Piecewise Constant Martingales and Lazy Clocks

Christophe Profeta, Universit d'Evry, France
christophe.profeta@univ-evry.fr
and
Frédéric Vrins*, Universit catholique de Louvain, Belgium
frederic.vrins@uclouvain.be

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Abstract

Conditional expectations (like e.g. discounted prices in financial applications) are martingales under an appropriate filtration and probability measure. When the information flow arrives in a punctual way, a reasonable assumption is to assume the latter to have piecewise constant sample paths between the random times of information updates. Providing a way to find and construct *piecewise constant martingales* evolving in a connected subset of $\mathbb{R}$ is the purpose of this paper. After a brief review of standard possible techniques, we propose a construction based on the sampling of latent martingales $\tilde{Z}$ with *lazy clocks* θ. These θ are time-change processes staying in arrears of the true time but that can synchronize at random times to the real clock. This specific choice makes the resulting time-changed process $Z_t = \tilde{Z}_{\theta_t}$ a martingale (called a *lazy martingale*) without any assumption on $\tilde{Z}$, and in most cases, the lazy clock θ is adapted to the filtration of the lazy martingale Z, so that sample paths of Z on $[0,T]$ only requires sample paths of $(\theta,\tilde{Z})$ up to T. This would not be the case if the stochastic clock θ could be ahead of the real clock, as typically the case using standard time-change processes. The proposed approach yields an easy way to construct analytically tractable lazy martingales evolving on (interval of) $\mathbb{R}$.

1 Introduction

Martingales play a central role in probability theory, but also in many applications. This is specifically true in mathematical finance where it can be used to model Radon- Nikodym derivative processes or discounted prices in arbitrage-free market models [10]. More generally, it is very common to deal with conditional expectation processes $Z = (Z_t, t \in [0,T])$, $Z_t := \mathbb{E}[Z_T|\mathcal{F}_t]$ where $(\mathcal{F}_t, t \in [0,T])$ is a reference filtration and $\mathbb{E}$ stands for the expectation operator associated to a given probability measure $\mathbb{P}$. Many different modeling setups have been proposed to represent the dynamics of Z (e.g. random walk, Brownian motion, Geometric Brownian motion, Jump diffusion, etc) depending on some assumptions about its range, pathwise continuity or continuous vs discrete-time setting. In many circumstances however, information can be considered to arrive at some random times, or in a partial (punctual way).

An interesting application in that respect is the modeling of *quoted recovery* rates. The recovery rate r of a firm corresponds to the ratio of the debt that will be recovered after the firm's default during an auction process. It is also a major factor driving the price of corporate bonds or other derivatives instruments likes credit default swaps or credit linked notes. In many standard models (like those suggested by the International Swaps and Derivatives Association, ISDA), the recovery rate process is assumed constant (see e.g. [12]). Many studies stressed the fact that r is in fact not a constant: it cannot be observed prior to the firm's default τ ; r is a $\mathcal{F}_\tau$ -measurable random variable in $[0,1]$. This simple observation can have serious consequences in terms of pricing and risk-management of credit sensitive products, and explains the development of stochastic recovery models [3, 2]. A further development in

*Corresponding author. Center for Operations Research and Econometrics, Voie du Roman Pays 34, 1348 Louvain-la-Neuve, Belgium

credit risk modeling is to take into account the fact that recovery rates can be "dynamized" [6]. Quoted recovery rates for instance can thus be modeled as a stochastic process $R = (R_t, t \geq 0)$ that gives the "market's view" of a firm's recovery rate as seen from time t. Hence, $R_t := \mathbb{E}[r|\mathcal{F}_t]$ can be seen as a martingale evolving in the unit interval. By correlating R with the creditworthiness of the firm, it becomes possible to account for a well-known fact in finance: recovery rate and default probability are statistically linked [1].

However, observations for the process R are limited: updates in recovery rate quotes arrive in a scarce and random way. Therefore, in contrast with the common setup, it is more realistic to represent R as a martingale whose trajectories remain constant for long period of times, but "changes" only occasionally, upon arrival of related information (e.g. when a dealer updates its view to specialized data providers). This very specific behavior may have important consequences in terms of hedging, for instance. More generally, such type of martingales could be used to model discounted price processes of financial instruments, observed under partial (punctual) information, e.g. at some random times, but also to represent price processes of illiquid products. Without additional information indeed, a reasonable approach may consist in assuming that discounted prices remain constant between arrivals of market quotes, and jump to the level given by the new quote when a new trade is done. Whereas discrete-time and continuous martingales have been extensively studied in the literature, very little work has been done with respect to martingales having piecewise constant sample paths.

In this paper, we propose a methodology to find and construct such type of martingales Z. The paper is organized as follows. In Section 2 we formally introduce the concept of piecewise constant martingales whereas Section 3 presents several routes to construct these processes. We then introduce a different approach in Section 4, where (Z_t) is modeled as a time-changed process $(\tilde{Z}_{\theta_t})$. This method proves to be very flexible as the time-changed Z and the latent processes $\tilde{Z}$ have the same range. We focus on specific time-change processes θ called lazy clocks. They are chosen in such a way that the stochastic clock always stays in arrears of the real clock ($\theta_t \leq t$ a.s.). By doing so, sampling Z over a fixed time horizon $[0, T]$ only requires the sampling of $(\tilde{Z}, \theta)$ over the same period, thereby limiting the computational effort for simulating trajectories of Z. Finally, as our objective is to provide a workable methodology, we derive the analytical expression for the distributions and moments in some particular cases, and provide efficient sampling algorithms for the simulations of such martingales.

2 Piecewise constant martingales

In the literature, *pure jump processes* defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t, 0 \leq t \leq T)$ and $\mathcal{F} := \mathcal{F}_T$, are often referred to as stochastic processes having no diffusion part. In this paper we are interested in a subclass of pure jump (PJ) processes: *piecewise constant (PWC) martingales* defined as follows.

Definition 1 (Piecewise constant martingale)**.** *A piecewise constant $\mathbb{F}$-martingale Z is a càdlàg $\mathbb{F}$-martingale whose jumps $\Delta Z_s = Z_s - Z_{s^-}$ are summable (i.e. $\sum_{s \leq T} |\Delta Z_s| < +\infty$ a.s.) and such that for every $t \geq 0$:*

$$Z_t = Z_0 + \sum_{s \leq t} \Delta Z_s.$$

In particular, the sample paths of $Z(\omega)$ for $\omega \in \Omega$ belong to the class of piecewise constant functions of time.

Note that an immediate consequence of this definition is that a PWC martingale has finite variation. Such type of processes may be used to represent martingales observed under partial (punctual) information, e.g. at some (random) times. One possible field of application is mathematical finance, where discounted price processes are martingales under an equivalent measure. Without additional information, a reasonable approach may consist in assuming that discounted prices remain constant between arrivals of market quotes, and jump to the level given by the new quote when a new trade is done. More generally, this could represent conditional expectation processes (i.e. "best guess") where information arrives in a discontinuous way.

Pure jump martingales can easily be obtained by taking the difference of a pure jump increasing process with a predictable, grounded, right-continuous process of bounded variation (called *compensator*). The simplest example is probably the compensated Poisson process of parameter λ defined by $(M_t = N_t - \lambda t,\ t \geq 0)$. This process is a pure jump martingale with piecewise *linear* sample paths, hence is not a PWC martingale as $\sum_{s\leq t} \Delta M_s = N_t \neq M_t$. Clearly, not all martingales having no diffusion term are piecewise linear. For example, the Azéma martingale M defined as

$$M_t := \mathbb{E}[W_t|\sigma(\mathrm{sign}(W_s))_{s\leq t}] = \mathrm{sign}(W_t)\sqrt{\frac{\pi}{2}}\sqrt{t-g_t^0(W)} \quad , \quad g_t^0(W) := \sup\{s \leq t, W_s = 0\} \tag{1}$$

where W is a Brownian motion, is essentially piecewise square-root (see e.g. Section 8 of [13] for a detailed analysis of this process). Similarly, the Geometric Poisson Process $\mathrm{e}^{N_t \log(1+\sigma)-\lambda\sigma t}$ is a positive martingale with piecewise negative exponential sample paths [18, Ex 11.5.2].

3 Construction schemes

Most of the "usual" martingales with no diffusion term fail to have piecewise constant sample paths. However, finding such type of processes is not difficult. We provide below three different methods to construct such type of processes. Yet, not all are equally powerful in terms of tractability. The last method proves to be quite appealing in that it yields PWC martingales whose range can be any connected set.

3.1 An autoregressive construction scheme

We start by looking at a subset of PWC martingales, namely step martingales. These are martingales whose paths belong to the space of step functions on any bounded interval. As a consequence, a step martingale Z admits a finite number of jumps on $[0,T]$ taking places at, say $(\tau_k,\ k \geq 1)$, and may be decomposed as (with $\tau_0 = 0$)

$$Z_t = Z_0 + \sum_{k=1}^{+\infty}(Z_{\tau_k} - Z_{\tau_{k-1}})1_{\{\tau_k \leq t\}}.$$

Looking at such decomposition, we see that step martingales may easily be constructed by an autoregressive scheme.

Proposition 1. *Let $(M_n, n \in \mathbb{N})$ be a martingale such that $\sup_{i\geq 1}\mathbb{E}[|M_i - M_{i-1}|] < +\infty$. Let $(\tau_k,\ k \geq 1)$ be an increasing sequence of random times, independent from M, and set $A_t := \sum_{k=1}^{+\infty} 1_{\{\tau_k \leq t\}}$. We assume that $\mathbb{E}[A_t] < +\infty$. Then, the process*

$$Z_t := M_0 + \sum_{k=1}^{+\infty}(M_k - M_{k-1})1_{\{\tau_k \leq t\}} = M_{A_t}$$

is a step martingale with respect to its natural filtration.

Proof. We first have

$$\mathbb{E}[|Z_t|] \leq \mathbb{E}[|M_0|] + \left(\sup_{i\geq 1}\mathbb{E}[|M_i - M_{i-1}|]\right)\sum_{k=1}^{+\infty}\mathbb{P}(\tau_k \leq t) < +\infty$$

which proves that Z_t is integrable. The martingale property is then an immediate consequence of the increasing time change A. □

Example 1. *Let N be a Cox process with intensity $\lambda = (\lambda_t)_{t\geq 0}$ and $\tau_1, \ldots, \tau_{N_t}$ be the sequence of jump times of N on $[0,t]$ with $\tau_0 := 0$. If $(Y_k, k \geq 1)$ is a family of independent and centered random variables, then*

$$Z_t := Z_0 + \sum_{k=1}^{\infty} Y_k 1_{\{\tau_k \leq t\}} = Z_0 + \sum_{k=1}^{N_t} Y_k \ , \quad Z_0 \in \mathbb{R}$$

is a PWC martingale. Note that we may choose the range of such a PWC martingale by taking bounded random variables. For instance, if $Z_0 = 0$ and for any $k \geq 1$,

$$\mathbb{P}\left(\frac{6a}{\pi^2 k^2} \leq Y_k \leq \frac{6b}{\pi^2 k^2}\right) = 1$$

with $a < 0 < b$, then for any $t \geq 0$, we have $Z_t \in [a, b]$ a.s.

The above construction scheme provides us with a simple method to construct PWC martingales. Yet, it suffers from two restrictions. First, the distribution of Z_t requires averaging the conditional distribution with respect to the Poisson distribution of rate λ, i.e. an infinite sum. Second, a control on the range of the resulting martingale requires strong assumptions. One might try to relax the i.i.d. assumption of the Y_i's. In Example 1 the Y_i's are independent but their support decreases as $1/k^2$. One could also draw Y_i from a distribution whose support is state dependent like $[a - Z_{\tau_{i-1}}, b - Z_{\tau_{i-1}}]$, then $Z_t \in [a, b]$ for all $t \in [0, T]$. By doing so however, we typically loose the tractability of the distribution. In the previous example, the characteristic function (CF) may read as an infinite sum (over the Poisson states) of products (of increasing size) of CF associated to the distribution of the random variables (Y_i). In the sequel, we address these drawbacks by proposing another construction scheme.

3.2 PWC martingales from PJ martingales with vanishing compensator

As hinted in the introduction, PWC martingales can be easily obtained by taking the difference of two pure jump processes whose compensators cancel out. We give below two examples.

Lemma 1. *Let Z^1, Z^2 be two i.i.d. Lévy processes with Lévy measure ν, and consider a measurable function f such that $f(0) = 0$ and $\int_{\mathbb{R}} |f(x)|\nu(dx) < +\infty$. Then the process*

$$M_t = \sum_{s \leq t} f(\Delta Z_s^1) - f(\Delta Z_s^2)$$

is a PWC symmetric martingale.

Proof. The proof is a straightforward consequence of the fact that $\sum_{s \leq t} f(\Delta Z_s^1) - t \int_{\mathbb{R}} f(x)\nu(dx)$ is a martingale. □

Remark 1. *A centered Lévy process Z is a PWC martingale if and only if it has no drift, no Brownian component and its Lévy measure ν satisfies the integrability condition $\int_{\mathbb{R}} |x|\nu(dx) < \infty$.*

As obvious examples, one can mention the difference of two independent Gamma or Poisson processes of same parameters. Note that stable subordinators are not allowed here, as they do not fulfill the integrability condition. We give below the PDF of these two examples :

Example 2. *Let N^1, N^2 be two independent Poisson processes with parameter λ. Then, $Z := N^1 - N^2$ is a step martingale taking integer values, with marginals given by the Skellam distribution with parameters $\mu_1 = \mu_2 = \lambda$:*

$$f_{Z_t}(k) = e^{-2\lambda t} I_{|k|}(2\lambda t), \qquad k \in \mathbb{Z} , \tag{2}$$

where I_k is the modified Bessel function of the first kind.

Example 3. *Let γ^1, γ^2 be two independent Gamma processes with parameters $a, b > 0$. Then, $Z := \gamma^1 - \gamma^2$ is a PWC martingale with marginals given by*

$$f_{Z_t}(z) = \frac{b}{\sqrt{\pi}\Gamma(at)} \left|\frac{bz}{2}\right|^{at-\frac{1}{2}} K_{\frac{1}{2}-at}\left(b|z|\right) , \tag{3}$$

where K_β denotes the modified Bessel function of the second kind with parameter $\beta \in \mathbb{R}$.

Proof. The PDFof Z_t is given, for $2at > 1$, by the inverse Fourier transform, see [8, p.349 Formula 3.385(9)] :

$$f_{Z_t}(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-iuz}}{\left(1 + i\frac{u}{b}\right)^{at} \left(1 - i\frac{u}{b}\right)^{at}} du .$$

The result then follows by analytic continuation. □

We conclude this section with an example of PWC martingale which does not belong to the family of Lévy processes but has the interesting feature to evolve in a time-dependent range.

Lemma 2. *Let R^1, R^2 be two independent squared Bessel processes of dimension $\delta \in (0,2)$. For $i = 1, 2$ set*

$$g_t^0(R^i) := \sup\{s \leq t,\ R_s^i = 0\}.$$

Then, $M := g^0(R^1) - g^0(R^2)$ is a 1-self-similar PWC martingale which evolves in the cone $\{[-t,t], t \geq 0\}$.

Proof. We denote by $L^0(R)$ the local time at 0 of R given by Tanaka's formula. Set

$$Y_t = (t - g_t^0(R))^{1-\frac{\delta}{2}}, \qquad t \geq 0.$$

By Rainer [14], the process $X = Y - \frac{1}{2^{2-\frac{\delta}{2}}\Gamma\left(2-\frac{\delta}{2}\right)} L_t^0(R)$ is a martingale. We shall prove that

$$\left(\frac{2}{2-\delta} g_t^0(X) - t,\, t \geq 0\right)$$

is also a martingale, hence, by difference so will be M. Applying Itô's formula to Y with the function $f(y) = y^{\frac{2}{2-\delta}}$, we obtain

$$t - g_t^0(R) = \int_0^t \frac{2}{2-\delta} Y_{s-}^{\frac{\delta}{2-\delta}} dY_s + \sum_{s\leq t} Y_s^{\frac{2}{2-\delta}} - Y_{s-}^{\frac{2}{2-\delta}} - \frac{2}{2-\delta}(s - g_{s-}^0(R))^{\frac{\delta}{2}} \Delta Y_s$$

Observe next that the jumps of Y, i.e. the event $\{s;\, g_s^0(R) \neq g_{s-}^0(R)\}$ is included in the set $\{s;\, Y_s = 0 \cap g_s^0(R) = s\}$. This yields the simplifications :

$$\begin{aligned} t - g_t^0(R) &= \frac{2}{2-\delta}\int_0^t Y_{s-}^{\frac{\delta}{2-\delta}} dY_s + \sum_{s\leq t} -(s - g_{s-}^0(R)) + \frac{2}{2-\delta}(s - g_{s-}^0(R))^{\frac{\delta}{2}}(s - g_{s-}^0(R))^{1-\frac{\delta}{2}} \\ &= \frac{2}{2-\delta}\int_0^t Y_{s-}^{\frac{\delta}{2-\delta}} dY_s + \left(\frac{2}{2-\delta} - 1\right)\sum_{s\leq t}(g_s^0(R) - g_{s-}^0(R)) \\ &= \frac{2}{2-\delta}\int_0^t Y_{s-}^{\frac{\delta}{2-\delta}} dY_s + \left(\frac{2}{2-\delta} - 1\right) g_t^0(R) \end{aligned}$$

and it remains to prove that the stochastic integral is a martingale. Since the support of $dL^0(R)$ is included in $\{s;\, R_s = 0\} \subset \{s; Y_s = 0\}$, and $L^0(R)$ is continuous, we deduce that

$$\int_0^t Y_{s-}^{\frac{\delta}{2-\delta}} dL_s^0(R) = \int_0^t Y_s^{\frac{\delta}{2-\delta}} dL_s^0(R) = 0$$

hence

$$\int_0^t Y_{s-}^{\frac{\delta}{2-\delta}} dY_s = \int_0^t Y_{s-}^{\frac{\delta}{2-\delta}} dX_s$$

which is indeed a martingale. Finally, the self-similarity of M comes from that of $g^0(R)$, which further implies that for $t \geq 0$:

$$M_t \sim t\,(g_1^0(R^1) - g_1^0(R^2)) \quad \in [-t,t]\,.$$

□

Remark 2. *When $\delta = 1$, we have $X = W^2$ where W is a standard Brownian motion. Using Lévy Arcsine law, the PDF of Z_1 is given by the convolution, for $z \in [0,1]$:*

$$f_{Z_t}(z) = \frac{1}{\pi^2}\int_0^{1-z} \frac{1}{\sqrt{x(1-x)}} \frac{1}{\sqrt{(z+x)(1-z-x)}} dx = \frac{2}{\pi^2} F\left(\frac{\pi}{2}, \sqrt{1-z^2}\right),$$

where F denotes the incomplete elliptic integral of the first kind, see [8, p.275, Formula 3.147(5)]. This yields, by symmetry and scaling :

$$f_{Z_t}(z) = \frac{2}{\pi^2}\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{t^2\cos^2(x) + z^2\sin^2(x)}}\, 1_{\{0<|z|\leq t\}}$$

Both the recursive and the vanishing compensators approaches are rather restrictive in terms of attainable range and analytical tractability. In the next section, we provide a more general method that can be used to build PWC martingales to any connected set of $\mathbb{R}$ in a simple and tractable way.

3.3 PWC martingales using time-changed techniques

In this section, we construct a PWC martingale Z by time-changing a latent $(\mathbb{P},\mathbb{F})$-martingale $\tilde{Z} = (\tilde{Z}_t)_{t\geq 0}$ with the help of a suitable *time-change process* θ.

Definition 2. *A $\mathbb{F}$ time-change process $\theta = (\theta_t)_{\in[0,T]}$ is a stochastic process satisfying*

- $\theta_0 = 0$
- *for any $t \in [0,T]$, θ_t is $\mathcal{F}_t$-measurable (i.e. θ is adapted to the filtration $\mathbb{F}$)*
- *the map $t \mapsto \theta_t$ is càdlàg a.s. non-decreasing*

Under mild conditions stated below, $Z := (\tilde{Z}_{\theta_t})_{t\geq 0}$ is proven to be a martingale with respect to its own filtration, with the desired piecewise constant behavior. Most results regarding time-changed martingales deal with continuous martingales time-changed with a continuous process [5, 10, 15]. This does not provide a satisfactory solution to our problem as the resulting martingale will obviously have continuous sample paths. On the other hand, it is obvious that not all time-changed martingales remain martingales, so that conditions are required on $\tilde{Z}$ and/or on θ.

Remark 3. *Every $\mathbb{F}$-semi-martingale time-changed with a $\mathbb{F}$-adapted process remains a semi-martingale but not necessarily a martingale. For instance, setting $\tilde{Z} = W$ and $\theta_t = \inf\{s : W_s > t\}$ then $\tilde{Z}_{\theta_t} = t$. Also, even if θ is independent from $\tilde{Z}$, Z may fail to be a martingale in the above filtration because of integrability issues. For example if $\tilde{Z} = W$ and θ is an independent α-stable subordinator with $\alpha = 1/2$ then the time-changed process Z is not integrable: $\mathbb{E}[|\tilde{Z}_{\theta_t}|\ |\theta_t] = \sqrt{\frac{2}{\pi}}\sqrt{\theta_t}$ and $\mathbb{E}[\sqrt{\theta_t}]$ is undefined.*

Proposition 2. *Let $\tilde{Z}$ be a martingale, and θ be a time-change process independent from $\tilde{Z}$. Assume that any of the following assumptions hold :*

1. *$\tilde{Z}$ is a positive martingale,*
2. *$\tilde{Z}$ is uniformly integrable,*
3. *there exists an increasing function k such that $\theta_t \leq k(t)$ a.s. for all t.*

Then $Z := (\tilde{Z}_{\theta_t})_{t\geq 0}$ is a martingale.

Proof. The martingale property of Z is obvious, and the proof that Z is integrable follows from the fact that $|\tilde{Z}|$ is a submartingale, which implies that $s \mapsto \mathbb{E}\left[|\tilde{Z}_s|\right]$ is a non-decreasing function of s. □

From a practical point of view, general time-changed processes θ that are unbounded on $[0,T]$ may cause some problems. Indeed, to simulate sample paths for Z on $[0,T]$, one needs to simulate sample paths for $\tilde{Z}$ on $[0,\theta_T]$. This is annoying as θ_T can take arbitrarily large values. Hence, the class of time changed processes θ that are bounded by some function k on $[0,T]$ for any $T < \infty$ whilst preserving analytical tractability prove to be quite interesting. This is of course violated by most of the standard time change processes (e.g. integrated CIR, Poisson, Gamma, or Compounded Poisson subordinators). A naive alternative consists in capping the later but this would trigger some difficulties. Using $\theta_t = N_t \wedge t$ would mean that $Z = Z_0$ on $[0,1]$ whilst if we choose $\theta_t = J_t \wedge t$ the resulting process may have linear pieces (hence not be piecewise constant). There exist however simple time change processes θ satisfying $\sup_{s\in[0,t]} \theta_s \leq k(t)$ for some functions k bounded on any closed interval and being piecewise constant, having stochastic jumps and having a non-zero possibility to jump in any time set of non-zero measure. Building PWC martingales using such type of processes is the purpose of next section.

4 Lazy martingales

We first present a stochastic time-changed process that satisfies this condition in the sense that the calendar time is always ahead of the stochastic clock that is, satisfies the boundedness requirement of the above lemma with the linear boundary $k(t) = t$. We then use the later to create PWC martingales.

4.1 Lazy clocks

We would like to define stochastic clocks that keep time frozen almost everywhere, can jump occasionally, but can't go ahead of the real clock. Those stochastic clocks would then exhibit the piecewise constant path and the last constraint has the nice feature that any stochastic process Z adapted to $\mathbb{F}$, $Z_t \in \mathcal{F}_t$ is also adapted to $\mathbb{F}$ enlarged with the filtration generated by θ. In particular, we do not need to know the value of Z after the real time t. As far as Z is concerned, only the sample paths of Z (in fact $\tilde{Z}$) up to $\theta_t \leq t$ matters. In the sequel, we consider a specific class of such processes, called *lazy clocks* hereafter, that have the specific property that the stochastic clock typically "sleeps" (i.e. is "on hold"), but gets synchronized to the calendar time at some random times.

Definition 3. *The stochastic process $\theta : \mathbb{R}^+ \to \mathbb{R}^+$, $t \mapsto \theta_t$ is a $\mathbb{F}$-*lazy clock *if it satisfies the following properties*

i) it is a time change process: in particular, it is grounded ($\theta_0 = 0$), càdlàg and non-decreasing;

ii) it has piecewise constant sample paths : $\theta_t = \sum_{s \leq t} \Delta\theta_s$;

iii) it can jump at any time and, when it does, it synchronizes to the calendar clock.

In the sense of this definition, Poisson and Compound Poisson processes are examples of subordinators that keep time frozen almost everywhere but are not lazy clocks however as nothing constraints them to reach t if they jump at t. Neither are their capped versions as there are some intervals during which θ cannot jump or grows linearly. We start by giving a simple example of lazy clocks.

Remark 4. *Note that for each $t > 0$, the random variable θ_t is a priori not a $(\mathcal{F}_s, s \geq 0)$-stopping time. In fact, defining*

$$C_t := \inf\{s \ ; \ \theta_s > t\}$$

then $(C_t, t \geq 0)$ is an increasing family of $\mathbb{F}$-stopping times. Conversely, for every $t \geq 0$, the lazy clock θ is a family of $(\mathcal{F}_{C_s}, s \geq 0)$-stopping times, see Revuz-Yor [15, Chapter V].

In the following, we shall show that lazy clocks are essentially linked with last passage times, as illustrated in the next proposition.

Proposition 3. *A process θ is a lazy clock if and only if there exists a càdlàg process A such that the set $\mathcal{Z} := \{s;\ A_{s^-} = 0 \text{ or } A_s = 0\}$ has a.s. zero Lebesgue measure and $\theta_t = g_t$ with*

$$g_t := \sup\{s \leq t;\ A_{s^-} = 0 \text{ or } A_s = 0\}$$

Proof. If θ is a lazy clock, then the result is immediate by taking $A_t = \theta_t - t$ which is càdlàg, and whose set of zeroes coincides with the jumps of θ, hence is countable. Conversely, fix a path ω. Since A is càdlàg, the set $\mathcal{Z}(\omega) = \{s; A_{s^-}(\omega) = 0 \text{ or } A_s(\omega) = 0\}$ is closed, hence its complementary may be written as a countable union of disjoint intervals. We claim that

$$\mathcal{Z}^c(\omega) = \bigcup_{s \geq 0}]g_{s^-}(\omega), g_s(\omega)[. \tag{4}$$

Indeed, observe first that since $s \longmapsto g_s(\omega)$ is increasing, its has a countable number of discontinuities, hence the union on the right hand side is countable. Furthermore, the intervals which are not empty are such that $A_s(\omega) = 0$ or $A_{s^-}(\omega) = 0$ and $g_s(\omega) = s$. In particular, if $s_1 < s_2$ are associated with non empty intervals, then $g_{s_1}(\omega) = s_1 \leq g_{s_2^-}(\omega)$ which proves that the intervals are disjoint.
Now, let $u \in \mathcal{Z}^c(\omega)$. Then $A_u(\omega) \neq 0$. Define $d_u(\omega) = \inf\{s \geq u, A_{s^-}(\omega) = 0 \text{ or } A_s(\omega) = 0\}$.

By right-continuity, $d_u(\omega) > u$. We also have $A_{u^-}(\omega) \neq 0$ which implies that $g_u(\omega) < u$. Therefore, $u \in]g_u(\omega), d_u(\omega)[$ which is non empty, and this may also be written $u \in]g_{d_u^-(\omega)}(\omega),\, g_{d_u(\omega)}(\omega)[$ which proves the first inclusion. Conversely, it is clear that if $u \in]g_{s^-}(\omega), g_s(\omega)[$, then $A_u(\omega) \neq 0$ and $A_{u^-}(\omega) \neq 0$. Otherwise, we would have $u = g_u(\omega) \leq g_{s^-}(\omega)$ which would be a contradiction. Equality (4) is thus proved. Finally, it remains to write :

$$g_t = \int_0^{g_t} 1_{\mathcal{Z}} ds + \int_0^{g_t} 1_{\mathcal{Z}^c} ds = \sum_{s\leq t} \Delta g_s$$

since $\mathcal{Z}$ has zero Lebesgue measure.

□

Remark 5. *Note that lazy clocks are naturally involved with PWC martingales. Indeed, if M is a PWC martingale, then $M_t = M_{g_t(M)}$ where $g_t(M) = \sup\{s \leq t,\ \Delta M_s \neq 0\}$ is a lazy clock.*

We give below a few examples of lazy clocks related to last passage times prior a given time t. Whereas some of these random variables (and corresponding distributions) have been studied in the literature, we use last passage times as clocks, i.e. in a dynamic way, as stochastic processes evolving with t.

[Figure 1 : about here. See uploaded files; title & legend provided at the end of the manuscript.]

[Figure 2 : about here. See uploaded files; title & legend provided at the end of the manuscript.]

4.1.1 Poisson Lazy clocks

Let $(X_n, n \geq 1)$ be strictly positive random variables and consider the counting process

$$\left(N_t := \sum_{k=1}^{+\infty} 1_{\{\sum_{i=1}^k X_i \leq t\}},\ t \geq 0\right).$$

Then the process $(g_t(N), t \geq 0)$ defined as the last jump time of N prior to t or zero if N did not jump by time t:

$$g_t(N) := \sup\{s \leq t,\ N_s \neq N_{s^-}\} = \sum_{k=1}^{+\infty} X_k 1_{\{\sum_{i=1}^k X_i \leq t\}}. \tag{5}$$

is a lazy clock. Its cumulative distribution function (CDF) is easily given, for $s \leq t$, by $\mathbb{P}(g_t(N) \leq s) = \mathbb{P}(N_t = N_s)$. If N is a Poisson process with intensity λ, i.e. when the r.v.'s $(X_k,\ k \geq 1)$ are i.i.d. with an exponential distribution of parameter λ, we obtain in particular $\mathbb{P}(g_t(N) \leq s) = e^{-\lambda(t-s)}$, see [19] for similar computations. Sample paths are shown on Fig. 1.

4.1.2 Diffusion Lazy clock

Another simple example is given by the last passage time $g_t^{(a)}(X)$ of a diffusion X to some level a before time t. Its CDF may be written, applying the Markov property :

$$\mathbb{P}(g_t^a(X) \leq s) = \mathbb{E}\left[\mathbb{P}_{X_s}(T_a > t - s)\right]$$

where $T_a = \inf\{u \geq 0 :\ X_u = a\}$.

- Let $b \in \mathbb{R}$ and consider the drifted Brownian motion $(X_t = B_t - bt,\ t \geq 0)$. Then, the probability density function (PDF) of $g_t^a(B - b)$ is given by (see for instance [16] or [11]) :

$$f_{\tilde{g}_t(B-b)}(s) = \frac{\phi\left(\frac{a+bs}{\sqrt{s}}\right)}{\sqrt{s}} \left(\frac{2}{\sqrt{t-s}}\phi\left(b\sqrt{t-s}\right) + 2b\Phi\left(b\sqrt{t-s}\right) - b\right) \quad , \quad 0 < s < t$$

  where Φ denotes the standard Normal CDF Φ and $\Phi' = \phi$. Note that when $a \neq 0$, the distribution of $g_t^{(a)}(B - b)$ may have a mass at 0, see [18, Corollary 7.2.2].

- Let R be a Bessel process with index $\nu \in (-1, 0)$. Then, the PDF of $g_t^0(R)$ is given by the generalized Arcsine law (see [7]) :

$$f_{g_t^0(R)}(s) = \frac{1}{\Gamma(|\nu|)\Gamma(1+\nu)}(t-s)^{\nu} s^{-1-\nu} \quad , \quad 0 < s < t.$$

4.1.3 Stable lazy clock

The generalized Arcsine law also appears when dealing with stable Lévy processes L with parameter $\alpha \in (1,2]$. Then, from Bertoin [4, Chapter VIII, Theorem 12], the PDF of $g_t^0(L)$ is given by :

$$f_{g_t^0(L)}(s) = \frac{\sin(\pi/\alpha)}{\pi} s^{-\frac{1}{\alpha}} (t-s)^{\frac{1}{\alpha}-1} \quad , \quad 0 < s < t.$$

4.2 Time-changed martingales with lazy clocks

In this section we consider a martingale $\tilde{Z}$ whose time is changed with an independent lazy clock θ to obtain a PWC martingale $Z = (Z_{\theta_t}, t \geq 0)$. Note that from item 3 of Proposition 2, the process Z is always a martingale.

We first show that (in most situations) the lazy clock is adapted to the filtration generated by Z. This is done by observing that the knowledge of θ amounts to the knowledge of its jump times, since the size of the jumps are always obtained as a difference with the calendar time. In particular, the properties of the lazy clocks allow one to reconstruct the trajectories of Z on $[0,t]$ only from past values of $\tilde{Z}$ and θ; no information about the future (measured according to the real clock) is required. We then provide the resulting distribution when the clock $g(N)$ is governed by Poisson, inhomogeneous Poisson or Cox processes.

Lemma 3. *Let $\tilde{Z}$ be a stochastic process independent from the lazy clock θ and assume that $\forall u \neq v$, $\mathbb{P}(\tilde{Z}_u = \tilde{Z}_v) = 0$. Then, θ is adapted to the filtration $(\mathcal{F}_t^Z, t \geq 0)$ and Z is a martingale in its natural filtration.*

Proof. Observe that the countable union

$$\mathcal{N} = \bigcup_{s \leq t, \theta_s = s} \{Z_s = Z_{s^-}\}$$

is of measure zero since Z and θ are independent, hence we have a.s. $\theta_t = \sup\{s \leq t;\ Z_s \neq Z_{s^-}\}$, which proves that θ_t is $\mathcal{F}_t^Z$-measurable.

We give below an intuitive interpretation of this result, i.e. by justifying that the sample paths of θ (both the jump times and the jump sizes) can be recovered from the sample paths of Z up to θ_t, hence up to t. First, the set of the jump times of θ on $[0,t]$ is given by $\{s \in [0,t] : \theta_s = s\}$. Moreover, the "synchronization events" $\{\theta_s = s\}$ coincide with the "jump events" $\{Z_s - Z_{s^-} > 0\}$ so that all jump times of θ are identified by the jumps of Z. But θ is constant between two jumps and jumps to a known value (the calendar time) each time Z jumps. This means that both θ_t and $\tilde{Z}_{\theta_t}$ are revealed in $\mathcal{F}_{\theta_t}^Z$ and, in particular, $\mathcal{F}_t^\theta \subseteq \mathcal{F}_{\theta_t}^Z$. The proof is concluded by noting that $\theta_t \leq t$, leading to $\mathcal{F}_{\theta_t}^Z \subseteq \mathcal{F}_t^Z$. □

Lemma 4. *Let $\tilde{Z}$ be a continuous martingale and N an independent Poisson process with intensity λ. Then, Z defined as $Z_t := \tilde{Z}_{g_t(N)}$ is a right-continuous pure jump process with same range as Z and piecewise constant sample paths. Moreover, it is a pure jump martingale with respect to its own filtration and its CDF is given by*

$$F_{Z_t}(z) = \mathbb{P}(Z_t \leq z) = e^{-\lambda t}\left(1_{\{Z_0 \leq z\}} + \lambda \int_0^t F_{\tilde{Z}_u}(z) e^{\lambda u} du\right) \tag{6}$$

Proof. This result is obvious from the independence assumption between $\tilde{Z}$ and N (i.e. $\theta_\cdot = \tau(\cdot)$),

$$F_{Z_t}(z) = \int_0^\infty F_{\tilde{Z}_u}(z) \mathbb{P}(\theta_t \in du) \,. \tag{7}$$

□

A similar result applies to the inhomogeneous Poisson and Cox cases. The proofs are very similar.

Corollary 1. *For inhomogeneous Poisson processes,*

$$F_{Z_t}(z) = e^{-\Lambda(t)}\left(1_{\{Z_0\le z\}} + \int_0^t \lambda(u)F_{\tilde{Z}_u}(z)e^{\Lambda(u)}du\right) \tag{8}$$

Corollary 2. *Let N be a inhomogeneous Poisson process with stochastic intensity (i.e. Cox process) independent from $\tilde{Z}$ and define $P(s,t) := \mathbb{E}[e^{-(\Lambda_t-\Lambda_s)}]$ where $\Lambda_t := \int_0^t \lambda_u du$. Then,*

$$F_{Z_t}(z) = \left(1_{\{Z_0\le z\}}P(0,t) + \int_0^t F_{\tilde{Z}_s}(z)dP(s,t)\right) . \tag{9}$$

Proof. We start from the inhomogeneous Poisson case, set as hazard rate function $\lambda(u)$ for all $u\in[0,T]$ a sample path $\lambda_u(\omega)$ of the stochastic intensity and take the expectation, which amounts to replace $\lambda(u)$ by λ_u (hence $\Lambda(u)$ by Λ_u) and take the expected value of the resulting CDF derived above with respect to the intensity paths:

$$\begin{aligned} F_{Z_t}(z) &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{P}(Z_t\le z)|\lambda(u)=\lambda_u,\ 0\le u\le t\right]\right] &(10)\\ &= 1_{\{Z_0\le z\}}\,\mathbb{E}\left[e^{-\Lambda_t}\right] + \mathbb{E}\left[\int_0^t \lambda_s F_{\tilde{Z}_s}(z)e^{-(\Lambda_t-\Lambda_s)}ds\right] &(11)\\ &= 1_{\{Z_0\le z\}}P(0,t) + \int_0^t F_{\tilde{Z}_s}(z)\,\mathbb{E}\left[\lambda_s e^{-(\Lambda_t-\Lambda_s)}\right]ds &(12)\end{aligned}$$

where in the last equality we have used Tonelli's theorem to exchange the integral and expectation operators when applied to non-negative functions as well as independence between λ and $\tilde{Z}$.

From Leibniz rule, $\lambda_s e^{-(\Lambda_t-\Lambda_s)} = \frac{d}{ds}e^{-(\Lambda_t-\Lambda_s)}$ so

$$\mathbb{E}\left[\lambda_s e^{-(\Lambda_t-\Lambda_s)}\right] = \frac{d}{ds}\,\mathbb{E}\left[e^{-(\Lambda_t-\Lambda_s)}\right] = \frac{d}{ds}P(s,t) . \tag{13}$$

□

Remark 6. *Notice that $P(s,t)$ does not correspond to the expectation of $e^{-\int_s^t \lambda_u du}$ conditional upon $\mathcal{F}_s$, the filtration generated by λ up to s as often the case e.g. in mathematical finance. It is an unconditional expectation that can be evaluated with the help of the tower law. In the specific case where λ is an affine process for example, $\mathbb{E}\left[e^{-\int_s^t \lambda_u du}|\lambda_s = x\right]$ takes the form $A(s,t)e^{-B(s,t)x}$ for some deterministic functions $A,\ B$ so that*

$$P(s,t) = \mathbb{E}\left[e^{-\int_s^t \lambda_u du}\right] = \mathbb{E}\left[\mathbb{E}\left[A(s,t)e^{-B(s,t)\lambda_s}\right]\right] = A(s,t)\varphi_{\lambda_s}(iB(s,t)) .$$

Example 4. *In the case λ follows a CIR process, $d\lambda_t = k(\theta-\lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t$ with $\lambda_0>0$ then $\lambda_s \sim r_s/c_s$ with $c_s = \nu/(\theta(1-e^{-ks}))$ and r_s is a non-central chi-squared random variable with non-centrality parameter $\nu = 4k\theta/\sigma^2$ and $\kappa = c_s\lambda_0 e^{-ks}$ the degrees of freedom. So, $\varphi_{\lambda_s}(u) = \mathbb{E}[e^{i(u/c_s)r_s}] = \varphi_{r_s}(u/c_s)$ where $\varphi_{r_s}(v) = \frac{\exp\left(\frac{\nu i v}{1-2iv}\right)}{(1-2iv)^{\kappa/2}}$.*

4.3 Some Lazy martingales without independence assumption

We have seen that when $\tilde{Z}$ is a martingale and θ an independent lazy clock, then $(Z_t = \tilde{Z}_{\theta_t},\ t\ge 0)$ is a PWC martingale. We now give an example where the lazy time-change θ is not independent from the latent process $\tilde{Z}$.

Proposition 4. *Let B and W be two correlated Brownian motions with coefficient ρ and f a continuous function. Define the Lazy clock :*

$$g_t^f(W) := \sup\{s : s\le t,\ W_s = f(s)\}.$$

Let $h(W)$ be a progressively measurable process with respect to W and assume that there exists a deterministic function ψ such that :

$$\int_0^{g_t^f(W)} h_u(W)dW_u = \psi(g_t^f(W))$$

Then, the process $\left(\int_0^{g_t^f(W)} h_u(W)dB_u - \rho\psi(g_t^f(W)),\ t\ge 0\right)$ is a Lazy martingale.

Proof. Let Z be a Brownian motion independent from W such that $B = \rho W + \sqrt{1-\rho^2}\, Z$. The time-change yields :

$$\begin{aligned}\int_0^{g_t^f(W)} h_u(W)dB_u - \rho\psi(g_t^f(W)) &= \int_0^{g_t^f(W)} h_u(W)dB_u - \rho \int_0^{g_t^f(W)} h_u(W)dW_u \\ &= \sqrt{1-\rho^2}\int_0^{g_t^f(W)} h_u(B)dZ_u \\ &= \sqrt{1-\rho^2} Z_{\int_0^{g_t^f(W)} h_u^2(B)du}\end{aligned}$$

which is a PWC martingale since $g^f(W)$ and $h(B)$ are independent from Z. □

It is interesting to point out here that the latent process $\tilde{Z}_t = \int_0^t h_u(W)dB_u - \rho\psi(t)$ is, in general, *not* a martingale (not even a local martingale). It becomes a martingale thanks to the lazy time-change.

Example 5. *We give below several examples of application of this proposition.*

1. *Take* $h_u = 1$. *Then,* $\psi = f$ *and* $\left(B_{g_t^f(W)} - \rho f(g_t^f(W)),\, t \geq 0\right)$ *is a PWC martingale.*
   More generally, we may observe from the proof above that if H *is a space-time harmonic function (i.e.* $(t,z) \to H(t,z)$ *is* $\mathcal{C}^{1,2}$ *and such that* $\frac{\partial H}{\partial t} + \frac{1}{2}\frac{\partial^2 H}{\partial z^2} = 0$*), then the process*
   $$\left(H\left(B_{g_t^f(W)} - \rho f(g_t^f(W)),\, (1-\rho^2)g_t^f(W)\right),\, t \geq 0\right)$$
   is a PWC martingale. Observe in particular that the latent process here is not, in itself, a martingale.
2. *Following the same idea, take* $h_u(W) = \frac{\partial H}{\partial z}(W_u, u)$ *for some harmonic function* H. *Then*
   $$\int_0^{g_t^f(W)} \frac{\partial H}{\partial z}(W_u,u)dW_u = H\left(W_{g_t^f(W)}, g_t^f(W)\right) - H(0,0) = H\left(f(g_t^f(W)), g_t^f(W)\right) - H(0,0)$$
   and the process $\left(\int_0^{g_t^f(W)} \frac{\partial H}{\partial z}(W_u,u)dB_u - \rho H\left(f(g_t^f(W)), g_t^f(W)\right),\, t \geq 0\right)$ *is a PWC martingale.*
3. *As a last example, take* $f = 0$ *and* $h_u = r(L_u^0)$ *where* r *is a* $\mathcal{C}^1$ *function and* L^0 *denotes the local time of* W *at 0. Then, integrating by parts :*
   $$\int_0^{g_t^f(W)} r(L_u^0)dW_u = r(L_{g_t^f(W)})W_{g_t^f(W)} - \int_0^{g_t^f(W)} W_u r'(L_u^0)dL_u^0 = 0$$
   since the support of dL *is included in* $\{u, W_u = 0\}$. *Therefore, the process* $\left(\int_0^{g_t^f(W)} r(L_u^0)dW_u,\, t \geq 0\right)$ *is a PWC martingale.*

5 Numerical simulations

In this section, we briefly sketch the construction schemes to sample paths of the lazy clocks discussed above. These procedures have been used to generate Fig. ??. Finally, we illustrate sample paths and distributions of a specific martingale in $[0,1]$ time-changed with a Poisson lazy clock.

5.1 Sampling of lazy clock and lazy martingales

By definition, the number of jumps of a lazy clock θ on $[0,T]$ is countable, but may be infinite. Therefore, except in some specific cases (such as the Poisson lazy clock), an exact simulation is impossible. Using a discretization grid, the simulated trajectories of a lazy clock θ on $[0,T]$ will take the form

$$\theta_t := \sup\{\tau_i, \tau_i \leq t\}$$

where $\tau_0 := 0$ and $\tau_1, \tau_2, \ldots$ are (some of) the synchronization times of the lazy clock up to time T. We can thus focus on the sampling times $\tau_1, \tau_2 \ldots$ whose values are no greater than T.

Poisson lazy clock

Trajectories of a Poisson lazy clock $\theta_t(\omega) = g_t(N(\omega))$ on a fixed interval $[0,T]$ are very easy to obtain thanks to the properties of Poisson jump times.

Algorithm 1 (Sampling of a Poisson lazy clock)**.**

1. *Draw a sample* $n = N_T(\omega)$ *for the number of jump times of* N *up to* T*:* $N_T \sim Poi(\lambda T)$*.*
2. *Draw* n *i.i.d. samples from a standard uniform* $(0,1)$ *random variable* $u_i = U_i(\omega)$*,* $i \in \{1,2,\dots,n\}$ *sorted in increasing order* $u_{(1)} \leq u_{(2)} \leq \dots \leq u_{(n)}$*.*
3. *Set* $\tau_i := Tu_{(i)}$ *for* $i \in \{1,2,\dots,n\}$*.*

Brownian lazy clock

Sampling a trajectory for a Brownian lazy clock requires the last zero of a Brownian bridge. This is the purpose of the following lemma.

Lemma 5. *Let* $W^{x,y,t}$ *be a Brownian bridge on* $[0,t]$ *,* $t \leq T$*, starting at* $W_0^{x,y,t} = x$ *and ending* $W_t^{x,y,t} = y$*, and define its last passage time at 0 :*

$$g_t(W^{x,y,t}) := \sup\{s \leq t,\ W_s^{x,y,t} = 0\}.$$

Then, the CDF $F(x,y,t;s)$ *of* $g_t(W^{x,y,t})$ *is given, for* $s \in [0,t]$ *by :*

$$\mathbb{P}(g_t(W^{x,y,t}) \leq s) = F(x,y,t;s) \quad := \quad 1 - e^{-\frac{xy}{t}} \left(d_+(x,y,t;s) + d_-(x,y,t;s)\right) , \tag{14}$$

$$\text{where} \qquad d_\pm(x,y,t;s) \quad := \quad e^{\frac{\pm|xy|}{t}}\, \Phi\left(\mp|x|\sqrt{\frac{t-s}{st}} - |y|\sqrt{\frac{s}{t(t-s)}}\right) . \tag{15}$$

In particular, the probability that $W^{x,y,t}$ *does not hit 0 during* $[0,t]$ *equals:*

$$\mathbb{P}(g_t(W^{x,y,t}) = 0) = F(x,y,t;0) = 1 - e^{-\frac{xy+|xy|}{t}}.$$

Note also the special case when $y = 0$ *:*

$$\mathbb{P}(g_t(W^{x,0,t}) = t) = 1.$$

Proof. Using time reversion and the absolute continuity formula of the Brownian bridge with respect to the free Brownian motion (see Salminen [17]), the PDF of $g_t(W^{x,y,t})$ is given, for $y \neq 0$, by :

$$\mathbb{P}(g_t(W^{x,y,t}) \in ds) = \frac{|y|\sqrt{t}}{\sqrt{2\pi}} e^{\frac{(y-x)^2}{2t}} \frac{1}{\sqrt{s}(t-s)^{3/2}} e^{-\frac{x^2}{2s}} e^{-\frac{y^2}{2(t-s)}}\, ds.$$

Integrating over $[0,t]$, we first deduce that

$$\frac{|y|\sqrt{t}}{\sqrt{2\pi}} \int_0^t \frac{e^{-\frac{x^2}{2s}}}{\sqrt{s}} \frac{e^{-\frac{y^2}{2(t-s)}}}{(t-s)^{3/2}}\, ds = \exp\left(\frac{(|y|+|x|)^2}{2t}\right). \tag{16}$$

We shall now compute a modified Laplace transform of F, and then invert it. Integrating by parts and using (16), we deduce that :

$$\lambda \int_0^t \frac{e^{-\frac{\lambda}{2s}}}{2s^2} F(x,y,t;s) ds = e^{-\frac{\lambda}{2t}} - e^{-\frac{\lambda}{2t}} \exp\left(-\frac{xy}{t} - \frac{|y|\sqrt{\lambda + x^2}}{t}\right).$$

Observe next that by a change of variable :

$$\lambda \int_0^t \frac{e^{-\frac{\lambda}{2s}}}{2s^2} F(x,y,t;s) ds = \lambda e^{-\frac{\lambda}{2t}} \int_0^{+\infty} e^{-\lambda v} F\left(x,y,t;\frac{1}{2v+1/t}\right) dv$$

hence

$$\int_0^{+\infty} e^{-\lambda v} F\left(x,y,t;\frac{1}{2v+1/t}\right) dv = \frac{1}{\lambda} - \frac{1}{\lambda}\exp\left(-\frac{xy}{t} - \frac{|y|\sqrt{\lambda+x^2}}{t}\right)$$

and the result follows by inverting this Laplace transform thanks to the formulae, for $a > 0$ and $b > 0$:

$$\frac{1}{\lambda}\exp\left(-a\sqrt{\lambda+x^2}\right) = \frac{a}{2\sqrt{\pi}}\int_0^{+\infty} e^{-\lambda v}\int_0^v e^{-ux^2}\frac{1}{u^{3/2}}e^{-\frac{a^2}{4u}}du\,dv$$

and

$$\int_0^z e^{-au-b/u}\frac{du}{u^{3/2}} = \frac{\sqrt{\pi}}{2\sqrt{b}}\left(e^{2\sqrt{ab}}\mathrm{Erfc}\left(\sqrt{\frac{b}{z}}-\sqrt{az}\right) + e^{-2\sqrt{ab}}\mathrm{Erfc}\left(\sqrt{\frac{b}{z}}+\sqrt{az}\right)\right).$$

□

Simulating a continuous trajectory of a Brownian lazy clock θ in a perfect way is an impossible task. The reason is that when a Brownian motion reaches zero at a specific time say s, it does so infinitely many times on $(s, s+\varepsilon]$ for all $\varepsilon > 0$. Consequently, it is impossible to depict such trajectories in a perfect way. Just like for the Brownian motion, one could only hope to sample trajectories on a discrete time grid, where the maximum stepsize provides some control about the approximation, and corresponds to a basic unit of time. By doing so, we disregard the specific jump times of θ, but focus on the supremum of the zeroes of a Brownian motion in these intervals. To do this, we proceed as follows.

Algorithm 2 (Sampling of a Brownian lazy clock)**.**

1. *Fix a number of steps n such that time step $\delta = T/n$ corresponds to the desired time unit.*
2. *Sample a Brownian motion $w = W(\omega)$ on the discrete grid $[0, \delta, 2\delta, \ldots, n\delta]$.*
3. *In each interval $((i-1)\delta, i\delta]$, $i \in \{1, 2, \ldots, n\}$, draw a uniform $(0,1)$ random variable $u_i = U_i(\omega)$*
   - *If $u_i < F(w_{(i-1)\delta}, w_{i\delta}, \delta; 0)$ then w does not reach 0 on the interval*
   - *Otherwise, set the supremum g_i of the last zero of w as the s-root of $F(w_{(i-1)\delta}, w_{i\delta}, \delta; s) - u_i$*
4. *Identify the k intervals ($1 \le k \le n$) in which w has a zero, and set $\tau_j := i_j\delta + g_{i_j}$, $j \in \{1, \ldots, k\}$ where $i_j\delta$ is the left bound of the interval.*

Example 6 (PWC martingale on $(0,1)$)**.** *Let N be a Poisson process with intensity λ and $\tilde{Z}$ be the Φ-martingale[9] with constant diffusion coefficient η,*

$$\tilde{Z}_t := \Phi\left(\Phi^{-1}(Z_0)e^{\eta^2/2t} + \eta\int_0^t e^{\frac{\eta^2}{2}(t-s)}dW_s\right) . \qquad (17)$$

Then, the stochastic process Z defined as $Z_t := \tilde{Z}_{g_t(N)}$, $t \geq 0$, is a pure jump martingale on $(0,1)$ with CDF

$$F_{Z_t}(z) = e^{-\lambda t}\left(1_{\{Z_0\le z\}} + \lambda\int_0^t \Phi\left(\frac{\Phi^{-1}(z) - \Phi^{-1}(Z_0)e^{\eta^2/2u}}{\sqrt{e^{\eta^2 u}-1}}\right)e^{\lambda u}du\right) . \qquad (18)$$

Some sample paths for $\tilde{Z}$ and Z are drawn on Fig. 3.Notice that all the martingales $\tilde{Z}$ given above can be simulated without error using the exact solution.

[Figure 3 a & 3 b : about here. See uploaded files; title & legend provided at the end of the manuscript.]

Figure 4 gives the CDF of Z and $\tilde{Z}$ where the later is a Φ-martingale. The main differences between these two sets of curves result from the fact that $\mathbb{P}(\tilde{Z}_t = Z_0) = 0$ for all $t > 0$ while $\mathbb{P}(Z_t = Z_0) = \mathbb{P}(\tilde{Z}_{g_t(N)} = Z_0) = \mathbb{P}(N_t = 0) > 0$ and that there is a delay resulting from the fact that Z_t correspond to some past value of $\tilde{Z}$.

[Figure 4 a - 4 d : about here. See uploaded files; title & legend provided at the end of the manuscript.]

6 Conclusion and future research

Many applications, like mathematical finance, extensively rely on martingales. In this context, discrete- or continuous-time processes are commonly considered. However, in some specific cases like when we work under partial information or when market quotes arrive in a scarce way, it is more realistic to assume that conditional expectations move in a more piecewise constant fashion. Such type of processes didn't receive attention so far, and our paper aims at filling this gap. We focused on the construction of piecewise constant martingales that is, martingales whose trajectories are piecewise constant. Such processes are indeed good candidates to model the dynamics of conditional expectations of random variables under partial (punctual) information. The time-changed approach proves to be quite powerful: starting with a martingale in a given range, we obtain a PWC martingale by using a piecewise constant time-change process. Among those time-change processes, *lazy* clocks are specifically appealing: these are time-change processes staying always in arrears to the real clock, and that synchronizes to the calendar time at some random times. This ensures that $\theta_t \leq t$ which is a convenient feature when one needs to sample trajectories of the time-change process. Such random times can typically be characterized as last passage times, and enjoy appealing tractability properties. The last jump time of a Poisson process before the current time for instance exhibits a very simple distribution. Other lazy clocks have been proposed as well, based on Brownian motions and Bessel processes, some of which rule out the probability mass at zero. We provided several martingales time-changed with lazy clocks (called *lazy martingales*) whose range can be any interval in $\mathbb{R}$ (depending on the range of the latent martingale) and showed that the corresponding distributions can be easily obtained in closed form. Finally, we presented algorithms to sample Poisson and Brownian lazy clocks, thereby providing the reader with a workable toolbox to efficiently use piecewise constant martingales in practice.

This paper paves the way for further research, in either fields of probability theory and mathematical finance. Tractability and even more importantly, the martingale property results from the independence assumption between the latent martingale and the time-change process. It might be interesting however to consider cases where the sample frequency (synchronization rate of the lazy clock θ to the real clock) depends on the level of the latent martingale Z. Finding a tractable model allowing for this coupling remains an open question at this stage. On the other hand, it is yet unclear how dealing with more realistic processes like piecewise constant ones would impact hedging strategies and model completeness in finance. In fact, investigating this route is the purpose of a research project that we are about to initiate.

Declarations

Availability of Data and Material

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Competing interests

The authors declare that they have no competing interests.

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Author's contribution

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Author's information

C.P. is with the *Laboratoire de Mathématiques et Modélisation d'Évry.* He is the author of the monograph *Peacocks and Associated Martingales, With Explicit Constructions* with F. Hirsh, B Roynette and M. Yor. F.V. served as a quantitative analyst on the trading floor of a major European bank, before moving back to academia. He is Research Director o the *Louvain Finance Center* (LFIN) and Faculty Member of the *Center for Operations Research and Econometrics* (CORE).

Abreviations

- PWC : Piecewise Constant Martingale
- CIR : Cox-Ingersoll-Ross
- PDF : Probability Distribution Function
- CDF : Cumulative Distribution Function

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Figure legends

- Figure 1 (Poisson Lazy clock) : Poisson Lazy clock ($\lambda = 3/2$, see Section 4.1.1)
- Figure 2 (Brownian Lazy clock) : Brownian Lazy clock (see Section 4.1.2)
- Figure 3 (Sample Paths of Z) : Four sample paths of $\tilde{Z}$ (circles) and Z (no marker) up to $T = 15$ years where $\tilde{Z}$ is the Φ-martingale with $Z_0 = 0.5$. Fig. 3.a ($\eta = 25\%$, $\lambda = 20\%$) and 3.b ($\eta = 15\%$, $\lambda = 50\%$)
- Figure 4 (CDF of $\tilde{Z}_t$) : CDF of $\tilde{Z}_t$ (circles) and Z_t (no marker) where $\tilde{Z}$ is the Φ-martingale with $Z_0 = 0.5$ and t in 0.5 (blue solid), 5 (red, dashed) and 40 (magenta, dotted) years. Fig. 4.a. ($Z_0 = 50\%$, $\eta = 25\%$, $\lambda = 20\%$), 4.b ($Z_0 = 50\%$ $\eta = 15\%$, $\lambda = 50\%$), 4.c ($Z_0 = 35\%$ $\eta = 15\%$, $\lambda = 50\%$) and 4.d. ($Z_0 = 35\%$, $\eta = 25\%$, $\lambda = 5\%$).