Pricing with Funding and Counterparty Risks

from arbitrage-free pricing to valuation adjustments

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Frontiers in Stochastic Modelling for Finance
Padova 10-11 February 2016

Talk Outline

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Talk Outline

1. Securities, Derivatives and Trading Strategies
   - Market Securities
   - Self-Financing Trading Strategies
   - Funding and Discounting
2. Arbitrage-Free Pricing
3. Wrong-Way Risk and Gap Risk in Derivative Contracts
4. Funding Costs
5. Funding Valuation Adjustments

Market Securities – I

- We start with the simple setting of a market with default-free securities.
  - ⟶ We later add counterparty credit risk, funding costs and collateralization.
- We assume that the market quotes the prices of some securities we name $\{S_t^1, \dots, S_t^n\}$.
- When holding a security we may face the possibility to receive or pay a quantity of cash.
  - ⟶ The owner of a bond receives coupons on a regular basis.
  - ⟶ Share holders receive dividends over time.
  - ⟶ Many bilateral contracts consists in a strip of random cash flows.

Market Securities – II

- We name $\{\gamma_{T_1}, \ldots, \gamma_{T_N}\}$ the coupons, dividends or cash flows received or paid while holding a security
- We define the cumulative dividend process as

$$D_t := \sum_{i=1}^{N} \gamma_{T_i} 1_{\{T_i \leq t\}}$$

- The profits and losses achieved holding a security are described by the gain process, which is defined as

$$G_t := S_t + D_t$$

Trading Portfolios and Total Wealth – I

- A trading strategy in the market securities consists in holding a portfolio of securities.
- We name $\{q^1_t, \ldots, q^n_t\}$ the quantities of each security held in the portfolio.
  - $\longrightarrow$ At each time the trader may change the composition of the portfolio.
  - $\longrightarrow$ The quantities q^i_t may be either positive or negative.
- The total wealth realized by the strategy can be computed by taking into account the profit and losses along time.
- If we can trade only on times $\{t_0 = 0, t_1, \ldots, t_m = t\}$, we can write the total wealth as

$$W_t := \sum_{i=1}^n q^i_{t_0} S^i_{t_0} + \sum_{k=1}^m \sum_{i=1}^n q^i_{t_{k-1}} (G^i_{t_k} - G^i_{t_{k-1}})$$

Trading Portfolios and Total Wealth – II

- We can substitute the definition of gain process in the total wealth formula to highlight how the dividends contribute to it.

$$W_t = \sum_{i=1}^{n} q^i_{t_0} S^i_{t_0} + \sum_{k=1}^{m} \sum_{i=1}^{n} q^i_{t_{k-1}} \left(S^i_{t_k} - S^i_{t_{k-1}} + \sum_{i=1}^{N} \gamma_{T_i} 1_{\{t_{k-1} < T_i \leq t_k\}} \right)$$

- A simple example of trading strategy is entering a position and never changing it, namely q_t does not depend on time.
  - $\longrightarrow$ In this case we should obtain that the total wealth is simply the sum of the gain processes of each security times their quantities.

Trading Portfolios and Total Wealth – III

- The wealth of a constant-quantity trading strategy

$$
\begin{aligned}
W_t &\doteq \sum_{i=1}^{n} q^i S^i_{t_0} + \sum_{i=1}^{n} q^i \sum_{k=1}^{m} \left(S^i_{t_k} - S^i_{t_{k-1}} + \sum_{i=1}^{N} \gamma_{T_i} 1_{\{t_{k-1} < T_i \leq t_k\}} \right) \\
&= \sum_{i=1}^{n} q^i S^i_{t_0} + \sum_{i=1}^{n} q^i \left(S^i_{t_m} - S^i_{t_0} + \sum_{i=1}^{N} \gamma_{T_i} 1_{\{T_i \leq t_m\}} \right) \\
&= \sum_{i=1}^{n} q^i \left(S^i_{t_m} + D_{t_m} \right) \\
&= \sum_{i=1}^{n} q^i G^i_t
\end{aligned}
$$

Self-Financing Trading Strategies – I

- An interesting class of trading strategies is given by the self-financing strategies.
- The wealth process of a self financing strategy is always equal to the liquidation value of the portfolio.

$$W_t \doteq \sum_{i=1}^{n} q_t^i S_t^i$$

- Which is the consequence of such constraint on the quantities q_t ?

Self-Financing Trading Strategies – II

- We focus on the increment in the wealth process over time.

$$W_{t_k} - W_{t_{k-1}} = \sum_{i=1}^{n} q^i_{t_{k-1}} (S^i_{t_k} - S^i_{t_{k-1}} + D^i_{t_k} - D^i_{t_{k-1}})$$

- If we require that the strategy is self-financing, we get

$$W_{t_k} - W_{t_{k-1}} = \sum_{i=1}^{n} (q^i_{t_k} S^i_{t_k} - q^i_{t_{k-1}} S^i_{t_{k-1}})$$

- If we equate the two expressions, we obtain

$$\sum_{i=1}^{n} q^i_{t_k} S^i_{t_k} = \sum_{i=1}^{n} q^i_{t_{k-1}} \left(S^i_{t_k} + D^i_{t_k} - D^i_{t_{k-1}} \right)$$

Self-Financing Trading Strategies – III

- Thus, the quantities are selected so that
  - $\longrightarrow$ dividends are re-invested in the strategy;
  - $\longrightarrow$ further cash is not required and no cash outflow is generated.
- In this sense the strategy is self-financing.
- Some examples are:
  - $\longrightarrow$ A strategy in shares of a company. This strategy is self-financing if, every time a dividend is paid, the trader buys more shares.
  - $\longrightarrow$ A strategy in a zero-coupon bond. At maturity the zero-coupon bond pays the notional, but we cannot re-invest in it since the contract is terminated. We need a second security to build a self-financing strategy.

Trading Strategies in Continuous Time

- In the following we use a continuous-time notation, and we express the cumulative dividend process as

$$D_t := D_0 + \int_0^t d\pi_u \;, \quad \pi_t := \sum_{i=1}^{N} \gamma_{T_i} 1_{\{T_i \leq t\}}$$

while the wealth process for the trading strategy q_t is given by

$$W_t := q_0 \cdot S_0 + \int_0^t q_u \cdot dG_u$$

where the internal products is in security space. If the strategy is self-financing we write

$$W_t \doteq q_t \cdot S_t$$

The Treasury Bank Account – I

- Implementing trading strategies requires to access some cash-paying (and cash-receiving) securities to fund (and to invest) dividends.
  - ⟶ For instance, if we have to pay at a future time T a unit of cash, we can buy a zero-coupon bond paying such cash at T.
- Since trading strategies have their own trading horizons, we wish to access cash-paying (and cash-receiving) securities without a maturity time.
- In practice we need a bank account.
  - ⟶ We can enter into a bank account by paying one unit of cash at inception, and receiving it back at any later time along with a compensation.
  - ⟶ On the other hand, we can also get one unit of cash at inception to pay it back at a later time along with a fee.
- Do bank accounts exist in the market ?

The Treasury Bank Account – II

- On the market we have saving accounts, but their are intended for retail operations.
- Traders may access a special bank account, named the treasury Bank Account (TBA), which is managed by the bank treasury department.
  - ⟶ The TBA is not a real security traded on the market, but it behaves as a security from the point of view of traders.
  - ⟶ The TBA is implemented by the treasury by issuing bonds, using collateral portfolios, accessing saving accounts, etc. . . .
- The compensation rate, received when borrowing cash, and the fees, required when lending cash, are decided by the treasury.

The Treasury Bank Account – III

- If we assume that the lending and borrowing rates are the same, name them r_t, we can calculate the price process B_t of the TBA as the solution of

$$dB_t = r_t B_t \, dt \; , \quad B_0 = 1$$

  namely

$$B_t = \exp\left\{ \int_0^t du \, r_u \right\}$$

- In the following we assume that the TBA is one of the security used to implement trading strategies.

  $\longrightarrow$ We discuss again this assumption when funding costs are introduced.

Price Deflators

- When we say that the price process of a security is given by S_t we are thinking of liquidating the security to obtain an amount of cash equal to S_t.

  $\longrightarrow$ Cash behaves as a unit of measure for prices.

- Yet, we cannot access cash without paying fees or receiving compensations, since we lend and borrow cash by means of the TBA.
- Thus, to take into account the cost of money, we need to express the wealth processes in term of the TBA, namely

$$\bar{W}_t := \frac{W_t}{B_t}$$

  where $\bar{W}$ is the deflated wealth.

- How can we define deflated price and cumulative dividend processes ?

Invariance of Self-Financing Trading Strategies – I

- We require that the property of a trading strategy of being self-financing is invariant under deflation.
  $\longrightarrow$ We define the deflated price and cumulative dividend processes to ensure this property.
- If q_t is a self-financing strategy $(W_t \doteq q_t \cdot S_t)$ we can write

$$\bar{W}_t = \frac{W_t}{B_t} = q_t \cdot \bar{S}_t$$

  where we define the deflated price process as

$$\bar{S}_t := \frac{S_t}{B_t}$$

- The definition of the deflated cumulative dividend process is less obvious, since we must consider that dividends are paid over time, and the TBA value depends on time too.

Invariance of Self-Financing Trading Strategies – II

- Starting from the definition of deflated wealth, we can write

$$\begin{aligned}
\bar{W}_t &= \bar{W}_0 + \int_0^t \left(\frac{dW_u}{B_u} - W_u r_u B_u \, du \right) \\
&= q_0 \cdot S_0 + \int_0^t q_u \cdot \left(\frac{dG_u}{B_u} - S_u r_u B_u \, du \right) \\
&= q_0 \cdot S_0 + \int_0^t q_u \cdot \left(\frac{dS_u}{B_u} - S_u r_u B_u \, du + \frac{dD_u}{B_u} \right) \\
&= q_0 \cdot S_0 + \int_0^t q_u \cdot d\bar{G}_u
\end{aligned}$$

where we define the deflated cumulative dividend and gain processes

$$\bar{D}_t := D_0 + \int_0^t \frac{dD_u}{B_u} \,, \quad \bar{G}_t := \bar{S}_t + \bar{D}_t$$

Invariance of Self-Financing Trading Strategies – III

- If the bank account is risky, as in a foreign-currency account, the definition of the deflated processes must take into account the covariation of the dividend process with the deflator.
- For a generic positive process Y_t (deflator) we can follow Duffie (2001) to write:

$$W_t^Y = q_0 \cdot S_0^Y + \int_0^t q_u \cdot dG_u^Y \ , \quad G_t^Y := S_t^Y + D_t^Y$$

where we define the deflated price and cumulative dividend processes

$$S_t^Y := Y_t S_t \ , \quad D_t^Y := Y_0 D_0 + \int_0^t (Y_u \, dD_u + d\langle Y, D\rangle_u)$$

Talk Outline

Arbitrages – I

- In efficient markets securities are always traded at their fair value.
  $\longrightarrow$ Investors can possibly obtain higher returns only by purchasing riskier investments.
- The possibility "to make money from nothing without risks" should be excluded from the set of possible trading strategies.
  $\longrightarrow$ We name arbitrages such strategies.
- A more formal definition of arbitrage is needed to going on.
- We refer again to Duffie (2001) for the huge literature on arbitrages and their relationship with martingale pricing.

Arbitrages – II

- We introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with
  - $\longrightarrow$ the standard filtration $\mathcal{F} := (\mathcal{F}_t)_{t\geq 0}$ generated by the security price processes, and
  - $\longrightarrow$ the physical probability measure $\mathbb{P}$ representing the actual distribution of supply-and-demand shocks on security prices.
- We can define arbitrages as a self-financing trading strategy q_t whose wealth at inception time t is non-positive, namely

$$W_t \leq 0$$

while at maturity T it is never negative, and it is strictly positive in some state, so that we can write

$$W_T \geq 0 \,, \quad \mathbb{P}\{\, W_T > 0 \,\} > 0$$

- To avoid arbitrages we can impose some conditions on the wealth process W.

Equivalent Martingale Pricing – I

- Given the TBA as price deflator, we can ensure the absence of arbitrages, if we can find a measure $\mathbb{Q}$, equivalent to the physical measure $\mathbb{P}$, such that the deflated gain process $\bar{G}_t$ is a martingale under such measure.

  $\longrightarrow$ The measure $\mathbb{Q}$ is known as risk-neutral measure.

- Arbitrages are forbidden even if we use a generic deflator Y_t.

  $\longrightarrow$ In this case the measure $\mathbb{Q}^Y$ depends on the choice of the deflator, and it is known as equivalent martingale measure.

- The reverse is not true in general.

Equivalent Martingale Pricing – II

- Under suitable technical conditions on the trading strategy q_t, the martingale condition allows us to write

$$\mathbb{E}\left[\,\bar{W}_T \,|\, \mathcal{F}_t\,\right] = \bar{W}_t + \int_t^T \mathbb{E}\left[\,q_u \cdot d\bar{G}_u \,|\, \mathcal{F}_t\,\right] = \bar{W}_t$$

  where the expectations are taken under the risk-neutral measure.

- If q_t is an arbitrage, we have $W_t \leq 0$ and

$$W_T \geq 0 \implies \bar{W}_T \geq 0 \implies \bar{W}_t = \mathbb{E}\left[\,\bar{W}_T \,|\, \mathcal{F}_t\,\right] \geq 0$$

  on the other hand, the equivalence between the measures implies

$$\mathbb{P}\{\,W_T > 0\,\} > 0 \implies \mathbb{Q}\{\,W_T > 0\,\} > 0 \implies \mathbb{Q}\left\{\,\bar{W}_T > 0\,\right\} > 0$$

  leading to $\bar{W}_t > 0$ which contradicts the hypothesis.

Equivalent Martingale Pricing – III

- If we assume the existence of a risk-neutral measure, we can price market securities with maturity date T by exploiting the martingale condition of deflated gain processes.

$$\bar{G}_t = \mathbb{E}\left[\, \bar{G}_T \;|\, \mathcal{F}_t \,\right]$$

- Then, we can expand the gain process to obtain the arbitrage-free pricing formula under $\mathbb{Q}$-expectation

$$S_t = B_t\, \mathbb{E}\left[\frac{S_T}{B_T} + \int_t^T \frac{dD_u}{B_u} \;|\, \mathcal{F}_t \right]$$

or for a generic deflator Y_t under $\mathbb{Q}^Y$-expectation

$$S_t = \frac{1}{Y_t} \mathbb{E}^Y\left[Y_T S_T + \int_t^T \left(Y_u\, dD_u + d\langle Y, D\rangle_u \right) \;|\, \mathcal{F}_t \right]$$

Replication of Derivative Contracts – I

- We can extend pricing formulae to derivative securities not traded on the market.
- We consider a derivative with price process V_t and cumulative dividend process Q_t.
- In order to replicate the derivative in terms of market securities, we can implement a strategy q_t to invest (or to fund) the dividends received (or paid) by the derivative, namely

$$Q_t \doteq W_t - q_t \cdot S_t$$

- Furthermore, we require that at maturity the price of the constituents of the strategy is equal to the price of the derivative.

$$V_T \doteq q_T \cdot S_T$$

Replication of Derivative Contracts – II

- The derivative price can be calculated at any time from the market security prices.
- We consider a trading strategy q' which invests in the market securities as the strategy q_t and shorts one unit of the derivative, namely

$$q'_t := (q_t, -1)$$

- The wealth generated by such strategy is given by

$$W'_t = q_0 \cdot S_0 - V_0 + \int_0^t (q_u \cdot dG_u - dV_u - dQ_u) = q_t \cdot S_t - V_t$$

so that we can conclude that the strategy q' is self-financing with null final wealth, $W'_T = 0$.

Replication of Derivative Contracts – III

- If we require absence of arbitrages, we obtain that at any time $t < T$ we must have

$$W'_T \geq 0 \implies W'_t \geq 0 \implies q_t \cdot S_t \geq V_t$$

On the other hand, we can consider the strategy $(-q_t, 1)$ leading to

$$q_t \cdot S_t \leq V_t$$

Thus, we have at any time t up to matuirty T that

$$V_t = q_t \cdot S_t$$

- We can write that the derivative gain process is equal to the wealth generated by the replicating strategy q_t.

$$W_t = V_t + Q_t$$

Replication of Derivative Contracts – IV

- If we assume the existence of a risk-neutral measure for the market securities, we have that the deflated gain process of the derivative is a martingale too, leading to the pricing equation

$$V_t = B_t \, \mathbb{E}\left[\frac{V_T}{B_T} + \int_t^T \frac{dQ_u}{B_u} \,|\, \mathcal{F}_t \right]$$

or for a generic deflator Y_t under $\mathbb{Q}^Y$-expectation

$$V_t = \frac{1}{Y_t} \mathbb{E}^Y \left[Y_T V_T + \int_t^T (Y_u \, dQ_u + d\langle Y, Q \rangle_u) \,|\, \mathcal{F}_t \right]$$

which can be solved once a terminal condition for V_T is selected.

Market and Enlarged Filtrations – I

- The next element we add to the pricing framework is the possibility of default of one of the counterparties of the contract.
- How can we deal with the default event under the risk-neutral measure?
  - $\longrightarrow$ We need to describe the filtration to adopt to calculate the risk-neutral expectations.
- Market risks for contracts with defaultable counterparties arise from the uncertainty both in default probabilities and in the default times.
  - $\longrightarrow$ We could add risks specific of the underlying asset and recoveries as well.
- As a first step we introduce the market filtration $\mathcal{F}_t$ representing all the observable market quantities but the default events.

Market and Enlarged Filtrations – II

- Then, we define the default events of the counterparty τ_C and of the investor τ_I along with the first default time

$$\tau := \tau_C \wedge \tau_I$$

- We define the enlarged filtration $\mathcal{G}$ containing also the default monitoring.
  $\longrightarrow$ See Bielecki and Rutkowski (2001) for details.

$$\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t^C \vee \mathcal{H}_t^I \supseteq \mathcal{F}_t$$

$$\mathcal{H}_t^k := \sigma(\{\tau_k \leq u\} : u \leq t) \, , \quad k \in \{C, I\}$$

Market and Enlarged Filtrations – III

- From the definition of $\mathcal{G}$, we can write

$$\forall g_t \in \mathcal{G}_t \; \exists f_t \in \mathcal{F}_t : g_t \cap \{\tau_C > t\} \cap \{\tau_I > t\} = f_t \cap \{\tau_C > t\} \cap \{\tau_I > t\}$$

  or simply

$$\forall g_t \in \mathcal{G}_t \; \exists f_t \in \mathcal{F}_t : g_t \cap \{\tau > t\} = f_t \cap \{\tau > t\}$$

- Thus, for any $\mathcal{G}$-adapted process x_t we can introduce the pre-default $\mathcal{F}$-adapted process $\tilde{x}_t$ such that

$$1_{\{\tau > t\}} x_t = 1_{\{\tau > t\}} \tilde{x}_t$$

- We can use this property for numerical implementations to express expectations under the enlarged $\mathcal{G}$ filtration as expectations under the market $\mathcal{F}$ filtration.

Trading Strategies with Defaultable Counterparties – I

- The counterparty credit risk is defined as the risk that the counterparty to a transaction could default before the final settlement of the transaction cash flows.
  - $\longrightarrow$ When one of the counterparty defaults the trade is terminated.
  - $\longrightarrow$ An economic loss would occur if the transaction with the counterparty has a positive economic value at the time of default.
- We can accommodate counterparty risk by terminating the dividend process at the first default event, and setting the terminal condition for the security price accordingly.

$$S_{T\wedge\tau} := 1_{\{\tau\leq T\}}\theta_\tau \ , \quad D_t := D_0 + \int_0^t 1_{\{\tau>u\}} d\pi_u$$

where θ_τ is the cash flow paid if the default occurs, and without loss of generality we set $1_{\{\tau>T\}}S_T \doteq 0$.

Trading Strategies with Defaultable Counterparties – II

- To avoid arbitrages we require that the deflated gain processes are martingale under the $\mathcal{G}$ filtration.
- The pricing equation becomes

$$S_t = B_t \, \mathbb{E}\left[1_{\{\tau \leq T\}} \frac{\theta_\tau}{B_\tau} + \int_t^T 1_{\{\tau > u\}} \frac{dD_u}{B_u} \,|\, \mathcal{G}_t \right]$$

- A similar expression holds for generic deflators Y_t.
- Since credit default risk introduces an element of non-predictability, we cannot implement a replication strategy to price derivative securities, but in simple cases.

  $\longrightarrow$ However, we can price them as any other market security.

Close-Out Netting Rules – I

- In case of default of one party, the surviving party should evaluate the transactions just terminated, due to the default event occurrence, to claim for a reimbursement after the application of netting rules to consolidate the transactions.

  $\longrightarrow$ The amount of the cash flow θ_τ results from such analysis.

- The cash flow θ_τ is described by the ISDA documentation as given by

$$
\begin{aligned}
\theta_\tau &:= 1_{\{\tau_C<\tau_I\}}\left(R_C\varepsilon_\tau^+ + \varepsilon_\tau^-\right) + 1_{\{\tau_I<\tau_C\}}\left(\varepsilon_\tau^+ + R_I\varepsilon_\tau^-\right) \\
&= \varepsilon_\tau \underbrace{- 1_{\{\tau_C<\tau_I\}}(1-R_C)\varepsilon_\tau^+}_{\text{CVA cash flow}} \underbrace{- 1_{\{\tau_I<\tau_C\}}(1-R_I)\varepsilon_\tau^-}_{\text{DVA cash flow}}
\end{aligned}
$$

  where R_C and R_I are the recovery rates, and ε_τ is the close-out amount representing the exposure measured by the surviving party on the default event.

Close-Out Netting Rules – II

- It is difficult to define the close-out amount, and also ISDA is not very assertive on the topic.

  $\longrightarrow$ See Brigo, Morini and Pallavicini (2013) for a review.

- You may have a risk-free close-out, where the residual deal is priced at mid market without any residual counterparty risk.

$$\varepsilon_\tau \doteq B_\tau \, \mathbb{E}\left[\int_{\tau\wedge T}^{T} \frac{d\pi_u}{B_u} \,|\, \mathcal{G}_\tau \right]$$

- You may have a replacement close-out, where the remaining deal is priced by taking into account the credit quality of the surviving party and of the party that replaces the defaulted one.
- A possible guess is the pre-default replacement close-out given by

$$\varepsilon_\tau \doteq \widetilde{S}_\tau$$

Close-Out Netting Rules – III

- The pre-default replacement close-out is the first example of non-linearities in the pricing equation.
- Indeed, if we write the pre-default price we get

$$1_{\{\tau>t\}}\widetilde{S}_t = 1_{\{\tau>t\}} B_t \, \mathbb{E}\left[1_{\{\tau\leq T\}} \frac{\theta_\tau(\widetilde{S}_\tau)}{B_\tau} + \int_t^T 1_{\{\tau>u\}} \frac{dD_u}{B_u} \,\middle|\, \mathcal{G}_t \right]$$

- The above expression is an implicit equation for the the pre-default price of the security, which could be without solutions.
- In the following, when we introduce collateralization and funding costs, we discuss again such problem.

Collateralization and Counterparty Credit Risk

- The growing attention on counterparty credit risk is transforming OTC derivatives money markets:
  - $\longrightarrow$ an increasing number of derivative contracts is cleared by CCPs, while
  - $\longrightarrow$ most of the remaining contracts are traded under collateralization.
- Both cleared and bilateral deals require collateral posting, along with its remuneration.
- Collateralized bilateral trades are regulated by ISDA documentation, known as Credit Support Annex (CSA).
- Centralized clearing is regulated by the contractual rules described by each CCP documentation.
- See Brigo et al. (2012) and Brigo and Pallavicini (2014) for a description of bilateral-traded and centrally-cleared contracts.

Trading Strategies with Margining Procedures – I

- We can include the margining procedure within arbitrage-free pricing by extending the definition of the gain and the cumulative dividend process.
- In general, a margining practice consists in a pre-fixed set of dates during the life of a deal when both parties post or withdraw collaterals, according to their current exposure, to or from an account held by the Collateral Taker.
- We consider that a positive collateral account C_t is held by the investor, otherwise by the counterparty. Moreover, as we set a null terminal condition for the security price, we set $C_T \doteq 0$.
- The Collateral Taker remunerates the account at rate c_t fixed by the collateralization agreement.

  $\longrightarrow$ The collateral rate may depend on the sign of the collateral account.

Trading Strategies with Margining Procedures – II

- Thus, the cumulative dividend process can be extended in the following way

$$D_t := D_0 + \int_0^t 1_{\{\tau > u\}} \left(d\pi_u + dC_u - c_u C_u \, du \right)$$

- Notice that including the collateral account in the cumulative dividend process means that we can re-hypothecate its content.
- Moreover, at trade termination we have to withdraw collateral assets kept in our accounts, so that the gain process can be re-defined as

$$G_t := S_t + D_t - C_t$$

Trading Strategies with Margining Procedures – III

- To avoid arbitrages we require that the deflated gain processes are martingale under the $\mathcal{G}$ filtration.
- Thus, we get

$$\bar{G}_t = \mathbb{E}\left[\,\bar{G}_{T\wedge\tau}\,|\,\mathcal{G}_t\right] \implies \bar{S}_t = \bar{C}_t + \mathbb{E}\left[\bar{S}_{T\wedge\tau} - \bar{C}_{T\wedge\tau} + \int_t^T 1_{\{\tau>u\}}\,d\bar{D}_u \,|\,\mathcal{G}_t\right]$$

- The integral over deflated dividends can be written as

$$\begin{aligned} \int_t^T 1_{\{\tau>u\}}\,d\bar{D}_u &= \int_t^T 1_{\{\tau>u\}}\left(\frac{d\pi_u}{B_u} + \frac{dC_u}{B_u} - \frac{c_u C_u\,du}{B_u}\right) \\ &= \frac{C_{T\wedge\tau}}{B_{T\wedge\tau}} - \frac{C_{t\wedge\tau}}{B_{t\wedge\tau}} + \int_t^T 1_{\{\tau>u\}}\left(\frac{d\pi_u}{B_u} + \frac{(r_u - c_u)C_u\,du}{B_u}\right) \end{aligned}$$

Trading Strategies with Margining Procedures – IV

- If we substitute the expression for the dividend integral, we get the pricing equation

$$1_{\{\tau>t\}}\widetilde{S}_t = 1_{\{\tau>t\}} B_t \, \mathbb{E}\left[1_{\{\tau\leq T\}} \frac{\theta_\tau}{B_\tau} + \int_t^T 1_{\{\tau>u\}} \left(\frac{d\pi_u}{B_u} + \frac{(r_u - c_u) C_u \, du}{B_u} \right) \mid \mathcal{G}_t \right]$$

- According to ISDA the definition of the on-default cash flow in presence of collateralization and re-hypothecation is given by

$$\theta_\tau := \varepsilon_\tau \underbrace{- 1_{\{\tau_C<\tau_I\}}(1-R_C)(\varepsilon_\tau - C_\tau)^+}_{\text{CVA cash flow}} \underbrace{- 1_{\{\tau_I<\tau_C\}}(1-R_I)(\varepsilon_\tau - C_\tau)^-}_{\text{DVA cash flow}}$$

- In the above equation we are assuming that the collateral account is continuous on default events.
  - $\longrightarrow$ Otherwise, we should evaluate the collateral account (not the close-out amount) just before the default event, namely at τ^-.

Trading Strategies with Margining Procedures – V

- The value of the collateral account is specified by the CSA contract.
  - $\longrightarrow$ Usually contracts with the same counterparty are grouped within one or more netting sets with the same CSA rules.
  - $\longrightarrow$ Collateral evaluations and CVA/DVA adjustments are calculated by summing (netting) the price of all contracts within a single netting set.
- A common approximation for the collateral process is setting it proportional to the pre-default price of the derivative.

$$C_t \doteq \alpha_t \widetilde{S}_t$$

where α_t is a $\mathcal{F}$-adapted process.

- This is another source of non-linearities in the pricing equation.
  - $\longrightarrow$ Pricing equations under such approximations are derived in Pallavicini and Brigo (2013).

Perfect Collateralization – I

- In order to further simplify the pricing equation, by removing all non-linearities, we adopt a further approximation known as "perfect collateralization".
- First, we assume a risk-free close-out, where the residual deal is priced at mid market without any residual counterparty risk, but with collateral costs.

$$\varepsilon_t \doteq B_t \, \mathbb{E}\left[\int_t^T \left(\frac{d\pi_u}{B_u} + \frac{(r_u - c_u) C_u \, du}{B_u} \right) \mid \mathcal{G}_t \right]$$

- Second, we assume that the collateral account is always able to remove all CVA/DVA risks.

$$\varepsilon_t \doteq C_t$$

- Such assumptions are usually holding for liquid non-credit-linked market instruments.

Perfect Collateralization – II

- We plug the expressions for C_t and ε_τ in the price equation to get

$$
\begin{aligned}
1_{\{\tau>t\}}\widetilde{S}_t &= 1_{\{\tau>t\}} B_t \,\mathbb{E}\left[1_{\{\tau\le T\}} \frac{C_\tau}{B_\tau} + \int_t^T 1_{\{\tau>u\}} \left(\frac{d\pi_u}{B_u} + \frac{(r_u - c_u) C_u \, du}{B_u} \right) \,|\, \mathcal{G}_t \right] \\
&= 1_{\{\tau>t\}} B_t \,\mathbb{E}\left[\int_t^T \left(\frac{d\pi_u}{B_u} + \frac{(r_u - c_u) C_u \, du}{B_u} \right) \,|\, \mathcal{G}_t \right] = 1_{\{\tau>t\}} C_t
\end{aligned}
$$

- Then, if we apply the Feynman-Kac theorem, we can write

$$
1_{\{\tau>t\}}\widetilde{S}_t = 1_{\{\tau>t\}} B^c_t \,\mathbb{E}\left[\int_t^T \frac{d\pi_u}{B^c_u} \,|\, \mathcal{G}_t \right] \;, \quad B^c_t := \exp\left\{ \int_0^t c_t \, du \right\}
$$

- Perfect collateralization means that we can “discount” at collateral rate discarding default events.
  - ⟶ This result is discussed in Piterbarg (2010) and in Brigo et al. (2012).

Perfect Collateralization – III

- Liquid market instruments are usually collateralized on a daily basis at overnight rate e_t.
- In particular, non-credit-linked instruments can be approximated with a continuous price process on investor or counterparty default event.
  - $\longrightarrow$ See Schönbucher and Schubert (2001) for a discussion of the impact of default events on FX derivatives.
  - $\longrightarrow$ See Brigo, Capponi and Pallavicini (2011) for a discussion of gap risk for CDS contracts.
- In the following, we use liquid market instruments as hedging instruments, and we price them under perfect collateralization assumption by disregarding these problems.

Perfect Collateralization – IV

- An possible exception are markets trading directly the underlying asset, or its future contract (stock, bond or commodity markets).
- In the first case borrowing and lending instruments (repurchase agreements, or repo) are usually actively traded to allow short positions.
  - ⟶ Repo contracts behave like collateralized contracts with remuneration rate h_t.
- In the second case contracts are traded on central financial exchange platforms requiring margin exchange.
  - ⟶ Future contracts behave like collateralized contracts with null remuneration rate.
- Hence, we always assume that hedging instruments can be priced as perfectly collateralized with the proper remuneration rate.

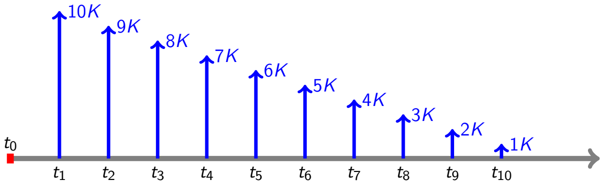
Talk Outline

Netted IRS Portfolios – I

- Here, we investigate the case of bilateral counterparty credit for netted portfolios of interest-rate swaps (IRS).
  - $\longrightarrow$ We discard margining and funding costs: $c_t \doteq r_t$ where r_t is the risk-free rate.
  - $\longrightarrow$ We consider a risk-free close-out amount.
- Interest-rates are modeled by means of a two-factor Gaussian model, while default intensities and the liquidity basis by means of a shifted CIR model.
- Market risks dependencies are created by correlating the driving Brownian motions, while default events are coupled by means of a Gaussian copula.
- A Least Square Monte Carlo simulation is used to value all the payoffs. See Brigo, et al. (2009,2011) and references therein for details.

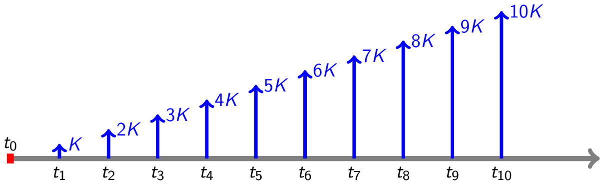
Netted IRS Portfolios – II

- **P1** – A portfolio of 10 swaps, where all the swaps start at date T_0 and the i-th swap matures i years after the starting date. The netting of the portfolio is equal to an amortizing swap with decreasing outstanding.



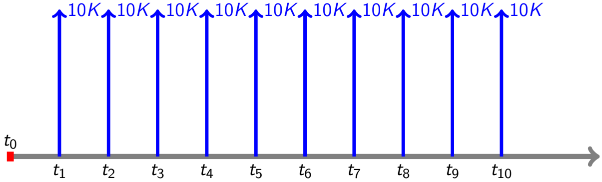
Netted IRS Portfolios – III

- **P2** – A portfolio of 10 swaps, where all the swaps mature in 10 years from date T_0, but they start at different dates, namely the i-th swap starts $i-1$ years from date T_0. The netting of the portfolio is equal to an amortizing swap with increasing outstanding.



Netted IRS Portfolios – IV

- **P3** – A portfolio of 10 swaps, where all the swaps start at date T_0 and mature in 10 years. The netting of the portfolio is equal to a swap similar to the ones in the portfolio but with 10 times larger notional.



Wrong-Way Risk

$\bar{\rho}_C$	$\bar{\rho}_I$	P1	P2	P3
-60%	**0%**	-117	-382	-237
-40%	**0%**	-74	-297	-138
-20%	**0%**	-32	-210	-40
0%	**0%**	-1	-148	31
20%	**0%**	24	-96	87
40%	**0%**	44	-50	131
60%	**0%**	57	-22	159

$\bar{\rho}_C$	$\bar{\rho}_I$	P1	P2	P3
-60%	**-60%**	-150	-422	-319
-40%	**-40%**	-98	-329	-197
-20%	**-20%**	-46	-230	-74
0%	**0%**	-1	-148	31
20%	**20%**	38	-77	121
40%	**40%**	75	-6	208
60%	**60%**	106	49	280

Bilateral credit valuation adjustment for three different receiver IRS portfolios for a maturity of ten years, using high-risk parameter set for the counterparty and mid-risk parameter set for the investor with uncorrelated default times. Every IRS has unitary notional. Prices are in basis points.

Changing the Parameter Set

$\bar{\rho}_C$	$\bar{\rho}_I$	H/M	H/H	M/H
-60%	**-60%**	-150	-76	47
-40%	**-40%**	-98	-12	97
-20%	**-20%**	-46	48	135
0%	**0%**	-1	110	187
20%	**20%**	38	173	241
40%	**40%**	75	239	297
60%	**60%**	106	304	361

$\bar{\rho}_C$	$\bar{\rho}_I$	H/M	H/H	M/H
-60%	**-60%**	-422	-284	-40
-40%	**-40%**	-329	-179	36
-20%	**-20%**	-230	-77	102
0%	**0%**	-148	16	179
20%	**20%**	-77	112	262
40%	**40%**	-6	218	351
60%	**60%**	49	315	450

Bilateral credit valuation adjustment, by changing the parameter set, for a decreasing (P1, left panel) and an increasing (P2, right panel) IRS portfolio for a maturity of ten years, with uncorrelated default times. Every IRS has unitary notional. Prices are in basis points.

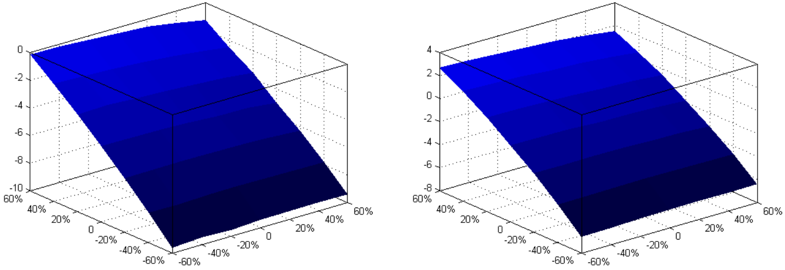
Changing the Default-Time Coupling

$\bar{\rho}_C$	$\bar{\rho}_I$	-80%	0%	80%
-60%	-60%	-150	-150	-169
-40%	-40%	-91	-98	-122
-20%	-20%	-33	-46	-72
0%	0%	18	-1	-34
20%	20%	61	38	-3
40%	40%	102	75	29
60%	60%	140	106	53

$\bar{\rho}_C$	$\bar{\rho}_I$	-80%	0%	80%
-60%	-60%	32	47	61
-40%	-40%	86	97	103
-20%	-20%	146	135	137
0%	0%	194	187	183
20%	20%	256	241	232
40%	40%	320	297	287
60%	60%	384	361	344

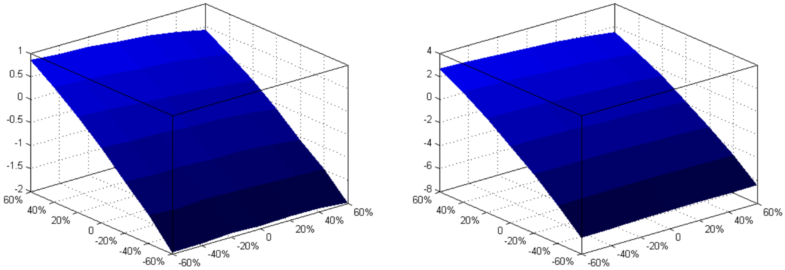
Bilateral credit valuation adjustment, by changing the Gaussian copula parameter ρ_G for a decreasing IRS portfolio (P1) for a maturity of ten years, using high-risk parameter set for the counterparty and mid-risk parameter set for the investor (left panel), and inverted settings (right panel). Every IRS has unitary notional. Prices are in basis points.

Re-Hypothecation vs. Segregation



Collateralized bilateral CVA for an IRS with ten year maturity and one year coupon tenor with different choices of interest-rate/credit-spread correlation (left-side axis) and default-time correlation (right-side axis) with collateral update intervals of three months with (left panel) or without (right panel) collateral re-hypothcation. H/M settings.

Collateral Update Frequency



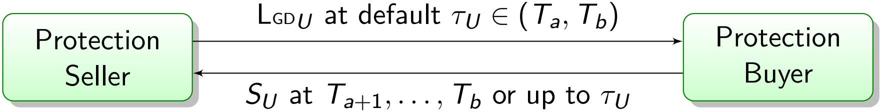
Collateralized bilateral CVA for an IRS with ten year maturity and one year coupon tenor with different choices of interest-rate/credit-spread correlation (left-side axis) and default-time correlation (right-side axis) with collateral update intervals of one week (left panel) and three months (right panel). H/M settings.

Gap Risk and Default Contagion

- Counterparty credit risk may be mitigated by margining practice, namely by using a collateral account as insurance against counterparty's default.
- Yet, there are contracts that cannot be completely collateralized, since their mark-to-market value jumps at default event.
- Credit derivatives, e.g. CDS. At default time the intensity of the reference name jumps, if its default time is correlated with counterparty's one.
  - → See Fujii and Takahashi (2011a); Bielecki, Cialenco and Iyigunler (2011); Brigo, Capponi and Pallavicini (2011).
- Cross-currency derivatives, e.g. CCS. At default time the FX rate may jump, as we can deduce from quotes of counterparty's CDS contracts in different currencies.
  - → See Ehler and Schönbucher (2006).
- Here, we focus on the CDS case.

CDS Payoff

- CDS are contracts that have been designed to offer protection $\text{LGD}_U := 1 - R_U$ against default of a reference name at τ_U in exchange for a periodic premium S_U.



- Thus, the coupon process for a receiver CDS is given by

$$
\begin{aligned}
d\pi_t^{\mathrm{CDS}} &:= S_U \sum_{i=a+1}^{b} (\min\{T_i, \tau_U\} - T_{i-1}) 1_{\{\tau_U > T_{i-1}\}} \delta_t(T_i)\, dt \\
&- \text{LGD}_U 1_{\{T_a < \tau < T_b\}} \delta_t(\tau_U)\, dt
\end{aligned}
$$

CDS Pricing

- The risk-neutral price of a receiver CDS, without taking into account counterparty risk or funding costs, is given by

$$
\begin{aligned}
V_0^{\mathrm{CDS}} &:= \int_0^{T_b} \mathbb{E}\left[\, D(0,t) d\pi_t^{\mathrm{CDS}} \,|\, \mathcal{G}_0 \right] \\
&= S_U \sum_{i=a+1}^{b} \mathbb{E}\left[\, D(0,T_i)(\min\{T_i,\tau_U\} - T_{i-1}) 1_{\{\tau_U > T_{i-1}\}} \,|\, \mathcal{G}_0 \right] \\
&- \mathbb{E}\left[\, D(0,\tau) \mathrm{L{\scriptstyle GD}}_U 1_{\{T_a < \tau_U < T_b\}} \,|\, \mathcal{G}_0 \right]
\end{aligned}
$$

  where we define $D(t,T) := B_T / B_t$.
- If we approximate the payments on a continuous basis we can write a simpler expression

$$
V_0^{\mathrm{CDS}} = \int_{T_a}^{T_b} \mathbb{E}\left[\, D(0,t) \left(S_U 1_{\{\tau_U > t\}} \, dt + \mathrm{L{\scriptstyle GD}}_U \, d1_{\{\tau_U > t\}} \right) \,|\, \mathcal{G}_0 \right]
$$

Bootstrapping the Survival Probabilities – I

- Survival probabilities can be bootstrapped from CDS quotes.
- Many approximations are required to avoid a model-dependent procedure.
  - ⟶ Recovery rates are uncertain and difficult to estimate.
  - ⟶ CDS contracts are collateralized, but counterparty risk is still relevant due to contagion effects.
  - ⟶ If CDS contracts are cleared via a CCP, funding costs may alter the quotes.
  - ⟶ Interest-rates are usually correlated to default probabilities, so that they may impact the quotes as well.
- Moreover, the default event may be poorly defined as the recent Greece case shown.
- Yet, CDS are still the best candidate for a bootstrap procedure.
  - ⟶ Rate agencies quotes default probabilities under historical measure in term of rating classes.

Bootstrapping the Survival Probabilities – II

- In the practice CDS are quoted with a deterministic recovery rate.
- Moreover, the analysis of Brigo and Alfonsi (2005) shows that we can safely assume independence of default probabilities from interest-rates when pricing CDS.
- Thus, since $\mathbb{Q}\{\,\tau_U > T \mid \mathcal{G}_0\,\} = \mathbb{Q}\{\,\tau_U > T \mid \mathcal{F}_0\,\}$, we can write

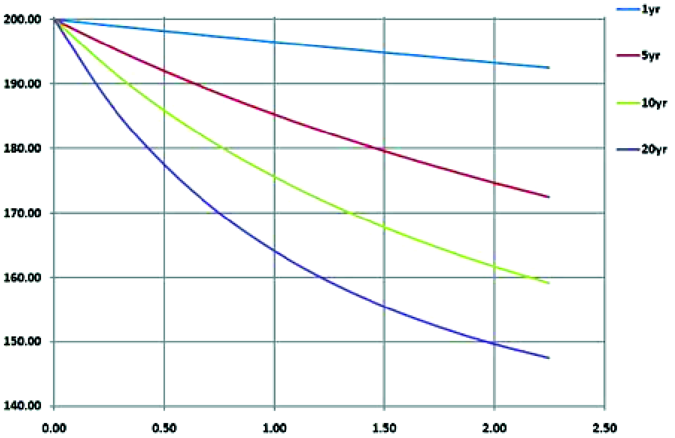
$$V_0^{\mathrm{CDS}} \doteq \int_{T_a}^{T_b} P_0(t)\,\big(S_U\,\mathbb{Q}\{\,\tau_U > t \mid \mathcal{F}_0\,\}\,dt + \mathrm{L}{\scriptstyle\mathrm{GD}}_U\,d\mathbb{Q}\{\,\tau_U > t \mid \mathcal{F}_0\,\}\big)$$

  where we define $P_t(T) := \mathbb{E}_t[\,D(t,T)\,]$.
- We can bootstrap the survival term structure as given by

$$T \mapsto \mathbb{Q}\{\,\tau_U > T \mid \mathcal{F}_0\,\}$$

- What happens if the protection seller defaults? Should we add a counterparty valuation adjustment?

Bootstrapping the Survival Probabilities – III



Change of par CDS spread for different maturities versus Clayton copula parameter. Details in Fujii and Takahashi (2011).

Contagion Effects in CDS Pricing – I

- The instantaneous gap risk on counterparty default is given by

$$\Delta V^{\mathrm{CDS}}_{\tau_C} := V^{\mathrm{CDS}}_{\tau_C} - V^{\mathrm{CDS}}_{\tau_C^-}$$

  and similarly for the investor case.

- The CDS price in our approximation depends only on default probabilities, so that it may jump only if

$$\mathbb{Q}\{\, \tau_U > t \mid \mathcal{G}_{\tau_C} \,\} \neq \mathbb{Q}\{\, \tau_U > t \mid \mathcal{G}_{\tau_C-} \,\}$$

  which happens only in presence of dependencies among the default times.

Contagion Effects in CDS Pricing – II

- What happens to the reference name default probabilities after the counterparty default event?
  - $\longrightarrow$ We assume that the counterparty defaults at time $t < u$, while the reference name defaults after u.
  - $\longrightarrow$ After time t we have a single-name market.
- Thus, given a $\mathcal{G}$-adapted process x_t, we can write

$$1_{\{\tau_U > u\}} 1_{\{\tau_C = t\}} x_u = 1_{\{\tau_U > u\}} 1_{\{\tau_C = t\}} \tilde{x}_u$$

where $\tilde{x}_t$ is the corresponding $\mathcal{F}$-adapted pre-default process.

Contagion Effects in CDS Pricing – III

- If we take expecations w.r.t. the market filtration we obtain

$$\tilde{x}_u \, \partial_v \mathbb{Q}\{ \tau_U > u, \tau_C > v \mid \mathcal{F}_u \}|_{v=t} = \mathbb{E}\left[1_{\{\tau_U > u\}} 1_{\{\tau_C = t\}} x_u \mid \mathcal{F}_u \right]$$

- We can consider the case $x_t \doteq \mathbb{E}[1_{\{\tau_U > T\}} \phi | \mathcal{G}_t]$, where ϕ is a $\mathcal{F}_T$-integrable random variable, to get the generalization of the filtration switching theorem to a two-name market.

$$1_{\{\tau_U > u\}} 1_{\{\tau_C = t\}} \mathbb{E}\left[1_{\{\tau_U > T\}} \phi \mid \mathcal{G}_u \right] =$$

$$1_{\{\tau_U > u\}} 1_{\{\tau_C = t\}} \frac{\mathbb{E}[\partial_v \mathbb{Q}\{ \tau_U > T, \tau_C > v \mid \mathcal{F}_u \}|_{v=t} \, \phi \mid \mathcal{F}_u]}{\partial_v \mathbb{Q}\{ \tau_U > u, \tau_C > v \mid \mathcal{F}_u \}|_{v=t}}$$

Contagion Effects in CDS Pricing – IV

Two-Name Default Probabilities

In a market with two defaultable names before any default event the default probabilities are given by

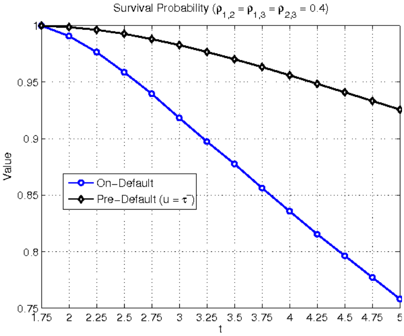
$$1_{\{\tau_U>t\}}1_{\{\tau_C>t\}}\mathbb{Q}\{\,\tau_U > T \mid \mathcal{G}_t\,\} = 1_{\{\tau_U>t\}}1_{\{\tau_C>t\}}\frac{\mathbb{Q}\{\,\tau_U > T, \tau_C > t \mid \mathcal{F}_t\,\}}{\mathbb{Q}\{\,\tau_U > t, \tau_C > t \mid \mathcal{F}_t\,\}}$$

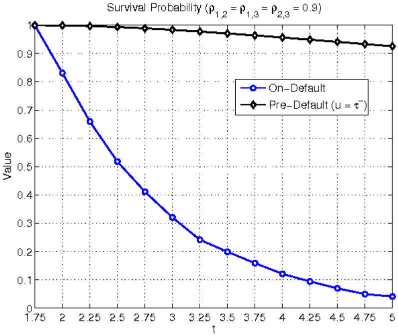
while on a default event the probabilities jump to

$$1_{\{\tau_U>\tau_C\}}\mathbb{Q}\{\,\tau_U > T \mid \mathcal{G}_{\tau_C}\,\} = 1_{\{\tau_U>\tau_C\}}\lim_{t\downarrow\tau_C}\frac{\partial_v\mathbb{Q}\{\,\tau_U > T, \tau_C > v \mid \mathcal{F}_t\,\}|_{v=t}}{\partial_v\mathbb{Q}\{\,\tau_U > t, \tau_C > v \mid \mathcal{F}_t\,\}|_{v=t}}$$

- The first part of the theorem can be obtained from the single-name case by defining the pre-default process w.r.t. the first default event.
- The theorem can be generalized to many names.

Default Probabilities On-Default Jump





Comparison between on-default survival probabilities and pre-default survival probabilities at 1.75 years. Left panel: Gaussian copula parameter is 40%. Right panel: Gaussian copula parameter is 40%. Details in Brigo, Capponi, Pallavicini (2011).

CDS Instantaneous Gap Risk

CDS Instantaneous Gap Risk

Collateralization cannot remove all counterparty risk from a CDS.

$$1_{\{\tau_C>t\}}\mathrm{CVA}_t^{\mathrm{CDS}} = -1_{\{\tau_C>t\}}\mathrm{L{\scriptstyle GD}}_C \int_t^{T_b} du\, P_t(u)\,\mathbb{E}\Big[\,1_{\{\tau_C\in du\}}\,(\Delta V_u^{\mathrm{CDS}})^+ \,|\, \mathcal{G}_t\Big]$$

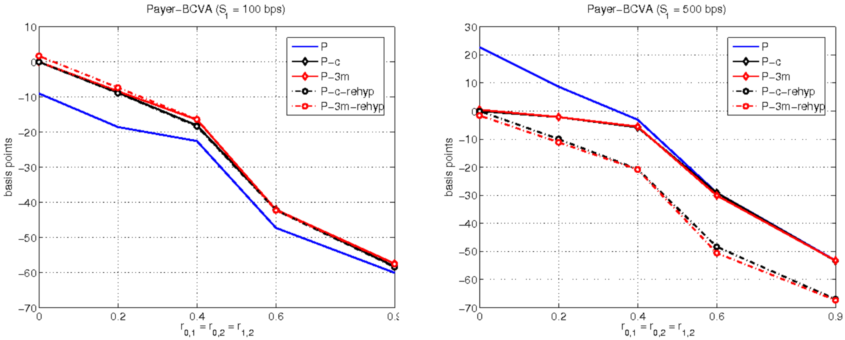
$$1_{\{\tau_U>t\}}\,V_t^{\mathrm{CDS}} = 1_{\{\tau_U>t\}}\int_t^{T_b} du\, P_t(u)\,(S_U\,\mathbb{Q}\{\,\tau_U>u\,|\,\mathcal{G}_t\,\} + \mathrm{L{\scriptstyle GD}}_U\, d\mathbb{Q}\{\,\tau_U>u\,|\,\mathcal{G}_t\,\})$$

$$1_{\{\tau_U>\tau_C\}}\mathbb{Q}\{\,\tau_U>T\,|\,\mathcal{G}_{\tau_C-}\,\} = 1_{\{\tau_U>\tau_C\}}\lim_{t\uparrow\tau_C}\frac{\mathbb{Q}\{\,\tau_U>T,\tau_C>t\,|\,\mathcal{F}_t\,\}}{\mathbb{Q}\{\,\tau_U>t,\tau_C>t\,|\,\mathcal{F}_t\,\}}$$

$$1_{\{\tau_U>\tau_C\}}\mathbb{Q}\{\,\tau_U>T\,|\,\mathcal{G}_{\tau_C}\,\} = 1_{\{\tau_U>\tau_C\}}\lim_{t\downarrow\tau_C}\frac{\partial_v\mathbb{Q}\{\,\tau_U>T,\tau_C>v\,|\,\mathcal{F}_t\,\}|_{v=t}}{\partial_v\mathbb{Q}\{\,\tau_U>t,\tau_C>v\,|\,\mathcal{F}_t\,\}|_{v=t}}$$

- The theorem can be generalized to include the investor default event.

CVA and DVA for CDS Contracts



Bilateral credit adjustment, namely the algebraic sum of CVA and DVA, versus default correlation under different collateralization strategies for a five-year payer CDS contract. Left panel: the CDS spread is 100bp. Right panel: the CDS spread is 500bp. Details in Brigo, Capponi, Pallavicini (2011).

Talk Outline

Funds Transfer Pricing – I

- Funds transfer pricing (FTP) is a process used in banking to adjust the performance of different units to reflect funding costs.
- A FTP policy is usually implemented by means of an intermediary unit, such as the treasury department, which centralizes cash lending and borrowing.
  - $\longrightarrow$ Lending and borrowing rates are issued by the treasury for all the other units of the bank.
  - $\longrightarrow$ These rates define the Treasury Bank Account (TBA).
- In the previous sections we introduced the TBA as a risk-free bank account available to traders to implement their activities.
  - $\longrightarrow$ The TBA rates are defined by the FTP policy adopted by the bank.

Funds Transfer Pricing – II

- Before the crisis derivatives are usually not subject to FTP because they were assumed to be products focusing on risks transfer rather than funds transfer.
- After the crisis, the increasing costs of funding and the advancing in the regulatory framework moved banks on including derivatives in FTP policies.
- Traders in their daily activity should be aware of lending and borrowing operations needed to fund their positions, since they are generating risks for the bank.
- Inspecting the needs of funding the netting set, or even each derivative within it, allow to reduce risks and to implement a profitability analysis.

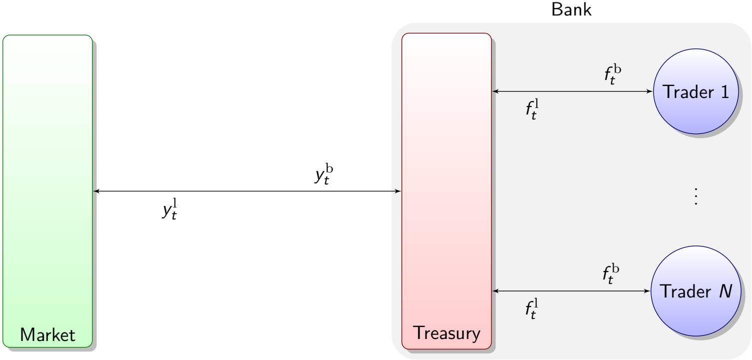
Treasury Funding Operations – I

- The TBA rate is determined by the treasury department according to the FTP policy:
  - $\longrightarrow$ trading positions may be netted before funding on the market (funding netting sets);
  - $\longrightarrow$ different rates may be applied to gauge the performances of different business units;
  - $\longrightarrow$ a maturity-transformation rule can be used to link portfolios to effective maturity dates;
  - $\longrightarrow$ many source of funding can be mixed.
- We refer to Castagna and Fede (2013) for a review of the activity of the treasury.
- Here, we follow Pallavicini, Perini, Brigo (2011) to describe in a stylized way the funding operations closed by the treasury to fund the trading desks so to explicitly define the TBA rate.

Treasury Funding Operations – II

- Starting from this section we reserve the symbol r_t for the risk-free rate, and we name f_t the TBA rate.
- Bank accounts are used by traders both for cash lending and borrowing.
  - ⟶ Trading strategies to borrow and to lend cash are differently implemented, leading to different bank accounts.
  - ⟶ See Bergman (1995), Crépey (2011), Pallavicini, Perini and Brigo (2011).
- We consider the following stylized procedure up to time t.
  - ⟶ Lending: a trading desk has a surplus of cash to be invested at time 0, at time t the desk gets the cash back with a premium.
  - ⟶ Borrowing: a trading desk needs cash at time 0, at time t the desk gives the cash back with a fee.

Treasury Funding Operations – III



The treasury department can lend money on the market at rate y_t^{l} while borrowing at rate y_t^{b}. Traders experience a TBA rate f_t^{l} for lending from the treasury and f_t^{b} for borrowing.

Lending Cash to the Market – I

- We start by discussing the lending case.
- In particular, we assume that a bank "I" invests cash in zero-coupon bonds of a counterparty "C".
- Along with the position in bonds the bank shall buy protection for losses due to the default of the counterparty.
  - $\longrightarrow$ The bank can buy a Credit Default Swap (CDS) for each bond in the strategy.
  - $\longrightarrow$ A CDS contract protects the bond owner from losses occurring on default time by paying a fee s_t^{I}.
- If the counterparty defaults the CDS covers all losses, and the bank may open a new position with another counterparty.
- The strategy can be implemented up to time t or up to the default of the bank. In particular, we assume to roll the positions on a time grid

$$\{t_0 = 0, t_1, \ldots, t_m = t\}$$

Lending Cash to the Market – II

- At time t_0 the bank buys a zero-coupon bond of the counterparty with maturity t_1 and notional

$$q^{\mathrm{l}}_{t_0} := \frac{1}{P^{\mathrm{l}}_{t_0}(t_1)}$$

where $P^{\mathrm{l}}_{t_0}(t_1)$ is the bond market price, so that we have a cash flow of

$$1_{\{\tau > t_0\}} \gamma^{\mathrm{buy}}_{t_0} := -1_{\{\tau > t_0\}} q^{\mathrm{l}}_{t_0} P^{\mathrm{l}}_{t_0}(t_1)$$

- At the same time the bank enters at par into a CDS contract with maturity t_1 on the same bond.

Lending Cash to the Market – III

- At time t_1 the notional of the bond is returned to the bank and the CDS fee is paid, if neither the bank nor the counterparty has defaulted between t_0 and t_1.

$$1_{\{\tau>t_1\}}\gamma_{t_1}^{\mathrm{receive}} := 1_{\{\tau>t_1\}} q_{t_0}^{\mathrm{l}} \,, \quad 1_{\{\tau>t_1\}}\gamma_{t_1}^{\mathrm{fee}} := -1_{\{\tau>t_1\}} q_{t_0}^{\mathrm{l}} s_t^{\mathrm{l}}(t_1 - t_0)$$

- If a default happens, and the defaulting party is the counterparty, the CDS covers all losses, and on the next time-step the position is opened with another counterparty.
- If the bank survives, all contracts are opened again with notional

$$q_{t_1}^{\mathrm{l}} := \frac{q_{t_0}^{\mathrm{l}}(1 - s_t^{\mathrm{l}}(t_1 - t_0))}{P_{t_1}^{\mathrm{l}}(t_2)}$$

so to build a self-financing strategy, namely

$$\gamma_{t_1}^{\mathrm{receive}} + \gamma_{t_1}^{\mathrm{fee}} + \gamma_{t_1}^{\mathrm{buy}} = 0$$

Lending Cash to the Market – IV

- Thus, we can sum all the contributions up to time t, or up to the default of the bank, to define the wealth generated by the investing strategy.

$$
\begin{aligned}
W^{\mathrm{l}}_{t\wedge\tau_l} &:= 1 + \sum_{k=0}^{m-1} 1_{\{\tau_l > t_k\}} \gamma^{\mathrm{buy}}_{t_k} + \sum_{k=1}^{m} 1_{\{\tau_l > t_k\}} \left(\gamma^{\mathrm{receive}}_{t_k} + \gamma^{\mathrm{fee}}_{t_k} \right) \\
&= \prod_{k=1}^{m} 1_{\{\tau_l > t_k\}} \frac{1 - s^{\mathrm{l}}_{t_k}(t_k - t_{k-1})}{P^{\mathrm{l}}_{t_{k-1}}(t_k)}
\end{aligned}
$$

- We can write the wealth of the strategy in continuous time as

$$
W^{\mathrm{l}}_{t\wedge\tau_l} = \exp\left\{ \int_0^{t\wedge\tau_l} (y^{\mathrm{l}}_u - s^{\mathrm{l}}_u)\, du \right\} , \quad y^{\mathrm{l}}_t := -\partial_T \log P^{\mathrm{l}}_t(T)|_{T=t}
$$

where y^{l}_t is the market yield of the bond issued by the counterparty.

The Lending Bank Account – I

- Up to the default of the bank (included) the wealth process generated by the lending operations is a locally risk-free bank account, whichever is the counterparty issuing the bonds.
- We can define the lending bank account as

$$B_t^{\mathrm{l}} := W_t^{\mathrm{l}} = \exp\left\{ \int_0^t (y_u^{\mathrm{l}} - s_u^{\mathrm{l}})\, du \right\}$$

- Moreover, since the lending bank account is locally risk-free, and obtained by means of a self-financing strategy, we have that all the bond/CDS bases must be equal to the risk-free rate r_t to avoid arbitrages

$$r_t = y_t^{\mathrm{l}} - s_t^{\mathrm{l}}$$

The Lending Bank Account – II

- In practice many factors, like bond and CDS market liquidity, CDS collateralization and gap risk, default event specification, etc..., prevent to extract r_t from bond and CDS quotes.
- For later convenience, we cast the bond/CDS basis as a spread ℓ^{l}_t over the overnight rate e_t, and we write

$$\ell^{\mathrm{l}}_t := y^{\mathrm{l}}_t - s^{\mathrm{l}}_t - e_t$$

- Thus, we can write

$$B^{\mathrm{l}}_t = B^e_t \exp\left\{\int_0^t \ell^{\mathrm{l}}_u \, du\right\} \ , \quad B^e_t := \exp\left\{\int_0^t e_t \, du\right\}$$

Borrowing Cash from the Market – I

- We continue the discussion with the borrowing case.
- In particular, we assume that a bank "I" obtains cash by issuing zero-coupon bonds.
- Notice that the bank cannot sell protection on herself when hedging its own default event.
- At time t_0 the bank issues a zero-coupon bond with maturity t_1 and notional

$$q^{\mathrm{b}}_{t_0} := \frac{1}{P^{\mathrm{b}}_{t_0}(t_1)}$$

where $P^{\mathrm{b}}_{t_0}(t_1)$ is the bond market price, so that we have a cash flow of

$$1_{\{\tau_I > t_0\}} \gamma^{\mathrm{issue}}_{t_0} := 1_{\{\tau_I > t_0\}} q^{\mathrm{b}}_{t_0} P^{\mathrm{b}}_{t_0}(t_1)$$

Borrowing Cash from the Market – II

- If the bank defaults, the strategy is terminated and the bond owner recovers only a fraction R_I^f of the notional.

$$1_{\{t_0<\tau_I\leq t_1\}}\gamma_{\tau_I}^{\mathrm{recovery}} := -1_{\{t_0<\tau_I\leq t_1\}}R_I^f q_{t_0}^{\mathrm{b}}$$

- If the bank survives, at time t_1 the notional of the bond is returned to the counterparty.

$$1_{\{\tau_I>t_1\}}\gamma_{t_1}^{\mathrm{pay}} := -1_{\{\tau_I>t_1\}}q_{t_0}^{\mathrm{b}}$$

  and all contracts are opened again with notional

$$q_{t_1}^{\mathrm{b}} := \frac{q_{t_0}^{\mathrm{b}}}{P_{t_1}^{\mathrm{b}}(t_2)}$$

  so to build a self-financing strategy (but on bank default event), namely

$$\gamma_{t_1}^{\mathrm{pay}} + \gamma_{t_1}^{\mathrm{issue}} = 0$$

Borrowing Cash from the Market – III

- Thus, we can sum all the contributions up to time t, or up to the default of the bank, to define the wealth generated by the funding strategy.

$$
\begin{aligned}
W^{\mathrm{b}}_{t\wedge\tau_I} &:= -1 + \sum_{k=0}^{m-1} 1_{\{\tau_I > t_k\}} \left(\gamma^{\mathrm{issue}}_{t_k} + 1_{\{\tau_I \le t_{k+1}\}} \gamma^{\mathrm{recovery}}_{\tau_I}\right) + \sum_{k=1}^{m} 1_{\{\tau_I > t_k\}} \gamma^{\mathrm{pay}}_{t_k} \\
&= -\prod_{k=1}^{m} 1_{\{\tau_I > t_k\}} \frac{1}{P^{\mathrm{b}}_{t_{k-1}}(t_k)} - R^f_I \sum_{k=0}^{m-1} 1_{\{t_k < \tau_I \le t_{k+1}\}} \prod_{j=1}^{k} \frac{1}{P^{\mathrm{b}}_{t_{j-1}}(t_j)}
\end{aligned}
$$

- We can write the wealth of the strategy in continuous time as

$$
W^{\mathrm{b}}_{t\wedge\tau_I} = 1_{\{\tau_I > t\}} \widetilde{W}^{\mathrm{b}}_t + 1_{\{t=\tau_I\}} R^f_I \widetilde{W}^{\mathrm{b}}_{\tau_I} \,, \quad \widetilde{W}^{\mathrm{b}}_t := -\exp\left\{ \int_0^t y^{\mathrm{b}}_u \, du \right\}
$$

where y^{b}_t is the market yield of the bond issued by the bank.

The Borrowing Bank Account – I

- Only up to the default of the bank (excluded) the wealth process is a locally risk-free bank account.
- Thus, we cannot build a locally risk-free bank account for borrowing cash as we did for the lending case.
- First, we need some algebra to re-write the wealth of the borrowing strategy as given by

$$W^{\mathrm{b}}_{t\wedge\tau_I} = \widetilde{W}^{\mathrm{b}}_{t\wedge\tau_I} - 1_{\{t=\tau_I\}}(1-R^f_I)\widetilde{W}^{\mathrm{b}}_{\tau_I}$$

- The first term is locally risk-free up to the default of the bank (included).
  - ⟶ Yet, we cannot replicate it by means of a self-financing strategy based on instruments available to the bank for trading.
- The second term can be interpreted as a funding benefit occurring on the bank default event.

The Borrowing Bank Account – II

- A possible solution is including the funding benefit in the cash flows of the netting set, and using the pre-default wealth generated by the borrowing strategy as a bank account.

$$B_t^{\mathrm{b}} := -\widetilde{W}_t^{\mathrm{b}} = \exp\left\{\int_0^t y_u^{\mathrm{b}}\,du\right\}$$

- For later convenience, we express the yield of bank bonds as a spread ℓ_t^{b} over the overnight rate e_t, and we write

$$\ell_t^{\mathrm{b}} := y_t^{\mathrm{b}} - s_t^I - e_t$$

where s_t^I is the CDS spread of the bank, so that we can write

$$B_t^{\mathrm{b}} = B_t^e \exp\left\{\int_0^t (s_u^I + \ell_u^{\mathrm{b}})\,du\right\}$$

The Borrowing Bank Account – III

- Yet, in such way we are not taking into account the funding risks generated by the trading activity, since we are offsetting them by means of the funding benefit.
- The FTP policy implemented by the treasury should adjust prices to make funding costs apparent.
  - $\longrightarrow$ The simplest way is removing the funding benefit from trading books.
  - $\longrightarrow$ We will see how to tune the FTP policy to avoid the non-linearities of the pricing equation.
- In the following section (i) we will derive an arbitrage-free pricing formula without FTP adjustments, (ii) we will check the conditions under which funding costs disappear, then (iii) we introduce FTP adjustments (funding valuation adjustments, or FVA).

Funding with Bonds in Different Seniorities – I

- The previous borrowing strategy based on trading a strip of zero-coupon bonds fails to produce a self-financing strategy on the bank default event.
- An alternative strategy is described in Burgard and Kjaer (2011) by means of bonds with different seniorities.
- At time t_0 the bank issues one zero-coupon bond with maturity t_1 and recovery R_I^1 and pay backs one bond with the same maturity but with recovery R_I^2. Their notionals are

$$q_{t_0}^{\mathrm{b},i} := \frac{1}{P_{t_0}^{\mathrm{b},i}(t_1)} \,, \quad i = \{1,2\}$$

where $P_{t_0}^{\mathrm{b},i}(t_1)$ are the bond market prices, so that we have a cash flow of

$$1_{\{\tau_I > t_0\}} \gamma_{t_0}^{\mathrm{issue}'} := 1_{\{\tau_I > t_0\}} \left(q_{t_0}^{\mathrm{b},1} P_{t_0}^{\mathrm{b},1}(t_1) - q_{t_0}^{\mathrm{b},2} P_{t_0}^{\mathrm{b},2}(t_1) \right)$$

Funding with Bonds in Different Seniorities – II

- If the bank survives, at time t_1 the notional of the bonds are returned.

$$1_{\{\tau_I > t_1\}} \gamma_{t_1}^{\mathrm{pay}'} := -1_{\{\tau_I > t_1\}} \left(q_{t_0}^{\mathrm{b},1} - q_{t_0}^{\mathrm{b},2} \right)$$

- Otherwise, if the bank defaults, the strategy is terminated with the following cash flow.

$$1_{\{t_0 < \tau_I \leq t_1\}} \gamma_{\tau_I}^{\mathrm{recovery}'} := -1_{\{t_0 < \tau_I \leq t_1\}} \left(R_I^1 q_{t_0}^{\mathrm{b},1} - R_I^2 q_{t_0}^{\mathrm{b},2} \right)$$

- The notionals are chosen so to build a self-financing strategy also on bank default event, when the debt must be paid to the funder.

$$\gamma_{t_1}^{\mathrm{recovery}'} = \gamma_{t_1}^{\mathrm{pay}'}$$

Funding with Bonds in Different Seniorities – III

- Thus, the quantities of the two bonds are linked by

$$q^{\mathrm{b},1}_{t_0} = -q^{\mathrm{b},2}_{t_0} \frac{1-R^2_I}{1-R^1_I}$$

so that if we are short in the first bond, then we must be long in the second one.

- If the bank survives, all contracts are opened again with notionals

$$q^{\mathrm{b},1}_{t_1} := \frac{1}{P^{\mathrm{b},1}_{t_0}(t_1) - \frac{1-R^1_I}{1-R^2_I} P^{\mathrm{b},2}_{t_0}(t_1)} \,, \quad q^{\mathrm{b},2}_{t_1} := \frac{1}{P^{\mathrm{b},2}_{t_0}(t_1) - \frac{1-R^2_I}{1-R^1_I} P^{\mathrm{b},1}_{t_0}(t_1)}$$

so to build a self-financing strategy, namely

$$\gamma^{\mathrm{pay}'}_{t_1} + \gamma^{\mathrm{issue}'}_{t_1} = 0$$

Default-Free Borrowing Bank Accounts – I

- Thus, we can sum all the contributions up to time t, or up to the default of the bank, to define the wealth generated by the funding strategy.

$$
\begin{aligned}
W^{\mathrm{b}'}_{t\wedge\tau_I} &:= -1 + \sum_{k=0}^{m-1} 1_{\{\tau_I > t_k\}} \left(\gamma^{\mathrm{issue}'}_{t_k} + 1_{\{\tau_I \leq t_{k+1}\}} \gamma^{\mathrm{recovery}'}_{\tau_I} \right) + \sum_{k=1}^{m} 1_{\{\tau_I > t_k\}} \gamma^{\mathrm{pay}'}_{t_k} \\
&= -\prod_{k=1}^{m} 1_{\{\tau_I > t_{k-1}\}} \frac{R_I^1 - R_I^2}{(1-R_I^2) P^{\mathrm{b},1}_{t_{k-1}}(t_k) - (1-R_I^1) P^{\mathrm{b},2}_{t_{k-1}}(t_k)}
\end{aligned}
$$

- We can write the wealth of the strategy in continuous time as

$$
W^{\mathrm{b}'}_{t\wedge\tau_I} = -\exp\left\{ \int_0^{t\wedge\tau_I} du \, \frac{(1-R_I^2) y^{\mathrm{b},1}_t - (1-R_I^1) y^{\mathrm{b},2}_t}{R_I^1 - R_I^2} \right\}
$$

where $y^{\mathrm{b},1}_t$ and $y^{\mathrm{b},2}_t$ are the yields of the bonds issued by the bank.

Default-Free Borrowing Bank Accounts – II

- Up to the default of the bank (included) the wealth process is a locally risk-free bank account, whatever are the bond recoveries.
- All these accounts are derived securities, so that, to avoid arbitrages, the accrual rate of these strategies must be equal to r_t.

$$r_t = \frac{(1 - R_I^2) y_t^{\mathrm{b},1} - (1 - R_I^1) y_t^{\mathrm{b},2}}{R_I^1 - R_I^2}$$

- The above equation must be valid for any choice of recovery rates with $0 \leq R_I^1 < R_I^2 \leq 1$, so that the market yields can be written as

$$y_t^{\mathrm{b},i} = r_t + s_t^{I,i} \ , \quad s^{I,i} := \lambda_t^I (1 - R_I^i) \ , \quad i \in \{1, 2\}$$

where λ_t^I is a proportionality factor which can be interpreted as the default intensity of the bank.

Default-Free Borrowing Bank Accounts – III

- As done for the previous funding strategy, in presence of liquidity basis we can write the market yields as

$$y_t^{\mathrm{b},i} = e_t + \lambda_t^I (1 - R_I^i) + \ell_t^{\mathrm{b},i} \,, \quad i \in \{1,2\}$$

- In practice a funding strategy based on trading own bonds is difficult to implement without restrictions on volumes and timings.
  $\longrightarrow$ See Castagna and Fede (2013).
- Furthermore, the bank cannot short selling its debt, but it can only buy back the bonds already issued.
  $\longrightarrow$ Yet, such problem may involve only institutions whose unique activity is derivative trading.

Netting Sets – I

- We can focus on a particular trading strategy in market or derived securities which is funded by the treasury on a netting base (funding netting set).
- The assignment of a security to a particular netting set is decided by the treasury.
  - ⟶ A possible choice is a netting set including all the trades of the bank.
  - ⟶ We assume that contracts of the same counterparty are not split among different netting sets.
  - ⟶ See Pallavicini, Perini and Brigo (2011), Albanese and Andersen (2015).
- In particular, we consider a netting set formed by N securities.
  - ⟶ We name V_t^f the adapted price process of the netting set.
  - ⟶ We name W_t^f the predictable wealth process generated (or consumed) by holding the netting set.

Netting Sets – II

- The treasury invests the wealth by lending cash to the market, while cash is borrowed to compensate consumed wealth.
- We recall that a positive price means that selling the netting set we get cash from the market, so that we assume that a positive wealth means the treasury is borrowing cash.
- As an example we consider that at inception the netting set is built by buying a non-collateralized call option on the market, so that we have $V_t^f = W_t^f > 0$.
- Then, at each time t we select the lending or the borrowing bank account according to the sign of the wealth process.

  $\longrightarrow$ If $W_t^f > 0$ the treasury needs borrowing cash.

  $\longrightarrow$ If $W_t^f \leq 0$ the treasury can lend cash.

Netting Sets – III

- We can consider the general case with collateral posting, and we can define the treasury bank account as

$$B^f_t := \exp\left\{\int_0^t f_u\,du\right\}\ ,\quad f_t := 1_{\{W^f_t > C^f_t\}} f^{\rm b}_t + 1_{\{W^f_t \leq C^f_t\}} f^{\rm l}_t$$

  where we set $C^f_t := \sum_{i=1}^N C^i_t$, while the borrowing and lending rates can be defined as

$$f^{\rm b}_t := e_t + s^I_t + \ell^{\rm b}_t\ ,\quad f^{\rm l}_t := e_t + \ell^{\rm l}_t$$

- If we extract the dependency on the overnight rate, we can also write

$$B^f_t = B^e_t\left(1_{\{W^f_t > C^f_t\}}\exp\left\{\int_0^t (s^I_t + \ell^{\rm b}_t)\,du\right\} + 1_{\{W^f_t \leq C^f_t\}}\exp\left\{\int_0^t \ell^{\rm l}_t\,du\right\}\right)$$

The Price-and-Hedge Problem – I

- For sake of simplicity we consider the simpler case of only one counterparty.

  $\longrightarrow$ The generalization is straightforward.
- For ease of notation we do not distinguish among market and derived securities, and we include both exotic trades and all the hedging instruments within the netting set.
- We name θ^i_t, π^i_t, C^i_t and c^i_t respectively the on-default cash flow, the cumulative coupon process, the collateral account and its accrual rate for each security.
- The terminal condition on the netting-set price process includes the funding benefit and the on-default cash flows.

$$V^f_{T\wedge\tau} := 1_{\{\tau\leq T\}} \sum_{i=1}^{N} \theta^i_\tau + 1_{\{\tau=\tau_I\leq T\}}(1-R^f_I)(W^f_{\tau_I} - C^f_{\tau_I})^+$$

The Price-and-Hedge Problem – II

- To avoid arbitrages we require that the gain processes, deflated by the TBA, are martingales under the $\mathcal{G}$ filtration.
  - ⟶ The equivalent martingale measure depends on the netting set, so that we are removing only the arbitrages within the netting set.
  - ⟶ See Bielecki and Rutkowski (2014) for a discussion of arbitrages.
- The pricing equation becomes

$$
\begin{aligned}
1_{\{\tau>t\}} \widetilde{V}_t^f \;&=\; B_t^f \, \mathbb{E}^f\left[\int_t^T 1_{\{\tau>u\}} \sum_{i=1}^N \left(\frac{d\pi_u^i}{B_u^f} + \frac{(f_u - c_u^i) C_u^i \, du}{B_u^f} \right) \Big| \, \mathcal{G}_t \right] \\
&+\; B_t^f \, \mathbb{E}^f\left[1_{\{t<\tau\le T\}} \sum_{i=1}^N \frac{\theta_\tau^i}{B_\tau^f} + 1_{\{t<\tau=\tau_I\le T\}} \frac{(1-R_I^f)(W_{\tau_I}^f - C_{\tau_I}^f)^+}{B_{\tau_I}^f} \, \Big| \, \mathcal{G}_t \right]
\end{aligned}
$$

where the f over the expectation symbols is a reminder of the dependency of the measure on the funding strategy.

The Price-and-Hedge Problem – III

- On the other hand, the wealth process has an initial condition given by the amount of cash needed to buy the netting set.

$$W_0^f := V_0^f$$

- Then, we can introduce a self-financing hedging strategy (q_t^f, q_t) in cash and hedging instruments S_t, where
  - ⟶ the quantity q_t^f invested in the cash account is the same we use to fund the netting set, namely $q_t^f B_t^f = W_t^f - C_t^f$, and
  - ⟶ the coupons paid by the netting set are invested in the strategy.

$$dW_t^f = W_t^f f_t \, dt - \sum_{i=1}^{N} \left(d\pi_t^i + (f_t - c_t^i) C_t^i \, dt \right) + q_t \cdot (dS_t + d\pi_t - S_t e_t \, dt)$$

where the hedging instruments are perfectly collateralized at overnight rate e_t, and π_t is their cumulated coupon process.

The Price-and-Hedge Problem – IV

- The resulting price-and-hedge problem requires to jointly solve
  - ⟶ the backward SDE for the netting set price V_t^f, and
  - ⟶ the forward SDE for the hedging strategy wealth W_t^f,

  by selecting a strategy q_t minimizing the hedging error $\rho_t^f := V_t^f - W_t^f$.
- We refer to Crépey (2011) for a broader discussion.
- In general, the existence of a solution for the price-and-hedge problem is difficult to prove.
  - ⟶ Here, we consider a simpler framework by following Pallavicini, Perini, Brigo (2011,2012).
  - ⟶ A similar approximation is considered in Burgard and Kjaer (2011,2013) and termed semi-replication.
- We assume that the wealth generated by the hedging strategy is equal to the pre-default value of the netting set, namely we set

$$W_t^f \doteq \widetilde{V}_t^f$$

Martingale Pricing – I

- We continue with the approximation $W_t^f \doteq \widetilde{V}_t^f$, and we apply the Feynman-Kac theorem to extract the dependency of funding rates on the overnight rate e_t.

$$
\begin{aligned}
1_{\{\tau>t\}} \widetilde{V}_t^f \; = \; & \sum_{i=1}^{N} \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} \left(d\pi_u^i - (c_u^i - e_u) C_u^i \, du + 1_{\{\tau \in du\}} \theta_u^i \right) \,|\, \mathcal{G}_t \right] \\
& - \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} (f_u - e_u) \left(\widetilde{V}_u^f - C_u^f \right) du \,|\, \mathcal{G}_t \right] \\
& + \int_t^T \mathbb{E}^f \left[1_{\{\tau=\tau_I \in du\}} \frac{B_t^e}{B_u^e} (1 - R_I^f) \left(\widetilde{V}_u^f - C_u^f \right)^+ \,|\, \mathcal{G}_t \right]
\end{aligned}
$$

Martingale Pricing – II

- We recall that funding rates, when the investor is alive, can be expressed in term of liquidity bases as

$$1_{\{\tau_I > t\}} f_t = 1_{\{\tau_I > t\}} \left(e_t + 1_{\{\widetilde{V}^f_t > C^f_t\}} \left(\ell^{\mathrm{b}}_t + s^I_t \right) + 1_{\{\widetilde{V}^f_t \le C^f_t\}} \ell^{\mathrm{l}}_t \right)$$

where s^I_t is the CDS spread of the bank, so that we can write

$$\begin{aligned} 1_{\{\tau_I > t\}} (f_t - e_t)\, dt &= 1_{\{\tau_I > t\}} 1_{\{\widetilde{V}^f_t > C^f_t\}} \left(\ell^{\mathrm{b}}_t\, dt + (1 - R^f_I)\, \mathbb{E}\left[1_{\{\tau_I \in dt\}} \,|\, \mathcal{G}_t \right] \right) \\ &+ 1_{\{\tau_I > t\}} 1_{\{\widetilde{V}^f_t \le C^f_t\}} \ell^{\mathrm{l}}_t\, dt \end{aligned}$$

- We can substitute the above expression in the pricing equation to gather funding costs.

Martingale Pricing – III

- We obtain the following expression for the netting set.

$$
\begin{aligned}
1_{\{\tau>t\}} \widetilde{V}_t^f \; = \; & \sum_{i=1}^{N} \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} \left(d\pi_u^i - (c_u^i - e_u) C_u^i \, du + 1_{\{\tau \in du\}} \theta_u^i \right) \, | \, \mathcal{G}_t \right] \\
& - \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} \left(1_{\{\widetilde{V}_u^f > C_u^f\}} \ell_u^{\mathrm{b}} + 1_{\{\widetilde{V}_u^f < C_u^f\}} \ell_u^{\mathrm{l}} \right) \left(\widetilde{V}_u^f - C_u^f \right) du \, | \, \mathcal{G}_t \right]
\end{aligned}
$$

- The first line represents the price of contractual coupons, collateralization costs, and CVA/DVA contributions.
- The second line collects the funding costs due to a mismatch between bond yields and CDS spreads (bond/CDS basis).
- Other dependencies on the funding strategy have disappeared.
  $\longrightarrow$ In particular, we notice the cancellation between the funding term depending on the CDS spread and the funding benefit.

Martingale Pricing – IV

Martingale Pricing of a Netting Set

A netting set, when (i) it is funded by means of a TBA with lending and borrowing bases ℓ^{l}_t and ℓ^{b}_t, (ii) it is hedged against market risks by means of perfectly collateralized instruments, and (iii) it generates a wealth equal to its pre-default value, can be priced as

$$
\begin{aligned}
1_{\{\tau>t\}} \widetilde{V}^f_t = & \sum_{i=1}^{N} \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B^e_t}{B^e_u} \left(d\pi^i_u - (c^i_u - e_u) C^i_u \, du + 1_{\{\tau \in du\}} \theta^i_u \right) \mid \mathcal{G}_t \right] \\
& - \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B^e_t}{B^e_u} \left((\widetilde{V}^f_u - C^f_u)^+ \ell^{\mathrm{b}}_u + (\widetilde{V}^f_u - C^f_u)^- \ell^{\mathrm{l}}_u \right) du \mid \mathcal{G}_t \right]
\end{aligned}
$$

- Since the bases can be asymmetric, the above expectation cannot be solved explicitly.

  $\longrightarrow$ We can write and numerically solve the corresponding backward SDE problem, as in Brigo and Pallavicini (2014).

Martingale Pricing – V

- We now consider the case of null lending and borrowing bases.
  $\longrightarrow$ In other terms there is not a liquidity basis between bond and CDS markets.
- In such case the netting-set pricing formula reduces to the usual formula with credit and collateral adjustments, see for instance Brigo et al. (2011).

$$1_{\{\tau>t\}} \widetilde{V}_t^f \doteq \sum_{i=1}^{N} \int_t^T \mathbb{E}\left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} \left(d\pi_u^i - (c_u^i - e_u) C_u^i \, du + 1_{\{\tau\in du\}} \theta_u^i \right) \mid \mathcal{G}_t \right]$$

where we drop the f from the expectation, since the dependency on the funding strategy has disappeared.

- Thus, under these assumptions, funding costs disappear, as shown by many papers in the literature.
  $\longrightarrow$ See for a discussion Brigo et al. (2013) or Crépey et al. (2014).

Martingale Pricing – VI

- In the following numerical examples is useful to write the pricing equation as

$$1_{\{\tau>t\}}\widetilde{V}_t^f = \sum_{i=1}^N \int_t^T \mathbb{E}^f\left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} \left(d\pi_u^i - (c_u^i - e_u) C_u^i \, du + 1_{\{\tau\in du\}} \theta_u^i \right) \,|\, \mathcal{G}_t \right] - \int_t^T \mathbb{E}^f\left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} (\widetilde{V}_u^f - C_u^f)(f_u^\ell - e_u) \, du \,|\, \mathcal{G}_t \right]$$

where we define f_t^ℓ as the component of the funding rate depending only on liquidity bases, namely

$$f_t^\ell := e_t + 1_{\{\widetilde{V}_t^f > C_t^f\}} \ell_t^{\mathrm{b}} + 1_{\{\widetilde{V}_t^f \leq C_t^f\}} \ell_t^{\mathrm{l}}$$

Effective Discount Approximation – I

- Here, we investigate the case of a netting set formed by a single interest-rate swap (IRS).

  $\longrightarrow$ For a lighter notation we omit any symbol f referring to the netting set.

- We follow Pallavicini and Brigo (2013), and we define the collateral process and the close-out amount as

$$C_t \doteq \alpha_t \widetilde{V}_t \ , \quad \varepsilon_\tau \doteq \beta_\tau \widetilde{V}_\tau$$

  where $\alpha_t \geq 0$ is the collateral fraction, and β_τ the devaluation factor.

- We have some special cases:

  $\longrightarrow$ no collateralization: $\alpha_t = 0$, e.g. IRS with a corporate;

  $\longrightarrow$ partial collateralization: $0 < \alpha_t < 1$, e.g. IRS with asymmetric CSA;

  $\longrightarrow$ perfect collateralization: $\alpha_t = 1$, e.g. standard IRS;

  $\longrightarrow$ over-collateralization: $\alpha_t > 1$, e.g. IRS with haircuts.

- In the numerical examples we set $\beta_\tau \doteq 1$.

Effective Discount Approximation – II

- We obtain after some algebra in case of $\mathcal{F}$-conditional independence between the default times

$$1_{\{\tau>t\}} \widetilde{V}_t = 1_{\{\tau>t\}} \int_t^T \mathbb{E}\left[\exp\left\{ - \int_t^u (f_v^\ell + \xi_v)\, dv \right\} d\pi_u \,\middle|\, \mathcal{F}_t \right]$$

where we define the spread ξ_t as

$$\begin{aligned} \xi_t \quad := & \quad -\alpha_t (f_t^\ell - e_t) + (\lambda_t^I + \lambda_t^C)(1 - \beta_t) \\ + & \quad (\beta_t - \alpha_t)^+ \left(\lambda_t^C \mathrm{L}_{\mathrm{GD}C} 1_{\{V_t>0\}} + \lambda_t^I \mathrm{L}_{\mathrm{GD}I} 1_{\{V_t<0\}} \right) \\ + & \quad (\beta_t - \alpha_t)^- \left(\lambda_t^I \mathrm{L}_{\mathrm{GD}I} 1_{\{V_t>0\}} + \lambda_t^C \mathrm{L}_{\mathrm{GD}C} 1_{\{V_t<0\}} \right) \end{aligned}$$

and the pre-default intensities are defined as

$$\lambda_t^I\, dt := \mathbb{Q}\{ \tau_I \in dt \mid \tau_I > t, \mathcal{F}_t \} \;, \quad \lambda_t^C\, dt := \mathbb{Q}\{ \tau_C \in dt \mid \tau_C > t, \mathcal{F}_t \}$$

Effective Discount Approximation – III

- We consider the following proxy for funding rates

$$f_t^{\ell} \doteq e_t + w\,\ell_t + 1_{\{(1-\alpha_t)\widetilde{V}_t>0\}} w^{\mathrm{b}} \lambda_t^I$$

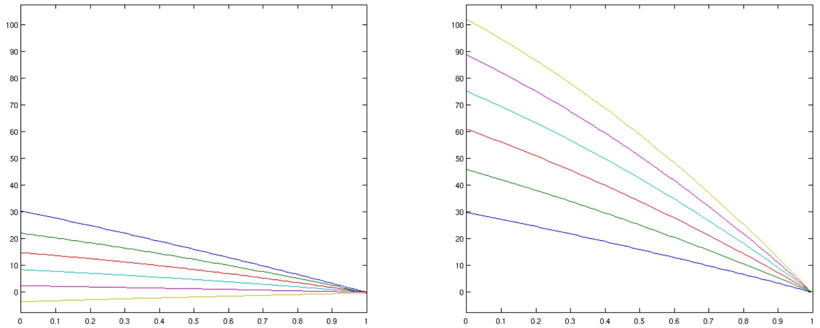
  where w and w^{b} are non-negative weights, e_t is the overnight rate, ℓ_t is a liquidity basis, and λ_t^I is the default intensity of the bank.

- Interest-rates are modeled by means of a two-factor Gaussian model, while default intensities and the liquidity basis by means of a shifted CIR model.

  $\longrightarrow$ Market data and calibration details on Pallavicini and Brigo (2013).

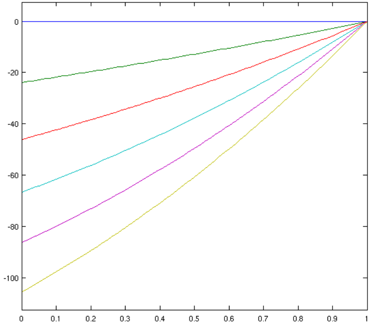
- Market risks dependencies are created by correlating the driving Brownian motions, while we assume $\mathcal{F}$-conditional independence of default times.

Funding Costs and Partial Collateralization



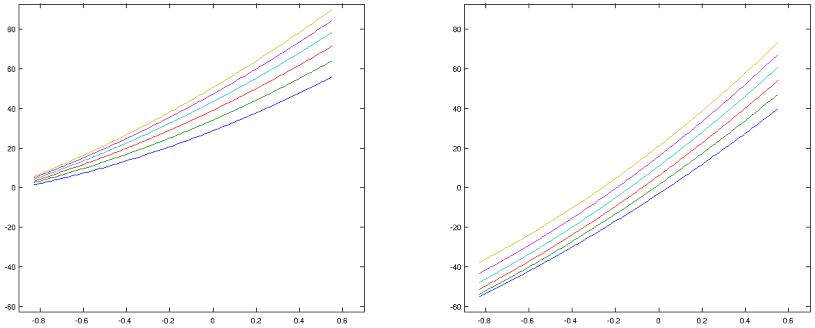
Price for a receiver IRS (left) and for shorting a payer IRS (right) vs. collateral fraction α for different borrowing rates, while keeping the lending rate equal to the overnight rate. $w^{\mathrm{b}} \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$, $w = 0$.

Bid-Ask Spreads and Partial Collateralization



Bid-ask spread for an IRS vs. collateral fraction α for different borrowing rates, while keeping the lending rate equal to the overnight rate.
$w^{\mathrm{b}} \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$, $w = 0$.

Funding Costs and Wrong-Way Risk



Price for a receiver IRS vs. correlation between credit-spreads and overnight rate for different funding rates. Collateralization is off ($\alpha = 0$).
Left: $w^{\mathrm{b}} = 0$, $w \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.
Right: $w^{\mathrm{b}} = 1$, $w \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

Talk Outline

Derivative Pricing in Non-Collateralized Markets – I

- The analysis of the previous section showed that funding costs do not appear in prices but for liquidity effects.
- On the other hand, prices in trader books should reflect funding costs to represent the risks produced by in the funding activity required by trading.
- The treasury department can implement a FTP policy to charge the trader book for funding costs.
  - ⟶ Derivative prices in the trader book are adjusted to reflect funding costs (funding valuation adjustment, or FVA).
- FTP policies can be designed to mimic the terms in the pricing equation offset by the funding benefit term.
  - ⟶ Usually approximations are introduced to avoid non-linearities.

Derivative Pricing in Non-Collateralized Markets – II

- Before analyzing possible design of FTP policies, we wish to discuss a relevant practical issues motivating bank attention to funding costs.
- Traders charged for FVA will mark losses in their books, unless they transfer these adjustments to clients.
  - $\longrightarrow$ Such adjustments are not implied by martingale arguments.
  - $\longrightarrow$ Naive FTP policies may result in arbitrages, see Hull and White (2014).
- However, if we search for non-collateralized markets, where funding costs are an issue, we find only non-liquid markets between banks and corporates, or retail markets.
  - $\longrightarrow$ Interbank markets are always collateralized.
  - $\longrightarrow$ Collateralization requires a great quantity of liquid assets usually detained only by banks or very large corporates.
- Arbitrages due to funding costs in non-collateralized markets are difficult to lock in by corporate or retail clients

Derivative Pricing in Non-Collateralized Markets – III

- As a consequence banks in non-collateralized markets usually include funding valuation adjustments in prices to clients.
  - ⟶ This can be controversial, since the client often has no transparency on the bank funding policy.
  - ⟶ In a more provocative way we could ask why clients are not charging their own funding cost to the banks.
- In the latest years many papers flourished in the practitioner literature on this subject without a conclusive answer.
  - ⟶ See for instance Hull and White (2012), Alavian (2014), Albanese and Andersen (2015).
- In July 2016 the European Money Market Institute will start a reform on Euribor rate to track all funding activity of a panel of banks to build a funding benchmark for EUR zone.

Design of a FTP Policy for Funding Costs

- We summarize two common choices adopted in the bank industry to formulate a FTP policy to define FVA.
- The first choice is a linearized approximation of the netting set pricing formula without explicitly offsetting the funding costs with the funding benefits.
- The second choice is a further approximation based on the fact that the treasury is usually short of cash.

Linear Approximation of the Pricing Formula – I

- We write the netting set pricing formula before offsetting the funding terms.

$$
\begin{aligned}
1_{\{\tau>t\}} \widetilde{V}_t^f \;=\; & \sum_{i=1}^{N} \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} \left(d\pi_u^i - (c_u^i - e_u) C_u^i \, du + 1_{\{\tau \in du\}} \theta_u^i \right) \,|\, \mathcal{G}_t \right] \\
& - \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} \ell_u^{\mathrm{l}} \left(\widetilde{V}_u^f - C_u^f \right)^- du \,|\, \mathcal{G}_t \right] \\
& - \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} (s_u^I + \ell_u^{\mathrm{b}}) \left(\widetilde{V}_u^f - C_u^f \right)^+ du \,|\, \mathcal{G}_t \right] \\
& + \int_t^T \mathbb{E}^f \left[1_{\{\tau=\tau_I \in du\}} \frac{B_t^e}{B_u^e} (1 - R_I^f) \left(\widetilde{V}_u^f - C_u^f \right)^+ \,|\, \mathcal{G}_t \right]
\end{aligned}
$$

Linear Approximation of the Pricing Formula – II

- We approximate the pre-default price $\widetilde{V}^f_t$ entering the right-hand side with the sum of all close-out amounts ε^i_t.
- Moreover, we define close-out amounts as overnight-based prices corrected by collateral costs.
- Thus, we set

$$\widetilde{V}^f_t \approx \varepsilon^f_t := \sum_{i=1}^{N} \varepsilon^i_t \,, \quad \varepsilon^i_t \doteq B^e_t \, \mathbb{E}\left[\int_t^T \left(\frac{d\pi^i_u}{B^e_u} - \frac{(c^i_u - e_u) C^i_u \, du}{B^e_u} \right) \,|\, \mathcal{G}_t \right]$$

- We split the pre-default price of the netting set according to the FTP policy by moving to the treasury the so called funding debit adjustment (FDA).

$$\mathrm{FDA}_t := \int_t^T \mathbb{E}^f\left[1_{\{\tau=\tau_I \in du\}} \frac{B^e_t}{B^e_u} (1 - R^f_I) \left(\sum_{i=1}^{N} \left(\varepsilon^i_u - C^f_i \right) \right)^+ \,|\, \mathcal{G}_t \right]$$

Linear Approximation of the Pricing Formula – III

- We can group all terms marked in the trader desk on the right.

$$1_{\{\tau>t\}}(\widetilde{V}_t^f - \text{FDA}_t) = 1_{\{\tau>t\}} \sum_{i=1}^{N} \varepsilon_t^i$$

$$\text{CVA}_t \qquad - \int_t^T \mathbb{E}^f \left[1_{\{\tau=\tau_C \in du\}} \frac{B_t^e}{B_u^e} (1 - R_C) \sum_{i=1}^{N} \left(\varepsilon_u^i - C_u^i\right)^+ \mid \mathcal{G}_t \right]$$

$$\text{DVA}_t \qquad - \int_t^T \mathbb{E}^f \left[1_{\{\tau=\tau_I \in du\}} \frac{B_t^e}{B_u^e} (1 - R_I) \sum_{i=1}^{N} \left(\varepsilon_u^i - C_u^i\right)^- \mid \mathcal{G}_t \right]$$

$$\text{FBA}_t \qquad - \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} \ell_u^{\mathrm{l}} \left(\sum_{i=1}^{N} \left(\varepsilon_u^i - C_u^i\right) \right)^- du \mid \mathcal{G}_t \right]$$

$$\text{FCA}_t \qquad - \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} (s_u^I + \ell_u^{\mathrm{b}}) \left(\sum_{i=1}^{N} \left(\varepsilon_u^i - C_u^i\right) \right)^+ du \mid \mathcal{G}_t \right]$$

Linear Approximation of the Pricing Formula – IV

Linear Approximation of Netting Set Pricing Formula

Under the Linear Approximation FTP policy the netting set price marked in the trader book is given by

$$P_t^{f,\mathrm{EFB}} := \mathrm{MtM}_t - \mathrm{CVA}_t + \mathrm{DVA}_t + \mathrm{FBA}_t - \mathrm{FCA}_t \,, \quad \mathrm{MtM}_t := 1_{\{\tau > t\}} \sum_{i=1}^{N} \varepsilon_t^i$$

while the FDA term is marked by the treasury.

- The above pricing equation is also known as External Funder Benefit approximation (EFB).
- The close-out amount can be defined alternatively without the collateral costs.

  ⟶ In this case an additional collateral adjustment appears in the pricing equation, sometimes quoted as LVA or ColVA.

Linear Approximation of the Pricing Formula – V

- If we discard liquidity bases we have that

$$P_t^{f,\mathrm{EFB}} \doteq \mathrm{MtM}_t - \mathrm{CVA}_t + \mathrm{DVA}_t - \mathrm{FCA}_t \,, \quad \mathrm{FDA}_t = \mathrm{FCA}_t$$

- Thus, we have the trader desk marking funding costs to compensate on a daily basis the charge paid by the treasury to borrow cash from the market.
- This approximation is discussed also in Burgard and Kjaer (2013), where is termed “strategy I”, and in Andersen and Albanese (2015), where is termed “FVA/FDA accounting”.
- The fair value of the netting set can be derived by summing up all the contributions.

$$V_t^{f,\mathrm{EFB}} := \mathrm{MtM}_t - \mathrm{CVA}_t + \mathrm{DVA}_t$$

Price Adjustments at Contract Level – I

- Funding adjustments in the linear approximation are defined at netting set level.
  - $\longrightarrow$ A recipe to split them on each single contract is needed to effectively implement the FTP policy.
- A possible approach is calculating the marginal contribution of each contract in the netting set.
  - $\longrightarrow$ When a new contract is added to the netting, we assign to it the increment in funding costs (and benefits) of the whole netting set.
- Thus, we can define the cash borrowed for a particular contract in the netting set as

$$F_u^{\mathrm{b},i} := \Big(\sum_{j=1}^{N} \big(\varepsilon_u^j - C_u^j \big) \Big)^+ - \Big(\sum_{j=1, j \neq i}^{N} \big(\varepsilon_u^j - C_u^j \big) \Big)^+$$

Price Adjustments at Contract Level – II

- We can now calculate the marginal FCA, namely the amount of FCA to be market on the trader book when a new contract is entered, as

$$\mathrm{FCA}_t^i := -\int_t^T \mathbb{E}^f\left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e}(s_u^l + \ell_u^{\mathrm{b}})\left(F_u^{\mathrm{b},i}\right)^+ du \mid \mathcal{G}_t \right]$$

- Similarly we can define the marginal FBA starting from

$$F_u^{\mathrm{l},i} := \left(\sum_{j=1}^N (\varepsilon_u^j - C_u^j) \right)^- - \left(\sum_{j=1, j\neq i}^N (\varepsilon_u^j - C_u^j) \right)^-$$

Large Netting Set Approximation – I

- We can further approximate the previous pricing equation by assuming that the netting is always short of cash.
  - $\longrightarrow$ This is a reasonable approximation if we consider a bank with only one netting set for all the trading activity.
- Under this approximation we can set

$$\widetilde{V}_t^f \approx \varepsilon_t^f > C_t^f$$

- A direct consequence is a null funding benefit term, while the funding cost term can be written as

$$\text{FCA}_t \doteq \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B_t^e}{B_u^e} (s_u^l + \ell_u^{\mathrm{b}}) \sum_{i=1}^{N} (\varepsilon_u^i - C_u^i) du \mid \mathcal{G}_t \right]$$

- We can continue by splitting the argument of the summation in its positive and negative parts.

Large Netting Set Approximation – II

- We can group all terms marked in the trader desk on the right.

$$1_{\{\tau>t\}}(\widetilde{V}^f_t - \mathrm{FDA}_t) = 1_{\{\tau>t\}} \sum_{i=1}^N \varepsilon^i_t$$

$$\mathrm{CVA}_t \qquad - \int_t^T \mathbb{E}^f \left[1_{\{\tau=\tau_C\in du\}} \frac{B^e_t}{B^e_u} (1-R_C) \sum_{i=1}^N \left(\varepsilon^i_u - C^i_u\right)^+ \,|\, \mathcal{G}_t \right]$$

$$\mathrm{DVA}_t \qquad - \int_t^T \mathbb{E}^f \left[1_{\{\tau=\tau_I\in du\}} \frac{B^e_t}{B^e_u} (1-R_I) \sum_{i=1}^N \left(\varepsilon^i_u - C^i_u\right)^- \,|\, \mathcal{G}_t \right]$$

double counting (DVA and FBA)

$$\mathrm{FBA}'_t \qquad - \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B^e_t}{B^e_u} (s^I_u + \ell^{\mathrm{b}}_u) \sum_{i=1}^N \left(\varepsilon^i_u - C^i_u\right)^- du \,|\, \mathcal{G}_t \right]$$

$$\mathrm{FCA}'_t \qquad - \int_t^T \mathbb{E}^f \left[1_{\{\tau>u\}} \frac{B^e_t}{B^e_u} (s^I_u + \ell^{\mathrm{b}}_u) \sum_{i=1}^N \left(\varepsilon^i_u - C^i_u\right)^+ du \,|\, \mathcal{G}_t \right]$$

Large Netting Set Approximation – III

- In this approximation we have succeeded in defining per-contract funding costs adjustments, so that we can leave the netting-set view.
- However, the DVA and FBA clearly produce a double counting of the investor credit charge on each single contract.
- Then, the FTP policy is modified to discard one of the two contributions from the price marked on the trading desk.
  - ⟶ Here, we assume that the DVA term is marked by the treasury, or by another trading desk.

Large Netting Set Approximation – IV

Large Netting Set Approximation of Linear Formula

Under the Large Netting Set approximation FTP policy the netting set price marked in the trader book is given by

$$P_t^{f,\mathrm{RBB}} := \mathsf{MtM}_t - \mathsf{CVA}_t + \mathsf{FBA}'_t - \mathsf{FCA}'_t$$

while the DVA and FDA term are marked by the treasury.

- The above pricing equation is also known as Reduced Borrowing Benefit approximation (RBB).
- The close-out amount can be defined alternatively without the collateral costs.
  - $\longrightarrow$ In this case an additional collateral adjustment appears in the pricing equation, sometimes quoted as LVA or ColVA.

Large Netting Set Approximation – V

- If we discard liquidity bases we have that

$$P_t^{f,\mathrm{RBB}} \doteq \mathrm{MtM}_t - \mathrm{CVA}_t + \mathrm{FBA}'_t - \mathrm{FCA}'_t$$

$$\mathrm{FDA}_t = \mathrm{FCA}'_t - \mathrm{FBA}'_t \,, \quad \mathrm{DVA}_t = \mathrm{FBA}'_t$$

- Thus, we have the trader desk marking
  - $\longrightarrow$ funding costs to compensate on a daily basis the charge paid by the treasury to borrow cash from the market.
  - $\longrightarrow$ funding benefits to compensate on a daily basis the profits coming from DVA trading.
- Similar approximations can be found in Burgard and Kjaer (2013) and in Andersen and Albanese (2015).
- The fair value of the netting set can be derived by summing up all the contributions.

$$V_t^{f,\mathrm{RBB}} := \mathrm{MtM}_t - \mathrm{CVA}_t + \mathrm{DVA}_t$$

Beyond the Effective Discount Approximation

- We continue the numerical investigations with the analysis of Brigo and Pallavicini (2014).
- We consider a netting set formed by a single IRS and FTP policy based on the RBB approach.
  - $\longrightarrow$ In Brigo and Pallavicini (2014) the linearization step is avoided and a full BSDE is numerically solved.
- In this example we extend the previous framework by allowing a delay in the default procedure.
  - $\longrightarrow$ At default time τ collateralization is stopped while the close-out amount is calculated at $\tau + \delta$ (gap risk).
  - $\longrightarrow$ Gap risk is reduced with additional collateralization (initial margin), which requires extra funding costs.

Variation and Initial Margin Estimates – I

- In presence of gap risks collateralization procedures require two different types of collaterals.
  - ⟶ A variation margin account M_t to track the mark-to-market movements of the contract up to the default event.
  - ⟶ Two initial margin accounts N^I_t and N^t_C to insure against the worst market movements of the exposure from default date τ up to the completion of the default procedure at $\tau + \delta$.
- Usually, the variation margin may be re-hypothecated, while the initial margin is segregated.
- We can approximate the variation margin, as we did before in case of plain collateralization.

$$M_t \doteq \alpha_t \, \varepsilon_t$$

- While initial margins should be linked to the variance of the exposure conditional on the default event.

Variation and Initial Margin Estimates – II

- Initial margins are strongly dependent on the particular asset class of the derivative.
- LCH, a CCP clearing IRS contracts, consider a single source of risk to estimate initial margins.
  - ⟶ Interest-rate uncertainty, analyzed in term of a historical metric.
- ICE, a CCP clearing CDS contracts, considers seven(!) different sources of risk to estimate initial margins.
  - ⟶ Credit-spread, interest-rate and recovery-rate uncertainties.
  - ⟶ Jump risk, namely default contagion effects.
  - ⟶ Basis risk, namely mismatches between particular contracts and market proxies.
  - ⟶ Liquidity risk, by observing bid/ask spreads and via price discovery.
  - ⟶ Concentration risk, namely systemic risk associated with large portfolios.

Variation and Initial Margin Estimates – III

- Here, we focus on interest-rate derivatives.
- The gap risk arising from the mark-to-market term is usually analyzed in terms of historical Value-at-Risk (VaR) or Expected Shortfall (ES).
- For instance, we can estimate the initial margin posted to protect from mark-to-market movements as the protection against the worst movement of the contract due to market risk within δ days at a confidence level q according to VaR risk metric.

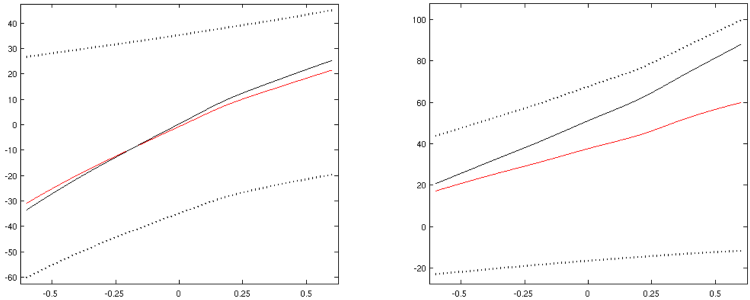
$$N_t^C \doteq \inf\{x \geq 0 : \mathbb{Q}\{\,\varepsilon_{t+\delta} - \varepsilon_t < x \mid \mathcal{F}_t\,\} > q\}$$

and only for bilateral contracts under CSA

$$N_t^I \doteq \sup\{x \leq 0 : \mathbb{Q}\{\,\varepsilon_{t+\delta} - \varepsilon_t > x \mid \mathcal{F}_t\,\} > q\}$$

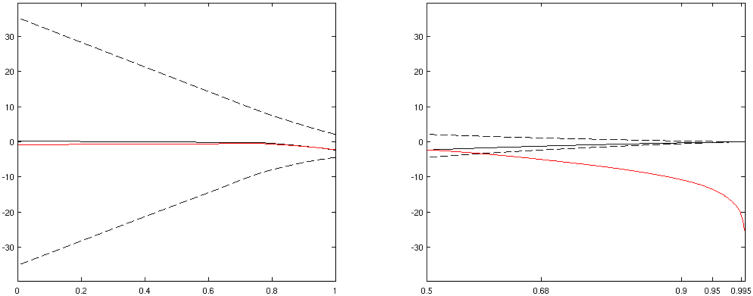
where we approximate the risk metric by using the pricing measure in spite of the physical measure, since we need to insert such estimates into a pricing equation.

Interest-Rate Swap: bilateral trades without margining



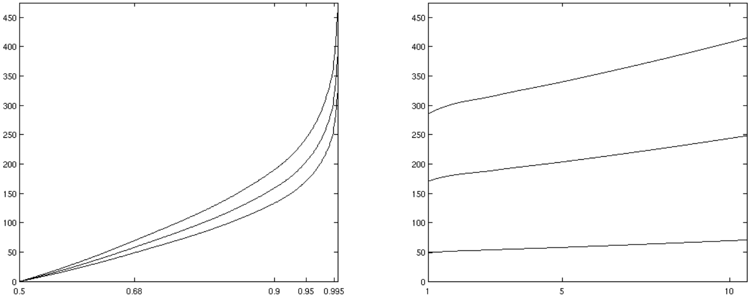
Prices of a ten-year receiver IRS, left "H/M", right "M/H". The black continuous line represents the price inclusive of CVA and DVA but not funding costs, with the dashed black lines representing separately CVA and DVA. The red continuous line is the price inclusive both of credit and funding costs. On the x-axis the correlation among market and credit risks.

Interest-Rate Swap: bilateral trades with margining – I



Prices of a ten-year receiver IRS, "H/M". Left: prices for different collateralization fractions α without initial margin. Right: prices with $\alpha = 1$ and initial margin posted at various confidence levels q. The black continuous line represents the price inclusive of CVA and DVA but not funding costs, with the dashed black lines representing separately CVA and DVA. The red continuous line is the price inclusive both of credit and funding costs.

Interest-Rate Swap: bilateral trades with margining – II



Amount of initial margin requested at contract inception for a ten-year receiver IRS, "H/M". Left: the x-axis lists different confidence levels, while the curves correspond to three different margin period of risks (1, 5 and 10 days). Right the x-axis lists different margin period of risks, while the curves correspond to three confidence levels (68%, 95% and 99.7%). Correlation is zero.

Probabilistic Interpretation of Pricing Equations – I

The Feynman-Kac Theorem

Consider a vector of Markov risk factors S_t with infinitesimal generator

$$\mathcal{L}_t^{\mu} := (\mu_t S_t) \cdot \partial_S + \frac{1}{2} \mathrm{Tr} \partial_t \langle S, S \rangle_t \partial_S^2$$

and assume that the derivative price V_t solves the PDE

$$(\partial_t + \mathcal{L}_t^{\mu} - \nu_t) V_t + \partial_t \pi_t = 0 \,, \quad V_T = 0$$

Hence, the solution of the PDE is given by

$$V_t = \int_t^T \mathbb{E}_t^{\mu} \left[\frac{B_t^{\nu}}{B_u^{\nu}} \, d\pi_u \right]$$

where under the pricing measure $\mathbb{Q}^{\mu}$ the risk factors grow at rate μ_t.

Probabilistic Interpretation of Pricing Equations – II

- A useful application of the theorem is changing the discount factor by adding a stream of coupons.

$$
\begin{aligned}
V_t &= \int_t^T \mathbb{E}_t^{\mu}\Big[\frac{B_t^{\nu}}{B_u^{\nu}}\, d\pi_u \Big] \\
&= \int_t^T \mathbb{E}_t^{\mu}\Big[\frac{B_t^{\rho}}{B_u^{\rho}}\, d\pi_u + (\mu_u - \rho_u) V_u \, du \Big] \\
&= \int_t^T \mathbb{E}_t^{\rho}\Big[\frac{B_t^{\rho}}{B_u^{\rho}}\, d\pi_u + (\mu_u - \rho_u) V_u \, du - (\nu_u - \rho_u) S_u \cdot \partial_S V_u \, du \Big]
\end{aligned}
$$

where under the pricing measure $\mathbb{Q}^{\rho}$ the risk factors grow at rate ρ_t.

Pricing Cash Flows Occurring before the Default Event – I

- For any $\mathcal{G}$-adapted process ϕ_t, we can consider the $\mathcal{G}$-adapted process

$$x_t \doteq \mathbb{E}\big[\, 1_{\{\tau>T\}}\phi_T \,|\, \mathcal{G}_t \,\big]$$

- If we observe x_t only before the default event, and we take the expectations of both side under $\mathcal{F}$ filtration, we get

$$\tilde{x}_t \, \mathbb{E}\big[\, 1_{\{\tau>t\}} \,|\, \mathcal{F}_t \,\big] = \mathbb{E}\big[\, 1_{\{\tau>t\}} \mathbb{E}\big[\, 1_{\{\tau>T\}}\phi_T \,|\, \mathcal{G}_t \,\big] \,|\, \mathcal{F}_t \,\big] = \mathbb{E}\big[\, 1_{\{\tau>T\}}\phi_T \,|\, \mathcal{F}_t \,\big]$$

On the other hand, we have from the definition of pre-default process

$$1_{\{\tau>t\}}\tilde{x}_t = 1_{\{\tau>t\}} \mathbb{E}\big[\, 1_{\{\tau>T\}}\phi_T \,|\, \mathcal{G}_t \,\big]$$

leading to

$$1_{\{\tau>t\}} \mathbb{E}\big[\, 1_{\{\tau>T\}}\phi_T \,|\, \mathcal{G}_t \,\big] = 1_{\{\tau>t\}} \frac{\mathbb{E}\big[\, 1_{\{\tau>T\}}\phi_T \,|\, \mathcal{F}_t \,\big]}{\mathbb{Q}\{\, \tau > t \,|\, \mathcal{F}_t \,\}}$$

Pricing Cash Flows Occurring before the Default Event – II

First Filtration Switching Lemma

In a market with defaultable names, where τ is the first default event, we can price cash flows occurring before the first default event by switching to the market filtration $\mathcal{F}$.

$$1_{\{\tau>t\}}\mathbb{E}\left[\,1_{\{\tau>T\}}\phi_T \,|\, \mathcal{G}_t\,\right] = 1_{\{\tau>t\}}\frac{\mathbb{E}\left[\,\mathbb{Q}\{\,\tau > T \,|\, \mathcal{F}_T\,\}\,\tilde{\phi}_T \,|\, \mathcal{F}_t\,\right]}{\mathbb{Q}\{\,\tau > t \,|\, \mathcal{F}_t\,\}}$$

where ϕ_t is a $\mathcal{G}$-adapted process, and ϕx_t is the corresponding pre-default process. In particular, we have also

$$1_{\{\tau>t\}}\mathbb{Q}\{\,\tau > T \,|\, \mathcal{G}_t\,\} = 1_{\{\tau>t\}}\frac{\mathbb{Q}\{\,\tau > T \,|\, \mathcal{F}_t\,\}}{\mathbb{Q}\{\,\tau > t \,|\, \mathcal{F}_t\,\}}$$

Pricing Cash Flows Occurring on the Default Event – I

- A second useful lemma can be derived for cash flows paid only if a default occurs.
- For any $\mathcal{G}$-adapted process ϕ_t we can proceed as before, but, now, we consider the $\mathcal{G}$-adapted process

$$x_t \doteq \mathbb{E}\left[\, 1_{\{\tau<T\}}\phi_\tau \,|\, \mathcal{G}_t \,\right]$$

leading to

$$1_{\{\tau>t\}}\mathbb{E}\left[\, 1_{\{\tau<T\}}\phi_\tau \,|\, \mathcal{G}_t \,\right] = 1_{\{\tau>t\}}\frac{\mathbb{E}\left[\, 1_{\{t<\tau<T\}}\phi_\tau \,|\, \mathcal{F}_t \,\right]}{\mathbb{Q}\{\, \tau > t \,|\, \mathcal{F}_t \,\}}$$

- As before we wish to remove the explicit dependency on the default event on the right-hand side.

Pricing Cash Flows Occurring on the Default Event – II

- We go on by localizing the default event, and we get

$$1_{\{\tau>t\}}\mathbb{E}\left[\, 1_{\{t<\tau<T\}}\phi_\tau \,|\, \mathcal{F}_t \,\right] = 1_{\{\tau>t\}} \int_t^T \mathbb{E}\left[\, 1_{\{\tau\in du\}}\phi_u \,|\, \mathcal{F}_t \,\right]$$

- To proceed further we require that ϕ_t is also predictable. We obtain

$$1_{\{\tau>t\}}\mathbb{E}\left[\, 1_{\{t<\tau<T\}}\phi_\tau \,|\, \mathcal{F}_t \,\right] = 1_{\{\tau>t\}} \int_t^T du\, \mathbb{E}\left[\, 1_{\{\tau>u\}}\lambda_u\phi_u \,|\, \mathcal{F}_t \,\right]$$

where we define the first-default intensity as the density of the compensator of $1_{\{\tau<t\}}$, namely

$$\lambda_t\, dt := \mathbb{E}\left[\, 1_{\{\tau\in dt\}} \,|\, \mathcal{G}_t \,\right]$$

Pricing Cash Flows Occurring on the Default Event – III

Second Filtration Switching Lemma – First Default

In a market with defaultable names, where τ is the first default event, we can price cash flows occurring on the first default event by switching to the market filtration $\mathcal{F}$.

$$1_{\{\tau>t\}}\mathbb{E}\big[\, 1_{\{\tau<T\}}\phi_T \,|\, \mathcal{G}_t \,\big] = 1_{\{\tau>t\}} \int_t^T du \, \frac{\mathbb{E}\big[\, \mathbb{Q}\{\, \tau > u \,|\, \mathcal{F}_u \,\} \, \tilde{\lambda}_u \tilde{\phi}_u \,|\, \mathcal{F}_t \,\big]}{\mathbb{Q}\{\, \tau > t \,|\, \mathcal{F}_t \,\}}$$

where λ_t is the first-default intensity and ϕ_t is a $\mathcal{G}$-predictable process, while $\tilde{\lambda}_t$ and $\tilde{\phi}_t$ are the corresponding pre-default processes.

Pricing Cash Flows Occurring on the Default Event – IV

- We can formulate the lemma also in the case of the default of one of the counterparty, while the other one remains alive.
- For instance, if the investor defaults before the counterparty we can write

$$x_t \doteq \mathbb{E}\left[\, 1_{\{\tau=\tau_I<T\}} \phi_\tau \,\middle|\, \mathcal{G}_t \,\right] = \mathbb{E}\left[\, 1_{\{\tau<T\}} 1_{\{\tau_I<\tau_C\}} \phi_\tau \,\middle|\, \mathcal{G}_t \,\right]$$

  and the proof follows as in the previous case with the first-default intensity substituted by the investor default intensity, defined as the density of the compensator of $1_{\{\tau=\tau_I<t\}}$, namely

$$\lambda^I_t \, dt := \mathbb{E}\left[\, 1_{\{\tau\in dt\}} 1_{\{\tau_I<\tau_C\}} \,\middle|\, \mathcal{G}_t \,\right]$$

- Notice that, unless some restrictions are set on the default time dependencies, the explicit calculation of the default intensities may be challenging.

Pricing Cash Flows Occurring on the Default Event – V

Second Filtration Switching Lemma – Single-Name Default

In a market with defaultable names, where τ is the first default event, we can price cash flows occurring on the default event of name I by switching to the market filtration $\mathcal{F}$.

$$1_{\{\tau>t\}}\mathbb{E}\left[\,1_{\{\tau=\tau_I<T\}}\phi_T \,\middle|\,\mathcal{G}_t\,\right] = 1_{\{\tau>t\}}\int_t^T du\,\frac{\mathbb{E}\left[\,\mathbb{Q}\{\,\tau>u\,|\,\mathcal{F}_u\,\}\,\tilde{\lambda}^I_u\tilde{\phi}_u\,\middle|\,\mathcal{F}_t\,\right]}{\mathbb{Q}\{\,\tau>t\,|\,\mathcal{F}_t\,\}}$$

where λ^I_t is the default intensity of name I, while other names are still alive, and ϕ_t is a $\mathcal{G}$-predictable process, while $\tilde{\lambda}^I_t$ and $\tilde{\phi}_t$ are the corresponding pre-default processes.

Pricing Cash Flows Occurring on the Default Event – VI

- When the process ϕ_t is adapted, but not predictable, we can proceed as in Duffie (2005).
- We apply the tower rule at a time v before the first default event.

$$x_t \doteq \mathbb{E}\big[\, 1_{\{\tau<T\}}\phi_\tau \,|\, \mathcal{G}_t \,\big] = \lim_{v\to(\tau\wedge T)^-} \mathbb{E}\big[\, \mathbb{E}\big[\, 1_{\{\tau<T\}}\phi_\tau \,|\, \mathcal{G}_v \,\big] \,|\, \mathcal{G}_t \,\big]$$

Then, by means of a lemma by Dellacherie and Meyer (1978) we have that there is a $\mathcal{G}$-predictable process ϕ°_t such that

$$\phi^\circ_{\tau\wedge T} = \lim_{v\to(\tau\wedge T)^-} \mathbb{E}\big[\, 1_{\{\tau<T\}}\phi_\tau \,|\, \mathcal{G}_v \,\big]$$

- Thus, we can proceed as before with

$$x_t = \mathbb{E}\big[\, 1_{\{\tau<T\}}\phi_\tau \,|\, \mathcal{G}_t \,\big] = \mathbb{E}[\, \phi^\circ_{\tau\wedge T} \,|\, \mathcal{G}_t \,] = \mathbb{E}\big[\, 1_{\{\tau<T\}}\phi^\circ_\tau \,|\, \mathcal{G}_t \,\big]$$

and both versions of the previous lemma are still valid with ϕ°_t replacing ϕ_t.

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