y_0; a_n) - \Pi_\cdot(t_0, y_0; a)$ converges to 0 in $\mathcal{C}^0_\theta(\overline{Q}_{t_0}) \cap W^{1,2}_p(Q_{t_0})$. But no subsequence of $a_n - a$ will converge to 0 strongly in $H^1(Q_{t_0})$. Therefore Π or $\Pi_{|\mathcal{F}}$ cannot be continuously invertible around $\widetilde{\Pi} = \Pi(a)$ or $\pi = \Pi_{|\mathcal{F}}(a)$.

Since $\underline{a} < \overline{a}$, and because a is continuous, there exists on open subset $\mathcal{R} \subset Q_{t_0}$ on which $\underline{a} + \varepsilon \le a$ or $a + \varepsilon \le \overline{a}$, for some well chosen $\varepsilon > 0$. Let us assume for instance that $a + \varepsilon \le \overline{a}$ on a rectangle $\mathcal{R} =]t_1, t_2[\times]0, \varepsilon[$, as well as on the union $\mathcal{T}$ of the two equilateral triangles adjacent to the time boundaries of $\mathcal{R}$, with $\mathcal{R} \cup \mathcal{T} \subset Q_{t_0}$. Let us define $a_n - a = u_n$ to be the continuous function on Q, such that:

1. On $\mathcal{R}$, u_n is a continuous function of the space variable y alone, that vanishes at both sides of the space interval $]0, \varepsilon[$, and oscillates between the values 0 and $1/2n$ inbetween. More precisely, $\partial_y u_n = -1$ or $+1$ according to whether $E\{2ny/\varepsilon\}$ is odd or even.

2. On the left and right of $\mathcal{R}$, u_n decreases to 0 at unit speed with respect to the time variable, then vanishes identically.

3. Outside $\mathcal{R} \cup \mathcal{T}$, u_n vanishes identically.

Therefore, u_n vanishes identically outside $\mathcal{R}$, except on a set of measure tending to 0 as $n \to \infty$. Moreover, for every n, we have on Q:

$$0 \le u_n \le \varepsilon/2n \le \varepsilon \ , \ |\partial_t u_n| \le 1 \ , \ |\partial_y u_n| \le 1 \ .$$

So, by construction, $u_n = a_n - a \in H^1(Q)$, and

$$a_n = (a_n - a) + (a - a_0) + a_0 \in a_0 + H^1_Q(\underline{a}, \overline{a}) \ .$$

Moreover, for $n \in \mathbb{N}^\star$, $|\partial_y u_n| \equiv 1$ on $\mathcal{R}$, so that no subsequence of u_n can converge to 0 strongly in $H^1(Q_{t_0})$. But u_n converges to 0 weakly in $H^1(Q)$. Indeed, for any regular test function $\psi(t, y)$, let us define $\phi(y) = \int_{t=t_1}^{t_2} \partial_y \psi \ dt$. Then

$$\int \int_{\mathcal{R}} (\partial_y u_n) (\partial_y \psi) \ dydt = \int_{y=0}^{\varepsilon} (\partial_y u_n) \, \phi(y) \ dy = -\int_{y=0}^{\varepsilon} u_n \phi'(y) \ dy \ ,$$

by integration by parts. Since $|u_n| \le \varepsilon/2n$, this converges to 0 when $n \to \infty$. The rest of the verification is straightforward.

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